Metric contact manifolds and their Dirac operators

Christoph Stadtmüller

Humboldt-Universität zu Berlin
Mathematisch-Naturwissenschaftliche Fakultät II
Institut für Mathematik

This is essentially my "Diplomarbeit" (roughly a master thesis), originally submitted in September 2011, with some changes and corrections made since, last changes made on May 31, 2013

Contents

Introduction

Motivation and summary of results

At the origin of the ideas discussed in this thesis are the closely related notions of CR and contact manifolds. The study of CR manifolds arose as a generalization of a real hypersurface of complex space. Such hypersurfaces were for the first time studied by Henri Poincaré [Poi07], who showed that two real hypersurfaces in $\mathbb{C}^2$ are, in general, not biholomorphically equivalent. This was later generalized to higher dimensions, among others by Chern and Moser [CM74]. An abstract generalization has been introduced since and CR geometry has become an important field of differential geometry, with links to many neighbouring fields. It has applications to partial differential equations but also within the field of geometry, it is, for instance, closely linked to conformal Lorentzian geometry through the Feffermann construction (for an introduction, compare [BJ10, chapter 2.7] or [Bau99]).
An abstract CR manifold is defined as an odd-dimensional manifold equipped with a subbundle of the tangent bundle $H \subset TM$ of codimension one[1] and an integrable almost-complex structure J on this subbundle. Now, if the CR manifold is assumed to be orientable, it admits one-form η whose kernel coincides with H. We can then form the Lévy form $L_\eta = d\eta(\cdot, J\cdot)$. One can show that if this form is nondegenerate, η is a contact form, i.e. $\eta \wedge (d\eta)^m$ vanishes nowhere. If the Lévy form is even positive-definite, we call the CR manifold *strictly pseudoconvex* and we can then equip the manifold with a Riemannian metric $g_\eta = L_\eta + \eta \odot \eta$ and this metric is compatible with the contact structure in a certain sense.
Now, on a strictly pseudoconvex CR manifold, we always have a connection (covariant derivative) which is uniquely determined by the fact that it is metric and the following conditions on its torsion:

$$\begin{aligned} T(X,Y) &= L_\eta(JX,Y)\xi, \\ T(\xi,X) &= -\frac{1}{2}([\xi,X] + J[\xi,JX]) \end{aligned}$$

for any $X, Y \in \Gamma(H)$, where ξ is the Reeb vector field of the contact structure, i.e. uniquely determined by the requirements $\eta(\xi) = 1$ and $\xi \lrcorner d\eta = 0$. The Tanaka-Webster connection was first introduced by Noboru Tanaka in [Tan75] and S.M. Webster in [Web78]. The aim of this thesis is to give alternative descriptions of this connection, particularly through Dirac operators, and to generalize it to general metric contact manifolds (i.e. manifold equipped with a contact structure, an almost-complex structure on the kernel of the contact form and a metric compatible with both) which do not come from CR manifolds.

A first generalization of the Tanaka-Webster connection to general metric contact manifolds is given in the paper [Tan89] by Shukichi Tanno. He describes it as the unique metric connection that makes the contact structure parallel and whose torsion fulfils certain equations. He also gives explicit formulæ for the Christoffel symbols. This generalization is picked up by Robert Petit in [Pet05]. In this paper, he describes how every metric contact manifold admits a canonical $Spin^c$ structure and describes their spinor bundles. He then studies the Dirac operator induced on this spinor bundle by the generalized Tanaka-Webster connection.
In this thesis, we adopt a different generalization, introduced by Liviu Nicolaescu in [Nic05]. This generalization is obtained in the following way: To a metric contact manifold, we associate a manifold $\hat{M} = \mathbb{R} \times M$ which carries an almost-hermitian structure $(\hat{g}, \hat{J})$ extending the one we

[1] A more general theory of CR manifolds with higher codimensions also exists.

have on the contact distribution of M. On this manifold, we can now use the theory of hermitian connections (i.e. metric connections with respect to which the almost-complex structure is parallel), as introduced by Pauline Libermann (cf. [Lib54]) and disucssed by André Lichnerowicz in [Lic55] (in English [Lic76]) and, more recently, by Paul Gauduchon in [Gau97]. Building on Gauduchon's work, Nicolaescu defines a class of basic connection ∇^b on $T\hat{M}$ which is closely related to the first canonical connection introduced in [Lic55]. In particular, he proves that there is one in this class that respects the splitting $T\hat{M} = \partial t \oplus TM$ and thus induces a connection on TM. This induced connection is a contact connection (i.e. it is hermitian and the almost-complex structure on the contact distribution is parallel with respect to it) and we have the following result:

Theorem
The Tanaka-Webster connection of a CR manifold is the restriction to M of a certain basic connection on $\hat{M}$ which respects the splitting $T\hat{M} = \mathbb{R}\partial t \oplus TM$.

Next, let N be any almost-hermitian manifold. Then it admits a canonical *Spin*^{c} structure, whose spinor bundle has a particular form described as follows: The complexified tangent bundle $TN_c = TN \otimes \mathbb{C}$ of the almost-hermitian manifold (N, J, g) splits into the $\pm i$ eigenspaces of the (extended) operator J, which we denote $TN^{1,0}$ and $TN^{0,1}$. We can then form the spaces of (p, q)-forms

$$\Lambda^{p,q}(T^*N) = \Lambda^p((TN^{1,0})^*) \wedge \Lambda^q((TN^{0,1})^*).$$

Now, the spinor bundle $\mathbb{S}^c$ of the canonical *Spin*^{c} structure can be identified with the bundle of $(0, q)$-forms and if N is spin, the spinor bundle of the spin structure has the form $\mathbb{S} \simeq \Lambda^{0,*}(T^*N) \otimes L$, where L is a square-root of the canonical bundle. On either bundle, we have the Hodge-Dolbeault operator $\mathcal{H} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$, where $\overline{\partial}\omega = \mathrm{proj}_{0,q+1} \circ (d\omega)$ for $\omega \in \Gamma(\Lambda^{0,q}(T^*N))$ (with a certain extension to L). Gauduchon compares the Hodge-Dolbeault operator on $\mathbb{S}$ with *geometric Dirac operators*, i.e. Dirac operators induced by a connection on the tangent bundle, in particular the Levi-Civtà connection and the canonical connections. It is then deduced that the Hodge-Dolbeault operator is, up to a Clifford multiplication term, equal to the Dirac operator induced by the Levi-Cività connection ([Gau97, section 3.5]). He then extends this result to the *Spin*^{c} spinor bundle as follows: He considers the *Spin*^{c} bundle locally as a product $\mathbb{S}^c = \mathbb{S} \otimes L^{-1}$, where L^{-1} is a square-root of the anti-canonical bundle and given a connection ∇ on TN (and thus, an induced connection on $\mathbb{S}$) and a connection ∇^L on L^{-1}, he induces one on $\mathbb{S}^c$ as their product. Any such connection then defines a geometric Dirac operator and a similar result as in the spin case is obtained for the Hodge-Dolbeault operator ([Gau97, section 3.6]).
Nicolaescu then again applies this theory to the almost-hermitian manifold $(\hat{M}, \hat{J}, \hat{g})$ associated to a metric contact manifold. He deduces that the Hodge-Dolbeault operator coincides with the Dirac operator induced by the basic connection on $T\hat{M}$ and a canonical connection on L^{-1}. The spinor bundle of M has a similar structure to the one of $\hat{M}$, where we take exterior powers of the complexified contact distribution and it can thus be regarded as a subbundle of the spinor bundle over $\hat{M}$. Nicolaescu then studies the Dirac operator induced by the generalized Tanaka-Webster connection on the spinor bundle over M and compares it with the Hodge-Dolbeault operator.

In this thesis, we follow the approach of Nicolaescu, also discussing the necessary results of Gauduchon. However, we take a different approach to connections on the *Spin*^{c} spinor bundle of an almost-complex manifold. Every *Spin*^{c} structure canonically induces a U_1-bundle P_1 over the same manifold: In fact, this bundle is the det-extension of the unitary frame bundle P_U. Then, a connection ∇ on the tangent bundle together with a connection form Z on P_1 induce a connection on the spinor bundle. In particular, the connection form Z can be induced by a hermitian connection ∇^z on the tangent bundle. Using this approach, we obtain the following result:

Theorem
On the spinor bundle associated to the canonical $Spin^c$ structure of an almost-hermitian manifold, the Hodge-Dolbeault operator coincides with the Dirac operator induced by a basic connection and the connection form Z induced by the same basic connection.

Starting from this relationship, we can deduce a result that compares any geometric Dirac operator with the Hodge-Dolbeault operator. In order to state the formula, we introduce the *potential* of a connection, which is the difference between any metric connection and the Levi-Cività connection.

Theorem
On the spinor bundle associated to the canonical $Spin^c$ structure of an almost-hermitian manifold, let $\mathcal{H}$ be the Hodge-Dolbeault operator and let $\mathcal{D}_c(\nabla, \nabla^z)$ be the geometric Dirac operator induced by the metric connection ∇ with potential A and the connection form Z which, in turn, is induced by the hermitian connection ∇^z with potential A^z. Then these operators satisfy the following formula;

$$\mathcal{D}_c(\nabla, \nabla^z) = \mathcal{H} - \frac{1}{2}c(\mathrm{tr}(A - A^b)) + \frac{1}{2}c(\mathfrak{b}(A - A^b)) - \frac{1}{2}c(\mathrm{tr}_c(A^z - A^b)),$$

where A^b is the potential of a basic connection and c denotes Clifford multiplication.

Explicit formulæ for the cases of the Levi-Cività connection and the canonical connections are deduced.
Again following Nicolaescu, we use this result in the case of a metric contact manifold and the almost-hermitian manifold associated to it. Connections on the spinor bundle associated to the canonical $Spin^c$ structure of a metric contact manifold can be obtained in the same way as for an almost-hermitian manifold. By comparing the spinor bundles and the Dirac operators of the two manifolds, we then deduce the following result:

Theorem
On a metric contact manifold, the Dirac operator induced by the generalized Tanaka-Webster connection and the connection form Z induced by it coincides with a Hodge-Dolbeault-type operator.
If the manifold is CR, the Tanaka-Webster connection is the only contact connection that induces this operator and whose torsion satisfies $g(X, T(Y, Z)) = 0$ for any $X, Y, Z \in \Gamma\mathcal{C}$.

In this thesis, we give a presentation of the above results, supplemented by an introduction to almost-hermitian, metric contact and CR manifolds and their canonical $Spin^c$ structures and a presentation of the results on hermitian connections as discussed in [Gau97].

Structure

In the first two chapters, we present the necessary background on almost-hermitian, metric contact and CR manifolds. We begin in the first chapter with the notion of almost-hermitian manifolds. A large section is dedicated to differential forms on such manifolds, discussing both the spaces of (p, q)-forms and some decompositions of the spaces of three-forms and TM-valued two-forms, which we will need later to describe hermitian connections. The second chapter introduces metric contact and CR manifolds, first seperately and then explaining the relationship between the two.

In the third chapter, we turn our attention to spin and $Spin^c$ structures and their spinor bundles. In the first two sections, we review some facts on spin representations and spin ($Spin^c$) structures. In the following section we then discuss connections on spinor bundles and their Dirac operators and, in particular, investigate how certain properties of the Dirac operators are reflected in the torsion of the connection. In a final section, we consider the cases which are important in this thesis: The canonical $Spin^c$ structures on almost-hermitian and metric contact manifolds. We prove their existence, describe their spinor bundles and discuss connections on the spinor bundles, which can not only be induced in the usual way for spinor bundles, but also as covariant derivatives of differential forms.
In the following chapter, we discuss the theory of hermitian connections, i.e. connections making the almost-complex structure parallel. These connections are completely described by their torsion, which splits into various parts, only some of which actually depend on the chosen connection. Having discussed the general theory, we then introduce canonical and basic connections. In a second section, we apply this theory to a metric contact manifold M and the almost-hermitian manifold $\hat{M}$ associated to it, obtaining a first description of the (generalized) Tanaka-Webster connection.
In the final chapter, we turn our attention to Dirac operators. As in the chapter on connections, we first consider the case of an almost-hermitian manifold and then apply this theory to a metric contact manifold. In particular, we compare the Hodge-Dolbeault operator with geometric Dirac operators.
The five chapters of the main text are supplemented by an appendix, where we collect some results on connections on principal bundles, which we use in the description of connections on spinor bundles.

Notation and conventions

In order to avoid confusion, we establish certain conventions and notation. To begin with, let M and N be manifolds. Throughout this thesis, all manifolds are understood to be smooth (i.e. differentiable of class C^∞) and any mapping $f : M \to N$ will be assumed to be smooth if nothing else is mentioned. The tangent bundle of a manifold M is denoted TM and the tangent space at a point $x \in M$ by T_xM. For the exterior differential on the space of differential forms we agree on the following convention: The exterior differential is defined in such a way that for a k-form ω and vector field $X_0, ..., X_k$

$$\begin{aligned} d\omega(X_0, ..., X_k) = & \sum_{i=0}^{k}(-1)^i X(\omega(X_0, ..., \hat{X}_i, ..., X_k)) \\ & + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k). \end{aligned}$$

Note that this convention agrees with that in most of our sources, but differs from that in [Bla02] and [KN69].

Let furthermore $\pi_P : P \to M$ and $\pi_Q : Q \to M$ be principal or vector bundles over M. in the case of vector bundles, when speaking of a vector bundle morphism $f : P \to Q$, we always mean a smooth map between the manifolds P and Q which is linear on each fibre and which satisfies $\pi_P = \pi_Q \circ f$.

The notation for the following spaces will be used without mention:

$C^\infty(M, N)$		smooth functions from M to N
$\Gamma(P)$	$= \Gamma(M, P)$	sections of P
$\Gamma(U, P)$		local sections over $U \subset M$
$\Gamma_{\text{comp}}(P)$		compactly supported sections
$\mathfrak{X}(M)$	$=\Gamma(TM)$	vector fields over M
$\Omega^k(M)$	$=\Gamma(\Lambda^k(T^*M))$	differential forms of order k on M
$\Omega^k(M, V)$	$= \Gamma(V \otimes \Lambda^k(T^*M))$	differential forms with values in the vector space or vector bundle V
$\Omega^k_{\text{comp}}(M)$		compactly supported differential forms

1

Almost-hermitian manifolds

This is an introductory chapter in which we discuss almost-hermitian structures. While the manifolds we are really interested in are metric contact and CR manifolds, which we will introduce in the following chapter, almost-hermitian structures play an important role because they are closely related to metric contact manifolds.
In the first section of this chapter, we begin with basic definitions of almost-complex structures and almost-hermitian metrics which are metrics compatible with the almost-complex structure and discuss local bases adapted to these structures. In the following section, we discuss the spaces of differential forms on an almost-hermitian manifold which are quite rich in structure and which will be important for describing connections on almost-hermitian manifolds. Finally, we apply this theory to the Kähler and the Lee form, two differential forms canonically associated to an almost-hermitian manifold.

1.1 Almost-complex Structures

Here we introduce the notion of an almost-complex structure, that is a manifold with a vector bundle morphism modelled on multiplication by i on a complex vector space. We begin with the definition of an almost-complex structure and then study the structure of the complexified tangent bundle, before we introduce almost-hermitian metrics, i.e. Riemannian metrics adapted to the almost-complex structure. Throughout this section, let M be a $2n$-dimensional differentiable[2] manifold.

1.1.1 Definition Let $E \to M$ be a real vector bundle over M. A vector bundle endomorphism

$$J : E \longrightarrow E$$

such that $J^2 = -Id$ is called an *almost-complex structure* on E.
The tupel (M, J) where J is an almost-complex structure on TM is then called an almost-complex manifold.

1.1.2 Remark

(1) As (pointwise) $\det J^2 = (-1)^{2m} = 1$, J is an isomorphism.

(2) Obviously, J is modelled on the multiplication by i in the case of a complex manifold. We shall bear this idea in mind for further ideas in connection with almost-complex structures.

Let M be a complex manifold[3]. Then, the multiplication by i induces an almost-complex structure on the (real) tangent bundle of the underlying real manifold. However, not every almost-complex structure is induced by a complex structure. The following tensor may tell us whether the almost-complex structure is indeed induced by an actual complex structure.

1.1.3 Definition Let (M, J) be an almost-complex manifold. The (2,1)-tensor field

$$N(X,Y) = \frac{1}{4}\left([JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]\right)$$

is called *Nijenhuis Tensor* of the almost-complex structure.

[2]By differentiable or smooth, we always mean of class C^∞.

[3]When speaking of a complex manifold we mean a smooth manifold whose transition functions are holomorphic.

Obviously, this tensor vanishes if J is induced by a complex structure. In fact, J being induced by a complex structure is not only sufficient but also necessary. For more details on this, compare section 1.2.1.

Given an almost-complex structure on the tangent bundle, TM may be considered as a complex vector bundle by setting

$$(a+bi)X := aX + bJX.$$

However, there is another way to equip M with a complex vector bundle which is more rich in structure by complexifying the tangent bundle:

$$TM_c = TM \otimes \mathbb{C}.$$

The real tangent bundle can be considered as a subbundle via the following embedding:

$$\begin{aligned} TM &\hookrightarrow TM_c \\ X &\mapsto X \otimes 1. \end{aligned}$$

Obviously, we can also complexify the cotangent bundle and then the following identity holds

$$T^*M_c := (TM_c)^* = T^*M \otimes \mathbb{C}. \tag{1.1}$$

We can now extend the almost-complex structure J to an almost-complex structure on TM_c by setting $J_c(X \otimes z) := (JX) \otimes z$ for $X \in TM, z \in \mathbb{C}$ and demanding it to be $\mathbb{C}$-linear. Note that if we restrict J_c to (the image of) TM it is equal to J.

As opposed to the case where we simply considered TM as a complex vector bundle, J does not coincide with the multiplication by i which gives us the following additional structure on TM_c: Because J_c is a complex-linear operator on TM_c with $(J_c)^2 = -Id$, it can only have eigenvalues $\{\pm i\}$ and TM_c splits (pointwise) into eigenspaces

$$TM_c = E(J_c, i) \oplus E(J_c, -i) =: TM^{1,0} \oplus TM^{0,1}. \tag{1.2}$$

Given a vector $X \in TM$, it splits into its (1,0) and (0,1)-parts as follows:

$$\begin{aligned} X^{1,0} &:= X - iJX \in TM^{1,0}, \\ X^{0,1} &:= X + iJX \in TM^{0,1}, \end{aligned}$$

and both bundles are completely given by elements of this type.

In the same way as the tangent bundle, the cotangent bundle splits as

$$T^*M_c = T^*M^{1,0} \oplus T^*M^{0,1},$$

where $T^*M^{1,0} = (TM^{1,0})^*$ and $T^*M^{1,0} = (TM^{0,1})^*$. Alternatively, one can extend the operator J to the cotangent space via the formula $(J\alpha)(X) = -\alpha(JX)$ for any $\alpha \in T^*M$ and $X \in TM$. Then, the splitting introduced above is again a decomposition into the $\pm i$-eigenspaces of J. Given an element $\alpha \in T^*M$, its parts are given by

$$\begin{aligned} \alpha^{1,0} &:= \alpha + iJ\alpha \in T^*M^{1,0}, \\ \alpha^{0,1} &:= \alpha - iJ\alpha \in T^*M^{0,1}. \end{aligned}$$

We can further introduce a complex conjugation on TM_c by setting

$$\overline{X \otimes z} = X \otimes \overline{z},$$

which gives us that $TM^{0,1}$ is the complex conjugate of $TM^{1,0}$.

1.1.4 Remark Alternatively, one can use such a splitting of the complexified tangent bundle to define an almost-complex structure, i.e. one defines an almost-complex structure as a subbundle $TM^{10} \subset TM_c$ such that $TM^{10} \cap \overline{TM^{10}} = \{0\}$ and $TM^{10} \oplus \overline{TM^{10}} = TM_c$. One then defines the operator J on TM_c by setting it equal to i on TM^{10} and equal to $-i$ on its conjugate. Then, $J^2 = -Id$ and it can be restricted to TM where it is an almost-complex structure in the sense of definition 1.1.1.

1.1.5 Remark An alternative way to define the complexification of TM is to define

$$T_xM_c = \{X + iY | X, Y \in T_xM\}$$

and to define the multiplication by complex numbers in the following, natural way:

$$(a + bi) \cdot (X + iY) = (aX - bY) + i \cdot (bX + aY).$$

We can identify the two spaces as follows

$$\begin{aligned} X + iY &\mapsto X \otimes 1 + Y \otimes i, \\ (aX) + i(bX) &\leftarrow\!\shortmid X \otimes (a + ib). \end{aligned}$$

Then, J extends as $J_c(X+iY) = J(X)+iJ(Y)$ to TM_c. The embedding of TM in this case is trivial and any local real basis of TM is a complex basis of TM_c. Furthermore, the conjugation is given canonically by

$$\overline{X + iY} = X - iY.$$

In what follows, we shall use whichever description of TM_c is more handy.

Almost-hermitian metrics

In order to do geometry on a manifold, we need to equip it with a metric. Bearing in mind that for a hermitian scalar product we have

$$< u, iv >= \bar{i} < u, v >= -i < u, v >= - < iu, v >,$$

we introduce the following notion of a metric compatible with an almost-complex structure:

1.1.6 Definition Let (M, J) be an almost-complex manifold. Then a metric g is called *almost-hermitian* if it is Riemannian and satisfies

$$g_x(X, JY) = -g_x(JX, Y) \quad \text{for any } x \in M,\ X, Y \in T_xM.$$

A tuple (M, g, J) is called an *almost-hermitian manifold* if J is an almost-complex structure on M and g an almost-hermitian metric.

Note that this implies

$$g(JX, JY) = -g(J^2X, Y) = g(X, Y)$$

and that by

$$g(X \otimes z, Y \otimes w) := z\overline{w}g(X, Y)$$

a hermitian metric is induced on TM_c by g.
The Riemannian duality $^\flat : TM \to T^*M$ given by

$$g(Y, X) = X^\flat(Y) \quad \forall X, Y \in T_xM$$

with inverse $^\natural : T^*M \to TM$ given by

$$g(\alpha^\natural, X) = \alpha(X) \qquad \text{for any } X \in T_xM, \alpha \in T_x^*M,$$

can be extended by $\mathbb{C}$-linearity to T_cM and T_c^*M, ie $(X+iY)^\flat = X^\flat + iY^\flat$.
There is a caveat in this extension of duality: It exchanges types, i.e. the dual of a vector of type $(1,0)$ is a covector of type $(0,1)$ and vice versae.

Before closing our discussion of almost-hermitian manifolds, we make some remarks about local coordinates on such a manifold and about the frame bundle of such a manifold. We begin with the local coordinates: Given an almost-complex structure on a vector bundle E, one can always form a so-called *J-adapted* basis of E, i.e. a basis of the form $e_1, f_1, ..., e_m, f_m$ where $Je_j = f_j$ and thus $Jf_j = -e_j$. Moreover, such a basis can always be chosen as an orthonormal one. This is done as follows: Given $e_1, ..., f_{j-1}$, chose e_j normalized and perpendicular to $\operatorname{span}\{e_1, ..., f_{j-1}\}$. Then we set $f_j := Je_j$ and because g is almost-hermitian, we have $g(e_j, f_j) = -g(f_j, e_j)$ and thus the two vectors are orthogonal. Furthermore, for any $k < j$ we have that

$$g(e_k, f_j) = -g(f_k, e_j) = 0 \qquad \text{and} \qquad g(f_k, f_j) = g(e_k, e_j) = 0$$

and therefore, a basis thus constructed is orthonormal and J-adapted.
Furthermore, interpreting e_j, f_j as $e_j \otimes 1, f_j \otimes 1$, such a basis (or, indeed, any basis of TM) can be considered as a basis of TM_c. However, we can also construct another basis of TM_c which splits into bases of $TM^{1,0}$ and $TM^{0,1}$. Define

$$\begin{aligned} z_k &:= \frac{1}{\sqrt{2}}(e_k - if_k) \quad (k = 1, \dots, m), \\ \overline{z_k} &:= \frac{1}{\sqrt{2}}(e_k + if_k) \quad (k = 1, \dots, m). \end{aligned}$$

Then (z_k) is a basis of $TM^{1,0}$ and $(\overline{z_k})$ is a basis of $TM^{0,1}$.
Concerning the dual spaces, denote $(e^1, f^1, ..., e^m, f^m)$ the dual of a J-adapted basis $e_1, ..., f_m$. Then a dual basis of $(z_k, \overline{z_k})$ is given by

$$\begin{aligned} z^k &:= \frac{1}{\sqrt{2}}(e^k + if^k) \quad (k = 1, \dots, m), \\ \overline{z^k} &:= \frac{1}{\sqrt{2}}(e^k - if^k) \quad (k = 1, \dots, m). \end{aligned}$$

We now turn our attention to the frame bundle of an almost-hermitian manifold (M, g, J). If we consider TM as a complex vector bundle, then the metric g is hermitian (this is due to the fact that the multiplication by the imaginary unit is given by J and g is almost-hermitian) and we can form unitary bases of TM. Given such a basis $e_1, ..., e_m$, define $f_j = Je_j$. Because (e_j) is a complex basis and multiplication by i is equivalent to applying J, the set $e_1, f_1, ..., e_m, f_m$ is a real basis of TM. Now, if we set

$$\begin{gathered} (P_U(M))_x = \{e = (e_1, ..., e_m) | e \text{ is a unitary basis of } T_xM\} \\ P_U(M) = \coprod_{x \in M} (P_U(M))_x \end{gathered}$$

and

$$\begin{aligned} f \colon P_U(M) &\longrightarrow P_{GL}(M) \\ (e_1, ..., e_m) &\longmapsto (e_1, Je_1, ..., e_m, Je_m), \end{aligned}$$

then (P_U, f) is a U_m-reduction of $P_{GL}(M)$ (for more details on reductions, compare appendix A.2).

Just like for the other structure groups, the tangent bundle, its dual and its exterior powers can be realized as vector bundles associated to $P_U(M)$ in the usual way, where we define the representation ρ on U_m as $\rho \circ \iota$, where ι is the inclusion of U_m in SO_{2m} and ρ the standard matrix representation on $\mathbb{R}^{2m}$ (cf. appendix A.3). If we want to consider TM as a complex vector bundle, then it is the associated vector bundle $P_U(M) \times_{\rho_c} \mathbb{C}^m$, where ρ_c is the standard matrix action on $\mathbb{C}^m$.

1.2 Differential forms on almost-hermitian manifolds

This is a rather technical section in which we discuss some results on the structure of the space of differential forms on an almost-hermitian manifold: We begin with a short introduction to the spaces of (p,q)-forms and then describe various decompositions of the spaces $\Omega^3(M)$ and $\Omega^2(M,TM)$ and the relationships between them following the work of Paul Gauduchon (cf. [Gau97, section 1]). We will later use these results to describe the torsion of a hermitian connection on an almost-hermitian manifold.
Throughout this section, the manifold considered is an almost-hermitian manifold (M,g,J) of dimension $n=2m$. At each point of M, we denote by $(e_i)_{i=1}^m, (f_i)_{i=1}^m$ a J-adapted basis which, for ease of notation, we shall sometimes call $(b_i)_{i=1}^{2m}$ where $b_{2k-1}=e_k$ and $b_{2k}=f_k$.

1.2.1 The spaces of (p,q)-forms and integrability

The space of differential forms on the complexified tangent bundle of an (almost) complex manifold can be decomposed into certain subspaces induced by the splitting into $\pm i$-eigenspaces of TM_c. To begin with, recall that the complexified cotangent space, splits in the same way as

$$T^*M_c = T^*M^{1,0} \oplus T^*M^{0,1},$$

where $T^*M^{1,0} = (TM^{1,0})^*$ or, alternatively, these two subspaces form the decomposition into $\pm i$-eigenspaces of the operator J extended to T^*M as explained in the preceding section. We shall call the elements of the respective spaces *1,0-forms* and *0,1-forms*. We extend this notion to exterior powers of higher order: The space of differential forms on TM_c

$$\Omega^k_c(M) = \Gamma(\Lambda^k(T^*M_c))$$

splits as follows:

1.2.1 Definition A form $\omega \in \Omega^k_c(M)$ is called of *type p,q* if it is an element of

$$\Omega^{p,q}(M) := \Gamma\left(\Lambda^p(T^*M^{1,0}) \wedge \Lambda^q(T^*M^{0,1})\right).$$

This gives us the decomposition into subspaces

$$\Omega^k_c(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M) \tag{1.3}$$

and, extending the metric to the space of differential forms in the usual way, these subspaces become orthogonal.
Just like the "ordinary" spaces of differential forms, the spaces $\Lambda^{p,q}(T^*M)$ can be realized as vector bundles associated to the frame bundle. In order to do so, we consider the representation

$$\rho_{\Lambda p,q}\colon U_m \to GL(\Lambda^{p,q}((\mathbb{R}^{2m})^*))$$
$$\rho_{\Lambda p,q}(A)(z^{j_1}\wedge ... \wedge z^{j_p} \wedge \overline{z^{k_1}} \wedge ... \wedge \overline{z^{k_q}}) = \rho^*(A)z^{j_1} \wedge ...\rho^*(A)z^{j_p} \wedge \rho^*(A)\overline{z^{k_1}} \wedge ... \wedge \rho^*(A)\overline{z^{k_q}},$$

where $\rho^*(A)$ is extended complex-linearly as follows:

$$\rho^*(A)z^j = \frac{1}{\sqrt{2}}\left(\rho^*(A)e^j + i\rho^*(A)f^j\right)$$

and analogously for $\overline{z^j}$. Then, the following identity holds:

$$\Lambda^{p,q}(T^*M) \simeq P_U(M) \times_{\rho_{\Lambda p,q}} \Lambda^{p,q}((\mathbb{R}^{2m})^*).$$

1.2.2 Remark (Local coordinates) Recall the discussion (cf. the end of section 1.1) of local coordinates (z_j) and $(\overline{z_j})$ for the spaces $TM^{1,0}$ and $TM^{0,1}$ respectively and their duals by (z^j) and $(\overline{z}^j)$. Then any form $\omega \in \Omega^{p,q}(M)$ has local coordinates

$$\omega = \sum_{|I|=p,|J|=q} \omega_{IJ} z^I \wedge \overline{z}^J \quad (\omega_{IJ} \in C^\infty(M, \mathbb{C})).$$

Now, the space of real k-forms can be embedded into the space of complex forms, and thus, every real form admits a decomposition into (p,q)-forms. Notice, however, that even if we begin with a real form, this is a decomposition into *complex* forms.
We shall now study the behaviour of the decomposition into (p,q)-forms under the exterior differential. First, note that the space of complex differential forms can be interpreted as follows:

$$\Omega_c^k(M) \simeq \Omega^k(M) \otimes \mathbb{C}. \tag{1.4}$$

This can be seen by remembering that any real basis of TM also forms a complex basis of TM_c. Then, taking exterior powers, the complex forms are just complex-linear combinations of real forms, which is precisely the meaning of $\Omega^k(M) \otimes \mathbb{C}$.
We can then extend the exterior differential on real k-forms to a complex-linear operator

$$d \otimes \mathrm{Id}_{\mathbb{C}} \colon \Omega_c^k(M) = \Omega^k(M) \otimes \mathbb{C} \longrightarrow \Omega^{k+1}(M) \otimes \mathbb{C} = \Omega_c^{k+1}(M),$$

which, by a slight abuse of notation, we shall again call d.
Now, if we restrict the exterior differential to the space of (p,q)-forms, its image lies in $\Omega_c^{p+q+1}(M)$ which again splits. We now define two operators:

$$\begin{aligned} \partial \colon \Omega^{p,q}(M) &\to \Omega^{p+1,q}(M) \\ w &\mapsto \mathrm{proj}_{\Omega^{p+1,q}}(d\omega), \\ \overline{\partial} \colon \Omega^{p,q}(M) &\to \Omega^{p,q+1}(M) \\ w &\mapsto \mathrm{proj}_{\Omega^{p,q+1}}(d\omega). \end{aligned}$$

It is now tempting to assume that $d = \partial + \overline{\partial}$. This is, however, not true in general. In fact, this property defines a certain class of almost-complex manifolds.

1.2.3 Definition An almost-complex manifold (M, J) (or the almost-complex structure J) is called *integrable* if $d = \partial + \overline{\partial}$

1.2.4 Proposition (cf. [Wel08, Theorem 3.7])
Let M be a complex manifold. Then the almost-complex structure induced by the complex structure is integrable.

In fact, this condition is not only necessary but also sufficient and closely linked to the Nijenhuis tensor as the following theorem by Newlander and Nirenberg ([NN57]) shows. A more recent proof can be found in [Hör66].

1.2.5 Theorem (Newlander-Nirenberg)
Let M be a differentiable manifold and J an almost-complex structure on M. Then the following statements are equivalent:

(1) J is integrable,

(2) $N_J \equiv 0$,

(3) J is induced by a complex structure.

1.2.2 TM-valued 2-forms and 3-forms on an almost-hermitian manifold

We shall study in great detail the spaces $\Omega^2(M,TM)$ and $\Omega^3(M)$ as they will be very useful for describing the torsion of hermitian connections on almost-complex manifolds. We begin by noting that the two spaces are closely related: Every three-form $\omega \in \Omega^3(M)$ defines a TM-valued two-form by

$$g(X,\omega(Y,Z)) = \omega(X,Y,Z) \qquad \text{for any } X,Y,Z \in \mathfrak{X}(M). \tag{1.5}$$

On the other hand, every form $B \in \Omega^2(M,TM)$ defines a trilinear real-valued mapping, skew-symmetric in the last two arguments, by the same formula. We will always take the point of view which seems more useful in the situation and, by a slight abuse of notation, will note both the two-form and the three-form by the same symbol. To avoid some confusion, we agree to separate the first argument from the others by a semicolon if the form is not skew-symmetric in all three arguments, i.e. we write $B(X;Y,Z) = g(X,B(Y,Z))$ for $B \in \Omega^2(M,TM)$.
One can then apply the *Bianchi operator* to make the form totally skew-symmetric:

$$\mathfrak{b}\colon \Omega^2(M,TM) \simeq \Gamma(T^*M \otimes \Lambda^2 T^*M) \longrightarrow \Omega^3(M)$$

which is given by

$$(\mathfrak{b}B)(X,Y,Z) := \frac{1}{3}\left(B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y)\right)$$

for any $X,Y,Z \in \mathfrak{X}(M)$. One immediately deduces the following elementary properties

1.2.6 Lemma
The operator $\mathfrak{b}$ is a projection, i.e. $\mathfrak{b}^2 = \mathfrak{b}$ and for local coordinates (b_j), the following formula holds:

$$\mathfrak{b}(b_j \otimes b^k \wedge b^l) = \frac{1}{3} b^j \wedge b^k \wedge b^l.$$

Proof: The first statement is immediate from the definition. The second statement can be easily checked by applying both forms to a 3-tuple of a local basis. □

The not totally skew-symmetric part of $B \in \Omega^2(M,TM)$ can be further decomposed as follows: Via the trace operator

$$\begin{aligned} \mathrm{tr}\colon \Omega^2(M,TM) &\longrightarrow \Omega^1(M) \\ B &\longmapsto \sum_{j=1}^{n} B(b_j;b_j,\cdot), \end{aligned}$$

we can associate a one-form to B. A one-form can then again be made into a TM-valued two-form by

$$\begin{aligned} \widetilde{\ }\colon \Omega^1(M) &\to \Omega^2(M,TM) \\ \widetilde{\alpha}(X,Y) &:= \frac{1}{n-1}\left(\alpha(Y)X - \alpha(X)Y\right). \end{aligned}$$

Therefore, we have a decomposition

$$\begin{aligned} \Omega^2(M,TM) &= \Omega^1(M) \oplus \Omega^3(M) \oplus \Omega^2_0(M,TM) \\ B &= \widetilde{\mathrm{tr}\, B} + \mathfrak{b}B + B_0, \end{aligned}$$

where

$$\Omega^2_0(M, TM) = \{B \in \Omega^2(M, TM) |\, \mathrm{tr}\, B = 0, \mathfrak{b}B = 0\},$$
$$B_0 := B - \widetilde{\mathrm{tr}\, B} - \mathfrak{b}B \in \Omega^2_0(M, TM).$$

The decomposition we have introduced so far only uses the manifold structure of M, whereas the decomposition we shall introduce now is induced by the almost-complex structure. In due course, we shall then compare the two structures.

1.2.7 Definition A TM-valued two-form $B \in TM \otimes \Lambda^2(T^*M)$ is called

- of <u>type (1,1)</u> if $B(JX, JY) = B(X, Y)$,
- of <u>type (2,0)</u> if $B(JX, Y) = JB(X, Y)$ and
- of <u>type (0,2)</u> if $B(JX, Y) = -JB(X, Y)$.

The respective subspaces of $TM \otimes \Lambda^2(T^*M)$ will be denoted by

$$TM \otimes \Lambda^{1,1}(T^*M),\ TM \otimes \Lambda^{2,0}(T^*M) \text{ and } TM \otimes \Lambda^{0,2}(T^*M)$$

and the respective spaces of sections by

$$\Omega^{1,1}(M, TM),\ \Omega^{2,0}(M, TM) \text{ and } \Omega^{0,2}(M, TM).$$

In the same way, the respective parts of B will be denoted by $B^{1,1}$, $B^{2,0}$ and $B^{0,2}$.

We quickly note some elementary properties of the forms of the various types.

1.2.8 Lemma

(1) Let $B \in \Omega^{1,1}(M, TM)$. Then the following equation holds:

$$B(JX, Y) = -B(X, JY).$$

(2) Let $B \in \Omega^{2,0}(M, TM)$. Then the following equations hold:

$$B(JX, JY) = -B(X, Y),$$
$$B(X, JY) = B(JX, Y).$$

Furthermore,

$$B \in \Omega^{2,0}(M, TM) \quad \textit{if and only if} \quad B(X; JY, Z) = -B(JX; Y, Z).$$

(3) Let $B \in \Omega^{0,2}(M, TM)$. Then the following equations hold:

$$B(JX, JY) = -B(X, Y),$$
$$B(X, JY) = B(JX, Y).$$

Furthermore,

$$B \in \Omega^{0,2}(M, TM) \quad \textit{if and only if} \quad B(X; JY, Z) = B(JX; Y, Z).$$

Proof: (1) $B(JX,Y) = -B(JX,J^2Y) = -B(X,JY)$.
(2) We have

$$B(JX,JY) = JB(X,JY) = -JB(JY,X) = -J^2B(Y,X) = -B(X,Y)$$

and $B(X,JY) = -B(JY,X) = -JB(Y,X) = JB(X,Y) = B(JX,Y)$. The equivalent formulation of being of type $(2,0)$ follows immediately from (1.5) and the fact that $g(J\cdot,\cdot) = -g(\cdot,J\cdot)$.
(3) is analogous to (2). $\square$

From the above lemma, we conclude that the various subspaces of $\Omega^2(M,TM)$ are given by the following local bases, where (e_k,f_k) is a local J-adapted frame in TM and (e^k,f^k) its dual.
$\boxed{\Omega^{1,1}(M,TM)}$ has a basis consisting of the following forms:

$$\begin{aligned} e_l &\otimes \left(e^j \wedge e^k + f^j \wedge f^k\right) \\ f_l &\otimes \left(e^j \wedge e^k + f^j \wedge f^k\right) \\ e_l &\otimes \left(e^j \wedge f^k - f^j \wedge e^k\right) \\ f_l &\otimes \left(e^j \wedge f^k - f^j \wedge e^k\right). \end{aligned} \qquad (j<k, l \in 1,...,m)$$

$\boxed{\Omega^{2,0}(M,TM)}$ has a basis consisting of the following forms:

$$\begin{aligned} e_l \otimes \left(e^j \wedge e^k - f^j \wedge f^k\right) + f_l \otimes \left(e^j \wedge f^k + f^j \wedge e^k\right) \\ f_l \otimes \left(e^j \wedge e^k - f^j \wedge f^k\right) - e_l \otimes \left(e^j \wedge f^k + f^j \wedge e^k\right). \end{aligned} \qquad (j<k, l \in 1,...,m)$$

$\boxed{\Omega^{0,2}(M,TM)}$ has a basis consisting of the following forms:

$$\begin{aligned} e_l \otimes \left(e^j \wedge e^k - f^j \wedge f^k\right) - f_l \otimes \left(e^j \wedge f^k + f^j \wedge e^k\right) \\ f_l \otimes \left(e^j \wedge e^k - f^j \wedge f^k\right) + e_l \otimes \left(e^j \wedge f^k + f^j \wedge e^k\right). \end{aligned} \qquad (j<k, l \in 1,...,m)$$

Using these bases, the following lemma is an easy corollary:

1.2.9 Lemma
The space $\Omega^2(M,TM)$ decomposes into a direct sum of orthogonal (with respect to the metric extended to forms in the usual way) subspaces as

$$\Omega^2(M,TM) = \Omega^{2,0}(M,TM) \oplus \Omega^{1,1}(M,TM) \oplus \Omega^{0,2}(M,TM)$$

and the (pointwise) dimensions of the subspaces are given by

$$\operatorname{rank} TM \otimes \Lambda^{1,1}(T^*M) = 2m^3$$

and

$$\operatorname{rank} TM \otimes \Lambda^{2,0}(T^*M) = \operatorname{rank} TM \otimes \Lambda^{0,2}(T^*M) = m^2(m-1).$$

The subspaces we just discussed are in close relationship with the following linear operator on $\Omega^2(M, TM)$:

$$\begin{aligned} \mathfrak{M}\colon \Omega^2(M, TM) &\to \Omega^2(M, TM) \\ \mathfrak{M}B(X, Y) &:= B(JX, JY). \end{aligned}$$

In fact, we have the following identities

$$\begin{aligned} \Omega^{1,1}(M, TM) &= E(\mathfrak{M}, 1), \\ \Omega^{2,0}(M, TM) \oplus \Omega^{0,2}(M, TM) &= E(\mathfrak{M}, -1), \end{aligned}$$

where the first equality is obvious and in the second equality, the inclusion $\supset$ is given by lemma 1.2.8 and the equality then follows from a dimension argument.

The following lemma, stated in [Gau97, formula (1.3.7)], describes the interplay of the operators $\mathfrak{M}$ and $\mathfrak{b}$:

1.2.10 Lemma
Any $\omega \in \Omega^{1,1}(M, TM) \oplus \Omega^{2,0}(M, TM)$ satisfies the following equality:

$$\omega = 3\mathfrak{b}\mathfrak{M}\omega \tag{1.6}$$

Proof: We have for $\omega \in \Omega^{1,1}(M, TM)$ that

$$\begin{aligned} 3(\mathfrak{b}\mathfrak{M}\omega)(X, Y, Z) &= \omega(X, JY, JZ) + \omega(Y, JZ, JX) + \omega(Z, JX, JY) \\ &= \omega(X, JY, JZ) + \omega(JX, Y, JZ) + \omega(JX, JY, Z) \\ &= \omega(X, Y, Z) + \omega(JY, JY, J^2Z) + \omega(JX, JY, Z) \\ &= \omega(X, Y, Z) \end{aligned}$$

and for $\omega \in \Omega^{2,0}(M, TM)$ that

$$\begin{aligned} 3(\mathfrak{b}\mathfrak{M}\omega)(X, Y, Z) &= \omega(X, JY, JZ) + \omega(Y, JZ, JX) + \omega(Z, JX, JY) \\ &= \omega(X, JY, JZ) + \omega(JX, Y, JZ) + \omega(JX, JY, Z) \\ &= -\omega(JX, Y, JZ) + \omega(JX, Y, JZ) - \omega(J^2X, Y, Z) \\ &= \omega(X, Y, Z). \end{aligned}$$

□

Now, consider a three-form $\omega \in \Omega^3(M)$. On the one hand, ω can be considered as a TM-valued 2-form and admits a decomposition as described above. On the other hand, we can also consider ω as a complex three-form and it thus admits a splitting into (p, q)-forms. However, this is a decomposition into *complex* forms. Yet, certain sums of these forms are again real[4] as we shall see in the sequel.

1.2.11 Definition Let $\omega \in \Omega^3(M)$. We then define

$$\begin{aligned} \omega^+ &:= \omega^{1,2} + \omega^{2,1}, \\ \omega^- &:= \omega^{0,3} + \omega^{3,0}. \end{aligned}$$

The following lemma then compares the decompositions of $\Omega^3(M)$ into forms of type $+/-$ and into forms of type $(1, 1)$, (2,0) and (0,2).

[4] We call a form $\omega \in \Omega^3_c(M)$ real iff $\omega(X, Y, Z) \in \mathbb{R}$ for all $X, Y, Z \in TM$

1.2.12 Lemma
Let $\omega \in \Omega^3(M)$. Then the following hold:

(1) ω^+ and ω^- are real three-forms.

(2) $\omega^{0,2}$ and $\omega^{1,1} + \omega^{2,0}$ are again skew-symmetric in all three arguments.

(3) We have the following identities:

$$\omega^+ = \omega^{2,0} + \omega^{1,1},$$
$$\omega^- = \omega^{0,2}.$$

Proof: (1) To show that $\omega^\pm$ is real, we write ω with respect to the following local basis, where (e_j, f_j) denotes a local J-adapted basis, (e^j, f^j) its dual and, as above, $z^j = \frac{1}{\sqrt{2}}(e^j + if^j)$:

$$\begin{aligned}\omega = \sum_{j<k<l} & \lambda_1^{jkl} z^j \wedge z^k \wedge z^l + \lambda_2^{jkl} \overline{z^j} \wedge z^k \wedge z^l + \lambda_3^{jkl} z^j \wedge \overline{z^k} \wedge z^l + \lambda_4^{jkl} z^j \wedge z^k \wedge \overline{z^l} \\ & + \lambda_5^{jkl} z^j \wedge \overline{z^k} \wedge \overline{z^l} + \lambda_6^{jkl} \overline{z^j} \wedge z^k \wedge \overline{z^l} + \lambda_7^{jkl} \overline{z^j} \wedge \overline{z^k} \wedge z^l + \lambda_8^{jkl} \overline{z^j} \wedge \overline{z^k} \wedge \overline{z^l} \\ + \sum_{j<k} & \lambda_1^{jk} z^j \wedge \overline{z^j} \wedge z^k + \lambda_2^{jk} z^j \wedge \overline{z^j} \wedge \overline{z^k} + \lambda_3^{jk} z^j \wedge \overline{z^k} \wedge z^k + \lambda_4^{jk} \overline{z^j} \wedge \overline{z^k} \wedge z^k.\end{aligned}$$

Obviously, the forms $\omega^\pm$ can be expressed in this basis as follows:

$$\begin{aligned}\omega^+ = \sum_{j<k<l} & \lambda_2^{jkl} \overline{z^j} \wedge z^k \wedge z^l + \lambda_3^{jkl} z^j \wedge \overline{z^k} \wedge z^l + \lambda_4^{jkl} z^j \wedge z^k \wedge \overline{z^l} \\ & + \lambda_5^{jkl} z^j \wedge \overline{z^k} \wedge \overline{z^l} + \lambda_6^{jkl} \overline{z^j} \wedge z^k \wedge \overline{z^l} + \lambda_7^{jkl} \overline{z^j} \wedge \overline{z^k} \wedge z^l \\ + \sum_{j<k} & \lambda_1^{jk} z^j \wedge \overline{z^j} \wedge z^k + \lambda_2^{jk} z^j \wedge \overline{z^j} \wedge \overline{z^k} + \lambda_3^{jk} z^j \wedge \overline{z^k} \wedge z^k + \lambda_4^{jk} \overline{z^j} \wedge \overline{z^k} \wedge z^k\end{aligned}$$

and

$$\omega^- = \sum_{j<k<l} \lambda_1^{jkl} z^j \wedge z^k \wedge z^l + \lambda_8^{jkl} \overline{z^j} \wedge \overline{z^k} \wedge \overline{z^l}.$$

First, we note that the sum with only two indices goes completely to ω^+ and therefore, that part of ω^+ must be real. Now, turning our attention to the parts with three indices, we note that a form $\eta \in \Omega_c^3(M)$ is real iff

$$\eta(V, W, Z) = \overline{\eta(\overline{V}, \overline{W}, \overline{Z})} \tag{1.7}$$

for any $V, W, Z \in TM_c$ (This can easily be checked by writing $V = X_V + iY_V$ with $X_V, Y_V \in TM$ and using linearity). Thus, because ω is real, $\overline{\lambda_1^{jkl}} = \lambda_8^{jkl}$ etc. These properties carry over to $\omega^\pm$ and thus these forms also fulfil (1.7) and are thus real.
(3) Consider ω^-. Then, for $X, Y, Z \in TM$ we have

$$\begin{aligned}0 &= \omega^-(\underbrace{X + iJX}_{\in TM^{0,1}}, \underbrace{Y - iJY}_{\in TM^{1,0}}, Z) \\ &= \omega^-(X, Y, Z) + \omega^-(JX, JY, Z) + i(\omega^-(JX, Y, Z) - \omega^-(X, JY, Z))\end{aligned}$$

and thus

$$\omega^-(JX, Y, Z) = \omega^-(X, JY, Z),$$

which implies $\omega^- \in \Omega^{0,2}(M, TM)$. On the other hand, by the same equation

$$\omega^{0,2}(X + iJX, Y - iJY, Z) = 0$$

and thus $\omega^{0,2}$ is of type $-$. Together, this yields $\omega^- = \omega^{0,2}$.
Now, we have $\omega^- \perp \omega^+$ and $\omega^{0,2} \perp \omega^{1,1} + \omega^{2,0}$ and therefore $\omega^+ = \omega^{1,1} + \omega^{2,0}$.
As $\omega^\pm$ are three-forms and thus skew-symmetric in every argument, (2) follows immediately from (3). □

Notation We denote the subspaces of $\Omega^3(M)$ consisting of forms of type $\pm$ by $\Omega^\pm(M)$.

Next, we want to compare the two decompositions we have just discussed with the following one:

$$\Omega^2(M, TM) = \Omega^1(M) \oplus \Omega^3(M) \oplus \Omega^2_0(M, TM).$$

In particular, we will prove three results, stated in [Gau97, lemmas 1-3 of section 1.4], that study the behaviour of the spaces $\Omega^{1,1}(M, TM)$, $\Omega^{2,0}(M, TM)$ and $\Omega^{0,2}(M, TM)$ under the Bianchi and trace operators.

1.2.13 Lemma
Let $B \in \Omega^{0,2}(M, TM)$. Then the following results hold:

(1) The trace of B vanishes: $\operatorname{tr} B = 0$.

(2) The parts B^0 and $\mathfrak{b}B$ are elements of $\Omega^{0,2}(M, TM)$.

Proof: (1) Using $B(e_i; e_i, \cdot) = -B(f_i; f_i, \cdot)$, we obtain

$$\begin{aligned}\operatorname{tr} B(X) &= \sum_{j=1}^{m} B(e_j; e_j, X) + B(f_j; f_j, X) \\ &= 0.\end{aligned}$$

(2) We have, using lemma 1.2.8, that

$$\begin{aligned}3\mathfrak{b}B(X, JY, Z) &= B(X; JY, Z) + B(JY; Z, X) + B(Z; X, JY) \\ &= B(JX, Y, Z) + B(Y; Z, JX) + B(Z; JX, Y) \\ &= 3\mathfrak{b}B(JX, Y, Z),\end{aligned}$$

which implies $\mathfrak{b}B \in \Omega^{0,2}(M, TM)$. Because $\operatorname{tr} B = 0 \in \Omega^{2,0}(M, TM)$ we get that

$$B^0 = B - \operatorname{tr} B - \mathfrak{b}B \in \Omega^{2,0}(M, TM)$$

and have thus proven everything. □

1.2.14 Lemma
Let $B \in \Omega^{2,0}(M, TM)$. Then the identity

$$B = \frac{3}{2}(\mathfrak{b}B - \mathfrak{M}\mathfrak{b}B) \tag{1.8}$$

holds and the mapping $\mathfrak{b}|_{\Omega^{2,0}} \colon \Omega^{2,0}(M, TM) \to \Omega^+(M)$ is an isomorphism.

Proof: We have that

$$\begin{aligned}\frac{3}{2}(\mathfrak{b}B - \mathfrak{M}\mathfrak{b}B)(X;Y,Z) &= \frac{1}{2}\left(B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y)\right.\\ &\quad \left.-B(X;JY,JZ) - B(JY;JZ,X) - B(JZ;X,JY)\right)\\ &= \frac{1}{2}\left(B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y)\right.\\ &\quad \left.+ B(X;Y,Z) + B(Y;J^2Z,X) + B(Z;X,J^2Y)\right)\\ &= B(X;Y,Z).\end{aligned}$$

This proves (1.8), which implies $\mathfrak{b}B = \frac{3}{2}(\mathfrak{b}B - \mathfrak{b}\mathfrak{M}\mathfrak{b}B)$, which, in turn, is equivalent to $\mathfrak{b}B = 3\mathfrak{b}\mathfrak{M}\mathfrak{b}B$. Hence, for $1 \leq j,k,l \leq m, k < l$, we have

$$\begin{aligned}\mathfrak{b}B(z_j,z_k,z_l) &= \mathfrak{b}B(z_j,Jz_k,Jz_l) + \mathfrak{b}B(z_k,Jz_l,Jz_j) + \mathfrak{b}B(z_l,Jz_j,Jz_k)\\ &\overset{Jz_\cdot = iz_\cdot}{=} -\mathfrak{b}B(z_j,z_k,z_l) - \mathfrak{b}B(z_k,z_l,z_j) - \mathfrak{b}B(z_l,z_j,z_k)\\ &= -3\mathfrak{b}B(z_j,z_k,z_l).\end{aligned}$$

This is equivalent to

$$\mathfrak{b}B(z_i,z_j,z_k) = 0.$$

Analogously, one shows that $\mathfrak{b}B(\overline{z_j},\overline{z_k},\overline{z_l}) = 0$ and thus $\mathfrak{b}B \in \Omega^+(M)$.
Now, $\mathfrak{b}|_{\Omega^{2,0}(M,TM)}$ is injective because

$$\begin{aligned}\mathfrak{b}B(X,JY,JZ) &= \frac{1}{3}(B(X;JY,JZ) + B(JY;JZ,X) + B(JZ,X,JY))\\ &= \frac{1}{3}(-B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y))\\ &= \mathfrak{b}B(X;Y,Z) - 2B(X;Y,Z)\end{aligned}$$

and thus if $\mathfrak{b}B$ is zero, so is B. To show that $\mathfrak{b}|_{\Omega^{2,0}(M,TM)}$ is onto $\Omega^+(M)$, define for $\omega \in \Omega^+(M)$ $B' := \frac{3}{2}(\omega - \mathfrak{M}\omega)$. Then,

$$\begin{aligned}\mathfrak{b}B' &= \frac{3}{2}(\omega - \mathfrak{b}\mathfrak{M}\omega)\\ &= \frac{3}{2}(\omega - \frac{1}{3}\omega) = \omega\end{aligned}$$

Since

$$B'(X;JY,JZ) = \frac{3}{2}(\omega(X,JY,JZ) - \omega(X,Y,Z)) = -B'(X;Y,Z),$$

we must have $B' \in \Omega^{2,0}(M,TM) \oplus \Omega^{0,2}(M,TM)$. Yet, $\mathfrak{b}\left((B')^{0,2}\right) \in \Omega^-(M)$ and $\omega \in \Omega^+(M)$ and therefore $\mathfrak{b}\left((B')^{0,2}\right) = 0$. This implies that if we set $B := (B')^{0,2}$, we have $\mathfrak{b}B = \mathfrak{b}B' = \omega$, which yields surjectivity. $\square$

Note that indeed B and B' as defined in the proof coincide because

$$B = \frac{3}{2}(\mathfrak{b}B - \mathfrak{M}\mathfrak{b}B) = \frac{3}{2}(\omega - \mathfrak{M}\omega) = B'.$$

We stress that we only know this a posteriori, because initially we did not know whether B' thus defined was in $\Omega^{2,0}(M,TM)$.

$\Omega^+(M)$ is not only isomorphic to $\Omega^{2,0}(M,TM)$ but also to a certain subspace of $\Omega^{1,1}(M,TM)$ which we now define:

$$\Omega_s^{1,1}(M,TM) := \{B \in \Omega^{1,1}(M,TM) | \mathfrak{b}B = 0\},$$
$$\Omega_a^{1,1}(M,TM) \text{ is its orthogonal complement in } \Omega^{1,1}(M,TM).$$

Then we have the following result.

1.2.15 Lemma
The mapping $\mathfrak{b}|_{\Omega_a^{1,1}} : \Omega_a^{1,1}(M,TM) \to \Omega^+(M)$ is an isomorphism and we have for any $A \in \Omega_a^{1,1}(M,TM)$ that

$$A = \frac{3}{4}(\mathfrak{b}A + \mathfrak{M}\mathfrak{b}A).$$

Proof: We first show that $\mathfrak{b}A \in \Omega^+(M)$ for any $A \in \Omega^{1,1}(M,TM)$ (note that this is a trivial statement for $A \in \Omega_s^{1,1}(M,TM)$). It is sufficient to prove this for the elements of the basis of $\Omega^{1,1}(M,TM)$ we introduced before. In particular, we show that the elements of that basis are zero on three-tuples of type (z_l, z_j, z_k) and $(\overline{z_l}, \overline{z_j}, \overline{z_k})$. To begin with, we have that

$$\mathfrak{b}\left(e_l \otimes \left(e^j \wedge e^k + f^j \wedge f^k\right)\right)(z_l, z_j, z_k) = \frac{1}{3} e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(e_l, z_j, z_k).$$

Using the definition of z_j, z_k, one obtains that for any $\omega \in \Omega^3(M)$, we have that

$$\omega(e_l, z_j, z_k) = \frac{1}{2}\left(\omega(e_l, e_j, e_k) - \omega(e_l, f_j, f_k) - i(\omega(e_l, e_j, f_k) + \omega(e_l, f_j, e_k))\right),$$
$$\omega(e_l, \overline{z_j}, \overline{z_k}) = \frac{1}{2}\left(\omega(e_l, e_j, e_k) - \omega(e_l, f_j, f_k) + i(\omega(e_l, e_j, f_k) + \omega(e_l, f_j, e_k))\right).$$

Using this, we obtain that

$$\begin{aligned} 2e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(z_l, z_j, z_k) = \quad & e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(e_l, e_j, e_k) \\ & - e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(e_l, f_j, f_k) \\ & - i\left(e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(e_l, e_j, f_k)\right. \\ & \quad \left. + e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(e_l, f_j, e_k)\right) \\ = 0 & \end{aligned}$$

Analogously, one shows

$$e^l \wedge \left(e^j \wedge e^k + f^j \wedge f^k\right)(\overline{z_l}, \overline{z_j}, \overline{z_k}) = 0$$

and that the other elements of the basis are also zero on these tuples. By linearity, this proves that $\mathfrak{b}A \in \Omega^+(M)$ for any $A \in \Omega^{1,1}(M,TM)$.
By definition of $\Omega_s^{1,1}(M,TM)$, we have that $\ker(\mathfrak{b}|_{\Omega^{1,1}} = \Omega_s^{1,1}(M,TM)$ and thus that

$$\mathfrak{b}|_{\Omega_a^{1,1}(M,TM)} : \Omega_a^{1,1}(M,TM) \longrightarrow \mathfrak{b}(\Omega_a^{1,1}(M,TM)) \subset \Omega^+(M)$$

is an isomorphism. To prove that it is an isomorphism onto $\Omega^+(M)$, we still need to prove surjectivity. Let $\omega \in \Omega^+(M)$ and define $A^\omega := \frac{3}{4}(\omega + \mathfrak{M}\omega)$. Then $\mathfrak{b}A^\omega = \frac{3}{4}(\omega + \mathfrak{b}\mathfrak{M}\omega)$ and thus, by lemma 1.2.10 we have $\mathfrak{b}A^\omega = \omega$. We have that A^ω lies in $\Omega^{1,1}(M,TM)$ because

$$\begin{aligned} A^\omega(X; JY, JZ) &= \frac{3}{4}(\omega(X, JY, JZ) + \omega(X, Y, Z)) \\ &= A^\omega(X; Y, Z). \end{aligned}$$

Then $(A^\omega)^{1,1}_a$ lies in $\Omega^{1,1}_a(M,TM)$ and because $\mathfrak{b}(A^\omega)^{1,1}_s$ is zero, we have that $\mathfrak{b}(A^\omega)^{1,1}_a = \omega$ which proves surjectivity. □

Just like in the case of $\Omega^{2,0}(M,TM)$ we deduce that the inverse of $\mathfrak{b}$ on $\Omega^{1,1}_a(M,TM)$ is indeed given by

$$\mathfrak{b}^{-1}\omega = \frac{3}{4}(\omega + \mathfrak{M}\omega).$$

Combining the above results, we obtain the following corollary.

1.2.16 Corollary
The operator $\mathfrak{b}$ *and the decompositions into types "commute" in the following way*

$$(\mathfrak{b}B)^- = (\mathfrak{b}B)^{0,2} = \mathfrak{b}(B^{0,2}) \quad \textit{and} \quad (\mathfrak{b}B)^+ = \mathfrak{b}(B^{1,1} + B^{2,0}) = \mathfrak{b}(B^{1,1}_a + B^{2,0}).$$

Using lemmas 1.2.14 and 1.2.15, we deduce that there exists an isomoprhism

$$\varphi\colon \Omega^{2,0}(M,TM) \longrightarrow \Omega^{1,1}_a(M,TM)$$

given by

$$\varphi(B) = \frac{3}{4}(\mathfrak{b}B + \mathfrak{M}\mathfrak{b}B), \tag{1.9}$$

$$\varphi^{-1}(C) = \frac{3}{2}(\mathfrak{b}C - \mathfrak{M}\mathfrak{b}C). \tag{1.10}$$

for any $B \in \Omega^{2,0}(M,TM)$ and $C \in \Omega^{1,1}_a(M,TM)$.
Finally, we introduce the following variant of the exterior differential that we will use later.

1.2.17 Definition We define the following operator:

$$\begin{aligned} d^c\colon \Omega^2(M) &\to \Omega^3(M) \\ d^c\omega &= -d\omega(J\cdot, J\cdot, J\cdot) \end{aligned}$$

1.3 Properties of the Kähler form and the Nijenhuis tensor

In this section, we introduce the Kähler and Lee form and discuss some of their properties. In particular, we prove a theorem that describes the various parts (as defined in the preceding section) of the Nijenhuis tensor and the covariant derivative of the Kähler form. These results will be important for the description of a hermitian connection later.

1.3.1 Definition The *Kähler form* of an almost-hermitian manifold (M, g, J) is the two-form given by

$$F(X, Y) := g(JX, Y).$$

The *Lee form* is the one-form defined by

$$\theta(X) := \frac{1}{2} \sum_{j=1}^{2m} dF(b_j, Jb_j, X),$$

where (b_j) is a local orthonormal basis.

1.3.2 Lemma
The Lee form is alternatively given by

$$\theta = \frac{1}{2} \operatorname{tr} \mathfrak{M}(d^c F)^+.$$

Proof: We have that

$$\begin{aligned}
\sum_{j=1}^{2m} dF(b_j, Jb_j, X) &= -\sum_{j=1}^{2m} dF(Jb_j, b_j, X) \\
&= -\sum_{j=1}^{2m} dF(Jb_j, J^2 b_j, J^2 X) \\
&= \sum_{j=1}^{2m} (d^c F)(b_j, Jb_j, JX).
\end{aligned}$$

Now recall $d^c F = (d^c F)^+ + (d^c F)^{0,2}$. Using the properties of forms of type $0, 2$ and a J-adapted basis (e_j, f_j), we compute

$$\begin{aligned}
\sum_{j=1}^{m} (d^c F)^{0,2}(e_j, f_j, JX) + (d^c F)^{0,2}(f_j, Jf_j, JX) &= \sum_{j=1}^{m} (d^c F)^{0,2}(f_j, e_j, JX) - (d^c F)^{0,2}(f_j, e_j, JX) \\
&= 0
\end{aligned}$$

and thus conclude

$$\begin{aligned}
\sum_{j=1}^{2m} dF(b_j, Jb_j, X) &= \sum_{j=1}^{2m} (d^c F)(b_j, Jb_j, JX) \\
&= \sum_{j=1}^{2m} (d^c F)^+(b_j, Jb_j, JX) \\
&= \operatorname{tr} \mathfrak{M}(d^c F)^+(X).
\end{aligned}$$

□

The following theorem is a collection of results on the type of the Nijenhuis tensor and the (Levi-Civita-)covariant derivative of the Kähler form, where we understand $\nabla^g F$ as a trilinear mapping, skew-symmetric in the last two arguments (or, alternatively as an element of $\Omega^2(M, TM)$, see the preceding section for this identification) via the following equality

$$(\nabla^g F)(X; Y, Z) = (\nabla^g_X F)(Y, Z).$$

1.3.3 Theorem (cf. [Gau97, Section 2, proposition 1])
For the Nijenhuis tensor the following statements hold:

(N1) N *is of type* $(0, 2)$.

(N2) N *is trace-free and therefore splits as* $N = \mathfrak{b}N + N_0$.

(N3) Applying the Bianchi operator to N *yields* $\mathfrak{b}N = \frac{1}{3}(d^c F)^-$.

For the covariant derivative of the Kähler form we have the following results:

(F1) The (1,1)-part vanishes: $(\nabla^g F)^{1,1} \equiv 0$.

(F2) $(\nabla^g F)^{0,2}$ *and* N *determine each other by*

$$\begin{aligned}(\nabla^g F)^{0,2}(X, Y, Z) &= 2N_0(JX, Y, Z) + \frac{1}{3}(dF)^-(X, Y, Z)\\ &= 2N(JX; Y, Z) + (dF)^-(X, Y, Z),\end{aligned} \tag{1.11}$$

or, equivalently, by

$$(\nabla^g F)^{0,2}(X; Y, Z) = N(JX, Y, Z) + N(JY, X, Z) - N(JZ, X, Y). \tag{1.12}$$

(F3) The (2,0)-part of $\nabla^g F$ *is given by*

$$(\nabla^g F)^{2,0} = \frac{1}{2}\left((dF)^+ - \mathfrak{M}(dF)^+\right).$$

Proof: We begin by proving the elementary properties (N1), (N2) and (F1): We compute

$$\begin{aligned}4N(JY, Z) &= [J^2Y, JZ] - [JY, Z] - J([J^2Y, Z] + [JY, JZ])\\ &= -J[JY, JZ] + J[Y, Z] - [Y, JZ] - [JY, Z] = -J4N(Y, Z),\end{aligned}$$

i.e. $N \in \Omega^{0,2}(M, TM)$ and we have thus proved (N1). That N is traceless follows immediately by Lemma 1.2.13 which proves (N2).
We furthermore compute

$$\begin{aligned}(\nabla^g F)(X; Y, Z) &= (\nabla^g_X F)(Y, Z)\\ &= X(F(Y, Z)) - F(\nabla^g_X Y, Z) - F(Y, \nabla^g_X Z)\\ &= X(g(JY, Z)) + g(\nabla^g_X Y, JZ) - g(JY, \nabla^g_X Z)\\ &= -X(g(Y, JZ)) + X(g(Y, JZ)) - g(Y, \nabla^g_X JZ) - X(g(JY, Z)) + g(\nabla^g_X JY, Z)\\ &= X(g(Y, JZ)) - g(Y, \nabla^g_X JZ) + g(J(\nabla^g_X JY), JZ)\\ &= -X(F(JY, JZ)) + F(\nabla^g_X JY, JZ) + F(JY, \nabla^g_X JZ)\\ &= -(\nabla^g F)(X; JY, JZ).\end{aligned}$$

This yields $\nabla^g F \in \Omega^{2,0}(M,TM) \oplus \Omega^{0,2}(M,TM)$, which proves (F1).
We now prove (F2): First, using the definition of N, we obtain that

$$\begin{aligned}
&4N(JX;Y,Z) + 4N(JY;X,Z) - 4N(JZ;X,Y)\\
=&g(JX,[JY,JZ] - [Y,Z] - J([JY,Z] + [Y,JZ]))\\
&+ g(JY,[JX,JZ] - [X,Z] - J([JX,Z] + [X,JZ]))\\
&- g(JZ,[JX,JY] - [X,Y] - J([JX,Y] + [X,JY])).
\end{aligned}$$

Using that ∇^g is torsion-free and reordering we further compute

$$\begin{aligned}
&4N(JX;Y,Z) + 4N(JY;X,Z) - 4N(JZ;X,Y)\\
=&g(JX,\nabla^g_{JY}JZ) - g(JX,\nabla^g_{JZ}JY) - g(JX,\nabla^g_Y Z) + g(JX,\nabla^g_Z Y) - g(JX,J(\nabla^g_{JY}Z))\\
&+ g(JX,J\nabla^g_Z JY) - g(JX,J\nabla^g_Y JZ) + g(JX,J\nabla^g_{JZ}Y) + g(JY,\nabla^g_{JX}JZ) - g(JY,\nabla^g_{JZ}JX)\\
&- g(JY,\nabla^g_X Z) + g(JY,\nabla^g_Z X) - g(JY,J(\nabla^g_{JX}Z)) + g(JY,J\nabla^g_Z JX) - g(JY,J\nabla^g_X JZ)\\
&+ g(JY,J\nabla^g_{JZ}X) - g(JZ,\nabla^g_{JX}JY) + g(JZ,\nabla^g_{JY}JX) + g(JZ,\nabla^g_X Y) - g(JZ,\nabla^g_Y X)\\
&+ g(JZ,J\nabla^g_{JX}Y) - g(JZ,\nabla^g_Y JX) + g(JZ,J\nabla^g_X JY) - g(JZ,J\nabla^g_{JY}X)\\
=&g(JX,\nabla^g_{JY}JZ) + g(\nabla^g_{JY}JX,JZ) - g(JX,\nabla^g_{JZ}JY) - g(\nabla^g_{JZ}JX,JY) - g(X,\nabla^g_{JY}Z)\\
&- g(\nabla^g_{JY}X,Z) + g(X,\nabla^g_Z JY) + g(\nabla^g_Z X,JY) - g(X,\nabla^g_Y JZ) - g(\nabla^g_Y X,JZ) + g(X,\nabla^g_{JZ}Y)\\
&+ g(\nabla^g_{JZ}X,Y) - g(JX,\nabla^g_Y Z) - g(\nabla^g_Y JX,Z) + g(JX,\nabla^g_Z Y) + g(\nabla^g_Z JX,Y) + g(JY,\nabla^g_{JX}JZ)\\
&- g(\nabla^g_{JX}JY,JZ) - g(Y,\nabla^g_{JX}Z) + g(\nabla^g_{JX}Y,Z) - g(Y,\nabla^g_X JZ) + g(\nabla^g_X Y,JZ)\\
&- g(JY,\nabla^g_X Z) + g(\nabla^g_X JY,Z)\\
=&JY(g(X,Z)) - JZ(g(X,Y)) + JX(g(Y,Z)) - 2g(JZ,\nabla^g_{JX}JY) - JY(g(X,Z)) + Z(g(X,JY))\\
&- Y(g(X,JZ)) + JZ(g(X,Y)) - Y(g(JX,Z)) + Z(g(JX,Y)) - JX(g(Y,Z)) + 2g(Z,\nabla^g_{JX}Y)\\
&- X(g(Y,JZ)) + 2g(JZ,\nabla^g_X Y) - X(g(JY,Z)) + 2g(Z,\nabla^g_X JY)\\
=&-2g(JZ,\nabla^g_{JX}JY) + 2g(Z,\nabla^g_{JX}Y) + 2g(JZ,\nabla^g_X Y) + 2g(Z,\nabla^g_X JY). \qquad (*)
\end{aligned}$$

On the other hand, consider $(\nabla^g F)^{2,0}$. We have that $(\nabla^g F)^{0,2}(X;JY,Z) = (\nabla^g F)^{0,2}(JX;Y,Z)$ and $(\nabla^g F)^{2,0}(X;JY,Z) = -(\nabla^g F)^{2,0}(JX;Y,Z)$. As $(\nabla^g F)^{1,1} = 0$, we obtain

$$\begin{aligned}
2(\nabla^g F)^{0,2}(X;JY,Z) &= (\nabla^g F)(X;JY,Z) + (\nabla^g F)(JX;Y,Z)\\
&= -X(g(Y,Z)) - g(J(\nabla^g_X JY),Z) + g(Y,\nabla^g_X Z)\\
&\quad + (JX)(g(JY,Z)) - g(J(\nabla^g_{JX}Y),Z) - g(JY,\nabla^g_{JX}Z)\\
&= -g(\nabla^g_X Y,Z) + g(\nabla^g_X JY,JZ) + g(\nabla^g_{JX}JY,Z) + g(\nabla^g_{JX}Y,JZ)
\end{aligned}$$

and thus

$$2(\nabla^g F)^{0,2}(X;Y,Z) = g(\nabla^g_X JY,Z) + g(\nabla^g_X Y,JZ) + g(\nabla^g_{JX}Y,Z) - g(\nabla^g_{JX}JY,JZ).$$

Comparing this with $(*)$ yields (1.12).
Next, we prove (N3). We begin by showing that

$$(d^c F)^{k,l}(X,Y,Z) = -(dF)^{k,l}(JX,JY,JZ) =: (dF)^{c,k,l} \quad (k+l=2) \tag{1.13}$$

i.e. in a manner of speaking ^{c} and ^{k,l} commute. One easily verifies that $(dF)^{c,k,l}$ is indeed of type (k,l). The identity

$$\begin{aligned}
&(dF)^{c,2,0}(X,Y,Z) + (dF)^{c,1,1}(X,Y,Z) + (dF)^{c,0,2}(X,Y,Z)\\
=&-(dF)^{2,0}(JX,JY,JZ) - (dF)^{1,1}(JX,JY,JZ) - (dF)^{0,2}(JX,JY,JZ)\\
=&-dF(JX,JY,JZ)\\
=&d^c F(X,Y,Z)
\end{aligned}$$

then proves (1.13). We now use the following result:

$$d\omega(X_0,...,X_k)=\sum_{j=0}^{k}(-1)^j(\nabla^g_{X_j}F)(X_0,..,\widehat{X_j},...,X_k).$$

Should the reader be unfamiliar with this result, a proof can be found at the beginning of section 5.1. Making use of this, we obtain in particular that

$$\begin{aligned}dF(X,Y,Z)&=(\nabla^g_XF)(Y,Z)-(\nabla^g_YF)(X,Z)+(\nabla^g_ZF)(X,Y)\\&=3(\mathfrak{b}\nabla^gF)(X,Y,Z).\end{aligned}\tag{1.14}$$

We then compute

$$\begin{aligned}(d^cF)^{0,2}(X,Y,Z)&=-(dF)^{0,2}(JX,JY,JZ)\\&=(dF)^{0,2}(JX,Y,Z)\\&=3(\mathfrak{b}\nabla^gF)^{0,2}(JX,Y,Z).\end{aligned}$$

Recall from corollary 1.2.16 that $\mathfrak{b}$ and ^{0,2} commute. Thus,

$$\begin{aligned}(d^cF)^{0,2}(X,Y,Z)&=3\mathfrak{b}(\nabla^gF)^{0,2}(JX,Y,Z)\\&\overset{(1.12)}{=}-N(X;Y,Z)+N(JY;JX,Z)-N(JZ;JX,Y)+N(JY;Z,JX)\\&\quad+N(JZ;Y,JX)+N(X;Y,Z)+N(JZ;JX,Y)\\&\quad-N(X;Z,Y)-N(JY;Z,JX)\\&=N(X;Y,Z)+\underbrace{N(JY;JX,Z)}_{=N(Y;Z,X)}+\underbrace{N(JZ;Y,JX)}_{=N(Z,X,Y)}\\&=3\mathfrak{b}N(X,Y,Z),\end{aligned}$$

which proves (N3).
Going back to (F2), we compute, using (1.12) and that N is of type (0,2), that

$$\begin{aligned}(\nabla^gF)^{0,2}(X;Y,Z)&=N(JX;Y,Z)+N(JY;X,Z)-N(JZ;X,Y)\\&=N(JX;Y,Z)-N(Y;Z,JX)-N(Z;JX,Y)\\&=-3\mathfrak{b}N(JX,Y,Z)+2N(JX;Y,Z).\end{aligned}$$

Because N is of type (0,2), so is $\mathfrak{b}N$ and we thus obtain

$$\begin{aligned}(\nabla^gF)^{0,2}(X;Y,Z)&=3\mathfrak{b}N(JX,JY,JZ)+2N(JX;Y,Z)\\&=(d^cF)^{0,2}(JX;JY,JZ)+2N(JX;Y,Z)\\&=(dF)^{0,2}(X,Y,Z)+2N(JX;Y,Z),\end{aligned}$$

which proves the second equation in (1.11). Continuing, we get

$$\begin{aligned}(\nabla^gF)^{0,2}(X;Y,Z)&=(dF)^{0,2}(X,Y,Z)+2N_0(JX;Y,Z)+2\mathfrak{b}N(JX,Y,Z)\\&\overset{(N3)}{=}(dF)^{0,2}(X,Y,Z)+2N_0(JX;Y,Z)+\frac{2}{3}(d^cF)^{0,2}(JX,Y,Z)\\&=(dF)^{0,2}(X,Y,Z)+2N_0(JX;Y,Z)-\frac{2}{3}(d^cF)^{0,2}(JX,JY,JZ)\\&=\frac{1}{3}(dF)^{0,2}(X,Y,Z)+2N_0(JX;Y,Z),\end{aligned}$$

which concludes the proof of (F2).
Finally, for (F3), lemma 1.2.14 yields that

$$(\nabla^g F)^{2,0} = \frac{3}{2}(\mathfrak{b}(\nabla^g F)^{2,0} - \mathfrak{M}\mathfrak{b}(\nabla^g F)^{2,0}).$$

Furthermore,

$$\mathfrak{b}(\nabla^g F)^{2,0} = (\mathfrak{b}\nabla^g F)^{2,0} = \frac{1}{3}(dF)^{2,0} = \frac{1}{3}(dF)^+,$$

where the first identity follows from lemma 1.2.16 and the second from (1.14). Note that the $(1,1)$-parts of $\nabla^g F$ and dF vanish. This yields the claim. □

This concludes our discussion of these forms and with that our introduction to almost-hermitian manifolds. In the following chapter, we will introduce manifolds that admit an almost-complex structure on a subbundle of their tangent bundle. The theory of differential forms on an almost-hermitian manifold developed in this and the preceding section will be used again in the chapter on hermitian connections, in order to describe their torsion.

2

Metric contact and CR manifolds

This is another introductory chapter, this time presenting the structures that will be central to this thesis. We begin with contact and metric contact manifolds. The latter are manifolds with a contact structure, an almost-complex structure on the contact distribution and a metric compatible with both. In a second section, we introduce CR manifolds, which we consider as structures in their own right before adopting the point of view that they are metric contact manifolds whose almost-complex structure is integrable.

2.1 Contact structures

This section serves to introduce contact and metric contact manifolds. It is this kind of manifolds that we will mainly be studying throughout this thesis. We begin by presenting contact structures, their Reeb vector fields and contact distributions and give some examples. We then study contact manifolds which admit a Riemannian metric compatible with the contact structure, the so-called metric contact manifolds. Metric contact manifolds by definition carry an almost-complex structure on their contact distribution and we consider the Lie derivative of this structure in some more detail. In this section, assume that M is a differentiable manifold of odd dimension $n = 2m + 1$.

2.1.1 Definition A *contact structure* on M is a one-form $\eta \in \Omega^1(M)$ such that

$$\eta \wedge (d\eta)^m \neq 0, \tag{2.1}$$

where $\neq$ is to be understood as nowhere vanishing and $(d\eta)^m$ means the wedge product of $(d\eta)$ with itself taken m times.
(M, η) is then called a contact manifold.

Because of the contact condition (2.1), we have in particular that $(d\eta)^m \neq 0$. Therefore, at each point $x \in M$ the dimension of the space $\{v \in T_xM | v \lrcorner d\eta_x = 0\}$ cannot exceed one. Moreover, at each point, we have $2m$ linearly independent vetors $v_1, ..., v_{2m}$ such that $(d\eta_x)^m(v_1, ..., v_{2m}) \neq 0$. Define $\alpha_i = v_i \lrcorner d\eta_x$. Then $\dim \ker \alpha_i = 2m$ and because all these kernels lie in a $2m+1$-dimensional space, their intersection must be at least of dimension one. An element ξ_x of this intersection fulfils $\xi_x \lrcorner d\eta_x = 0$. Furthermore, because of the contact condition, $\eta_x(\xi_x) \neq 0$ and demanding $\eta(\xi_x) = 1$ then uniquely defines ξ_x. The vectors $(\xi_x)_x$ define a smooth vector field, because all the conditions are smooth. The vector field we have just described plays an important role in contact geometry and we shall therefore give it a name.

2.1.2 Definition The *Reeb vector field* of a contact manifold (M, η) is the vector field uniquely determined by

$$\eta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner d\eta = 0.$$

Furthermore, η induces a distribution $\mathcal{C} \subset TM$ ($\mathcal{C}_x = \ker \eta_x$) that we shall call the *contact distribution*. Recall (Frobenius Theorem) that $\mathcal{C}$ is integrable if and only if $\eta \wedge d\eta = 0$. Therefore, we can consider the condition (2.1) as meaning that the contact distribution is "as unintegrable as possible". Note that this means, in particular, that $\mathcal{C}$ is not involutive.
Given $\mathcal{C}$ as the kernel of η and ξ such that $\eta(\xi) = 1$, we note that we can split the tangent bundle into $TM = \mathcal{C} \oplus \mathbb{R}\xi$.

We now consider a first example:

2.1.3 Example We consider $\mathbb{R}^{2m+1}$ whose coordinates we shall call $(x_1, y_1, \ldots, x_m, y_m, z)$ equipped with the one-form

$$\eta = dz - \sum_{i=1}^{m} x_i dy_i.$$

Then, we have

$$\begin{aligned} d\eta &= ddz - \sum_{i=1}^{m} dx_i \wedge dy_i - \sum_{i=1}^{m} x_i ddy_i \\ &= -\sum_{i=1}^{m} dx_i \wedge dy_i \end{aligned}$$

and thus

$$\eta \wedge d\eta = (-1)^m m! dz \wedge dx_1 \wedge dy_1 \ldots \wedge dx_m \wedge dy_m \neq 0. \qquad \Diamond$$

In fact, this contact structure on the standard real space is exemplary for all contact structures as the following theorem shows.

2.1.4 Theorem (Darboux, cf. [Bla02, Theorem 3.1])
Let (M, η) be a contact manifold. Then, locally around every point $p \in M$ there exist local coordinates $(x_1, y_1, \ldots, x_m, y_m, z)$ such that locally

$$\eta = dz - \sum_{i=1}^{m} x_i dy_i.$$

Let us consider one further example:

2.1.5 Example (cf. [Bla02, example 2.3.6])
We consider the three-dimensional torus $\mathbb{T}^3 \simeq {\mathbb{R}^3}/{(2\pi\mathbb{Z})^3}$. First, we consider the following form on $\mathbb{R}^3$:

$$\eta = \sin y \; dx + \cos y \; dz \;\; \in \Omega^1(\mathbb{R}^3).$$

This form is 2π-periodic in every coordinate and thus induces a one-form on the torus. Next, we calculate

$$d\eta = \cos y \; dy \wedge dx - \sin y \; dy \wedge dz$$

and thus

$$\eta \wedge d\eta = -\sin^2 y \; dx \wedge dy \wedge dz + \cos^2 y dz \wedge dy \wedge dx = -dx \wedge dy \wedge dz.$$

This implies that η is indeed a contact form. $\Diamond$

In order to study geometric properties on a contact manifold, we need to introduce a metric on it which we demand to be compatible with the contact structure in the following sense:

2.1.6 Definition A *metric contact manifold* is a tuple (M, g, η, J) with g a Riemannian metric on M, $\eta \in \Omega^1(M)$ and $J \in \mathrm{End}(TM)$ such that

(i) $\|\eta_x\| = 1$ for any $x \in M$,

(ii) $d\eta(X, Y) = g(JX, Y)$ for any $X, Y \in \mathfrak{X}(M)$ and

(iii) $J^2 X = -X + \eta(X)\xi$ for any $X \in \mathfrak{X}(M)$, where ξ is the metric dual of η.

2.1.7 Lemma
Let (M, g, η, J) be a metric contact manifold. Then

$$\eta \wedge (d\eta)^n \neq 0,$$

i.e. (M, η) is contact: Furthermore, we have that $\xi = \eta^\natural$ is the Reeb vector field and fulfils that $J\xi = 0$.

Proof: We first prove that ξ fulfils the conditions for a Reeb vector field. Obviously $\eta(\xi) = \|\eta\|^2 = 1$. Furthermore, we have that $J^2\xi = -\xi + \eta(\xi)\xi = 0$. From this we conclude that $0 = g(J^2\xi, \xi) = -g(J\xi, J\xi)$ implying that $J\xi = 0$. Thus, we obtain $d\eta(\xi, \cdot) = g(J\xi, \cdot) = 0$.
Let, as above, $\mathcal{C}_x = \ker \eta_x$. Then for $X \in \mathcal{C}$ there holds $J^2X = -X$, i.e. J is an almost-complex-structure on $\mathcal{C}$. We pick a local basis $(e_1, f_1 ..., e_m, f_m, \xi)$ such that the first $2m$ elements are orthonormal and J-adapted and denote $(e^1, f^1, ..., e^m, f^m, \eta)$ its dual. We then have

$$d\eta(e_i, f_j) = g(Je_i, f_j) = g(f_i, f_j) = \delta_{ij}.$$

As $\xi \lrcorner d\eta = 0$, we have

$$d\eta = \sum_{i=1}^{m} e^i \wedge f^i$$

and thus

$$\eta \wedge (d\eta)^m = m!\eta \wedge e^1 \wedge ... \wedge f^m \neq 0.$$ □

On a contact manifold, one can also define a Nijenhuis tensor:

2.1.8 Definition The *contact Nijenhuis tensor* is defined by

$$N(X, Y) = \frac{1}{4}\left([JX, JY] + J^2[X, Y] - J([JX, Y] + [X, JY])\right).$$

Note that because $J^2 \neq -Id$, this tensor differs slightly from the one defined for almost-complex structures. We stated that the Nijenhuis tensor of an almost-complex manifold vanishes if and only if the almost-complex structure is integrable. No such "easy" interpretation can be given in the case of a contact manifold and we refer the reader to the following section for an introduction to CR manifolds, which are, in a certain way, contact manifolds on which the almost-complex structure on the contact distribution is "integrable".
The relationship between metric and contact structure on such a manifold is very close as the following lemma shows:

2.1.9 Lemma
On a metric contact manifold (M, g, η, J), the metric g is completely determined by η and J by

$$g = \eta \otimes \eta + d\eta(\cdot, J\cdot).$$

Proof: Fix $x \in M$. Then for $u, v \in \mathcal{C}_x$ we see that

$$\eta \otimes \eta(u, v) + d\eta(u, Jv) = -d\eta(Jv, u) = -g(J^2v, u) = g(v, u).$$

Furthermore, for $u \in T_xM$, we have

$$\eta \otimes \eta(\xi, u) + d\eta(\xi, Ju) = \underbrace{\eta(\xi)}_{=1}\eta(u) = g(\xi, u).$$

An analogous argument for u in the first argument concludes the proof. □

We conclude this section with some auxiliary results on the operator J and its Lie derivative $\phi = \mathcal{L}_\xi J$ which will be useful later. Recall that the Lie derivative of an endomorphism F of the tangent bundle in the direction of a vector field $X \in \mathfrak{X}(M)$ is defined as follows

$$\mathcal{L}_X F(Y) = \mathcal{L}_X(F(Y)) - F(\mathcal{L}_X Y) = [X, F(Y)] - F([X, Y]).$$

2.1.10 Lemma (cf. [Bla02, Lemma 6.1 and Corollary 6.1])
Let (M, g, η, J) be a metric contact manifold. Then, for the Levi-Civita-covariant derivative of J the following formula holds:

$$2g((\nabla^g_X J)Y, Z) = g(JX, 4N(Y, Z)) + d\eta(JY, X)\eta(Z) + d\eta(X, JZ)\eta(Y).$$

In particular, we have $\nabla^g_\xi J = 0$.

Proof: Recall that $J^2 X = -X + \eta(X)\xi$ and thus

$$\begin{aligned} g(X, Y) &= g(JX, JY) + \eta(X)\eta(Y) \\ &= d\eta(X, JY) + \eta(X)\eta(Y) = d\eta(Y, JX) + \eta(X)\eta(Y). \end{aligned}$$

Then, using the Koszul formula for ∇^g and the relationship between g and $d\eta$, we obtain

$$\begin{aligned} 2g((\nabla^g_X J)Y, Z) &= 2g(\nabla^g_X(JY), Z) + 2g(\nabla^g_X Y, JZ) \\ &= X(g(JY, Z)) + JY(g(X, Z)) - Z(g(X, JY)) \\ &\quad + g([X, JY], Z) + g([Z, X], JY) - g([JY, Z], X) \\ &\quad + X(g(Y, JZ)) + Y(g(X, JZ)) - JZ(g(X, Y)) \\ &\quad + g([X, Y], JZ) + g([JZ, X], Y) - g([Y, JZ], X) \\ &= X(d\eta(Y, Z)) + JY(d\eta(X, JZ)) + JY(\eta(X)\eta(Z)) - Z(d\eta(Y, X)) \\ &\quad + d\eta([X, JY], JZ) + \eta([X, JY])\eta(Z) + d\eta(Y, [Z, X]) \\ &\quad - d\eta(X, J[JY, Z]) - \eta(X)\eta([JY, Z]) + X(d\eta(Z, Y)) + Y(d\eta(Z, X)) \\ &\quad - JZ(d\eta(X, JY)) - JZ(\eta(X)\eta(Y)) + d\eta(Z, [X, Y]) + d\eta([JZ, X], JY) \\ &\quad + \eta([JZ, X])\eta(Y) - d\eta([Y, JZ], JX) - \eta([Y, JZ])\eta(X). \end{aligned}$$

Now, using that

$$\begin{aligned} 0 = dd\eta(A, B, C) &= A(d\eta(B, C)) - B(d\eta(A, C)) + C(d\eta(A, B)) \\ &\quad - d\eta([A, B], C) + d\eta([A, C], B) - d\eta([B, C], A) \end{aligned}$$

for any vector fields $A, B, C \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned} 2g((\nabla^g_X J)Y, Z) &= d\eta([Y, Z], X) - d\eta([JY, JZ], X) + JY(\eta(X)\eta(Z)) - JZ(\eta(X)\eta(Y)) \\ &\quad + \eta([X, JY])\eta(Z) - d\eta(X, J[JY, Z]) - \eta([JY, Z])\eta(X) + \eta([JZ, X])\eta(Y) \\ &\quad - d\eta([Y, JZ], JX) - \eta([Y, JZ])\eta(X) \\ &= d\eta([Y, Z], X) - d\eta([JY, JZ], X) + JY(\eta(X))\eta(Z) + \eta(X)JY(\eta(Z)) \\ &\quad - JZ(\eta(X))\eta(Y) - \eta(X)JZ(\eta(Y)) + \eta([X, JY])\eta(Z) - d\eta(X, J[JY, Z]) \\ &\quad - \eta([JY, Z])\eta(X) + \eta([JZ, X])\eta(Y) - d\eta([Y, JZ], JX) - \eta([Y, JZ])\eta(X). \end{aligned}$$

Then using that for any vector fields $A, B \in \mathfrak{X}(M)$

$$d\eta(A, JB) = A(\underbrace{\eta(JB)}_{=0}) - JB(\eta(A)) - \eta([A, JB]),$$

we obtain

$$\begin{aligned}2g((\nabla^g_X J)Y,Z) &= d\eta([Y,Z],X) - d\eta([JY,JZ],X) + d\eta(JY,X)\eta(Z) + d\eta(JY,Z)\eta(X)\\ &\quad - JZ(\eta(X))\eta(Y) + d\eta(Y,JZ)\eta(X) - d\eta(X,J[JY,Z]) + \eta([JZ,X])\eta(Y)\\ &\quad - d\eta([Y,JZ],JX)\\ &= -g(JX,[Y,Z]) + g(JX,[JY,JZ]) + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y)\\ &\quad \eta(X)(d\eta(JY,Z) + d\eta(Y,JZ)) - d\eta(X,J[JY,Z]) + d\eta(J[Y,JZ],X)\\ &= -g(JX,[Y,Z]) + g(JX,[JY,JZ]) - g(JX,J[JY,Z]) - g(JX,J[Y,JZ])\\ &\quad + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y)\end{aligned}$$

Then, because $J^2 = -\operatorname{Id} + \eta \otimes \xi$, we obtain

$$\begin{aligned}2g((\nabla^g_X J)Y,Z) &= g(JX,J^2[Y,Z]) - \eta([Y,Z])\eta(JX) + g(JX,[JY,JZ]) - g(JX,J[JY,Z])\\ &\quad - g(JX,J[Y,JZ]) + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y)\\ &= g(JX,4N(Y,Z)) + d\eta(JY,X)\eta(Z) + d\eta(X,JZ)\eta(Y),\end{aligned}$$

which proves the claim. In particular, choosing $X = \xi$ we obtain

$$2g((\nabla^g_\xi J)(Y),Z) = g(J\xi,4N(Y,Z)) + d\eta(JY,\xi)\eta(Z) + d\eta(\xi,JZ)\eta(Y) = 0$$

and have thus proven everything. □

2.1.11 Lemma (cf. [Bla02, Lemma 6.2])
Let (M,g,η,J) be a metric contact manifold. Then the operator $\phi = \mathcal{L}_\xi J$ is trace-free and symmetric and we have $J\phi = -\phi J$.

Proof: We first prove an auxiliary result: $\nabla^g_\xi \xi = 0$. Note that

$$\mathcal{L}_\xi \eta = d(\eta(\xi)) + \xi \lrcorner d\eta = 0$$

and thus

$$\begin{aligned}0 = \mathcal{L}_\xi\eta(X) &= \xi(\eta(X)) - \eta([\xi,X])\\ &= g(\nabla^g_\xi\xi,X) + g(\xi,\nabla^g_\xi X) - \eta(\nabla^g_\xi X - \nabla^g_X\xi)\\ &= g(\nabla^g_\xi\xi,X) - g(\xi,\nabla^g_X\xi).\end{aligned}$$

Noting that ξ is a vector field of constant length and thus $g(\xi,\nabla^g_X\xi) = 0$, this yields the claimed equation.
Furthermore, we have $\nabla^g_\xi J = 0$ and thus

$$\begin{aligned}g((\mathcal{L}_\xi J)(X),Y) &= g(\nabla_\xi(JX) - \nabla^g_{JX}\xi - J(\nabla^g_\xi X) + J(\nabla^g_X\xi),Y)\\ &= g(\underbrace{(\nabla^g_\xi J)(X)}_{=0} - \nabla^g_{JX}\xi + J(\nabla^g_X\xi),Y).\end{aligned}$$

If $X = \xi$, this is zero. The same holds for $Y = \xi$ because

$$g(-\nabla^g_{JX}\xi + J\nabla^g_X\xi,\xi) = -(JX)(\underbrace{g(\xi,\xi)}_{=const}) + \underbrace{g(\xi,\nabla^g_{JX}\xi)}_{=0} - g(\nabla^g_X\xi,\underbrace{JX}_{=0}) = 0.$$

Thus, we now consider $X, Y \in \xi^\perp$. Then, we have

$$\begin{aligned} g((\mathcal{L}_\xi J)(X), Y) &= g(-\nabla^g_{JX}(\xi), Y) - g(\nabla^g_X \xi, JY) \\ &= -(JX)(g(\xi, Y)) + g(\xi, \nabla^g_{JX} Y) - X(g(\xi, JY)) + g(\xi, \nabla^g_X JY) \\ &= \eta(\nabla^g_{JX} Y) + \eta(\nabla^g_X JY). \end{aligned}$$

Because $X, Y \in \xi^\perp = \mathcal{C}$, we have that

$$\begin{aligned} d\eta(X, Y) &= X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \\ &= -\eta([X, Y]). \end{aligned}$$

Thus, we have

$$\begin{aligned} \eta([JX, Y] + [X, JY]) &= -d\eta(JX, Y) - d\eta(X, JY) \\ &= 0 \end{aligned}$$

and hence

$$\begin{aligned} g((\mathcal{L}_\xi J)(X), Y) &= \eta(\nabla^g_{JX} Y) + \eta(\nabla^g_X JY) \\ &= \eta(\nabla^g_Y JX) + \eta(\nabla^g_{JY} X). \end{aligned}$$

Arguing as above, the right hand side is equal to $g(X, (\mathcal{L}_\xi J)(Y))$, which proves symmetry. Next, by the preceding lemma, we have

$$\begin{aligned} 2g((\nabla^g_X J)(\xi), Z) &= g(JX, 4N(\xi, Z)) + d\eta(X, JZ) \\ &= g(JX, J^2[\xi, Z] - J[\xi, JZ]) + d\eta(X, JZ) \\ &= -g(JX, J(\mathcal{L}_\xi J)(Z)) + g(JX, JZ) \end{aligned}$$

Using the formula for J^2, we deduce that

$$g(JX, JY) = -g(J^2 X, Y) = g(X, Y) - \eta(X) g(\xi, Y) = g(X, Y) - \eta(X)\eta(Y)$$

and use this to compute

$$\begin{aligned} 2g((\nabla^g_X J)(\xi), Z) &= -g(X, (\mathcal{L}_\xi J)(Z)) + \eta(X)\eta((\mathcal{L}_\xi J)(Z)) + g(Z, X) - \eta(Z)\eta(X) \\ &= -g((\mathcal{L}_\xi J)(X), Z) + g(Z, X) - g(\eta(X)\xi, Z), \end{aligned}$$

where the last equation follows because the symmetry of ϕ implies that

$$\eta((\mathcal{L}_\xi J)(Z)) = g((\mathcal{L}_\xi J)(\xi), Z) = 0.$$

Therefore, we obtain the following equivalent statements:

$$\begin{aligned} (\nabla^g_\xi J)(Y) &= -\frac{1}{2}(\mathcal{L}_\xi J)(X) + \frac{1}{2}X - \frac{1}{2}\eta(X)\xi, \\ -J(\nabla^g_X \xi) &= -\frac{1}{2}(\mathcal{L}_\xi J)(X) + \frac{1}{2}X - \frac{1}{2}\eta(X)\xi - \nabla^g_X(\underbrace{J\xi}_{=0}), \\ \nabla^g_X \xi &= -\frac{1}{2}J(\mathcal{L}_\xi J)(X) + \frac{1}{2}JX + \underbrace{\eta(\nabla^g_X \xi)}_{=g(\xi, \nabla^g_X \xi)=0}\ \xi, \\ \nabla^g_X \xi &= \frac{1}{2}J\phi X + \frac{1}{2}JX. \end{aligned} \tag{2.2}$$

Therefore, we obtain

$$\begin{aligned}
g(X, JY) &= d\eta(Y, X) \\
&= Y(\eta(X)) - X(\eta(Y)) - \eta([Y, X]) \\
&= g(\nabla^g_Y X, \xi) + g(X, \nabla^g_Y \xi) - g(\nabla^g_X Y, \xi) - g(Y, \nabla^g_X \xi) - g(\nabla^g_Y X, \xi) + g(\nabla^g_X Y, \xi) \\
&= g(X, \nabla^g_Y \xi) - g(Y, \nabla^g_X \xi) \\
&\overset{(2.2)}{=} \frac{1}{2} g(X, J\phi Y + JY) - g(Y, J\phi X + JX) \\
&= \frac{1}{2}(g(X, J\phi Y) - g(Y, J\phi X)) + g(X, JY),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &= (g(X, J\phi Y) - g(Y, J\phi X)) \\
&= g(X, J\phi Y) + g(JY, \phi X) = g(X, J\phi Y) + g(X, \phi JY),
\end{aligned}$$

which implies $J\phi = -\phi J$.
Concerning the trace, assume that λ is an eigenvalue of ϕ with eigenvector X. Then $\phi JX = -J\phi X = -\lambda JX$, i.e. $-\lambda$ is also an eigenvalue of ϕ, with eigenvector JX. This implies the tracelessness. □

This concludes our introduction to contact manifold. In the following section, we introduce CR manifolds, which may be considered as contact manifold whose almost-complex structure on the contact distribution is integrable. We will come back to metric contact manifolds in later chapters, when we discuss their spin structures, connections and Dirac operators.

2.2 CR structures

CR manifolds are manifolds with an integrable almost-complex structure on a subbbundle of their tangent space. They are modelled on a real hypersurface in standard complex space $\mathbb{C}^n$. In this section, we will introduce these structures, first through the motivating example of a real hypersurface and then as an abstract structure. We will pay particular attention to the relationship between CR and contact manifolds. Much of this introduction is inspired by the first chapter of [Jac90].
Let $M^{2m+1} \subset \mathbb{C}^{m+1}$ be a real hypersurface. We would then like to induce an (almost-)complex structure on its tangent space. However, TM is not stable under the complex structure of the surrounding complex space. Therefore, we want to find a certain subspace of the tangent space that is stable under the complex structure: We consider the *stable tangent space* of M:

$$H_x := T_xM \cap J(T_xM),$$

where J denotes the standard almost-complex structure of the complex space $\mathbb{C}^{m+1}$. Then we have the following result:

2.2.1 Lemma
Let $M^{2m+1} \subset \mathbb{C}^{m+1}$ be a real hypersurface. Then the following properties of its stable tangent space hold:

(1) $\dim_{\mathbb{R}} H_x = 2m$

(2) $J(H) \subset H$ *and* $(J|_H)^2 = -Id$

(3) For all $X, Y \in \Gamma(H)$ *we have that* $[JX, Y] + [X, JY] \in \Gamma(H)$ *and*

$$J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]. \tag{2.3}$$

Proof: (1) As J is an isomorphism of $\mathbb{C}^{m+1}$, the dimensions of T_xM and $J(T_xM)$ must agree. T_xM cannot be preserved by J as it is of odd (real) dimension and thus, by a dimension argument, the intersection of T_xM and its image under J must be $2m$. (2) is obvious.
(3) The second equation follows from the integrability of J. Obviously, $[JX, Y] + [X, JY] \in \Gamma(TM)$. But from (2.3), on sees that it is also in $\Gamma(J(TM))$ and thus in $\Gamma(H)$ □

One now uses these properties to define an abstract CR manifold:

2.2.2 Definition A *(real) CR structure* on a smooth manifold of odd dimension $n = 2m + 1$ is a pair (H, J) such that

(i) $H \subset TM$ is a subbundle of rank $2m$

(ii) $J : H \to H$ is an almost-complex structure

(iii) For any $X, Y \in \Gamma(H)$, the following holds:

- $[X, JY] + [JX, Y] \in \Gamma(H)$,
- $J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0$.

As an example we consider the so-called Sasakian manifolds. They play an important role in the study of Killing spinors, as every manifold which is Sasaki, Einstein and spin admits a real Killing spinor.

2.2.3 Example A Riemannian manifold (M^{2m+1}, g) together with a Killing vector field ξ is called a *Sasaki manifold* if it satisfies the following conditions:

(a) $g(\xi,\xi)=1$,

(b) $\psi := -\nabla^g\xi$ satisfies $\psi^2 X = -X + g(X,\xi)\xi$,

(c) $(\nabla^g_X\psi)(Y) = g(X,Y)\xi - g(Y,\xi)X \quad$ for any $X, Y \in \mathfrak{X}(M)$.

Then, setting $H = \xi^\perp$ and $J = \psi|_H$, we obtain a CR structure (H, J). This can be seen as follows: Obviously, H is a subbundle of rank $2m$. Furthermore, if $X \in \Gamma(H)$, then $g(X,\xi) = 0$ and thus, (b) implies that J is an almost-complex structure on H. It remains to check (iii) in the definition of a CR manifold. Let $X, Y \in \Gamma(H)$. Then we obtain that

$$g([X, JY] + [JX, Y], \xi) = g(\nabla^g_X JY - \nabla^g_{JY}X + \nabla^g_{JX}Y - \nabla^g_Y JX, \xi).$$

We have that[5] $g(\nabla^g_X JY) = g((\nabla^g_X J)Y + J(\nabla^g_X Y), \xi)$. But, for any vector field $Z \in \mathfrak{X}(M)$, we have that $g(J(Z),\xi) = g(\nabla^g_Z\xi,\xi) = 0$, because ξ has constant length. Therefore, we have $g(\nabla^g_X JY) = g((\nabla^g_X J)Y, \xi)$ and hence, we obtain

$$g([X, JY] + [JX, Y], \xi) = g((\nabla^g_X J)Y - (\nabla^g_Y J)X - \nabla^g_{JY}(X) + \nabla^g_{JX}Y, \xi).$$

Now, we use property (c) and obtain

$$\begin{aligned} g([X, JY] + [JX, Y], \xi) &= g(g(X,Y)\xi - g(Y,\xi)X - g(X,Y)\xi + g(X,\xi)Y - \nabla^g_{JY}(X) + \nabla^g_{JX}Y, \xi) \\ &= -g(Y,\xi)g(X,\xi) + g(X,\xi)g(Y,\xi) - g(\nabla^g_{JY}X - \nabla^g_{JX}Y, \xi) \\ &= -g(\nabla^g_{JY}X - \nabla^g_{JX}Y, \xi). \end{aligned}$$

Using that ∇^g is metric, $H = \xi^\perp$ and the definition of ψ, we deduce

$$\begin{aligned} -g(\nabla^g_{JY}X - \nabla^g_{JX}Y, \xi) &= g(X, \nabla^g_{JY}\xi) - g(Y, \nabla^g_{JX}\xi) \\ &= -g(X, J^2Y) + g(Y, J^2X) \\ &= +g(X,Y) - g(Y,X), \end{aligned}$$

where the last equality follows because J is an almost-complex structure on H. Thus, $g([X, JY] + [JX, Y], \xi) = 0$, i.e. $[X, JY] + [JX, Y] \in \Gamma(H)$ for any $X, Y \in \Gamma(H)$.
We still need to prove that the integrability condition is fulfilled. Because $(\nabla^g_X J)Y = \nabla^g_X(JY) - J(\nabla^g_X Y)$, we obtain

$$(\nabla^g_X J)Y - (\nabla^g_Y J)X = J([Y,X]) + \nabla^g_X(JY) - \nabla^g_Y(JX).$$

Analogously,

$$(\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X = [JX, JY] - J(\nabla^g_{JX}Y - \nabla^g_{JY}X).$$

Therefore, we have

$$\begin{aligned} &(\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X - J((\nabla^g_X J)Y - (\nabla^g_Y J)X) \\ =&[JX, JY] + J^2[X,Y] - J(\nabla^g_{JX}Y - \nabla^g_{JY}X + \nabla^g_X JY - \nabla^g_Y JX) \\ =&[JX, JY] - [X,Y] - J([JX,Y] + [X,JY]) + g([X,Y],\xi)\xi. \end{aligned}$$

[5] In the computations, we will apply J to some vectors which are not necessarily in H. In that case, we assume J to be extended by ψ for the purpose of this calculation.

On the other hand, by property (c), we have

$$\begin{aligned}
&(\nabla^g_{JX}J)Y-(\nabla^g_{JY}J)X-J((\nabla^g_X J)Y-(\nabla^g_Y J)X)\\
=&g(JX,Y)\xi-g(Y,\xi)JX-g(JY,X)\xi+g(X,\xi)JY-J(g(X,Y)\xi-g(Y,\xi)X\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-g(X,Y)\xi+g(X,\xi)Y)\\
=&\left(g(JX,Y)-g(JY,X)\right)\xi\\
=&\left(-g(\nabla^g_X\xi,Y)+g(\nabla^g_Y\xi,X)\right)\xi.
\end{aligned}$$

Then, because ∇^g is metric and $X,Y\perp\xi$, we have

$$\left(-g(\nabla^g_X\xi,Y)+g(\nabla^g_Y\xi,X)\right)=-g([X,Y],\xi).$$

Using all the above, we deduce

$$\begin{aligned}
&[JX,JY]-[X,Y]-J([JX,Y]+[X,JY])\\
=&(\nabla^g_{JX}J)Y-(\nabla^g_{JY}J)X-J((\nabla^g_X J)Y-(\nabla^g_Y J)X)-g([X,Y],\xi)\xi\\
=&0,
\end{aligned}$$

which proves that the integrability condition is fulfilled. $\Diamond$

Just like an almost-complex structure can be defined in terms of subspaces of its complexified tangent space, so can a CR manifold. We go back to the example of a real hypersurface. The space $T(\mathbb{C}^{m+1})^{0,1}$ can be written as

$$T(\mathbb{C}^{m+1})^{0,1}=\operatorname{span}\{\frac{\partial}{\partial\overline{z_1}},\ldots,\frac{\partial}{\partial\overline{z_{m+1}}}\},$$

where we note $z_k:=x_k+iy_k$ the coordinates of $\mathbb{C}^{m+1}$ and set

$$\frac{\partial}{\partial\overline{z_k}}:=\frac{1}{2}\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right).$$

Now, we set

$$H^{01}_pM:=(TM_c)_p\cap\left(T\mathbb{C}^{m+1}\right)^{0,1}.$$

This space has the following properties inherited from those of $\left(T\mathbb{C}^{m+1}\right)^{0,1}$:

- $H^{01}\cap\overline{H}^{01}=\{0\}$,
- $\dim_{\mathbb{C}}H^{01}_p=m$,
- H^{01} is involutive, i.e. $[H^{01},H^{01}]\subset H^{01}$.

We use these properties to define a complex CR structure on M.

2.2.4 Definition A (complex) CR structure on an odd-dimensional manifold M^{2m+1} is a subbundle $H^{01}\subset T_cM$ of complex rank m such that

(i) $H^{01}\cap\overline{H^{01}}=\{0\}$,

(ii) $[H^{01},H^{01}]\subset H^{01}$.

2.2.5 Remark This approach motivates the name Cauchy-Riemann or CR manifold: Recall that a function on $\mathbb{C}^{m+1}$ is holomorphic if and only if it is zero under all Cauchy-Riemann operators $\frac{\partial}{\partial \overline{z_k}}$. Thus, in a certain sense, a CR structure on M is given by a space of Cauchy-Riemann operators. Indeed, one can show that in the case of a real-analytic hypersurface in complex space, a real analytic function $f \in C^\infty(M, \mathbb{C})$ is induced by a holomorphic function on $\mathbb{C}^{m+1}$ if and only if $V(f) = 0$ for any $V \in \Gamma(H^{01})$ (for more details on this, compare chapter 1, paragraph 3 of [Jac90]).

Having given two definitions of a CR structure we now need to show that these are equivalent.

2.2.6 Lemma
A manifold M^{2m+1} has a CR structure in the sense of definition 2.2.2 if and only if it admits one in the sense of definition 2.2.4.

Proof: Given a real CR structure, one extends J to the complexification of H and defines H^{01} as the $-i$-eigenspace of this extension. All required properties follow immediately.
Conversely, given a complex CR structure, choose a (local) basis $\{L_k = X_k + iY_k\}$ of H^{01}. Then, $\{X_1, Y_1, ..., X_m, Y_m\}$ are pointwise linearly independent over the reals.
This can be seen as follows: Assume there exist $\lambda_k, \mu_k \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= \sum_{k=1}^m \lambda_k X_k + \mu_k Y_k \\ &= \sum_{k=1}^m \frac{\lambda_k}{2}(L_k + \overline{L_k}) - \frac{i\mu_k}{2}(L_k - \overline{L_k}) \\ &= \sum_{k=1}^m \frac{\lambda_k - i\mu_k}{2} L_k + \frac{\lambda_k + i\mu_k}{2}\overline{L_k}, \end{aligned}$$

which implies $\lambda_k - i\mu_k = 0$ and $\lambda_k + i\mu_k = 0$ for all k and thus $\lambda_k, \mu_k = 0$ which proves the claimed independence.
Now, going back to the main proof, define $H = \mathrm{span}_{\mathbb{R}}\{X_k, Y_k | k = 1, ..., m\}$ and $JX_k = Y_k$ (and thus $JY_k = -X_k$). One verifies that J is independent of the choice of basis by extending it to $H \otimes \mathbb{C}$. One then sees that H^{01} and $\overline{H^{01}}$ are the $\mp i$-eigenspaces of J which uniquely determines J on $H \otimes \mathbb{C}$ and thus on H. It remains to check the integrability condition: For $X, Y \in H$, we have that $[X + iJX, Y + iJY] \in \Gamma(H^{01})$ and thus

$$\begin{aligned} [X + iJX, Y + iJY] &= \sum_{k=1}^m \alpha_k (X_k + iY_k) \\ &= \sum_{k=1}^m \alpha_k X_k + i\alpha_k Y_k \end{aligned}$$

and also

$$[X + iJX, Y + iJY] = [X, Y] - [JX, JY] + i([JX, Y] + [X, JY]).$$

This implies

$$[X, Y] - [JX, JY] = \sum_{k=1}^m \alpha_k X_k, \tag{2.4}$$

$$\begin{aligned} [JX, Y] + [X, JY] &= \sum_{k=1}^m \alpha_k Y_k \\ &= J(\sum_{k=1}^m \alpha_k X_k). \end{aligned} \tag{2.5}$$

Then, (2.5) proves that $[JX,Y]+[X,JY]\in\Gamma(H)$. Furthermore, (2.4) and (2.5) imply that

$$\begin{aligned}[JX,JY]-[X,Y]-J([X,JY]+[JX,Y]) &= -\sum_{k=1}^{m}\alpha_k X_k - J^2\sum_{k=1}^{m}\alpha_k X_k\\ &= 0.\end{aligned}$$

This yields the claim. □

We now want to investigate the link between CR manifolds and contact manifolds. Let an orientable CR manifold be given. We can then define a form $\eta\in\Omega^1(M)$ which is nonzero, and vanishes on H. This is possible globally because, since M is oriented, there exists a global vector field in the complement of H on which we set η to be one and zero on H which completely determines η. We then define the *Lévy form* L_η on H as

$$L_\eta(X,Y):=d\eta(X,JY)$$

for any $X,Y\in\Gamma(H)$.

2.2.7 Definition (M,H,J,η) is called a *nondegenerate* CR manifold, if the Lévy form is nondegenerate. If L_η is additionally positive definite, the CR manifold is called *strictly pseudoconvex*.

In the case of a strictly pseudoconvex CR manifold, recalling Lemma 2.1.9, we define a Riemannian metric on M by

$$g_\eta=L_\eta+\eta\odot\eta.$$

Let ξ be the metric dual of η. We then extend J by setting $J\xi=0$. Then, by construction, we have $\|\eta_x\|=1$ and $g_\eta(JX,Y)=d\eta(X,Y)$. As $\eta(H)=0$ and $\eta(\xi)=1$, we have $J^2X=-X+\eta(X)\xi$. Thus, the tuple (M,g_η,η,J) is a metric contact manifold.

2.2.8 Remark In fact, a nondegenerate Lévy form is enough for η to be a contact form. We consider only the strictly pseudoconvex case here, because it is this case that gives us a (Riemannian) metric contact manifold.

Conversely, if (M,g_η,η,J) is a metric contact manifold, then $(\mathcal{C},J|_\mathcal{C})$ fulfill conditions (i) and (ii) of the definition of a (real) CR structure while we need an extra condition to ensure integrability. To this end, we have the following result, stated in [Nic05, section 3.1]:

2.2.9 Lemma
A metric contact manifold is CR if and only if the following condition is satisfied:

$$JN(X,Y)=0\quad \textit{for all } X,Y\in\Gamma(\mathcal{C})$$

for its contact Nijenhuis tensor. In that case, the CR manifold is strictly pseudoconvex.

Proof: Note that on a metric contact manifold we have for all $X,Y\in\Gamma(\mathcal{C})$ that

$$\begin{aligned}d\eta(X,Y)&=X(\eta(Y))-Y(\eta(X))-\eta([X,Y])\\&=-\eta([X,Y]).\end{aligned}$$

Thus we have

$$\begin{aligned}\eta([JX,Y]+[X,JY])&=-d\eta(JX,Y)-d\eta(X,JY)\\&=0\end{aligned}$$

and thus $[JX, Y] + [X, JY] \in \Gamma(\mathcal{C})$. Now we compute

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$$

which is equivalent to

$$\begin{aligned} &4N(X, Y) - \eta([X, Y])\xi = 0 \text{ and to} \\ &4N(X, Y) + d\eta(X, Y)\xi = 0. \end{aligned}$$

Noting that

$$\begin{aligned} d\eta(X, Y) = d\eta(JX, JY) &= -\eta([JX, JY]) \\ &\overset{\eta\circ J=0}{=} \eta(4N(X, Y)), \end{aligned}$$

one obtains the following equivalent equations:

$$\begin{aligned} [JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) &= 0, \\ 4N(X, Y) - \eta(4N(X, Y))\xi &= 0, \\ -J^2 N(X, Y) &= 0, \\ JN(X, Y) &= 0, \end{aligned}$$

where the last equivalence follows because the image of J lies in $\mathcal{C}$ on which J acts as an isomorphism. This yields the claim. $\square$

We summarize that every strictly pseudoconvex CR manifold is also a contact manifold and conversely, a metric contact manifold is CR (and then automatically strictly pseudoconvex) if and only if its Nijenhuis tensor fulfils $J \circ N = 0$. In the rest of this thesis, we will mostly consider the more general case of a contact manifold and restrict our discussion to the case of a CR manifold where necessary. Whenever we mention a CR manifold in the sequel, this is to be understood as a metric contact manifold which is also CR.

3

Spinor bundles, connections and geometric Dirac operators

In this section, we discuss the spinor bundles of a *Spin*- or $Spin^c$-manifold and the connections and Dirac operators they carry. In particular, we discuss how a connection on the tangent bundle TM induces a connection and a Dirac operator on the spinor bundle and how certain properties of the Dirac operator induced are reflected in the torsion of the connection. Our focus is on $Spin^c$ structures and in particular on the canonical $Spin^c$ structures on an almost-hermitian or metric contact manifold.
In a first section, we review some facts about the spin groups and the representations of Clifford algebras and spin groups and, in particular, give a description of the spinor module as a space of exterior forms. In the following section, we move on from the purely algebraic viewpoint to spin structures on manifolds and their spinor bundles. Having introduced those, we then discuss the differential geometric core of this chapter, the connections induced on the spinor bundle by connections on TM and the Dirac operators defined by them. The theory developed so far is then applied in the last section to the case of the canonical $Spin^c$ structure on almost-hermitian and metric contact manifolds.

3.1 Some algebraic facts on $Spin^c$ and spinor representations

This section serves as a short introduction to the complex spin group $Spin^c$ and to the theory of representations of Clifford algebras with a particular focus on induced representations of $Spin^c$ and their relationship with representations of the unitary group U_m.
We assume that the reader is familiar with the spin group and will therefore discuss it only where it serves as a background for understanding the respective theory for the complex spin group $Spin^c$. Also, because the theory of representations of Clifford algebras is well-known, we only state the results we need without proof and refer the reader to sections I.5 and I.6 of [LM89] for further details.

Let $Cl_n = Cliff(\mathbb{R}^n, x_1^2 + \cdots + x_n^2)$ and $\mathbb{C}l_n = Cliff(\mathbb{C}^n, z_1^2 + \cdots z_n^2)$ be the Clifford algebras of the standard real and complex space respectively. The group $Spin_n$ is contained in Cl_n and in $\mathbb{C}l_n \simeq Cl_n \otimes \mathbb{C}$, we can consider the group generated by $Spin_n$ and the unit sphere S^1:

3.1.1 Definition The *complex spin group* is defined as

$$Spin_n^c = {}^{(Spin_n \times S^1)}\!\big/_{\{\pm 1\}} = Spin_n^c \times_{\mathbb{Z}_2} S^1.$$

There are a number of mappings that give links between the $Spin^c$ group and other groups: Noting $\lambda\colon Spin_n \to SO_n$ the two-fold covering, we define the following:

$$\begin{aligned} \lambda^c\colon Spin^c &\longrightarrow SO_n & \lambda^c([g,z]) &= \lambda(g),\\ l\colon Spin_n^c &\longrightarrow S^1 & l([g,z]) &= z^2,\\ i\colon Spin_n &\longrightarrow Spin_n^c & i(g) &= [g,1] \end{aligned}$$

and

$$j\colon S^1 \longrightarrow Spin_n^c \qquad\qquad j(z) = [1,z].$$

Finally, defining $sq\colon S^1 \longrightarrow S^1$ by $sq(z) = z^2$, we obtain the following commutative diagram, where the row and the column are exact (cf. [Fri00, section 1.6])

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & S^1 \arrow[d, "j"] \arrow[dr, "sq"] & & \\
1 \arrow[r, "i"] & Spin_n \arrow[r, "i"] \arrow[dr, "\lambda"'] & Spin_n^c \arrow[r, "l"] \arrow[d, "\lambda^c"] & S^1 \arrow[r] & 1 \\
 & & SO_n \arrow[d] & & \\
 & & 1 & &
\end{tikzcd} \tag{3.1}$$

Furthermore, we obtain a two-fold covering mapping

$$\begin{aligned} p\colon Spin_n^c &\longrightarrow SO_n \times S^1 \\ [g,z] &\longmapsto (\lambda(g), z^2). \end{aligned}$$

We will later use these maps in the discussion of representations and in the definition of $Spin^c$ structures.

We now want to discuss the representations of Clifford algebras and the representations they induce on the (complex) spin group. A *Clifford representation* is an algebra homomorphism

$$\rho\colon \mathbb{C}l_n \longrightarrow End_{\mathbb{C}}(V)$$

where V is some complex vector space. As it turns out, there are not many "different" Clifford representations if we restrict ourselves to the "smallest" representations possible. We now explain what we mean by that:

3.1.2 Definition A Clifford representation $\rho\colon \mathbb{C}l_n \to End_{\mathbb{C}}(V)$ is called *irreducible* if no decomposition $V = V_1 \oplus V_2$ such that $\rho(\mathbb{C}l_n)(V_i) \subset V_i$ exists.
Two representations $\rho_i\colon \mathbb{C}l_n \to End_{\mathbb{C}}(V_i)$ $(i = 1,2)$ are called *equivalent* if there exists a vector space isomorphism $F\colon V_1 \to V_2$ such that for any $\varphi \in \mathbb{C}l_n$

$$F \circ \rho_1(\varphi) = \rho_2(\varphi) \circ F$$

The following theorem collects the results on representations of $\mathbb{C}l_n$ we need:

3.1.3 Theorem (cf. [LM89, sections I.5 and I.6])
(1) $\mathbb{C}l_{2m} \simeq \mathcal{M}(\mathbb{C}, 2^m)$ *and* $\mathbb{C}l_{2m+1} \simeq \mathcal{M}(\mathbb{C}, 2^m) \oplus \mathcal{M}(\mathbb{C}, 2^m)$.

(2) The trivial representation

$$\mathbb{C}l_{2m} \simeq \mathcal{M}(\mathbb{C}, 2^m) \longrightarrow \mathrm{End}(\mathbb{C}^{2^m})$$

is, up to equivalence, the only irreducible representation of $\mathbb{C}l_{2m}$.

(3) Up to equivalence, the only irreducible representations of $\mathbb{C}l_{2m+1}$ are given by

$$\mathbb{C}l_{2m+1} \simeq \mathcal{M}(\mathbb{C}, 2^m) \oplus \mathcal{M}(\mathbb{C}, 2^m) \longrightarrow \mathrm{End}(\mathbb{C}^{2^m}) \oplus \mathrm{End}(\mathbb{C}^{2^m}) \xrightarrow{pr} \mathrm{End}(\mathbb{C}^{2^m})$$

where pr is the projection onto the first or second component.

Each representation of $\mathbb{C}l_n$ induces one of its subgroups $Spin_n$ and $Spin^c_n$, which we shall discuss now. We begin with $Spin$:

3.1.4 Definition The *spinor representation* is the restriction of (one of) the irreducible representation(s) of $\mathbb{C}l_n$ to the spin group. We will note it

$$\kappa\colon Spin_n \subset Cl_n \subset \mathbb{C}l_n \longrightarrow \mathrm{End}(\Delta_n).$$

This is well-defined by the following result:

3.1.5 Proposition (cf. [LM89, Proposition I.5.15])
In the case where $n = 2m+1$ is odd, the restrictions of the two irreducible Clifford representations to $Spin_n$ coincide and give an irreducible representation of the spin group.
In the even case, the restriction to $Spin_n$ splits into two irreducible representations

$$\Delta_{2m} = \Delta^+_{2m} \oplus \Delta^-_{2m}.$$

As the spinor representation comes from a mapping defined on all of the Clifford algebra, the Clifford algebra acts on the spinor module in the obvious way. This action is called *Clifford multiplication.*

3.1.6 Proposition (cf. [Fri00, section 1.5 (p. 24)])
There exists a positive definite hermitian scalar product $(\cdot,\cdot)$ on Δ_n such that

$$(x.\varphi, \psi) = -(\varphi, x.\psi)$$

for all $x \in \mathbb{R}^n$ and all $\varphi, \psi \in \Delta_n$.

One can now use the same theory for the complex spin group as well. The spinor representation extends to $Spin^c$ as follows:

3.1.7 Definition The *spinor representation* on $Spin^c$ is given by

$$\begin{aligned} \kappa^c\colon Spin^c_n \simeq Spin_n \times_{\mathbb{Z}_2} S^1 &\longrightarrow \mathrm{End}(\Delta_n) \\ [g,z] &\longmapsto z\kappa(g). \end{aligned}$$

All the above results carry over to $Spin^c$, i.e. the complex spinor representation is well-defined, it is irreducible if n is odd and admits a decomposition into the subspaces $\Delta^\pm_n$ if n is even.
In the even case, we have an alternative description of the spinor representation which we shall later use to describe the spinor bundle on almost-hermitian manifolds.

3.1.8 Proposition (cf. [Mor96, Lemma 3.4.3])
The irreducible representation of $\mathbb{C}l_n$ for even $n = 2m$ is induced by the following mapping

$$\begin{aligned} cl\colon \mathbb{R}^{2m} \times \Lambda^{0,*}((\mathbb{R}^{2m})^*) &\longrightarrow \Lambda^{0,*}((\mathbb{R}^{2m})^*) \\ (v,\omega) &\longmapsto \sqrt{2}\left((v^{1,0})^\flat \wedge \omega - v^{0,1} \lrcorner \omega\right), \end{aligned}$$

where $v^{1,0}$ and $v^{0,1}$ denote the respective parts of v considered as an element of the complexification $\mathbb{R}^{2m} \otimes \mathbb{C}$ and where we equip $\mathbb{R}^{2m}$ with the standard almost-complex structure.

Proof: We begin by proving $cl^2(v) = -\|v\|^2$:

$$\begin{aligned}
cl^2(v)(\omega) &= \sqrt{2}cl(v)\left((v^{1,0})^\flat \wedge \omega - v^{0,1}\lrcorner \wedge \omega\right)\\
&= 2\left((v^{1,0})^\flat \wedge (v^{1,0})^\flat \wedge \omega - v^{0,1}\lrcorner((v^{1,0})^\flat \wedge \omega) - (v^{1,0})^\flat \wedge (v^{0,1}\lrcorner\omega) + v^{0,1}\lrcorner v^{0,1}\lrcorner\omega\right)\\
&= 2\left(-(v^{1,0})^\flat(v^{0,1})\omega + (v^{1,0})^\flat \wedge (v^{0,1}\lrcorner\omega) - (v^{1,0})^\flat \wedge (v^{0,1}\lrcorner\omega)\right)\\
&= -2(v^{1,0})^\flat(v^{0,1})\omega.
\end{aligned}$$

The last line is equal to $-\|v\|^2$ which can be seen as follows: We have that $v^{1,0} = \frac{1}{2}(v - iJv)$ and $v^{0,1} = \frac{1}{2}(v + iJv)$ and thus

$$\begin{aligned}
(v^{1,0})^\flat(v^{0,1}) &= \frac{1}{4}(v^\flat(v) + (Jv)^\flat(Jv) - i((Jv)^\flat(v) + v^\flat(Jv)))\\
&= \frac{1}{2}\|v\|^2.
\end{aligned}$$

Thus, this mapping extends to an action of Cl_n and, extending by $\mathbb{C}$-linearity to one of $\mathbb{C}l_n$ which we will denote $\widetilde{cl}$. Because the dimension of $\Lambda^{0,*}((\mathbb{R}^{2m})^*)$ is 2^m, $\widetilde{cl}$ must be the irreducible representation. □

Recall that $Spin_n$ is a two-fold covering of SO_n, the structure group of an oriented Riemannian manifold. We are thus led to ask whether there is a link between the complex spin group and the unitary group U_m which is the structure group of an almost-hermitian manifold. Indeed, one has the following result:

3.1.9 Lemma
Let $m \in \mathbb{N}$ and $n \in \{2m, 2m+1\}$ and let $f\colon U_m \to SO_n \times S^1$ be given by $f(A) = (\iota A, \det A)$ where ι is the inclusion map. Then there exists exactly one group homomorphism F such that the following diagram commutes:

$$\begin{tikzcd}
 & Spin^c_{2m} \arrow[d, "p"] \\
U_m \arrow[ur, "\exists! F"] \arrow[r, "f"] & SO_{2m} \times S^1
\end{tikzcd}$$

Proof: By the theory of covering spaces, we need to show that $f_\#(\pi_1(U_k)) \subset p_\#(\pi_1(Spin^c_{2m}))$. We have that $\pi_1(Spin^c_{2m}) \simeq \mathbb{Z}$. Let α be a generating element of that fundamental group. Then, $p_\#(\alpha) = \lambda^c_\#(\alpha) + l_\#(\alpha)$. Recalling the exactness of the column in (3.1), we deduce that $\beta = \lambda^c_\#(\alpha)$ must generate all of $\pi_1(SO_n)$. The row is also exact and $\pi_1(Spin_n) = 1$, therefore $l_\#$ must be bijective, i.e. $\gamma = l_\#(\alpha)$ must generate $\pi_1(S^1) \simeq \mathbb{Z}$. Thus, $p_\#(\alpha)$ generates the whole fundamental group of $SO_{2k} \times S^1$ and thus, the condition is trivial. □

While the proof using covering theory we have just given is short and elegant, we can also give an explicit formula for F which will be useful later: Let $A \in U_m$. Then there exist unique $\theta_1, ..., \theta_m \in [0, 2\pi)$ and a unitary basis $e_1, ..., e_m$ of $\mathbb{C}^m$ with respect to which the matrix has the form

$$\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_m} \end{pmatrix},$$

whose inclusion[6] in SO_{2m} has the following form

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_m & -\sin\theta_m \\ & & & \sin\theta_m & \cos\theta_m \end{pmatrix}$$

with respect to the basis $e_1, f_1, ..., e_m, f_m$ where $f_j = Je_j$ and J is the almost-complex structure induced on $\mathbb{R}^{2m}$ by the complex structure of $\mathbb{C}^m$. The form of ιA implies that it is the product of rotations:

$$\iota A = D_{\theta_1}^{\langle e_1,f_1\rangle} \circ \cdots \circ D_{\theta_m}^{\langle e_m,f_m\rangle},$$

where $D_\theta^{\langle u,v\rangle}$ denotes the rotation around the origin by the angle θ in the plain spanned by the vectors u and v. Under the covering λ we have for its preimage

$$\begin{aligned} \lambda^{-1}(D_{\theta_j}^{\langle e_j,f_j\rangle}) &\ni (\cos\theta_j e_j + \sin\theta_j f_j)(\cos\theta_j e_j - \sin\theta_j f_j) \\ &= -\cos^2\theta_j - 2\sin\theta_j\cos\theta_j e_j f_j + \sin^2\theta_j \\ &= \cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2}\cdot e_j f_j. \end{aligned}$$

On the other hand, $\det A = \exp(i\sum_j \theta_j)$ and $p([g,z]) = \lambda(g) \times z^2$ and thus, setting

$$F(A) := \left(\prod_{j=1}^m (\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2}\cdot e_j f_j)\right) \times e^{\frac{i}{2}\sum_j \theta_j}$$

fulfils the conditions of the above lemma.
Now, in the case, where n is even, we prove a result stated in [Mor96, Lemma 3.4.4], comparing the representation $\widetilde{cl} \circ F$ with the standard representation of U_m on $\Lambda^{0,*}((\mathbb{R}^{2m})^*)$ that we described in section 1.2.1:

3.1.10 Lemma
The U_m-representations $\rho_\Lambda \colon U_m \to \Lambda^{0,*}((\mathbb{R}^{2m})^*)$ and $\widetilde{cl} \circ F$ coincide.

Proof: Let $A \in U_m$ and $e_1, f_1, ..., e_m, f_m$ as described above. Then $z_j = \frac{1}{\sqrt{2}}(e_j - if_j)$ and their conjugates $\overline{z_j} = \frac{1}{\sqrt{2}}(e_j - if_j)$ form a basis of $\mathbb{R}^{2m} \otimes \mathbb{C}$ with $\overline{z_j} \in (\mathbb{R}^{2m})^{0,1}$. Denote z^j and $\overline{z^j}$ their duals. We then have $e_j = \frac{1}{\sqrt{2}}(z_j + \overline{z_j})$ and thus $e_j^{1,0} = \frac{1}{\sqrt{2}}z_j$ and $e_j^{0,1} = \frac{1}{\sqrt{2}}\overline{z_j}$. Analogously, we have $f_j^{1,0} = \frac{i}{\sqrt{2}}z_j$ and $f_j^{0,1} = -\frac{i}{\sqrt{2}}\overline{z_j}$. Thus, we obtain

$$\begin{aligned} \widetilde{cl}(e_j f_j)(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) &= \sqrt{2}\widetilde{cl}(e_j)((f_j^{1,0})^\flat \wedge - f_j^{0,1}\lrcorner)\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} \\ &= i\widetilde{cl}(e_j)(\overline{z^j} \wedge + \overline{z_j}\lrcorner)\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}. \end{aligned}$$

The following calculations depend on whether j is an element of $I = \{i_1, ..., i_k\}$ or not. We first consider the case where $j = i_\mu$. In this case we obtain:

$$\begin{aligned} \widetilde{cl}(e_j f_j)(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) &= i(-1)^{\mu-1}(\overline{z^j} \wedge - \overline{z_j})\overline{z^{i_1}} \wedge ... \wedge \widehat{\overline{z^{i_\mu}}} \wedge ... \wedge \overline{z^{i_k}} \\ &= i\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}. \end{aligned}$$

[6] We only consider the case of $n = 2m$ here. The considerations for $n = 2m + 1$ are analogous and the result obtained is the same.

In the other case, i.e. $j \notin I$ we have

$$\begin{aligned}\widetilde{cl}(e_jf_j)(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) &= i\widetilde{cl}(\overline{z^j} \wedge - \overline{z_j} \lrcorner)\overline{z^j} \wedge \overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} \\ &= -i\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\widetilde{cl}\left(\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2}e_jf_j\right)\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} &= \begin{cases}\left(\cos\frac{\theta_j}{2} + i\sin\frac{\theta_j}{2}\right)\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} & \text{if } j \in I \\ \left(\cos\frac{\theta_j}{2} - i\sin\frac{\theta_j}{2}\right)\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} & \text{if } j \notin I\end{cases} \\ &= \begin{cases} e^{i\theta_j/2}\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} & \text{if } j \in I \\ e^{-i\theta_j/2}\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} & \text{if } j \notin I\end{cases}\end{aligned}$$

and thus,

$$\begin{aligned}\widetilde{cl}(F(A))(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) &= e^{\frac{i}{2}\sum_{j=1}^m \theta_j} e^{-\frac{i}{2}\sum_{j\notin I}\theta_j} e^{\frac{i}{2}\sum_{j=1}^k \theta_{i_j}}\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}} \\ &= e^{\sum_{j=1}^k i\theta_{i_j}}\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}.\end{aligned}$$

On the other hand, by the defintion of ρ_Λ, we have that

$$\rho_\Lambda(A)(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) = \rho^*(A)\overline{z^{i_1}} \wedge ... \wedge \rho^*(A)\overline{z^{i_k}}.$$

Writing A as a real matrix and writing $\overline{z^j} = \frac{1}{\sqrt{2}}(e^j - if^j)$, we obtain that

$$\begin{aligned}\rho^*(A)\overline{z^j} &= \frac{1}{\sqrt{2}}\left(\cos\theta_j e^j - \sin\theta_j f^j + i\sin\theta_j e^j - i\cos\theta_j f_j\right) \\ &= \frac{1}{\sqrt{2}}\left(e^{i\theta_j}e^j - ie^{i\theta_j}f^j\right) \\ &= e^{i\theta_j}\overline{z^j}.\end{aligned}$$

This implies that

$$\rho_\Lambda(A)(\overline{z^{i_1}} \wedge ... \wedge \overline{z^{i_k}}) = e^{i\sum_{j=1}^k \theta_{i_j}}$$

which yields the claim. ☐

This concludes our discussion of representations of the even-dimensional complex spin group. Here we used the almost-complex structure of $\mathbb{R}^{2m}$ to describe the spinor representation. We now turn to the odd-dimensional case. In that case, we can use an almost-complex structure on a $2m$-dimensional subspace of $\mathbb{R}^{2m+1}$ and fix a transversal direction, thus creating a structure on $\mathbb{R}^{2m+1}$ that resembles that on the tangent space of a contact manifold. The following proposition, similarly stated in [Pet05, proposition 3.2], describes the spinor representation in the odd-dimensional case.

3.1.11 Proposition

Let $\mathbb{R}^{2m+1} = \{(x_1, y_1, ..., x_m, y_m, z)\}$ and set $V = \{(x_1, y_1, ..., x_m, y_m, 0) \in \mathbb{R}^{2m+1}\}$ and define an almost-complex structure on V by setting

$$J(x_1, y_1, ..., x_m, y_m, 0) = (-y_1, x_1, ..., -y_m, x_m, 0).$$

Furthermore, write $\xi = (0, ..., 0, 1)$ and $\eta = \xi^\flat$. Let furthermore $V^{1,0}$ and $V^{0,1}$ be the $\pm i$ eigenspaces of J on $V \otimes \mathbb{C}$. Then an irreducible Clifford representation of $\mathbb{C}l_{2m+1}$ is induced by

$$\begin{aligned} cl\colon \mathbb{R}^{2m+1} \times \Lambda^{0,*}(V^*) &\longrightarrow \Lambda^{0,*}(V^*) \\ (u, \omega) &\longmapsto \sqrt{2}((u^{1,0})^\flat \wedge - u^{0,1}\lrcorner)\omega + i(-1)^{\deg\omega+1}\eta(u)\omega\end{aligned}$$

where $u^{1,0}$ and $u^{0,1}$ are the respective parts of the orthogonal projection of u onto V.

Proof: We need to show that $cl^2(u) = -\|u\|^2$ for any $u \in \mathbb{R}^{2m+1}$. Then cl induces a representation of $\mathbb{C}l_{2m+1}$ and the dimension of $\Lambda^{0,*}(V^*)$ implies that it must be one of the irreducible ones. So let $u = v + \lambda\xi \in \mathbb{R}^{2m+1}$ with $v \in V$ and $\omega \in \Lambda^{0,k}(V^*)$. Then, we have that

$$cl(u)(\omega) = \sqrt{2}\left((v^{1,0})^\flat \wedge \omega - v^{0,1} \lrcorner \omega\right) + i(-1)^{k+1}\lambda\omega$$

and thus

$$\begin{aligned}
cl^2(u)(\omega) &= \sqrt{2} cl(u)\left((v^{1,0})^\flat \wedge \omega - v^{0,1} \lrcorner \omega\right) + i(-1)^{k+1}\lambda cl(u)(\omega) \\
&= 2\left(v^{0,1} \lrcorner ((v^{1,0})^\flat \wedge \omega) - (v^{1,0})^\flat \wedge (v^{0,1} \lrcorner \omega)\right) + \sqrt{2} i(-1)^{k+2}\lambda (v^{1,0})^\flat \wedge \omega \\
&\quad + \sqrt{2} i(-1)^k \lambda v^{0,1} \lrcorner \omega + \sqrt{2} i(-1)^{k+1}\lambda\left(\sqrt{2}(v^{1,0})^\flat \wedge \omega - \sqrt{2} v^{0,1} \lrcorner \omega + i(-1)^{k+1}\lambda\omega\right) \\
&= -2(v^{1,0})^\flat (v^{0,1})\omega - \lambda^2\omega = -\|v\|^2\omega.
\end{aligned}$$

This yields the claim. □

Just like in the even-dimensional case, we have the following result:

3.1.12 Lemma
The U_m-representations $\rho_\Lambda \colon U_m \to \mathrm{End}_{\mathbb{C}}\left(\Lambda^{0,*}\left((\mathbb{R}^{2m})^*\right)\right)$ and $\widetilde{cl} \circ F$ coincide.

Proof: The proof from the even-dimensional case carries over, because the image $F(U_m)$ is only generated by the first $2m$ basis vectors of $\mathbb{R}^{2m+1}$. □

This concludes our discussion of representations of the spin groups. We will use the general theory in the following section to introduce spinor bundles and will apply the more detailed analysis of representations of $\mathbb{C}l_n$ to spinor bundles of almost-hermitian and metric contact manifolds in the last section of this chapter.

3.2 *Spin* and *Spin*^{c} structures and their spinor bundles

Having discussed the algebraic structure of the spin groups and their representations, we now consider the "extension" of these concepts to manifolds, i.e. spin and $Spin^c$ structures on (oriented Riemannian) manifolds. This section serves to give a short review of these structures and the spinor bundles associated to them. The discussion will be short and serves mainly to establish notation, because we assume that the reader is familiar with these structures, at least in the spin case.

Spin structures

A spin structure on a manifold is the existence of a principal *Spin*-bundle together with a map that extends the two fold covering $\lambda\colon Spin_n \to SO_n$ to a two fold-covering of $P_{SO}(M)$. More formally, we have the following definition:

3.2.1 Definition A *spin manifold* is an oriented Riemannian manifold (M, g) whose frame bundle $P_{SO}(M)$ admits a *spin strcuture*, i.e. a principal $Spin_n$-bundle $P_{Spin}(M)$ together with a smooth map $f\colon P_{Spin}(M) \to P_{SO}(M)$ that commutes with the projections onto M such that the following diagram commutes:

$$\begin{tikzcd} P_{Spin}(M) \times Spin_n \arrow[r] \arrow[d, "f\times\lambda"] & P_{Spin}(M) \arrow[d, "f"] \\ P_{SO}(M) \times SO_n \arrow[r] & P_{SO}(M), \end{tikzcd}$$

where $\lambda\colon Spin_n \to SO_n$ is the two-fold covering map and the arrows in the lines denote the group actions.

To this bundle we can then associate a vector bundle through the spinor representation:

3.2.2 Definition The *spinor bundle* of a spin manifold is the following vector bundle associated to $P_{Spin}(M)$:

$$\mathbb{S} = P_{Spin}(M) \times_\kappa \Delta_n,$$

where κ is the spinor representation.

Because κ is defined on all of $Cl_n \subset \mathbb{C}l_n$, we obtain a Clifford multiplication mapping

$$\begin{aligned} c\colon Cl_n \times S_n &\longrightarrow S_n \\ (X, v) &\longmapsto c(X)(v) = \Delta_n(X)(v). \end{aligned}$$

Considering the Clifford bundle

$$Cl(M, g) = \coprod_{x\in M} Cl(T_xM, g_x),$$

the Clifford multiplication carries over to a *Clifford multiplication* defined on $Cl(M, g)$ and $\mathbb{S}$:

$$c\colon Cl(M, g) \times \mathbb{S} \longrightarrow \mathbb{S},$$

which for ease of notation we shall also write $X.\phi = c(X)\phi$. It is compatible with the bundle structure, i.e. the following diagram commutes:

$$\begin{tikzcd} Cl(M,g) \times \mathbb{S} \arrow[r, "c"] \arrow[d, "\pi_{Cl}\times\pi_{\mathbb{S}}"'] & \mathbb{S} \arrow[dl, "\pi_{\mathbb{S}}"] \\ M & \end{tikzcd}$$

where π_{Cl} and $\pi_{\mathbb{S}}$ denote the projections onto M for the respective bundle. $\mathbb{S}$ is therefore also called a *Clifford module*.
One can also extend the Clifford multiplication map to Clifford multiplication of forms

$$c\colon \Omega^*(M) \times \mathbb{S} \longrightarrow \mathbb{S}$$

given by

$$c(\omega)(\phi) = \sum_{i_1<...<i_k} \omega(s_{i_1}, ..., s_{i_k}) s_{i_1} ... s_{i_k}.\phi,$$

where (s_i) is a local orthonormal basis.
Recall that Δ_n carries a postive definite hermitian scalar product $(\cdot,\cdot)$. This product carries over to a bundle metric on $\mathbb{S}$ by setting

$$\begin{aligned}(\cdot,\cdot)_x\colon \mathbb{S}_x \times \mathbb{S}_x &\longrightarrow \mathbb{C}\\ \phi,\psi &\longmapsto (\phi,\psi)_x.\end{aligned}$$

This bundle metric then induces an L^2 scalar product

$$(\phi,\psi)_{L^2} = \int_M (\phi(x),\psi(x))_x dM(x) \quad \forall \phi,\psi \in \Gamma_0(\mathbb{S})$$

on the space of compactly supported sections of $\mathbb{S}$.

Spin^c structures

The notion of spin structure has a complex analogue which is the notion of a *Spin*^{c} structure:

3.2.3 Definition A *Spin*^{c} *structure* on an oriented Riemannian manifold (M^n, g) is a $Spin^c_n$-principal bundle $P_{CS}(M)$ together with a smooth map $f\colon P_{CS}(M) \to P_{SO}(M)$ such that the following diagram commutes:

$$\begin{tikzcd} P_{CS}(M) \times Spin^c_n \arrow[r] \arrow[d, "f\times\lambda^c"'] & P_{CS}(M) \arrow[d, "f"] \arrow[dr] & \\ P_{SO}(M) \times SO_n \arrow[r] & P_{SO}(M) \arrow[r] & M \end{tikzcd}$$

where again the horizontal arrows on the left stand for the group action and the horizontal and diagonal arrows on the right for the projections onto M.

Given a *Spin*^{c} structure (P_{CS}, f), we can associate the following bundles to it:

(1) An SO_n-principal bundle ${P_{CS}}/{S^1}$ which is isomorphic to $P_{SO}(M)$,

(2) An S^1-principal bundle $P_1 = {P_{CS}}/{Spin_n}$.

The bundle P_1 will become important later and we will therefore consider it in some more detail. We obtain that the projection map $\xi\colon P_{CS} \longrightarrow P_{SO} \times P_1$ is a two-fold covering map. This covering can actually be used as an alternative definition of a *Spin*^{c} structure by demanding the existence of a $Spin^c_n$-bundle P_{CS} together with a S^1-bundle P_1 and a two-fold covering map $\xi\colon P_{CS} \longrightarrow P_{SO} \times P_1$. For a proof of the equivalence, see again [Fri00, section 2.4].

3.2.4 Definition The *determinant line bundle* of a *Spin*^{c} strcuture is the complex line bundle

$$\mathcal{L} = P_1 \times_{U_1} \mathbb{C} = P_{CS} \times_{Spin^c} \mathbb{C}.$$

As for a spin structure, we want to associate a vector bundle to a $Spin^c$ structure. Recall that the mapping $\kappa\colon Spin_n \longrightarrow SO(\Delta_n)$ can be extended to a representation of $Spin_n^c$ by setting

$$\begin{aligned} \kappa^c\colon Spin_n^c &\longrightarrow U(\Delta_n) \\ [g,z] &\longmapsto z\kappa(g) \end{aligned}$$

and we can then associate a vector bundle to the $Spin^c$-bundle

$$\mathbb{S}^c = P_{CS} \times_\kappa \Delta_n$$

which we will call the *spinor bundle* associated to the $Spin^c$ structure. In the same way as for the real case, $\mathbb{S}^c$ and $\Gamma_0(\mathbb{S}^c)$ carry scalar products.
In the next section, we shall see how any connection on TM induces one on $\mathbb{S}$ and $\mathbb{S}^c$ respectively and how they determine a first-order differential operator on the spaces of sections of the spinor bundles.

3.3 Basic properties of connections and geometric Dirac operators

In this section, we shall study connections and the Dirac operators they induce. We begin with a short introduction to connections and their torsion and potential. We then move on to consider how a metric connection on TM induces one on the spinor bundles $\mathbb{S}$ and $\mathbb{S}^c$ and how certain properties of a connection are related to properties of the Dirac operator it induces. In particular, we prove relationships between the torsion of the connection and the self-adjointness of the Dirac operator and how a comparison of the torsion of two connections can show whether they induce the same Dirac operator.

By a connection on TM, we understand a linear operator

$$\nabla \colon \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$$

satisfying

$$\nabla(fX) = df \otimes X + f \cdot (\nabla X)$$

for any $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. As it is well know, any Riemannian manifold (M, g) admits exactly one connection which is both *metric*, i.e.

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(X, \nabla_X Z)$$

and *torsion-free*, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

That connection is called the Levi-Civita-connection and will be noted ∇^g. If we drop the requirement that the connection be torsion-free, we obtain the much larger class of metric connections, which we shall note $\mathcal{A}(M, g)$. These connections are described by the following data:

3.3.1 Definition Let ∇ be a metric connection on (M, g). Then the (2,1)-tensor T defined by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is called the *torsion* of ∇.
The $(2, 1)$-tensor A defined by

$$A_X Y := \nabla_X Y - \nabla^g_X Y$$

is called the *potential* of ∇.

We can consider A and T as elements of $\Omega^2(M, TM)$ as follows:

$$\begin{aligned} T(X; Y, Z) &= g(X, T(Y, Z)), \\ A(X; Y, Z) &= g(A_X Y, Z), \end{aligned}$$

where the conventions for writing two-forms as trilinear mappings from section 1.2.2 are used. We stress that T is already a TM-valued two-form (in the classical sense) by its original definition, while A is not. Therefore, the conventions for understanding them as two-forms differ. It is obvious that ∇ is completely described by its potential. However, it is also completely described by its torsion as the following result shows:

3.3.2 Lemma
The potential and torsion of a metric connection $\nabla \in \mathcal{A}(M, g)$ are related as follows:

$$\begin{aligned} T &= -A + 3\mathfrak{b}A, \\ A &= -T + \frac{3}{2}\mathfrak{b}T. \end{aligned}$$

Proof: To begin with, note that we have

$$\begin{aligned} T(X;Y,Z) &= g(X, \nabla_Y Z - \nabla_Z Y - [Y,Z]) \\ &= g(X; \nabla^g_Y Z + A_Y Z - \nabla^g_Z Y - A_Z Y - [Y,Z]) \\ &= g(X, A_Y Z - A_Z Y) \\ &= A(Y;Z,X) + A(Z;X,Y). \end{aligned} \tag{*}$$

Then, the first identity follows immediately from (*) because

$$A(Y;Z,X) + A(Z;X,Y) = 3\mathfrak{b}A(X;Y,Z) - A(X;Y,Z).$$

Concerning the second identity, we calculate

$$\begin{aligned} A(X;Y,Z) &\overset{(*)}{=} T(Z;X,Y) + A(Y;X,Z) \\ &= T(Z;X,Y) - A(Y;Z,X) \\ &\overset{(*)}{=} T(Z;X,Y) - T(X;Y,Z) - A(Z;Y,X) \\ &= T(Z;X,Y) - T(X;Y,Z) + A(Z;X,Y) \\ &\overset{(*)}{=} T(Z;X,Y) - T(X;Y,Z) + T(Y;Z,X) + A(X;Z,Y) \\ &= T(Z;X,Y) - T(X;Y,Z) + T(Y;Z,X) - A(X;Y,Z), \end{aligned}$$

which is equivalent to

$$\begin{aligned} A(X;Y,Z) &= \frac{1}{2}(T(Z;X,Y) - T(X;Y,Z) + T(Y;Z,X)) \\ &= -T(X;Y,Z) + \frac{3}{2}\mathfrak{b}T(X;Y,Z). \end{aligned}$$

□

Any metric connection on TM defines a connection on the spinor bundle associated to a spin structure as follows. First, let M be a spin manifold. Then, every metric connection on M induces a connection on $\mathbb{S}$ which we now describe: Let ∇ be a metric connection on TM, then it induces a connection one-form $C^\nabla \in \Omega^1(P_{SO}(M), \mathfrak{so}_n)$ on $P_{SO}(M)$, locally given by

$$(C^\nabla)^s(X) = (C^\nabla)(ds(X)) = \sum_{i<j} g(\nabla_X s_i, s_j)E_{ij}, \tag{3.2}$$

where $s\colon U \subset M \to P_{SO}(M)$ is a local section in the frame bundle and $E_{ij} \in \mathbb{R}^{n\times n}$ given by $(E_{ij})_{kl} = -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ (for more details on this compare appendix A.3). Under the two-sheeted coverings

$$\begin{aligned} f\colon P_{Spin}(M) &\longrightarrow P_{SO}(M), \\ \lambda\colon \quad Spin_n &\longrightarrow SO_n, \end{aligned}$$

this connection form lifts to a connection one-form $\widetilde{C^\nabla} \in \Omega^1(P_{Spin}(M), \mathfrak{spin}_n)$ such that the following diagram commutes:

$$\begin{tikzcd} TP_{Spin}(M) \arrow[r, "\widetilde{C^\nabla}"] \arrow[d, "df"'] & \mathfrak{spin}_n \arrow[d, "\lambda_*"] \\ TP_{SO}(M) \arrow[r, "C^\nabla"] & \mathfrak{so}_n \end{tikzcd}$$

As $\mathbb{S}$ is a vector bundle associated to $P_{Spin}(M)$, the connetion form $\widetilde{C^\nabla}$ induces a connection $\widetilde{\nabla}$ on $\mathbb{S}$. By proposition A.9, it is given locally on $U \subset M$ by

$$\widetilde{\nabla}_X \phi|_U = [\widetilde{s}, X(v) + \sum_{i<j} g(\nabla_X s_i, s_j) s_i.s_j.v], \tag{3.3}$$

where $\phi|_U = [\widetilde{s}, v]$ with $s = (s_1, ..., s_n) \in \Gamma(U, P_{SO}(M)), v \in C^\infty(U, \Delta_n)$ and $\widetilde{s}$ is a lifting of s to $P_{Spin}(M)$. This connection has the following properties with respect to the Clifford mutiplication and the scalar product on $\mathbb{S}$:

3.3.3 Lemma
Let X, Y be vector fields and let $\varphi \in \Gamma(\mathbb{S})$. Then for the connection $\widetilde{\nabla}$ induced on $\mathbb{S}$ by any metric connection ∇ on TM, we have

$$\widetilde{\nabla}_X(Y.\varphi) = (\nabla_X Y).\varphi + Y.\widetilde{\nabla}_X \varphi.$$

Furthermore, $\widetilde{\nabla}$ is metric with respect to the hermitian scalar product on $\mathbb{S}$.

Proof: A proof of these facts can be found in [Fri00, section 3.1, pp. 58f]. It is stated there for the connection induced by the Levi-Cività connection, but holds for any metric connection. □

Now, any connection on $\mathbb{S}$ together with Clifford multiplication defines a first order differential operator.

3.3.4 Definition Let M be spin and ∇ a metric connection on TM. Then the first-order differential operator

$$\mathcal{D}(\nabla) \colon \Gamma(\mathbb{S}) \xrightarrow{\widetilde{\nabla}} \Gamma(T^*M \otimes \mathbb{S}) \xrightarrow{c} \Gamma(\mathbb{S}), \tag{3.4}$$

where c denotes Clifford multiplication is called the *geometric Dirac operator* associated to ∇. The operator $\mathcal{D}^g = \mathcal{D}(\nabla^g)$ is called the *Riemannian Dirac operator*

These operators will be considered in great detail in the rest of this thesis. We begin in this chapter with some basic properties and show how these properties are reflected in the torsion of the connection that was used to define the Dirac operator. We begin by defining self-adjoint operators.

3.3.5 Definition A Dirac operator is called *formally self-adjoint*, if

$$(\mathcal{D}(\phi), \psi)_{L^2} = (\phi, \mathcal{D}(\psi))_{L^2} \quad \text{for any } \phi, \psi \in \Gamma_{\text{comp}}(\mathbb{S}).$$

Not all Dirac operators have this property and we want to investigate which ones do. We begin with a well-known result.

3.3.6 Lemma (cf. [LM89, Proposition II.5.3])
The Riemannian Dirac operator is formally self-adjoint.

Proof: Fix some point $x \in M$. We can then use a local orthornomal basis $(s_1, ..., s_n)$ that is x-synchronous, i.e. a basis that is obtained from an orthonormal basis of T_xM by parallel transport along radial geodesics. In particular, such a basis fulfils $\nabla^g_{s_j} s_k(x) = 0$.

We then obtain that

$$\begin{aligned}
(\mathcal{D}^g\varphi(x),\psi(x)) &= \sum_{j=1}^n((s_j.\widetilde{\nabla^g}_{s_j}\varphi)(x),\psi(x)) \\
&= -\sum_{j=1}^n(\widetilde{\nabla^g}_{s_j}\varphi(x),s_j.\psi(x)) \\
&= -\sum_{j=1}^n[s_j((\varphi,s_j.\psi))(x)-(\varphi(x),\widetilde{\nabla^g}_{s_j}(s_j.\psi)(x))] \\
&= -\sum_{j=1}^n[s_j((\varphi,s_j.\psi))(x)-(\varphi(x),(\nabla^g_{s_j}s_j).\psi(x))-(\varphi(x),s_j.\widetilde{\nabla^g}_{s_j}\psi(x))] \\
&= (\varphi(x),\mathcal{D}^g\psi(x))-\sum_{j=1}^n s_j((\varphi,s_j.\psi))(x).
\end{aligned}$$

We can define a vector field V uniquely by demanding that

$$g_x(V(x),W(x))=(\varphi(x),W(x).\psi(x)) \quad \text{for any } W\in\mathfrak{X}(M), x\in M.$$

Then, using that $\nabla^g_{s_j}s_j(x)=0$, we obtain

$$\begin{aligned}
\sum_{j=1}^n s_j((\varphi,s_j.\psi))(x) &= \sum_{j=1}^n[s_j(g(V,s_j))(x)-g(V,\nabla^g_{s_j}s_j)(x)] \\
&= \sum_{j=1}^n g(\nabla^g_{s_j}V,s_j)(x) \\
&= div(V)(x).
\end{aligned}$$

Integrating over M and using Stoke's theorem then yields the claim. □

We now introduce two properties of connections relating to their Dirac operators:

3.3.7 Definition A metric connection ∇ on TM is called *nice*, if the geometric Dirac operator it induces is formally self-adjoint.
Two connections ∇^1 and ∇^2 are called *Dirac equivalent* if they induce the same geometric Dirac operator.

These properties are reflected in the structure of the torsion. To see this, we first prove an auxiliary result stated in [Nic05, formula (1.4)].

3.3.8 Lemma
Let M be spin and ∇ a metric connection on M. Then the following formula holds for geometric Dirac operators:

$$\mathcal{D}(\nabla)=\mathcal{D}^g-\frac{1}{2}c(\operatorname{tr}A)+\frac{3}{2}c(\mathfrak{b}A),$$

where A is the potential of ∇, i.e. $\nabla=\nabla^g+A$.

Proof: This is proven using the local formula for $\widetilde{\nabla}$. With the usual conventions, we have

$$\begin{aligned}
\widetilde{\nabla}_X\varphi &= [\widetilde{s},X(v)+\sum_{j<k}g(\nabla_X s_j,s_k)s_j.s_k.v], \\
\widetilde{\nabla^g}_X\varphi &= [\widetilde{s},X(v)+\sum_{j<k}g(\nabla^g_X s_j,s_k)s_j.s_k.v].
\end{aligned}$$

Now, the Dirac operator can locally be written as $\mathcal{D}(\nabla)\varphi = \sum_{k=1}^{n} s_k.\widetilde{\nabla}_{s_k}\varphi$. Using all this, we obtain that locally

$$(\mathcal{D}(\nabla) - \mathcal{D}^g)\varphi = \sum_{l=1}^{n} [\widetilde{s}, \frac{1}{2}\sum_{j<k} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v].$$

Using that $s_j.s_k. = -s_k.s_j.$ and (using that both connections are metric) that $g(A_{s_l}s_j, s_k) = -g(s_j, A_{s_k}s_l)$, we can rewrite this as

$$(\mathcal{D}(\nabla) - \mathcal{D}^g)\varphi = [\widetilde{s}, \frac{1}{4}\sum_{l,j,k=1}^{n} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v].$$

First, we only consider the terms where $l = j$. For these we obtain

$$\begin{aligned} &[\widetilde{s}, \frac{1}{4}\sum_{j,k=1}^{n} g(A_{s_j}s_j, s_k)\underbrace{s_j.s_j.}_{=-1}s_k.v] \\ =&-[\widetilde{s}, \frac{1}{4}\sum_{j,k=1}^{n} A(s_j; s_j, s_k)s_k.v] \\ =&-[\widetilde{s}, \frac{1}{4}\sum_{k=1}^{n} \operatorname{tr} A(s_k)s_k.v] \\ =&-[\widetilde{s}, \frac{1}{4}\sum_{k=1}^{n} \operatorname{tr} A.v] \\ =&-\frac{1}{4}\operatorname{tr} A.\varphi. \end{aligned}$$

Next, for $k = l$ we have that

$$g(A_{s_k}s_j, s_k)s_k.s_j.s_k. = -g(s_j, A_{s_k}s_k)s_k.s_j.s_k. = g(s_j, A_{s_k}s_k)s_j.s_k.s_k.$$

and thus the same calculations as for $j = l$ can be applied and both cases together yield $-\frac{1}{2}\operatorname{tr} A.\varphi$. Finally, for $l \neq j, k$, we go back to considering only indices $j < k$ and obtain for that part of the sum

$$\begin{aligned} &[\widetilde{s}, \frac{1}{2}\sum_{l=1}^{n}\sum_{\substack{l\neq j,k \\ j<k}} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v] \\ =&\left[\widetilde{s}, \frac{1}{2}\left(\sum_{j<k<l} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v + \sum_{j<l<k} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v + \sum_{l<j<k} g(A_{s_l}s_j, s_k)s_l.s_j.s_k.v\right)\right] \\ =&\left[\widetilde{s}, \frac{1}{2}\left(\sum_{j<k<l} A(s_l, s_j, s_k)s_l.s_j.s_k.v + \sum_{j<k<l} A(s_k, s_j, s_l)s_k.s_j.s_l.v + \sum_{j<k<l} A(s_j, s_k, s_l)s_j.s_k.s_l.v\right)\right] \\ =&[\widetilde{s}, \frac{3}{2}\sum_{j<k<l} (\mathfrak{b}A)(s_j, s_k, s_l)s_j.s_k.s_l.v] \\ =&\frac{3}{2}(\mathfrak{b}A).\varphi. \end{aligned}$$

The claim follows putting together the above facts. □

From the above lemma we deduce the following result:

3.3.9 Corollary
Assume that M is spin and let ∇ be a metric connection on TM. The the following statements hold:

(1) ∇ is nice if and only if its torsion satisfies $\operatorname{tr} T = 0$.

(2) Assume ∇ is nice. Then we have

$$\mathcal{D}(\nabla) = \mathcal{D}^g + \frac{3}{2}c(\mathfrak{b}A) = \mathcal{D}^g + \frac{3}{4}c(\mathfrak{b}T).$$

(3) Assume ∇^1 and ∇^2 are nice. Then they are Dirac-equivalent if and only if $\mathfrak{b}T^1 = \mathfrak{b}T^2$.

Proof: We use the above formula. The Riemannian Dirac operator is symmetric. Recall that the Clifford multiplication by forms can be calculated by

$$\omega.\phi = \sum_{i_1<...<i_k} \omega(s_{i_1}, ..., s_{i_k}) s_{i_1} ... s_{i_k}.\phi$$

for some orthonormal basis $(s_1, ..., s_n)$. We know that Clifford multiplication by such vectors is skew-symmetric. Thus Clifford multiplication by a one-form is skew-symmetric whereas multiplication by a three-form is symmetric (bringing over all vectors gives one minus, reordering them another one). Thus, for $\mathcal{D}(\nabla)$ to be symmetric $c(\operatorname{tr} A)$ must vanish. Yet, because $c(\operatorname{tr} A)^2 = -\|\operatorname{tr} A\|^2$ this implies that $\operatorname{tr} A$ itself must vanish, which by lemma 3.3.2 is equivalent to the vanishing of $\operatorname{tr} T$. This proves (1). (2) follows immediately from (1) and then, (3) is an immediate consequence. $\square$

We have the following converse:

3.3.10 Corollary
Let $P = \mathcal{D}^g + \omega$ be an operator on $\Gamma(\mathbb{S})$ with $\omega \in \Omega^3(M)$. Then, P is the geometric Dirac operator induced by $\nabla^g + A$ where $A = \frac{2}{3}\omega$.

Proof: Using the above results, we deduce that

$$\mathcal{D}(\nabla^g + \frac{2}{3}\omega) = \mathcal{D}^g - \frac{1}{3}\underbrace{\operatorname{tr}\omega}_{=0} + \mathfrak{b}\omega = P\omega = \mathcal{D}^g + \omega.$$

$\square$

Note that this is only a partial converse of lemma 3.3.8, as obviously not all geometric Dirac operators will have the form $\mathcal{D}^g + \omega$ with a three-form ω. More precisely, this captures exactly the nice connections. Furthermore, note that there may be many other connections inducing the same Dirac operator.

In the case of a *Spin*^{c} structure, we can induce a connection on $\mathbb{S}^c$ as follows: We fix a metric connection ∇ on TM and the connection one-form C^∇ it induces on P_{SO}. As opposed to the case of a spin structure, this is insufficient for inducing a connection form on the principal *Spin*^{c}-bundle, because $P_{CS}(M)$ is not a covering of P_{SO}. Therefore, we need to fix an auxiliary connection form Z on P_1. Together, they form a connection form $C^\nabla \times Z$ on $P_{SO} \times P_1$, which now lifts to $\widetilde{C^\nabla \times Z}$ such that the following diagram commutes:

$$\begin{array}{ccc} TP_{CS}(M) & \xrightarrow{\widetilde{C^\nabla \times Z}} & \mathfrak{spin}_n^c \simeq \mathfrak{spin}_n \oplus i\mathbb{R} \\ \Big\downarrow{\scriptstyle d\xi} & & \Big\downarrow{\scriptstyle p_*} \\ T(P_{SO}(M) \times P_1) & \xrightarrow{C^\nabla \times Z} & \mathfrak{so}_n \oplus i\mathbb{R}. \end{array} \tag{3.5}$$

As above, $\widetilde{C^\nabla \times Z}$ induces a connection $\widetilde{\nabla}^Z$ on $\mathbb{S}^c$. Using the expression from proposition A.9, we deduce that locally, it can be described as follows: Let $\phi \in \Gamma(\mathbb{S}^c)$ be localy described by $\phi|_U = [\widetilde{s \times e}, v]$, where $s \in \Gamma(U, P_{SO})$ and $e \in \Gamma(U, P_1)$ and $\widetilde{s \times e}$ is a lifting to $\Gamma(U, P_{CS})$ and finally, $v \in C^\infty(U, \Delta_n)$. Then, we have that

$$\begin{aligned}
(\widetilde{\nabla}^Z_X \phi)(x) &= [\widetilde{s \times e}, X(v) + \kappa^c_* \left((\widetilde{A^\nabla \times Z})^{\widetilde{s \times e}}(X) \right) .v] \\
&= [\widetilde{s \times e}, X(v) + \kappa^c_* \left((\widetilde{A^\nabla \times Z})(d\widetilde{s \times e}(X)) \right) .v] \\
&= [\widetilde{s \times e}, X(v) + \kappa^c_* \left((p_*^{-1}((C^\nabla \times Z)^{s \times e}(X)) \right) .v] \quad \text{(using (3.5))} \\
&= [\widetilde{s \times e}, X(v) + \kappa^c_* \left(\frac{1}{2} \sum_{j<k} g(\nabla_X s_j, s_k)(\lambda_*^{-1}(E_{jk}) + \frac{1}{2} Z^e(X) \right) .v] \\
&= [\widetilde{s \times e}, X(v) + \frac{1}{2} \sum_{j<k} g(\nabla_X s_j, s_k) s_j.s_k.v + \frac{1}{2} Z^e(X).v].
\end{aligned}$$

As for a real spin structure, we can define geometric Dirac operators.

3.3.11 Definition Let M admit a $Spin^c$ structure and let ∇ be a metric connection on M and Z a connection on P_1. Then the first-order differential operator

$$\mathcal{D}_c(\nabla, Z) \colon \Gamma(\mathbb{S}^c) \xrightarrow{\widetilde{\nabla}^Z} \Gamma(T^*M \otimes \mathbb{S}^c) \xrightarrow{c} \Gamma(\mathbb{S}) \tag{3.6}$$

is called the ($Spin^c$-)*geometric Dirac operator* associated to ∇ and Z.
The operator $\mathcal{D}^g_c(Z) = \mathcal{D}(\nabla^g, Z)$ is called the *Riemannian Dirac operator*.

The notion of self-adjointness is defined as in the spin case. The next lemma collects some results on connections on $\mathbb{S}^c$ and their Dirac operators. It is proven as in the spin case.

3.3.12 Lemma
Let M be a $Spin^c$-manifold and let ∇ be any metric connection on TM and Z a connection form on P_1. Let furthermore $\widetilde{\nabla}^Z$ be the connection induced on the spinor bundle $\mathbb{S}^c$ by ∇ and Z. Then, we have the following results:

(1) For any vector fields $X, Y \in \mathfrak{X}(M)$ and spinor field $\varphi \in \mathbb{S}^c$, the following formula holds:

$$\widetilde{\nabla}^Z_X(Y.\varphi) = (\nabla_X Y).\varphi + Y.\widetilde{\nabla}^Z_Y.\varphi.$$

(2) The connection $\widetilde{\nabla}^Z$ is metric with respect to the hermitian scalar product on $\mathbb{S}^c$.

(3) The Riemannian Dirac operator $\mathcal{D}_c(Z)$ for any connection Z is formally self-adjoint.

The Dirac operator $\mathcal{D}_c(\nabla, Z)$ does of course depend on ∇ and Z. However, the difference between it and the Riemannian Dirac operator induced by the same connection Z on P_1 depends only on ∇ as the following lemma shows:

3.3.13 Lemma
Let M admit a $Spin^c$ structure (P_{CS}, P_1, f), let ∇ be a metric connection on TM and Z a connection on P_1. Then the following formula holds for the geometric Dirac operators:

$$\mathcal{D}_c(\nabla, Z) = \mathcal{D}^g_c(Z) - \frac{1}{2} c(\operatorname{tr} A) + \frac{3}{2} c(\mathfrak{b} A).$$

Proof: This is proven exactly as in the case of a spin structure. The additional connection Z appears in the local formulæ of both $\widetilde{\nabla}^Z$ and $\widetilde{\nabla^g}^Z$ and thus the difference $\mathcal{D}_c(\nabla, Z) - \mathcal{D}_c^g(Z)$ looks exactly as in the real case. $\square$

One easily deduces the following results from the above formula:

3.3.14 Corollary
Let M be a manifold admitting a $Spin^c$ structure and let ∇^1, ∇^2 be metric connections on TM and Z_1, Z_2 connections on P_1. Then the following statements hold:

(1) $\mathcal{D}_c(\nabla^1, Z_1)$ is formally self-adjoint if and only if $\mathcal{D}_c(\nabla^1, Z_2)$ is, which holds if and only if $\operatorname{tr} T^1 = 0$.

(2) $\mathcal{D}_c(\nabla^1, Z_1) = \mathcal{D}_c(\nabla^2, Z_1)$ if and only if $\mathcal{D}_c(\nabla^1, Z_2) = \mathcal{D}_c(\nabla^2, Z_2)$.

(3) Assume ∇^1, ∇^2 are nice. Then $\mathcal{D}_c(\nabla^1, Z_1) = \mathcal{D}_c(\nabla^2, Z_1)$ if and only if $\mathfrak{b}T^1 = \mathfrak{b}T^2$.

Assume now that M is also spin. Then $\mathcal{D}_c(\nabla^1, Z_1)$ is formally self-adjoint if and only $\mathcal{D}(\nabla^1)$ is. Furthermore, two connections ∇^1, ∇^2 are Dirac equivalent if and only if $\mathcal{D}_c(\nabla^1, Z_1) = \mathcal{D}_c(\nabla^2, Z_1)$.

And we can also extend the following corollary to the complex case:

3.3.15 Corollary
Let $P = \mathcal{D}_c^g(Z) + \omega$ be an operator on $\Gamma(\mathbb{S}^c)$ with $\omega \in \Omega^3(M)$. Then, P is the geometric Dirac operator $\mathcal{D}_c(\nabla^g + A, Z)$ with $A = \frac{2}{3}\omega$.

We can now extend the notion of nice and Dirac equivalent connections to the case where M admits only a $Spin^c$ structure.

3.3.16 Definition Let M be a manifold admitting a spin or a $Spin^c$ structure and let ∇ be a metric connection on TM. We call ∇ *nice* if $\mathcal{D}(\nabla)$ or $\mathcal{D}_c(\nabla, Z)$ (for any connection Z on P_1) is formally self-adjoint.
We call two connections *Dirac equivalent* if the Dirac operators induced by them on $\mathbb{S}$ or $\mathbb{S}^c$ are equal.

By the above results, this is well-defined. Recalling the condition for Dirac equivalence in the case of nice connections, we introduce the following notion:

3.3.17 Definition Two metric connections ∇^1 and ∇^2 are called *quasi-equivalent* if $\mathfrak{b}T^1 = \mathfrak{b}T^2$.

Note that two nice connections are quasi-equivalent if and only if they are Dirac equivalent. In the general case, the notion of quasi-equivalence is, as the name suggests, less strict and in particular (as we shall see) a nice connection can be quasi-equivalent to one that is not nice.
This concludes our discussion of connections and Dirac operators for the moment. In the following section we will study the existence of $Spin^c$ structures on almost-complex and contact manifolds.

3.4 *Spin^c* structures on almost-complex and contact manifolds

There are many more manifolds admitting a *Spin^c* structure than manifolds admitting a *Spin*-structure. In the words of H.B. Lawson and M.-L. Michelsohn "it requires some searching about to find an oriented manifold which is not *Spin^c*" (cf. [LM89, p. 393]). Among the manifolds admitting a *Spin^c* structure are those whose frame bundle admits a U_m-reduction, in particular almost-hermitian and metric contact manifolds.
In this section, we shall examine the *Spin^c* structures of these manifolds, paying particular attention to the structure of their spinor bundles and the connections on them. We begin with the more well-known case of an almost-hermitian manifold and then proceed to analogous constructions on a metric contact manifold.

3.4.1 Lemma
Let M be a manifold admitting a U_k-reduction of its SO frame bundle. Then M admits a Spin^c-structure.

Proof: The existence of a U_k-reduction means that there exists a U_k-bundle Q together with a mapping $h_Q : Q \to P_{SO}(M)$ and one then has $P_{SO} = Q \times_{U_k} SO_{2k}$. By lemma 3.1.9 there exists a mapping $F : U_k \to Spin^c_{2k}$ such that $p \circ F = f$. We define a *Spin^c* bundle by

$$P_{CS} = Q \times_F Spin^c_{2k}$$

and

$$\begin{aligned} h_{CS} : P_{CS} &\longrightarrow P_{SO} \\ [q,g] &\longmapsto h_Q(q)\lambda^c(g). \end{aligned}$$

Then
$$h_{CS}([q,g]g\prime) = h_{CS}([q,gg\prime]) = h_Q(q)\lambda^c(gg\prime) = h_Q(q)\lambda^c(g)\lambda^c(g\prime),$$
which proves that (P_{CS}, h_{CS}) is a *Spin^c* structure. □

The case of almost-hermitian manifolds

In particular, any almost-hermitian manifold of dimension $n = 2m$ admits a U_m-reduction $P_U(M)$ of its frame bundle (compare the discussion in section 1.1). Thus, any almost-hermitian manifold admits a *Spin^c* structure. The *Spin^c* structure described above

$$P_{CS}(M) = P_U(M) \times_F Spin^c_n$$

is called the *canonical Spin^c structure* of the almost-hermitian manifold. Because we can always find an almost hermitian metric on an almost-complex manifold[7], any almost-complex manifold admits a *Spin^c* structure (however, it is not canonical any more, because we first have to choose an almost-hermitian metric).
We want to study the canonical *Spin^c* structure in some more detail: Recall that with each *Spin^c* structure, we have a U_1-bundle $P_1 = P_{CS}/Spin$. In the case of the canonical *Spin^c*-structure, this bundle can be written as

$$P_1 = {}^{\textstyle (P_U \times_F (Spin_n \times_{\mathbb{Z}_2} S^1))}\!\big/_{\textstyle Spin_n} = P_U \times_{\det} S^1,$$

[7] As it is well known, there always exists a Riemannian metric h. Then $g = h + h(J\cdot, J\cdot)$ is almost-hermitian.

i.e. it is a S^1-extension of P_U. The equality is immediate from the definiton of F. We can then give a more detailed description of the two-fold covering ξ:

$$\begin{aligned}\xi\colon P_{CS}(M) = P_U(M) \times_F Spin^c_n &\longrightarrow P_{SO} \times (P_U(M) \times_{\det} S^1)\\ [s,[g,z]] &\longmapsto (s\lambda(g)) \times [s, z^2].\end{aligned}$$

Similarly, the canonical line bundle is given by

$$\mathcal{L} = P_{CS} \times_l \mathbb{C} = P_U \times_F Spin^c_{2m} \times_l \mathbb{C}.$$

Because $l = pr_2 \circ p$, we obtain from the definition of F that $l \circ F = \det$. Thus, we have that

$$\mathcal{L} = P_{U_k} \times_{\det} \mathbb{C} = \Lambda^k_c(TM), \tag{3.7}$$

where Λ^k_c indicates that the exterior powers are taken over $\mathbb{C}$ and that TM is considered as a complex vector bundle.
Moreover, the associated spinor bundle can also be given a more explicit description:

3.4.2 Proposition
Let (M, J, g) be an almost-hermitian manifold and let $\mathbb{S}^c$ be the spinor bundle associated to the canonical $Spin^c$-structure on M. Then we have $\mathbb{S}^c \simeq \Lambda^{0,*}(T^*M)$ and Clifford multiplication is given by

$$X.\varphi = \sqrt{2}\left((X^{1,0})^\flat \wedge \varphi - X^{0,1} \lrcorner \varphi\right)$$

for any $X \in T_xM$ and $\varphi \in \mathbb{S}^c_x$.

Proof: We have that that

$$\mathbb{S}^c = P_{CS} \times_{\widetilde{cl}} \Lambda^{0,*}((\mathbb{R}^{2m})^*) = P_{U_k} \times_F Spin^c_{2m} \times_{\widetilde{cl}} \Lambda^{0,*}((\mathbb{R}^{2m})^*).$$

By lemma 3.1.10, the representations ρ_Λ and $\widetilde{cl} \circ F$ coincide and we thus have (cf. proposition A.11) that

$$\mathbb{S}^c = P_U(M) \times_{\rho_\Lambda} \Lambda^{0,*}((\mathbb{R}^{2m})^*) = \Lambda^{0,*}(T^*M).$$

By the definition of Clifford multplication on M, the above formula for elements of $TM \subset Cl(M, g)$ comes from the formula for cl as described in proposition 3.1.8. $\square$

Using the above formula, one obtains that Clifford multiplication for one-forms is then given by

$$\alpha.\varphi = \sqrt{2}(\alpha^{0,1} \wedge \varphi - (\alpha^{1,0})^\natural \lrcorner \varphi).$$

Now, note that we have two ways of defining a covariant derivative on $\mathbb{S}^c$. The first way is the one described in the previous section, possible for any spinor bundle associated to a $Spin^c$-structure. However, due to the special form of $\mathbb{S}^c$ here, we have a second way which is inducing a covariant derivative on forms, given by the following formula

$$(\nabla_X \omega)(X_1, ..., X_k) = X(\omega(X_1, ..., X_k)) - \sum_{j=1}^k \omega(X_1, ..., \nabla_X X_j, ..., X_k). \tag{3.8}$$

We now discuss the relationship between the two covariant derivatives. In order to do so, we first introduce the notion of a connection compatible with the almost-hermitian structure.

3.4.3 Definition Let (M, g, J) be an almost-hermitian manifold. Then a connection ∇ on TM is called *hermitian* it it is metric and parallelizes the almost-complex structure: $\nabla J = 0$.

3.4.4 Lemma
Let (M^{2m}, g, J) be an almost-hermitian manifold and ∇ a hermitian connection on TM. Then the connection form C induced by ∇ on the frame bundle restricts to a connection form on the unitary frame bundle $P_U(M)$.

Proof: A priori, $C \in \Omega^1(TP_{GL}(M), \mathfrak{gl}_{2m})$. Obviously, its restriction to P_U is still a connection form. We only need to show that $C(X) \in \mathfrak{u}_m$, i.e. that

$$C(X) = -\overline{C(X)}^T \tag{3.9}$$

as a complex matrix. As a real matrix $C(ds(X)) = (\omega_{jk}(X))$ where the ω_{jk} are defined by $\nabla s_j = \sum_k \omega_{kj} \otimes s_k$ for some basis $s_1, ..., s_{2m}$.
If $C(X)$ is to come from a complex matrix, $(\omega_i j)$ must fulfil that

$$\begin{pmatrix} \omega_{2j-1,2k-1} & \omega_{2j-1,2k} \\ \omega_{2j,2k-1} & \omega_{2j,2k} \end{pmatrix} = \begin{pmatrix} \omega_{2j-1,2k-1} & \omega_{2j-1,2k} \\ -\omega_{2j-1,2k} & \omega_{2j-1,2k-1} \end{pmatrix} \tag{3.10}$$

and the condition (3.9) translates as

$$\begin{pmatrix} \omega_{2j-1,2k-1} & \omega_{2j-1,2k} \\ -\omega_{2j-1,2k} & \omega_{2j-1,2k-1} \end{pmatrix} = \begin{pmatrix} -\omega_{2k-1,2j-1} & \omega_{2k-1,2j} \\ -\omega_{2k-1,2j} & omega_{2k-1,2j-1} \end{pmatrix}. \tag{3.11}$$

Now let $e_1, ..., e_m$ be a unitary basis of TM understood as a complex vector space, then we can form a real basis $\{s_1, ..., s_{2m}\}$ by setting $s_{2k-1} = e_k$ and $s_{2k} = Je_k$. From $(\nabla J)e_k = \nabla(Je_k) - J(\nabla e_k)$ we obtain that $\nabla s_{2k} = J(\nabla s_{2k-1})$. Thus, we obtain

$$\begin{aligned} \sum_{j=1}^{m} \omega_{2j,2k}(X)s_{2j} + \omega_{2j-1,2k}(X)s_{2j-1} &= \nabla_X s_{2k} \\ &= J(\nabla_X s_{2k-1}) \\ &= J(\sum_{j=1}^{m} \omega_{2j,2k-1}(X)s_{2j} + \omega_{2j-1,2k-1}(X)s_{2j-1}) \\ &= (\sum_{j=1}^{m} -\omega_{2j,2k-1}(X)s_{2j-1} + \omega_{2j-1,2k-1}(X)s_{2j}), \end{aligned}$$

from which we conclude $\omega_{2j,2k-1} = \omega_{2j-1,2k}$ and $\omega_{2j-1,2k-1} = \omega_{2j,2k}$, exactly fulfilling (3.10). We already know that C is a connection on $P_{SO}(M)$ and hence that C fulfils $C = -C^T$, i.e. $\omega_{j,k} = -\omega_{k,j}$. Together with the above, this yields (3.11) and the claim is thus proved. □

Now, starting with a hermitian connection ∇, we prove that the covariant derivative defined by (3.8) can be alternatively described as follows: Because

$$\Lambda^{p,q}(T^*M) \simeq P_U(M) \times_{\rho_\Lambda} \Lambda^{p,q}((\mathbb{R}^m)^*)$$

(cf. discussion in section 1.2.1), we can induce a connection on $\Lambda^{0,*}$ by considering the connection form C^∇ induced on $P_U(M)$ and then the covariant derivative induced on the associated vector bundle $\Lambda^{0,*}$, which we shall denote ∇^C. The following proposition compares the two connections in a somewhat more general setting.

3.4.5 Proposition
Let ∇ be a connection on TM. Then the covariant derivatives induced on $\Omega^*(M)$ by (3.8) and the covariant derivative ∇^C induced on the sections of the associated vector bundle $\Lambda^*(T^*M)$ by the connection form C^∇ on $P_{GL}(M)$ coincide.
The same result holds in the case of an almost-hermitian manifold and a hermitian connection for the covariant derivative induced on $\Omega^{0,*}(M)$ by formula (3.8) and the one induced by C^∇ on $P_U(M)$ and the representation ρ_Λ.

Proof: Both covariant derivatives must satisfy the Leibniz rule

$$\nabla(f\alpha) = df \otimes \alpha + f\nabla\alpha.$$

Furthermore, it is easily seen that both covariant derivatives satisfy the following rule with respect to the exterior product:

$$\nabla(\alpha \wedge \beta) = (\nabla\alpha) \wedge \beta + \alpha \wedge (\nabla\beta).$$

Therefore, it is sufficient to verify that the two coincide on a basis of one-forms. Now let $(s_1, ..., s_n)$ be a local basis over $U \subset M$ and let $(s^1, ..., s^n)$ be its dual. Then, by proposition A.9, ∇^C is given over $U \subset M$ by

$$\nabla^C_X(s^j) = \left[s, de^j(X) + d_e\rho^*((C^\nabla)^s(X))(e^j)\right],$$

where $s^j = [s, e^j]$. In order to proceed, we calculate $d\rho^*$. Recall that $\rho(B)$ can be interpreted as left multiplication by B and $\rho^*(B)$ as right multiplication by B^{-1}. Thus, the differential is right multiplication by $-B$. Now recall the local formula for $(C^\nabla)^s$: If $\nabla s_k = \sum_{l=1}^n \omega_{lk} \otimes s_l$, then $(C^\nabla)^s(X)$ is the matrix $(\omega_{lk}(X))$. Thus, we obtain that

$$d\rho^*((C^\nabla)^s(X)) = -(\omega_{lk}(X)).$$

Therefore, considering e^j as a row vector, we have that

$$\begin{aligned} d\rho^*((C^\nabla)^s(X))e^j &= -e^j \cdot (\omega_{lk}(X)) \\ &= -\sum_{l=1}^n \omega_{jl} e^l \end{aligned}$$

and therefore

$$\nabla^C_X(s^j) = [s, de^j(X) - \sum_{l=1}^n \omega_{jl}(X)e^l].$$

Testing this on the basis (s_k) gives

$$\begin{aligned} \nabla^C_X(s^j)(s_k) &= X(s^j(s_k)) - \sum_{l=1}^n \omega_{jl}(X)s^l(s_k) \\ &= -\omega_{jk}(X). \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla_X s^j(s_k) &= \overbrace{X(s^j(s_k))}^{=0} - s^j(\nabla_X s_k) \\ &= -\sum_{l=1}^n s^j(\omega_{lk} s_l) \\ &= -\omega_{jk}(X), \end{aligned} \tag{3.12}$$

so that the two are equal. □

3.4.6 Corollary

Let ∇ be a metric connection on a Riemannian manifold. Then, for the connection extended to one-forms the following relationship holds for any $\alpha \in \Omega^1(M)$:

$$\nabla_X \alpha = (\nabla_X \alpha^\natural)^\flat$$

Proof: It is again sufficient to prove this for elements of a basis (s_j) and its dual (s^j). In this case, we have $s_j^\flat = s^j$ and $(s^j)^\natural = s_i$. Furthermore, using (3.12), we deduce that

$$\nabla_X s^j = -\sum_{k=1}^n \omega_{jk}(X) s^k,$$

where the ω_{jk} are defined by $\nabla_X s_j = \sum_{k=1}^n \omega_{kj}(X) s_k$. Because ∇ is metric, ω is skew-symmetric and we thus obtain

$$(\nabla_X s_j)^\flat = -(\sum_{k=1}^n \omega_{jk}(X) s_k)^\flat = -\sum_{k=1}^n \omega_{jk}(X) s^k.$$

This yields the claim. $\square$

Next, we want to use this result to compare the two connections on $\mathbb{S}^c$. Recall that P_{CS} and P_1 are extensions of the unitary frame bundle P_U and thus P_U is a reduction of those two bundle with the reduction maps given by

$$\begin{aligned} \phi_{CS} \colon P_U &\longrightarrow P_{CS} \\ p &\longmapsto [p,1]_F \\ \phi_1 \colon P_U &\longrightarrow P_1 \\ p &\longmapsto [p,1]_{\mathrm{det}}. \end{aligned}$$

Now, if we choose a hermitian connection ∇ on TM, we obtain a connection on the unitary frame bundle. A connection form C on P_U however, admits a det-extension, i.e. a connection form Z on P_1 uniquely determined by the requirement

$$\phi_1^* Z = (\det_{\mathbb{C}})_* \circ C = \operatorname{tr} C$$

(cf. proposition A.14). Thus, every hermitian connection ∇ on TM induces a connection form Z^∇ on P_1 as described above.

In what follows, we consider two hermitian connections ∇^c and ∇^z. Let C the connection form induced by ∇^c on $P_U(M)$ and $P_{SO}(M)$ and C^z the connection form induced by ∇^z and furthermore Z the connection form induced by ∇^z (or C^z) on the U_1-bundle P_1. Together, C and Z induce a connection form $\widetilde{C \times Z}$ (compare section 3.3) which, in turn, induces a covariant derivative on the spinor bundle which we denote $\widetilde{\nabla}^Z$. Recall that we have a local formula for this covariant derivative, which uses a representation of a local spinor field $\varphi \in \Gamma(U, \mathbb{S}^c)$ as $\varphi = [\widetilde{s \times e}, v]$, where $s \in \Gamma(U, P_{SO}(M))$ and $e \in \Gamma(U, P_1)$ and $\widetilde{s \times e}$ is a lifting of the product to a local section of $P_{CS}(M)$.

Due to the special structure of P_{CS} which we have for the canonical *Spin*^{c}-structure, we can consider a particular type of local sections:

3.4.7 Lemma

Let (M, g, J) be almost-hermitian and let $P_{CS}(M)$ be its canonical Spin^c structure with spinor bundle $\mathbb{S}^c$. Let $s \in \Gamma(U, P_{SO}(M))$ be a local section of the SO-frame bundle and define $e = \phi_1 \circ s$. Then the following holds:

$$\widetilde{s \times e} = \phi_{CS}(s).$$

Proof: We need to show that $\xi(\phi_{CS}(s)) = s \times e$. Recall that the two-fold covering map $\xi\colon P_{CS} \to P_{SO}(M) \times P_1$ is given by $\xi([u,[g,z]_{\mathbb{Z}_2}]_F) = u\lambda(g) \times [u, z^2]_{\det}$ and thus

$$\xi(\phi_{CS}(s)) = \xi([s,[1,1]]_F) = s\lambda(1) \times [s,1] = s \times \phi_1(s) = s \times e.$$

□

Then, the local formula for $\widetilde{\nabla}^Z$ is given by

$$\begin{aligned}\widetilde{\nabla}^Z_X \varphi &= [\phi_{CS}(s), dv(X) + \kappa_* (\widetilde{C \times Z}^{\phi_{CS}(s)}(X))v] \\ &= [\phi_{CS}(s), dv(X) + \kappa_* \phi_{CS}^*(\widetilde{C \times Z}(ds(X)))v]\end{aligned} \tag{3.13}$$

and we therefore consider $\phi_{CS}^*(\widetilde{C \times Z})$ in some more detail:

$$\begin{aligned}\phi_{CS}^* \widetilde{C \times Z} &= p_*^{-1} \phi_{CS}^*(\widetilde{C \times Z} \circ d\xi) \\ &= p_*^{-1}(\widetilde{C \times Z} \circ d(\xi \circ \phi_{CS})).\end{aligned}$$

Recall from the proof above that $\xi \circ \phi = \mathrm{id} \times \phi_1$. Taking the derivative, we obtain

$$\phi_{CS}^* \widetilde{C \times Z} = p_*^{-1}(C + \phi_1^* Z)$$

and, using that Z is the det-extension of C^z, we obtain

$$\phi_{CS}^* \widetilde{C \times Z} = p_*^{-1}(C + (\det)_* C^z).$$

We note that det is the complex determinant here (i.e. we understand C^z as taking values in the complex space $\mathfrak{u}_m$). Its derivative is the complex trace $\mathrm{tr}_{\mathbb{C}}$. Thus, we obtain

$$\begin{aligned}\phi_{CS}^* \widetilde{C \times Z} &= p_*^{-1}(C + \mathrm{tr}_{\mathbb{C}}\, C + \mathrm{tr}_{\mathbb{C}}\, C^z - \mathrm{tr}_{\mathbb{C}}\, C) \\ &= F_*(C) + \frac{1}{2}\mathrm{tr}_{\mathbb{C}}(C^z - C).\end{aligned}$$

Then, continuing from (3.13), we obtain

$$\begin{aligned}\widetilde{\nabla}^Z_X \varphi &= [\phi(s), dv(X) + \kappa_* \phi_{CS}^*(\widetilde{C \times Z}(ds(X)))v] \\ &= [\phi(s), dv(X) + \kappa_*(F_*(C^s(X))v + \frac{1}{2}\mathrm{tr}_{\mathbb{C}}((C^z)^s(X) - C^s(X))v] \\ &= [\phi(s), (\rho_\Lambda)_*(C^s(X))v + \frac{1}{2}\mathrm{tr}_{\mathbb{C}}((C^z)^s(X) - C^s(X))v].\end{aligned} \tag{3.14}$$

We know the first part of the above formula to be equal to the covariant derivative induced on forms and will now consider the second part in some more detail. Locally, $C^s(X)$ is a matrix given by $\omega_{jk}(X)$ which are given by $\nabla^c_X s_j = \sum_k \omega_{kj}(X) s_k$ and the trace is to be understood in the sense of that matrix (considered as a complex matrix). Now, we obtain that

$$\nabla^c_X s_j - \nabla^z_X s_j = \sum_k (\omega_{kj}(X) - \omega^z_{kj}(X)) s_j.$$

On the other hand, we have that

$$\nabla^c_X s_j - \nabla^z_X s_j = A^c_X s_j - A^z_X s_j,$$

where A^c and A^z denote the potential of ∇^c and ∇^z respectively. Then, we have that

$$\omega_{kj}(X) - \omega^z_{kj}(X) = g((\nabla^c_X - \nabla^z_X)s_j, s_k) = A^c(X; s_j, s_k) - A^z(X, s_j, s_k).$$

Now, A is a real form and we will need to translate the complex trace onto such a form. A complex $n \times n$-matrix $B = (z_{jk})$ is represented by a real $2n \times 2n$ matrix $B_{\mathbb{R}}$ given by

$$B_{\mathbb{R}} = (Z_{jk}) \quad \text{where each } Z_{jk} \text{ is a block} \quad Z_{jk} = \begin{pmatrix} \mathrm{Re}(z_{jk}) & -\mathrm{Im}(z_{jk}) \\ \mathrm{Im}(z_{jk}) & \mathrm{Re}(z_{jk}) \end{pmatrix}$$

The complex trace is given by summing over all z_{jj}, thus we have

$$\mathrm{tr}_{\mathbb{C}} B = \sum_{j=1}^{m} (B_{\mathbb{R}})_{2j,2j} + i \sum_{j=1}^{m} (B_{\mathbb{R}})_{2j,2j-1}.$$

In our case, we have matrices in $\mathfrak{so}$ or forms skew-symmetric in the last two arguments and are thus left with the imaginary part only. We are therefore led to define

$$\mathrm{tr}_c \, \Omega(X) = i \sum_{j=1}^{m} \Omega(X; b_{2j}, b_{2j-1})$$

for any $\Omega \in \Omega^2(M, TM)$ and an adapted basis (b_j) and thus have

$$[\phi_{CS}(s), \mathrm{tr}_{\mathbb{C}}((C^z)^s(X) - C^s(X))v] = \mathrm{tr}_c(A^c - A^z)(X).$$

In particular, this means that $[\phi_{CS}(s), \mathrm{tr}((C^z)^s(X) - C^s(X))v]$ is indeed well-defined, i.e. independent of the choice of s and we can split (3.14) into two parts and obtain

$$\widetilde{\nabla}^Z_X \varphi = \nabla^c_X \varphi + \mathrm{tr}(A^c - A^z)(X) \cdot \varphi.$$

We summarize our results in the following theorem:

3.4.8 Theorem
Let (M, g, J) be an almost-hermitian manifold and let it be equipped with its canonical Spin^c structure, noting $\mathbb{S}^c \simeq \Lambda^{0,*}(T^*M)$ the associated spinor bundle. Let furthermore two hermitian connections ∇^c and ∇^z be given on TM with potentials A^c and A^z. We still note ∇^c the connection induced by ∇^c on the bundle of exterior forms.
Let C and C^z be the connection forms induced on $P_U(M)$ and let Z be the det*-extension of C^z. Furthermore, note $\widetilde{\nabla}^Z$ the covariant derivative induced on $\mathbb{S}^c$ by the connection form $\widetilde{C \times Z}$. Then the following formula holds for all vector fields $X \in \mathfrak{X}(M)$ and all spinor fields $\varphi \in \Gamma(\mathbb{S}^c)$:*

$$\widetilde{\nabla}^Z_X \varphi = \nabla^c_X \varphi + \frac{1}{2} \mathrm{tr}_c(A^c - A^z)(X) \cdot \varphi.$$

The case of metric contact manifolds

The case of a metric contact manifold is very much analogous to that of an almost-hermitian manifold, due to the almost-hermitian structure we have on the contact distribution. Extending the results to a metric contact manifold, one only needs to find a way to deal with the additional vector field ξ which is perpendicular to the contact distribution. In this section, we describe the *Spin^c* structure of a metric contact manifold and its spinor bundle and connections on this bundle.

As discussed in section 2.1, any contact manifold of dimension $2m+1$ admits a U_m-reduction of its frame bundle. Therefore, it admits a canonical $Spin^c$ structure. Its spinor bundle can be given a more detailed description, described in [Pet05, section 3], which we discuss in the following. In order to do so, we introduce spaces of (p,q)-forms on a metric contact manifold: We know that such a manifold admits a contact distribution $\mathcal{C}$ which carries an almost-hermitian structure (J,g). Complexifying $\mathcal{C}$ and splitting the complexified space into the $\pm i$-eigenspaces of J, we obtain a decomposition

$$\mathcal{C} \otimes \mathbb{C} = \mathcal{C}^{1,0} \oplus \mathcal{C}^{0,1}$$

just like in the case of an almost-hermitian manifold. Extending this decomposition to the dual by setting $(\mathcal{C}^*)^{1,0} = (\mathcal{C}^{1,0})^*$ and taking exterior powers, we obtain the bundles

$$\Lambda^{p,q}(\mathcal{C}^*) = \Lambda^p((\mathcal{C}^*)^{1,0}) \wedge \Lambda^q((\mathcal{C}^*)^{0,1}).$$

The sections are denoted

$$\Omega^{p,q}(\mathcal{C}) = \Gamma\left(\Lambda^{p,q}(\mathcal{C}^*)\right).$$

3.4.9 Proposition
Let (M,g,η,J) be a metric contact manifold. Then M admits a canonical $Spin^c$-structure which is given by

$$P_{CS}(M) = P_U(M) \times_F Spin^c,$$

where F is the mapping described in lemma 3.1.9. The associated spinor bundle then has the form

$$\mathbb{S}^c \simeq \Lambda^{0,*}(\mathcal{C}^*)$$

and Clifford multiplication is given by

$$X.\varphi = \sqrt{2}\left((X^{1,0})^\flat \wedge \varphi - X^{0,1} \lrcorner \varphi\right) + i(-1)^{\deg\varphi+1}\eta(X)\varphi$$

for any $X \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(\mathbb{S}^c) \simeq \Omega^{0,*}(\mathcal{C})$. The (0,1) and (0,1)-parts are taken of the projection of X onto $\mathcal{C}$.

Proof: The first statement is immediate from lemma 3.4.1. The second statement follows because

$$\begin{aligned}\mathbb{S}^c &= P_{CS}(M) \times_{\widetilde{cl}} \Lambda^{0,*}(\mathcal{C}^*) \\ &= P_U(M) \times_F Spin^c \times_{\widetilde{cl}} \Lambda^{0,*}(\mathcal{C}^*).\end{aligned}$$

Just like in the almost-hermitian case, one proves that

$$P_U(M) \times_{\rho_\Lambda} \Lambda^{0,*}((\mathbb{R}^{2m})^*) \simeq \Lambda^{0,*}(\mathcal{C}^*).$$

By lemma 3.1.12 the representations ρ_Λ and $\widetilde{cl} \circ F$ coincide and we obtain the second statement by proposition A.11. The formula for the Clifford multiplication follows from proposition 3.1.11. □

Just like in the almost-hermitian case, given a metric connection ∇ on TM, we have two ways of inducing a connection on $\mathbb{S}^c$: As the extension of the connection to forms or via the $Spin^c$ structure. The analogue of a hermitian connection is here played by the so-called contact connection:

3.4.10 Definition Let (M,g,η,J) be a metric contact manifold. Then a connection ∇ is called *contact* if it is metric and J is parallel with respect to it: $\nabla J = 0$.

Note that a contact connection also parallelizes the Reeb vector field. This can be seen as follows: We have that $0 = (\nabla J)\xi = \nabla(J\xi) - J(\nabla\xi)$. Because $J\xi \equiv 0$, this implies that $J(\nabla\xi) = 0$, i.e. $\nabla\xi = \lambda\xi$ with $\lambda \in C^\infty(M)$. However, because ξ is of constant length, we have $g(\nabla\xi, \xi) = 0$ and thus $\nabla\xi = 0$.
The following result is then an easy consequence of lemma 3.4.4

3.4.11 Lemma
Let (M, g, η, J) be a metric contact manifold and ∇ a contact connection. Then the connection form induced by ∇ on the frame bundle restricts to a connection form on the unitary frame bundle.

The result for the induced connections on $\mathbb{S}^c$ is also analogous to the one for almost-hermitian connections as we state in the following theorem.

3.4.12 Theorem
Let (M, g, η, J) be a metric contact manifold and let it be equipped with its canonical $Spin^c$ structure, noting $\mathbb{S}^c \simeq \Lambda^{0,*}(\mathcal{C}^*)$ the associated spinor bundle. Let furthermore two contact connections ∇^c and ∇^z be given on TM with potentials A^c and A^z. We still note ∇^c the connection induced by ∇^c on the bundle of exterior forms.
Let C and C^z be the connection forms induced on $P_U(M)$ and let Z be the det*-extension of C^z. Furthermore, note $\widetilde{\nabla}^Z$ the covariant derivative induced on $\mathbb{S}^c$ by the connection form $\widetilde{C \times Z}$. Then the following formula holds for all vector fields $X \in \mathfrak{X}(M)$ and all spinor fields $\varphi \in \Gamma(\mathbb{S}^c)$:*

$$\widetilde{\nabla}^Z_X\varphi = \nabla^c_X\varphi + \frac{1}{2}\,\mathrm{tr}_c(A^c - A^z)(X) \cdot \varphi.$$

Proof: In the proof of theorem 3.4.8 we only used the facts: We have a vector bundle with an almost-complex structure carrying two covariant derivatives that parallelize this structure. These connections induce a connection form on the bundle of unitary frames of the vector bundle P_U and thus on the P_1-bundle. Therefore, the result holds in the case of a metric contact manifold as well. □

Note that the results carry over so nicely because the frame bundle $P_U(M)$ consists of frames of the contact distribution only and the transversal direction ξ does not play any role. This is mirrored by the mapping $F : U_m \to Spin^c_{2m+1}$ whose image lies in the subgroup $Spin^c_{2m}$ and by the spinor module which is the bundle of exterior powers of the contact distribution only. It is only in the Clifford multiplication that we need to take the additional direction into account.

With this, we close our discussion of the canonical $Spin^c$ structures of almost-hermitian and metric contact manifolds. The explicit description of their spinor bundles and the connections on them will be picked up again in chapter 3.4, where we describe Dirac operators in these bundles.

4

Connections on almost-hermitian and metric contact manifolds

In this chapter, we will study connections on contact manifolds which are induced by connections on an almost-hermitian manifold obtained from the contact manifold by taking the cartesian product with the reals. We begin by discussing connections on general almost-hermitian manifolds, describing how these connections are completely determined by certain parts of their torsion and introducing certain distinguished sets of hermitian connections. In the following section, we apply this theory to the almost-hermitian manifold $\hat{M} = \mathbb{R} \times M$ associated to a metric contact manifold M. In particular, we consider a hermitian connection that restricts to a connection on M which, in the case where M is CR, coincides with the Tanaka-Webster connection.

4.1 Hermitian connections

We now develop the theory of hermitian connections described by their torsion as developed by Paul Gauduchon in [Gau97]. We begin with some introductory definitions and elementary results, before proving the main theorem, which describes the structure of the torsion of a hermitian connection. In this section, we assume that (M^{2m}, J) is an almost-complex manifold with almost-hermitian metric g. Recall that a hermitian connection is a metric connection fulfilling $\nabla J = 0$.
We know from section 3.3 that any metric connection is defined by its potential and thus, that the space of metric connections forms an affine space directed by $\Omega^2(M, TM)$. The space of hermitian connections is directed by a certain subspace as the following lemma shows:

4.1.1 Lemma
The space $\mathcal{A}(M, g, J)$ of hermitian connections is an affine space directed by $\Omega^{1,1}(M)$.

Proof: Let $\nabla^1, \nabla^2 \in \mathcal{A}(M, g, J)$. We consider $\nabla^1 - \nabla^2$ as an element of $\Omega^2(M, TM)$ in the same way as for the potential. We then have

$$\begin{aligned}(\nabla^1 - \nabla^2)(X; JY, JZ) &= g(\nabla^1_X JY, JZ) - g(\nabla^2_X JY, JZ)\\ &= g((\nabla^1_X J)Y + J(\nabla^1_X Y), JZ) - g((\nabla^2_X J)Y + J(\nabla^2_X Y), JZ)\\ &= g(J(\nabla^1_X Y), JZ) - g(J(\nabla^2_X Y), JZ)\\ &= (\nabla^1 - \nabla^2)(X; Y, Z)\end{aligned}$$

and thus, the difference is in $\Omega^{1,1}(M, TM)$. □

As we already mentioned, we will now analyse the torsion of a hermitian connection. As the following theorem shows, some parts of the torsion do not depend on the choice of ∇ and the connection is therefore completely determined by the remaining parts.

4.1.2 Theorem (cf. [Gau97, section 2.3, proposition 2])
Let ∇ be a hermitian connection on an almost-hermitian manifold (M, g, J) and let T be its torsion, considered as an element of $\Omega^2(M, TM)$. Then the following hold

(1) $T^{0,2}$ is independent of ∇ and given by

$$T^{0,2} = N.$$

(2) The component $\mathfrak{b}(T^{2,0} - T_a^{1,1})$ is independent of ∇ and given by

$$\mathfrak{b}(T^{2,0} - T_a^{1,1}) = \frac{1}{3}(d^cF)^+. \tag{4.1}$$

Equivalently, one has

$$T^{2,0} - \varphi^{-1}(T_a^{1,1}) = \frac{1}{2}\left((d^cF)^+ - \mathfrak{M}(d^cF)^+\right) = (\nabla^g F)^{2,0}(J\cdot,\cdot,\cdot) \tag{4.2}$$

with φ as defined in (1.9) *and* (1.10)*.*

(3) T is entirely determined by its components $T_s^{1,1}$ and $(\mathfrak{b}T)^+$ which can be chosen arbitrarily. More precisely, for any given three-form $\omega^+ \in \Omega^+(M)$ and two-form $B \in \Omega_s^{1,1}(M, TM)$, there exists exactly one hermitian connection whose torsion satisfies $T_s^{1,1} = B$ and $(\mathfrak{b}T)^+ = \omega^+$. One then has

$$T^{2,0} = \frac{3}{4}\omega^+ + \frac{1}{4}(d^cF)^+ - \frac{3}{4}\mathfrak{M}\omega^+ - \frac{1}{3}\mathfrak{M}(d^cF)^+, \tag{4.3}$$

$$T_a^{1,1} = \frac{3}{8}\omega^+ - \frac{1}{8}(d^cF)^+ + \frac{3}{8}\mathfrak{M}\omega^+ - \frac{1}{8}\mathfrak{M}(d^cF)^+, \tag{4.4}$$

and the complete torsion is thus given by

$$T = N + \frac{1}{8}(d^cF)^+ - \frac{3}{8}\mathfrak{M}(d^cF)^+ + \frac{9}{8}\omega^+ - \frac{3}{8}\mathfrak{M}\omega^+ + B. \tag{4.5}$$

Proof: **First step:** We show that ∇ is hermitian if and only if

$$A(X; JY, Z) + A(X; Y, JZ) = -(\nabla^g F)(X; Y, Z) \tag{4.6}$$

Let ∇ be hermitian. Then

$$\begin{aligned}
A(X; JY, Z) + A(X; Y, JZ) &= g(A_X JY, Z) + g(A_X Y, JZ) \\
&= g(\nabla_X JY, Z) + g(\nabla_X Y, JZ) - g(\nabla^g_X JY, Z) - g(\nabla^g_X Y, JZ) \\
&= -g(\underbrace{(\nabla_X J)}_{=0} Y, Z) + \underbrace{g(J(\nabla_X Y), Z) + g(\nabla_X Y, JZ)}_{=0} \\
&\quad - g(\nabla^g_X JY, Z) - g(\nabla^g_X Y, JZ) \\
&= -g(\nabla^g_X JY, Z) - g(\nabla^g_X Z, JY) - g(\nabla^g_X Y, JZ) + g(\nabla^g_X Z, JY) \\
&= -X(g(JY, Z)) + g(J(\nabla^g_X Y), Z) + g(\nabla^g_X Z, JY) \\
&= -(\nabla^g F)(X; Y, Z).
\end{aligned}$$

On the other hand, let (4.6) hold. Then, by the same calculations we obtain

$$\begin{aligned}
(\nabla^g F)(X; Y, Z) &= g(\nabla^g_X JY, Z) + g(\nabla^g_X Y, JZ), \\
A(X; JY, Z) + A(X; Y, JZ) &= -g((\nabla_X J)Y, Z) - g(\nabla^g_X JY, Z) - g(\nabla^g_X Y, JZ)
\end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Hence, we have $\nabla J = 0$.

Second step: We use $A = -T + \frac{3}{2}\mathfrak{b}T$ to reformulate (4.6) as a condition on the torsion, which yields

$$T(X; JY, Z) + T(X; Y, JZ) - \frac{3}{2}(\mathfrak{b}T(X, JY, Z) + \mathfrak{b}T(X; Y, JZ)) = (\nabla^g F)(X; Y, Z). \tag{4.7}$$

Using that $T^{1,1}(X;JY,Z)+T^{1,1}(X;Y,JZ)=0$, we obtain that

$$T(X;JY,Z)+T(X;Y,JZ)=T^{2,0}(X;JY,Z)+T^{2,0}(X;Y,JZ)+T^{0,2}(X;JY,Z)+T^{0,2}(X;Y,JZ).$$

Using the properties of $(0,2)$- and $(2,0)$-forms and the facts that $(\mathfrak{b}T)^{0,2}=(\mathfrak{b}T)^{-}$, $(\mathfrak{b}T)^{1,1}+(\mathfrak{b}T)^{2,0}=(\mathfrak{b}T)^{+}$ as well as $(\nabla^g F)^{1,1}=0$, one obtains that (4.7) is equivalent to

$$2T^{0,2}(JX;Y,Z)-3(\mathfrak{b}T)^{-}(JX;Y,Z)=(\nabla^g F)^{0,2}(X;Y,Z), \tag{4.8}$$

$$-2T^{0,2}(JX;Y,Z)-\frac{3}{2}((\mathfrak{b}T)^{+}(X;JY,Z)+(\mathfrak{b}T)^{+}(X;Y,JZ))=(\nabla^g F)^{2,0}(X;Y,Z). \tag{4.9}$$

Third step: We now prove the actual claims.
From (4.8) and theorem 1.3.3 we obtain the following:

$$2T^{0,2}(JX;Y,Z)=2N(JX;Y,Z)+(dF)^{-}(X,Y,Z)+3(\mathfrak{b}T)^{-}(JX,Y,Z). \tag{4.10}$$

Using the well-known formula for the exterior derivative of F, wo obtain

$$\begin{aligned} dF(X,Y,Z) &= X(F(Y,Z))-Y(F(X,Z))+Z(F(X,Y)) \\ &\quad -F([X,Y],Z)+F([X,Z],Y)-F([Y,Z],X) \\ &= X(g(JY,Z))-Y(g(JX,Z))+Z(g(JX,Y)) \\ &\quad -g(J[X,Y],Z)+g(J[X,Z],Y)-g(J[Y,Z],X). \end{aligned}$$

Because ∇ is metric and by the definition of T, this can be seen to be equal to

$$\begin{aligned} dF(X,Y,Z) &= g(\nabla_X JY,Z)+g(JY,\nabla_X Z)-g(\nabla_Y JX,Z)-g(JX,\nabla_Y Z)+g(\nabla_Z JX,Y) \\ &\quad +g(JX,\nabla_Z Y)+g(\nabla_X Y,JZ)-g(\nabla_Y X,JZ)-T(JZ;X,Y)-g(\nabla_X Z,JY) \\ &\quad +g(\nabla_Z X,JY)+T(JY;X,Z)+g(\nabla_Y Z,JX)-g(\nabla_Z Y,JX)-T(JX;Y,Z) \\ &= g(\nabla_X JY,Z)-g(\nabla_Y JX,Z)+g(\nabla_Z JX,Y)+g(\nabla_X Y,JZ)-g(\nabla_Y X,JZ) \\ &\quad -T(JZ;X,Y)+g(\nabla_Z X,JY)+T(JY;X,Z)-T(JX;Y,Z). \end{aligned}$$

Using that $\nabla J=0$, we then obtain

$$\begin{aligned} dF(X,Y,Z) &= g(J(\nabla_X Y),Z)-g(J(\nabla_Y X),Z)+g(J(\nabla_Z X),Y)-g(J(\nabla_X Y),Z) \\ &\quad +g(J(\nabla_Y X),Z)-T(JZ;X,Y)-g(J(\nabla_Z X),Y)+T(JY;X,Z)-T(JX;Y,Z) \\ &= -T(JZ;X,Y)+T(JY;X,Z)-T(JX;Y,Z). \end{aligned}$$

For ease of notation, we introduce the operator $\mathfrak{N}:\Omega^2(M,TM)\to\Omega^2(M,TM)$ given by $\mathfrak{N}B(X;Y,Z)=B(JX;Y,Z)$. With this convention, we have

$$dF=-3\mathfrak{b}\mathfrak{N}T$$

and therefore

$$(dF)^{-}=-3(\mathfrak{b}\mathfrak{N}T)^{-}=-3\mathfrak{b}(\mathfrak{N}T)^{0,2}$$

by the results of section 1.2.2. Furthermore, $(\mathfrak{N}T)^{0,2}=\mathfrak{N}T^{0,2}$, which can be seen as follows: First, using the properties of (0,2)-forms, one sees that $\mathfrak{N}T^{0,2}\in\Omega^{0,2}(M,TM)$ and $\mathfrak{N}T^{2,0}\in\Omega^{2,0}(M,TM)$. Furthermore, obviously $\mathfrak{N}$ and $\mathfrak{M}$ commute and thus $\mathfrak{N}T^{1,1}\in\Omega^{1,1}(M,TM)$, which yields the required result.
Therefore, we have

$$(dF)^{-}(X;Y,Z)=-3\mathfrak{b}\mathfrak{N}T^{0,2}(X;Y,Z)=-3\mathfrak{b}T^{0,2}(JY;Y,Z)=-3(\mathfrak{b}T)^{-}(JY;Y,Z).$$

Together with (4.10), this yields (1).
Next, using (4.9), we obtain that

$$T^{2,0}(X;Y,Z) = \frac{1}{2}(\nabla^g F)^{2,0}(JX;Y,Z) + \frac{3}{4}\left((\mathfrak{b}T)^+ - \mathfrak{M}(\mathfrak{b}T)^+\right)(JX;JY,Z).$$

By the results of section 1.2.2, we have that $(\mathfrak{b}T)^+ = \mathfrak{b}T^{2,0} + \mathfrak{b}T^{1,1}$ and thus

$$\begin{aligned} T^{2,0}(X;Y,Z) &= \frac{1}{2}(\nabla^g F)^{2,0}(JX;Y,Z) + \frac{3}{4}(\mathfrak{b}T^{2,0} - \mathfrak{M}\mathfrak{b}T^{2,0})(JX;JY,Z) \\ &\quad + \frac{3}{4}(\mathfrak{b}T_a^{1,1} - \mathfrak{M}\mathfrak{b}T_a^{1,1})(JX;JY,Z) \\ &= \frac{1}{2}\left((\nabla^g F)^{2,0}(JX;Y,Z) + T^{2,0}(JX;JY,Z) + \varphi^{-1}(T_a^{1,1})(JX;JY,Z)\right), \end{aligned}$$

where the second equality follows from lemma 1.2.14 and from (1.10). This yields

$$(T^{2,0} - \varphi^{-1}(T_a^{1,1}))(X;Y,Z) = (\nabla^g F)^{2,0}(JX;Y,Z). \tag{4.11}$$

Together with theorem 1.3.3, this yields (4.2).
Furthermore, by defintion of φ and using (4.2), we obtain

$$\begin{aligned} \mathfrak{b}(T^{2,0} - T_a^{1,1}) &= \mathfrak{b}(T^{2,0} - \varphi^{-1}(T_a^{1,1})) \\ &= \frac{1}{2}(\mathfrak{b}(d^cF)^+ - \mathfrak{b}\mathfrak{M}(d^cF)^+) \\ &\overset{(1.6)}{=} \frac{1}{2}(d^cF)^+ - \frac{1}{6}(d^cF)^+ \\ &= \frac{1}{3}(d^cF)^+. \end{aligned}$$

This proves (2).
Using the computations above, one sees that ∇ is hermitian if and only if the conditions set in (1) and (2) are satisfied. Therefore, the remainig parts of T can be chosen freely. Now, if we let $T_s^{1,1} = B$ and $(\mathfrak{b}T)^+ = \omega^+$, then $\omega^+ = \mathfrak{b}(T_a^{1,1} + T^{2,0})$. Using (2), one then obtains that

$$\begin{aligned} \mathfrak{b}(T^{2,0}) &= \frac{1}{2}(\omega^+ + \frac{1}{3}(d^cF)^+), \\ \mathfrak{b}(T_a^{1,1}) &= \frac{1}{2}(\omega^+ - \frac{1}{3}(d^cF)^+). \end{aligned}$$

Then, using lemmas 1.2.14 and 1.2.15, we obtain

$$\begin{aligned} T^{2,0} &= \frac{3}{2}(\mathfrak{b}(T^{2,0}) - \mathfrak{M}\mathfrak{b}T^{2,0}) \\ &= \frac{3}{4}\omega^+ + \frac{1}{4}(d^cF)^+ - \frac{3}{4}\mathfrak{M}\omega^+ - \frac{1}{3}\mathfrak{M}(d^cF)^+ \\ T_a^{1,1} &= \frac{3}{4}(\mathfrak{b}(T_a^{1,1}) + \mathfrak{M}\mathfrak{b}T_a^{1,1}), \\ &= \frac{3}{8}\omega^+ - \frac{1}{8}(d^cF)^+ + \frac{3}{8}\mathfrak{M}\omega^+ - \frac{1}{8}\mathfrak{M}(d^cF)^+. \end{aligned}$$

Putting together all the parts of T one then obtains the formula for T claimed in (3). This concludes the proof. $\square$

In the sequel, we shall denote the hermitian connection defined by $T_s^{1,1} = B$ and $(\mathfrak{b}T)^+ = \omega^+$ by $\nabla(B, \omega^+)$.

We will now make use of the above results to introduce certain distinguished hermitian connections by manipulating the defining data B and ω^+. We begin by introducing the class of canonical connections, which are characterized by B being zero and ω^+ a certain multiple of $(d^cF)^+$.

4.1.3 Definition A hermitian connection $\nabla^t = \nabla(0, \frac{2t-1}{3}(d^cF)^+)$ with $t \in \mathbb{R}$ is called a *canonical connection.*

The torsion of a canonical connection is then given by

$$T^t = N + \frac{3t-1}{4}(d^cF)^+ - \frac{t+1}{4}\mathfrak{M}(d^cF)^+. \tag{4.12}$$

Of particular interest for us are the canonical connections for the parameters 0 and 1 which we consider now.

4.1.4 Definition The canonical connection ∇^0 is called the *first canonical connection.*

Using (4.12), we deduce that its torsion is

$$T^0 = N - \frac{1}{4}((d^cF)^+ + \mathfrak{M}(d^cF)^+).$$

4.1.5 Remark This connection is distinguished among the canonical connections as it is the projection of the Levi-Civita connection onto the space of hermitian connections in the following sense (cf. [Gau97, section 2.5]): The space of metric connections $\mathcal{A}(M,g)$ is an affine space directed by $\Omega^2(M,TM)$. If we choose ∇^g as the zero element, then $\mathcal{A}(M,g) \simeq \Gamma(TM \otimes \Lambda^2(T^*M))$. Recalling the proof of the above theorem, one sees that under this identification

$$\begin{aligned}\mathcal{A}(M,g,J) &\simeq \Gamma_J(TM \otimes \Lambda^2(T^*M)) \\ &:= \{A \in \Gamma(TM \otimes \Lambda^2(T^*M)) | A(X;JY,Z) + A(X;Y,JZ) = -(\nabla^g F)(X;Y,Z)\}.\end{aligned}$$

Then, the first canonical connection is the image of the Levi-Civita-Connection (the zero element) under the projection of $\Omega^2(M,TM)$ onto $\Gamma_J(TM \otimes \Lambda^2(T^*M))$.

We note that ∇^0 is completely characterized in $\mathcal{A}(M,g,J)$ by the conditions

$$T_s^{1,1} = 0 \quad \text{and} \quad T^{2,0} = 0.$$

That these conditions are fulfilled is easily deduced from (4.3) and (4.4). The converse is immediate from (3) of theorem 4.1.2.
The other canonical connection that we will use in the sequel is the following one:

4.1.6 Definition The *second fundamental connection* or *Chern connection* ist the canonical connection with parameter $t = 1$.

The torsion of this connection is given by

$$T^1 = N + \frac{1}{2}\left((d^cF)^+ - \mathfrak{M}(d^cF)^+\right).$$

In general, the canonical connections are not nice: Using that $\operatorname{tr} N = 0$, one obtains that

$$\begin{aligned}\operatorname{tr} T^t &= \operatorname{tr} N + \frac{3t-1}{4}\operatorname{tr}(d^cF)^+ - \frac{t+1}{4}\operatorname{tr}\mathfrak{M}(d^cF)^+ \\ &= -\frac{t+1}{4}\operatorname{tr}\mathfrak{M}(d^cF)^+ \\ &= -\frac{t+1}{2}\theta,\end{aligned}$$

where $\theta = \frac{1}{2}\operatorname{tr}\mathfrak{M}(d^cF)^+$ is the Lee form. In general, the Lee form does not vanish. To overcome this, we introduce another set of connections:

4.1.7 Proposition (cf. [Nic05, Lemma 3.2])
For every $B \in \Omega_s^{1,1}(M, TM)$ such that $\operatorname{tr} B = \frac{1}{2}\theta$, there exists a hermitian connection $\nabla^b(B)$ uniquely determined by the following conditions:

(i) ∇^b *is nice,*

(ii) ∇^b *is quasi-equivalent to* ∇^0*,*

(iii) $(T^b)_s^{1,1} = B$.

4.1.8 Definition A hermitian connection $\nabla^b(B)$ as described in the above proposition is called a *basic connection.*

Proof: We must have that $\nabla^b(B) = \nabla(B, \omega^+)$ for some $\omega^+ \in \Omega^+(M)$. The second condition implies (using theorem 1.3.3) that $\mathfrak{b}T^b = \mathfrak{b}T^0$. We know that

$$\begin{aligned}\mathfrak{b}(T^0) &= \mathfrak{b}N - \frac{1}{4}\left((d^cF)^+ + \mathfrak{b}\mathfrak{M}(d^cF)^+\right)\\ &= \frac{1}{3}(d^cF)^- - \frac{1}{4}\left((d^cF)^+ + \mathfrak{b}\mathfrak{M}(d^cF)^+\right)\\ &\overset{(1.6)}{=} \frac{1}{3}\left((d^cF)^- - (d^cF)^+\right).\end{aligned}$$

On the other hand, by theorem 4.1.2, we have that

$$T^b = N + \frac{1}{8}(d^cF)^+ - \frac{3}{8}\mathfrak{M}(d^cF)^+ + \frac{9}{8}\omega^+ - \frac{3}{8}\mathfrak{M}\omega^+ + B.$$

Hence, using $\mathfrak{b}N = \frac{1}{3}(d^cF)^-$ and $\mathfrak{b}B = 0$ as well as lemma 1.2.10, we obtain

$$\begin{aligned}\mathfrak{b}T^b &= \frac{1}{3}(d^cF)^+ + \frac{1}{8}(d^cF)^+ - \frac{1}{8}(d^cF)^+ + \frac{9}{8}\omega^+ - \frac{1}{8}\mathfrak{M}\omega^+\\ &= \frac{1}{3}(d^cF)^- + \omega^+.\end{aligned}$$

Therefore, we see that $\mathfrak{b}T^0 = \mathfrak{b}T^b$ is fulfilled if we chose $\omega^+ = -\frac{1}{3}(d^cF)^+$. If we do so, we have

$$T^b = N - \frac{1}{4}(d^cF)^+ - \frac{1}{4}\mathfrak{M}(d^cF)^+ + B.$$

This implies, because N is trace-free, that

$$\begin{aligned}\operatorname{tr} T^b &= \operatorname{tr} N + \operatorname{tr} B_s - \frac{1}{4}(\operatorname{tr}(d^cF)^+ + \operatorname{tr}\mathfrak{M}(d^cF)^+)\\ &= \frac{1}{2}\theta - \frac{1}{4}\operatorname{tr}\mathfrak{M}(d^cF)^+\\ &= \frac{1}{2}\theta - \frac{1}{2}\theta = 0\end{aligned}$$

and thus, the connection is nice. □

The torsion of such a connection is then given by

$$T^b = N - \frac{1}{4}((d^cF)^+ + \mathfrak{M}(d^cF)^+) + B. \tag{4.13}$$

This concludes our discussion of hermitian connections. In the following section, we will apply this theory to an almost-hermitian manifold associated to a metric contact manifold.

4.2 Connections on contact manifolds

We now want to use the theory developed in the previous section to describe contact connections on a metric contact manifold. Recall that a connection is called contact if it is metric and fulfils $\nabla J = 0$. We pay particular attention to the case where the manifold is CR. In this case we have the following connection on the strictly pseudoconvex CR manifold M:

4.2.1 Definition Let (M, g, η, J) be a metric contact manifold that is CR. Then the metric connection ∇^T uniquely determined by the requirements

(i) $T^T(X, Y) = L_\eta(JX, Y)\xi$

(ii) $T^T(X, \xi) = -\frac{1}{2}([\xi, X] + J[\xi, JX])$

is called the *Tanaka-Webster connection* of the CR manifold M.

4.2.2 Remark The definition we have given here, using the real CR structure, follows the approach in [BJ10, section 2.7]. One can also characterize the Tanaka-Webster connection through its torsion on the complexified tangent space TM_c. For more details on this, see [DT06, section 1.2].

In what follows, we want to describe the Tanaka-Webster connections using the theory of hermitian connections. To this end, we associate an almost-hermitian manifold to the contact manifold in the following way: Let (M^{2m+1}, g, η, J) be a metric contact manifold. We set

$$\hat{M} := \mathbb{R} \times M,$$
$$\hat{g} := dt^2 + g,$$

and define $\hat{J} \in \mathrm{End}(TM)$ by setting

$$\hat{J}|_{\mathcal{C}} = J|_{\mathcal{C}}, \quad \hat{J}\xi = -\partial t \text{ and } \hat{J}\partial t = \xi.$$

It is easily seen that $(\hat{M}, \hat{g}, \hat{J})$ is almost-hermitian. We will now use connections on $\hat{M}$ to describe connections on M. In particular, we want to describe the Tanaka-Webster connection of M by a certain basic connection on $\hat{M}$. This section is based on work by Liviu Nicolaescu [Nic05, section 3.1].
We begin by describing the almost-hermitian structure on $\hat{M}$ and its relation with the contact structure on M in some more detail, starting with a result on the Kähler form, stated in [Nic05, section 3.1].

4.2.3 Lemma
On $\hat{M}$, the Kähler form staistfies the following identity:

$$\widehat{F} = dt \wedge \eta + d\eta.$$

Proof: We have that $T\hat{M} = \mathbb{R}\partial t \oplus \mathbb{R}\xi \oplus \mathcal{C}$. It is therefore enough to prove the claimed relation for combinations of vectors of the aforementioned subspaces. To begin with, let $X, Y \in TM$. Then, $\widehat{F}(X, Y) = \hat{g}(\hat{J}X, Y) = g(JX, Y) - g(\eta(X)\xi, Y) = d\eta(X, Y)$. On the other hand, $dt \wedge \eta(X, Y) = 0$.
Next, let $X \in \mathcal{C}$. Then, we have $\widehat{F}(X, \partial t) = \hat{g}(\hat{J}X, \partial t) = 0$. On the other hand, $(dt \wedge \eta + d\eta)(X, \partial t) = 0$. Continuing, we see that $\widehat{F}(\xi, \partial t) = \hat{g}(-\partial t, \partial t) = -1$ and on the other hand $(dt \wedge \eta + d\eta)(\xi, \partial t) = -1$. Finally, $\widehat{F}(\partial t, \partial t) = 0 = (dt \wedge \eta + d\eta)(\partial t, \partial t)$. $\square$

We use this result to calculate the derivative of the Kähler form. We obtain

$$\hat{d}\widehat{F} = \hat{d}(dt \wedge \eta) + \hat{d}d\eta = -dt \wedge d\eta.$$

To calculate $\hat{d}^c\widehat{F}$, we note that $dt\circ\hat{J}=\eta$ and $\eta\circ\hat{J}=-dt$. Furthermore, $d\eta(\hat{J}\cdot,\hat{J}\cdot)=d\eta(J\cdot,J\cdot)=d\eta$, and we deduce that

$$\hat{d}^c\widehat{F}=-\eta\wedge d\eta. \tag{4.14}$$

Let $(e_i,f_i)_{i=1}^m$ be a J-adapted frame of $\mathcal{C}$. We can extend it to a $\hat{J}$-adapted frame on $\hat{M}$ by setting

$$e_0:=\partial t \quad\text{and}\quad f_0:=\xi.$$

Next, we consider the Nijenhuis tensor of $\hat{M}$:

4.2.4 Lemma (cf [Bla02, section 6.1])
For the Nijenhuis tensor $\widehat{N}$ of the almost-complex manifold $(\hat{M},\hat{J})$ and the Nijenhuis tensor N of the metric contact manifold (M,g,η,J), the following formulæ hold for any $X,Y\in\mathfrak{X}(M)$:

$$\begin{aligned}\widehat{N}(X,Y)&=N(X,Y)+\frac{1}{4}d\eta(X,Y)\xi,\\ \widehat{N}(\partial t,X)&=\frac{1}{4}(\mathcal{L}_\xi J)(X).\end{aligned}$$

Proof: To begin with, note that by the structure of $\hat{M}$, we have that

$$[X,Y]_{\hat{M}}=[X,Y]_M$$

for any $X,Y\in\mathfrak{X}(M)$. Furthermore, for $X\in\mathfrak{X}(M)$, write $X=X_{\mathcal{C}}+\eta(X)\xi$ where $X_{\mathcal{C}}$ is the projection onto the contact distribution of X. Thus, one obtains that $\hat{J}X=JX-\eta(X)\partial t$. Thus, we obtain

$$\begin{aligned}4\widehat{N}(X,Y)&=[\hat{J}X,\hat{J}Y]-[X,Y]-\hat{J}([\hat{J}X,Y]+[X,\hat{J}Y])\\ &=[JX,JY]-[JX,\eta(Y)\partial t]-[\eta(X)\partial t,JY]+[\eta(X)\partial t,\eta(Y)\partial t]-[X,Y]\\ &\quad-\hat{J}[JX,Y]+\hat{J}[\eta(X)\partial t,Y]-\hat{J}[X,JY]+\hat{J}[X,\eta(Y)\partial t]\\ &=[JX,JY]-[JX,\eta(Y)\partial t]-[\eta(X)\partial t,JY]+[\eta(X)\partial t,\eta(Y)\partial t]-[X,Y]-J[JX,Y]\\ &\quad+\eta([JX,Y])\partial t+\hat{J}[\eta(X)\partial t,Y]-J[X,JY]+\eta([X,JY])\partial t+\hat{J}[X,\eta(Y)\partial t].\end{aligned}$$

Now, recall that $[fX,Y]=f[X,Y]-Y(f)X$ and $-[X,Y]=J^2[X,Y]-\eta([X,Y])\xi$. Obviously, $[X,\partial t]=0$. Then, we obtain

$$\begin{aligned}4\widehat{N}(X,Y)&=4N(X,Y)-\eta([X,Y])\xi-JX(\eta(Y))\partial t+JY(\eta(X))\partial t+\eta([JX,Y])\partial t\\ &\quad+\eta([X,JY])\partial t-\hat{J}(Y(\eta(X))\partial t)+\hat{J}(X(\eta(Y))\partial t)\\ &\quad+\eta(X)[\partial t,\eta(Y)\partial t]-\eta(Y)\partial t(\eta(X))\partial t\\ &=4N(X,Y)-\eta([X,Y])\xi-JX(\eta(Y))\partial t+JY(\eta(X))+\eta([JX,Y]\partial t)+\eta([X,JY])\partial t\\ &\quad-\hat{J}(Y(\eta(X))\partial t)+\hat{J}(X(\eta(Y))\partial t)+\eta(X)\underbrace{\partial t(\eta(Y))}_{=0}\partial t-\eta(Y)\underbrace{\partial t(\eta(X))}_{=0}\partial t.\end{aligned}$$

Using that $\mathcal{L}_A\eta(B)=L_A(\eta(B))-\eta(\mathcal{L}_AB)=A(\eta(B))-\eta([A,B])$ for any vector fields A,B, we obatain

$$\begin{aligned}4\widehat{N}(X,Y)&=4N(X,Y)+(\mathcal{L}_{JY}\eta)(X)\partial t-(\mathcal{L}_{JX}\eta)(Y)\partial t-Y(\eta(X))\xi+X(\eta(Y))\xi-\eta([X,Y])\xi\\ &=4N(X,Y)+(\mathcal{L}_{JY}\eta)(X)-(\mathcal{L}_{JX}\eta)(Y)+d\eta(X,Y)\xi.\end{aligned}$$

Finally, note that for any vector field U, one has $\mathcal{L}_U\eta=U\lrcorner d\eta+d(U\lrcorner\eta)$. Now, $\eta\circ J=0$ and thus we obtain that

$$(\mathcal{L}_{JY}\eta)(X) - (\mathcal{L}_{JX}\eta)(Y) = d\eta(JY, X) - d\eta(JX, Y) = d\eta(JY, X) + d\eta(Y, JX) = 0,$$

which proves the first equation.
To prove the second equation, we calculate

$$\begin{aligned}
4\widehat{N}(\partial t, X) &= -[\partial t, X] + [\xi, \hat{J}X] - \hat{J}([\xi, X] + [\partial t, \hat{J}X]) \\
&= -[JX, \xi] - [\xi, \eta(X)\partial t] - J([\xi, X]) + \eta([\xi, X])\partial t - \hat{J}[\partial t, JX - \eta(X)\partial t] \\
&= -[JX, \xi] - [\xi, \eta(X)\partial t] - J([\xi, X]) + \eta([\xi, X])\partial t + \hat{J}(\partial t(\eta(X))\partial t) \\
&= (\mathcal{L}_\xi J)(X) - \xi(\eta(X))\partial t + \eta([\xi, X])\partial t \\
&= (\mathcal{L}_\xi J)(X) - (\mathcal{L}_\xi \eta)(X)\partial t.
\end{aligned}$$

However, we have that

$$\mathcal{L}_\xi \eta = d(\eta(\xi)) + \xi \lrcorner d\eta = 0, \tag{4.15}$$

which yields the claim. $\square$

Furthermore, we have the following results stated in [Nic05]:

4.2.5 Lemma
On $\hat{M}$, the following identities hold:

(1) The Lee form is given by $\hat{\theta} = -mdt$.

(2) The Nijenhuis tensor takes the following form: $\widehat{N}|_M = N + \frac{1}{4}\eta \otimes d\eta$.

Proof: (1): Using the definition of $\hat{\theta}$ and a J-adapted basis (e_i, f_i) of $\mathcal{C}$, we calculate

$$\begin{aligned}
\hat{\theta} &= \frac{1}{2}\left(-dt \wedge d\eta(\partial t, \xi, \cdot) + dt \wedge d\eta(\xi, \partial t, \cdot)\right) + \frac{1}{2}\sum_{j=1}^{m} -dt \wedge d\eta(e_j, f_j, \cdot) + dt \wedge d\eta(f_j, e_j, \cdot) \\
&= \frac{1}{2}\sum_{j=1}^{m} -dt \wedge d\eta(e_j, f_j, \cdot) + dt \wedge d\eta(f_j, e_j, \cdot).
\end{aligned}$$

The last part can only be nonzero if the last argument is ∂t. In that case, one obtains

$$\begin{aligned}
\hat{\theta}(\partial t) &= \frac{1}{2}\sum_{j=1}^{m} -dt \wedge d\eta(e_j, f_j, \partial t) + dt \wedge d\eta(f_j, e_j, \partial t) \\
&= \frac{1}{2}\sum_{j=1}^{m} -d\eta(e_j, f_j) + d\eta(f_j, e_j) \\
&= -m,
\end{aligned}$$

which proves (1). (2) follows immediately from the preceding lemma. $\square$

Furthermore, we know from theorem 1.3.3 that $\mathfrak{b}\widehat{N} = \frac{1}{3}(\hat{d}^c\widehat{F})^-$. However, we have that $(\hat{d}^c\widehat{F}) = -\eta \wedge d\eta = -3\mathfrak{b}(\eta \otimes d\eta)$. Now, $\eta \otimes d\eta$ is of type 1,1 and thus $(\hat{d}^c\widehat{F})$ is of type $+$ and therefore, $\mathfrak{b}\widehat{N}$ vanishes. This implies

$$0 = \mathfrak{b}\widehat{N}|_M = \mathfrak{b}N + \frac{1}{4}\mathfrak{b}(\eta \otimes d\eta) = \mathfrak{b}N + \frac{1}{12}\eta \wedge d\eta,$$

which is equivalent to

$$\mathfrak{b}N = -\frac{1}{12}\eta \wedge d\eta.$$

This concludes the preliminary remarks on the structure of $\hat{M}$ and some data associated with it. We now move on to actually consider connections on $\hat{M}$, where we focus on basic connections. Recall that these connections are determined by $B \in \Omega^{1,1}_s(\hat{M}, T\hat{M})$ satisfying $B = \frac{1}{2}\hat{\theta} = -\frac{m}{2}dt$. Because we want this connection to induce a connection on M, it will need to preserve the splitting $T\hat{M} = \mathbb{R}\partial t \oplus TM$. To see what conditions we will need to impose on ∇^b for this to hold, we first discuss the behaviour of a general basic connection with respect to this splitting. Recall that the torsion of ∇^b is given by

$$\begin{aligned} T^b &= \widehat{N} - \frac{1}{4}((\hat{d}^c\widehat{F})^+ + \mathfrak{M}(\hat{d}^c\widehat{F})^+) + B \\ &= \widehat{N} + B + \frac{1}{4}(\eta \wedge d\eta + \mathfrak{M}\eta \wedge d\eta). \end{aligned} \tag{4.16}$$

Now, we know that $\nabla^b = \nabla^g + A^b$ where $A^b = -T^b + \frac{3}{2}\mathfrak{b}T^b$ and we can thus calculate ∇^b. First we deduce that

$$\begin{aligned} \mathfrak{b}T^b &= \underbrace{\mathfrak{b}\widehat{N} + \mathfrak{b}B}_{=0} + \frac{1}{4}(\eta \wedge d\eta + \mathfrak{b}\mathfrak{M}\eta \wedge d\eta) \\ &= \frac{1}{3}\eta \wedge d\eta, \end{aligned} \tag{4.17}$$

where we used that $\mathfrak{b}\mathfrak{M}\eta \wedge d\eta = \frac{1}{3}\eta \wedge d\eta$ (cf lemma1.2.10). Hence, we obtain that

$$A^b = -\widehat{N} - B + \frac{1}{4}\eta \wedge d\eta - \frac{1}{4}\mathfrak{M}\eta \wedge d\eta.$$

We are now ready to start calculating with ∇^b, where we assume $B(\partial t, \cdot, \cdot) = 0$. We begin by considering $\nabla^b_{\partial t}X$, where $X \in \mathfrak{X}(M)$. We then obtain that

$$\begin{aligned} \hat{g}(\nabla^b_{\partial t}X, Y) &= \overbrace{\hat{g}(\nabla^{\hat{g}}_{\partial t}X, Y)}^{=0} + A(\partial t; X, Y) \\ &= -\widehat{N}(\partial t; X, Y) + \frac{1}{4}\eta \wedge d\eta(\partial t; X, Y) - \frac{1}{4}\eta \wedge d\eta(\partial t; \hat{J}X, \hat{J}Y) \end{aligned}$$

for any $Y \in \mathfrak{X}(M)$. From lemma 4.2.4, we know that $\widehat{N}(\partial t, X, Y) = 0$ and because $\partial t \lrcorner \eta, \partial t \lrcorner d\eta = 0$, we have

$$\hat{g}(\nabla^b_{\partial t}X, Y) = 0.$$

Furthermore,

$$\begin{aligned} \hat{g}(\nabla^b_{\partial t}X, \partial t) &= A(\partial t; X, \partial t) \\ &= -\widehat{N}(\partial t; X, \partial t) + \frac{1}{4}\eta \wedge d\eta(\partial t; X, \partial t) - \eta \wedge d\eta(\partial t; \hat{J}X, \xi) \\ &= 0, \end{aligned}$$

because by lemma 4.2.4 we know that $\widehat{N}(X, \partial t) \in \mathfrak{X}(M)$ and thus $\widehat{N}(\partial t; X, \partial t) = 0$. Therefore, altogether, we obtain

$$\nabla^b_{\partial t}X = 0 \quad \text{for any } X \in \mathfrak{X}(M). \tag{4.18}$$

Furthermore, we have

$$\hat{g}(\nabla^b_{\partial t}\partial t, X) = A(\partial t; \partial t, X) = -A(\partial t; X, \partial t) = 0. \tag{4.19}$$

Next, because $\hat{g}$ is metric we obtain that $\hat{g}(\nabla^b_{\cdot}\partial t, \partial t) = \cdot(\hat{g}(\partial t, \partial t)) - \hat{g}(\partial t, \nabla^b_{\cdot}\partial t)$ implying

$$\hat{g}(\nabla^b_{\cdot}\partial t, \partial t) = 0. \tag{4.20}$$

Together with (4.19), this implies that $\nabla^b_{\partial t}\partial t = 0$.
Next, we have that

$$\begin{aligned}
\hat{g}(\nabla^b_X\partial t, Y) &= A(X; \partial t, Y) \\
&= -\widehat{N}(X; \partial t, Y) - B(X; \partial t, Y) + \frac{1}{4}\eta \wedge d\eta(X; \partial t, Y) - \frac{1}{4}\eta \wedge d\eta(X, \xi, JY - \eta(Y)\partial t) \\
&= -\hat{g}(X, \frac{1}{4}\phi(Y)) - B(X; \partial t, Y) - \frac{1}{4}\eta \wedge d\eta(X, \xi, JY).
\end{aligned}$$

We know that $\eta \wedge d\eta(X; \xi, JY) = -d\eta(X, JY) = -g(JY, JX) = -g(X_{\mathcal{C}}, Y_{\mathcal{C}})$ where $X_{\mathcal{C}}$ is the projection of X on $\mathcal{C}$ and the last identity follows because $g(JY, JX) = g(J(Y_{\mathcal{C}} + \eta(Y)\xi), J(X_{\mathcal{C}} + \eta(X)\xi)) = g(JY_{\mathcal{C}}, JX_{\mathcal{C}}) = g(Y_{\mathcal{C}}, X_{\mathcal{C}})$ (be reminded that because J is not an almost complex structure on all of TM, the identity $g(JX, JY) = g(X, Y)$ would be wrong in general). Thus, we obtain

$$\hat{g}(\nabla^b_X\partial t, Y) = -\frac{1}{4}(\hat{g}(X, \phi Y) - g(X_{\mathcal{C}}, Y_{\mathcal{C}})) - B(X; \partial t, Y). \tag{4.21}$$

Next, we consider

$$\begin{aligned}
\hat{g}(\nabla^b_X Y, \partial t) &= A^b(X; Y, \partial t) = -A(X; \partial t, Y) \\
&= \frac{1}{4}(\hat{g}(X, \phi Y) - g(X_{\mathcal{C}}, Y_{\mathcal{C}})) + B(X; \partial t, Y).
\end{aligned} \tag{4.22}$$

We summarize the above results: Any basic connection $\nabla^b(B)$, where $B(\partial t; \cdot, \cdot) = 0$, satisfies the following equations:

$$\nabla^b_{\partial t} X = 0, \tag{4.23}$$

$$\hat{g}(\nabla^b_{\partial t}\partial t, X) = 0, \tag{4.24}$$

$$\hat{g}(\nabla^b_{\cdot}\partial t, \partial t) = 0, \tag{4.25}$$

$$\hat{g}(\nabla^b_X\partial t, Y) = -\frac{1}{4}(\hat{g}(X, \phi Y) - g(X_{\mathcal{C}}, Y_{\mathcal{C}})) - B(X; \partial t, Y), \tag{4.26}$$

$$\hat{g}(\nabla^b_X Y, \partial t) = \frac{1}{4}(\hat{g}(X, \phi Y) - g(X_{\mathcal{C}}, Y_{\mathcal{C}})) + B(X; \partial t, Y) \tag{4.27}$$

for any $X, Y \in \mathfrak{X}(M)$. The fourth and fifth equations give us conditions that B m,ust satisfy if $\nabla^b(B)$ is to respect the splitting. We now prove that such a B does exist.

4.2.6 Lemma (cf. [Nic05, Lemma 3.2])
There exists a form $B \in \Omega^{1,1}_s(\hat{M}, T\hat{M})$ such that $\operatorname{tr} B = -\frac{m}{2}dt$ fulfilling

(i) $B(\partial t; \cdot, \cdot) = 0$,

(ii) $B(X; Y, \partial t) = \frac{1}{4}(\hat{g}(X, \phi Y) - g(X_{\mathcal{C}}, Y_{\mathcal{C}}))$ *for any* $X, Y \in \mathfrak{X}(M)$.

Proof: To simplify the notation, we introduce a wedge product

$$\begin{aligned}\wedge : \operatorname{End}(TM) \times \Omega^1(\hat{M}) &\to \Omega^2(T\hat{M})\\ (F \wedge \alpha)(X; Y, Z) &:= ((FX)^\flat \wedge \alpha)(Y, Z)\end{aligned}$$

where $\cdot^\flat$ denotes the $\hat{g}$-dual. For this product the following holds:
Lemma: For $F \in \operatorname{End}(T\hat{M})$ and $\alpha \in \Omega^1(\hat{M})$ define

$$F_+ = \frac{1}{2}(F + F^*) \quad \text{and} \quad F_- = \frac{1}{2}(F - F^*).$$

Then, for a local basis (b_i), the following formulæ hold:

$$\begin{aligned}\operatorname{tr}(F \wedge \alpha) &= \operatorname{tr} F \cdot \alpha - \sum_{i=1}^{2m+2} \alpha(b_i)(Fb_i)^\flat,\\ \mathfrak{b}(F \wedge \alpha) &= \frac{2}{3}(g(F_-\cdot, \cdot) \wedge \alpha).\end{aligned}$$

Proof: We have

$$\begin{aligned}\operatorname{tr}(F \wedge \alpha)(X) &= \sum_{i=1}^{2m+2} ((Fb_i)^\flat \wedge \alpha)(b_i, X)\\ &= \sum_{i=1}^{2m+2} g(Fb_i, b_i)\alpha(X) - g(Fb_i, X)\alpha(b_i)\\ &= \operatorname{tr}(F)\alpha(X) - \sum_{i=1}^{2m+2} \alpha(b_i)(Fb_i)^\flat(X),\end{aligned}$$

which proves the first identity. Furthermore, we have

$$\begin{aligned}\mathfrak{b}(F \wedge \alpha)(X, Y, Z) &= \frac{1}{3}\Big(((FX)^\flat \wedge \alpha)(Y, Z) + ((FY)^\flat \wedge \alpha)(Z, X) + ((FZ)^\flat \wedge \alpha)(X, Y)\Big)\\ &= \hat{g}(FX, Y)\alpha(Z) - \hat{g}(FX, Z)\alpha(Y) + \hat{g}(FY, Z)\alpha(X) - \hat{g}(FY, X)\alpha(Z)\\ &\quad + \hat{g}(FZ, X)\alpha(Y) - \hat{g}(FZ, Y)\alpha(X)\\ &= \frac{1}{3}\left(\alpha(X)(\hat{g}(FY, Z) - \hat{g}(F^*Y, Z)) + \alpha(Y)(\hat{g}(FX, Z) - \hat{g}(F^*X, Z))\right.\\ &\quad \left.+\alpha(Z)(\hat{g}(FX, Y) - \hat{g}(F^*X, Y))\right)\\ &= \frac{2}{3}(\hat{g}(F_-\cdot, \cdot) \wedge \alpha)(X, Y, Z),\end{aligned}$$

which yields the claim.

Now, going back to the main proof, we define

$$\begin{aligned}B_0 &:= \frac{1}{4}(\phi \wedge dt + (J\phi) \wedge \eta),\\ B_1 &:= -\frac{1}{4}(P_{\mathcal{C}} \wedge dt + (JP_{\mathcal{C}}) \wedge \eta),\\ B &:= B_0 + B_1 + \frac{1}{2}\eta \otimes d\eta,\end{aligned}$$

where $P_{\mathcal{C}}$ denotes the projection onto $\mathcal{C}$.
We use the basis $b_1 = \partial t, b_2 = \xi, b_{2k+1} = e_k, b_{2k+2} = f_k (k \geq 1)$ where (e_k, f_k) is a J-adapted

basis of $\mathcal{C}$. We then have, because ϕ and $J\phi$ are trace-free, that

$$\begin{aligned} 4\operatorname{tr} B_0 &= \operatorname{tr}(\phi \wedge dt) + \operatorname{tr}((J\phi) \wedge \eta) \\ &= -\sum_{i=1}^{2m+2} \hat{g}(\phi b_i, \cdot)dt(b_i) + \hat{g}(J\phi b_i, \cdot)\eta(b_i) \\ &= -\hat{g}(\phi\partial t, \cdot) - \hat{g}((J\phi)\xi, \cdot). \end{aligned}$$

Using that $\phi\partial t = 0$, $J\phi = -\phi J$ and $J\xi = 0$ this trace can be seen to be zero. Furthermore, because ϕ and $J\phi$ are symmetric, we obtain that $(B_0)_-$ vanishes and thus $\mathfrak{b}(B_0) = 0$. Next, we show that $B_i \in \Omega^{1,1}(\hat{M}, T\hat{M})$. We begin with B_0.
To begin with, let $X \in \mathfrak{X}(M), Y, Z \in \Gamma(\mathcal{C})$. Then, because dt and η are zero on Y and Z, we have

$$B_0(X; \hat{J}Y, \hat{J}Z) = 0 = B_0(X; Y, Z).$$

Next, we have

$$\begin{aligned} 4B_0(X, \hat{J}\xi, \hat{J}Y) &= -4B_0(X, \partial t, JY) \\ &= -(\phi X)^\flat \wedge dt(\partial t, JY) - (J\phi X)^\flat \wedge \eta(\partial t, JY). \end{aligned}$$

Because η is zero on ∂t and JY, we obtain

$$4B_0(X, \hat{J}\xi, \hat{J}Y) = -(\phi X)^\flat \wedge dt(\partial t, JY) = \hat{g}(\phi X, JY).$$

Analogously, one obtains

$$4B_0(X, \xi, Y) = (J\phi X)^\flat \wedge \eta(\xi, Y) = -\hat{g}(J\phi X, Y) = \hat{g}(\phi X, JY).$$

The definition of $\hat{J}$ inplies

$$\mathfrak{M}B_0(X, \xi, \partial t) = -B_0(X, \partial t, \xi) = B_0(X, \xi, \partial t). \tag{4.28}$$

Finally, we have

$$\begin{aligned} 4\mathfrak{M}B_0(X, \partial t, Y) = 4B_0(X, \xi, JY) &= (J\phi X)^\flat \wedge \eta(\xi, JY) \\ &= -\hat{g}(\phi X, Y) \end{aligned}$$

and

$$4B_0(X, \partial t, Y) = (\phi X)^\flat \wedge dt(\partial t, Y) = -\hat{g}(\phi X, Y),$$

which proves that $\mathfrak{M}B_0 = B_0$, i.e. $B_0 \in \Omega^{1,1}(\hat{M}, T\hat{M})$.
For B_1, we have

$$B_1(X; Y, Z) = 0 = \mathfrak{M}B_1(X; Y, Z).$$

With calculations analogous to those for B_0 one sees that

$$4\mathfrak{M}B_1(X; \xi, Y) = -(X_\mathcal{C})^\flat \wedge dt(\partial t, JY) = \hat{g}(X_\mathcal{C}, JY) = (JX_\mathcal{C})^\flat \wedge \eta(\xi, Y) = 4B_1(X, \xi, Y)$$

and

$$4\mathfrak{M}B_1(X; \partial t, Y) = (JX_\mathcal{C})^\flat \wedge \eta(\xi, JY) = -\hat{g}(JX_\mathcal{C}, JY) = (X_\mathcal{C})^\flat \wedge \partial t(\partial t, Y) = 4B_1(X; \partial t, Y)$$

and, as above, by definition of $\hat{J}$ on ξ and ∂t, the required identity follows in the last case. Obviously, $\eta \otimes d\eta \in \Omega^{1,1}(\hat{M}, T\hat{M})$ and thus $B \in \Omega^{1,1}(\hat{M}, T\hat{M})$.

Next, we compute the trace of B_1:

$$\operatorname{tr}(B_1) = -\frac{1}{4}(\operatorname{tr}(P_{\mathcal{C}})dt + \operatorname{tr}(JP_{\mathcal{C}})\eta) - \underbrace{\sum_{i=1}^{2m+2} \hat{g}(P_{\mathcal{C}}b_i, \cdot)dt(b_i) + \hat{g}(P_{\mathcal{C}}b_i, \cdot)\eta(b_i)}_{=0}.$$

because any b_i that does not vanish under dt or η is perpendicular to $\mathcal{C}$.
We furthermore have that

$$\operatorname{tr} P_{\mathcal{C}} = \sum_{j=1}^{2m+2} \hat{g}(P_{\mathcal{C}}b_j, b_j) = \sum_{j=1}^{m} \hat{g}(e_j, e_j) + \hat{g}(f_j, f_j) = 2m$$

and $\operatorname{tr} JP_{\mathcal{C}} = 0$. Therefore, we have $\operatorname{tr} B_1 = -\frac{m}{2}dt$. Using that $\mathfrak{b}(\eta \otimes d\eta) = \frac{1}{3}\eta \wedge d\eta$ and $\operatorname{tr}(\eta \otimes d\eta) = 0$ and putting together the above facts, we obtain that B is in $\Omega^{1,1}_s(\hat{M}, T\hat{M})$ and that $\operatorname{tr} B = -\frac{m}{2}dt$.
Obviously, we have $B(\partial t, \cdot, \cdot) = 0$.
It remains to show that B satisfies (ii). Using the above results we see that

$$\begin{aligned} B(X; Y, \partial t) &= -B_0(X, \partial t, Y) - B_1(X; \partial t, 1) \\ &= -\frac{1}{4}\hat{g}(\phi X, Y) + \frac{1}{4}\hat{g}(X_{\mathcal{C}}, Y_{\mathcal{C}}), \end{aligned}$$

which concludes the proof. $\square$

Now, using the results (4.23) to (4.27), we deduce that for the basic connection $\nabla^b(B)$ with B as described in the lemma above, we have

$$\begin{aligned} &\nabla^b_{\partial t} X = 0 \quad \text{for any } X \in \mathfrak{X}(M), \\ &\nabla^b_X Y \in \mathfrak{X}(M) \quad \text{for any } X, Y \in \mathfrak{X}(M), \\ &\nabla^b_{\cdot}\partial t = 0 \quad \text{i.e. } \partial t \text{ is parallel wirth repsect to } \nabla^b. \end{aligned} \tag{4.29}$$

We will denote this connection by $\hat{\nabla}^b$. Because it respects the splitting $T\hat{M} = \mathbb{R}\partial t \oplus TM$, it induces a connection on M which we shall denote ∇^{TW} and call the *generalized Tanaka-Webster connection*. As $\hat{\nabla}^b$ is hermitian, ∇^{TW} is contact. It is also nice, because $\hat{\nabla}^b$ is and thus the trace of its torsion is zero and therefore also zero on M.
We want to describe its torsion in some more detail. Recall that $\widehat{N}|_M = N + \frac{1}{4}\eta \otimes d\eta$ and, because $\eta \circ J = 0$ and $d\eta(J\cdot, J\cdot) = d\eta$, we have $\mathfrak{M}(\eta \wedge d\eta)|_M = \eta \otimes d\eta$. Using the explicit description in the proof of the lemma, we deduce

$$\begin{aligned} B|_M &= \frac{1}{4}((J\phi) \wedge \eta - (JP_{\mathcal{C}}) \wedge \eta) + \frac{1}{2}\eta \otimes d\eta \\ &= \frac{1}{4}((J\phi) \wedge \eta - J \wedge \eta) + \frac{1}{2}\eta \otimes d\eta. \end{aligned}$$

Thus, using (4.16), we deduce that

$$\begin{aligned} T^{TW} &= \widehat{N}|_M + B|_M + \frac{1}{4}(\eta \wedge d\eta + \mathfrak{M}\eta \wedge d\eta)|_M \\ &= N + \frac{1}{4}\eta \otimes d\eta + \frac{1}{4}((J\phi) \wedge \eta - J \wedge \eta) + \frac{1}{2}\eta \otimes d\eta + \frac{1}{4}\eta \wedge d\eta + \frac{1}{4}\eta \otimes d\eta \\ &= N + \eta \otimes d\eta + \frac{1}{4}((J\phi - J) \wedge \eta + \eta \wedge d\eta). \end{aligned}$$

Now, assume that the metric contact structure on M fulfils $J \circ N = 0$, i.e. we have a strictly pseudoconvex CR structure on M. We want to show that in this case, ∇^{TW} coincides with the Tanaka-Webster connection. Due to the additional restriction on N, the Nijenhuis tensor can be written as

$$N = \frac{1}{4}((J\phi) \wedge \eta - \eta \otimes d\eta).$$

This can be seen as follows: As J is an isomorphism on $\mathcal{C}$, any part of N that is already in $\mathcal{C}$ must be zero. Furthermore, the image of J on TM is in $\mathcal{C}$ and thus any part of N that is an image under J must be zero. Therefore, we obtain $N(Y, Z) = [JY, JZ]$, or, as a trilinear form, $N(X; Y, Z) = g(X, [JY, JZ])$. Now, write $[JY, JZ] = P_{\mathcal{C}}([JY, JZ]) + \eta([JY, JZ])\xi$. By the same arguments as above, $P_{\mathcal{C}}([JY, JZ])$ must be zero and thus $N(X; Y, Z) = 0$ for any $X \in \Gamma(\mathcal{C})$. Now, recall from the proof of lemma 2.2.9 that

$$N(Y, Z) = -\frac{1}{4} d\eta(Y, Z)\xi \quad \text{for any } Y, Z \in \Gamma(\mathcal{C}). \tag{4.30}$$

Furthermore,

$$\begin{aligned} N(\xi, Y) &= \frac{1}{4}([J\xi, Y] + J^2[\xi, Y] - J([J\xi, Y] + [\xi, JY])) \\ &= \frac{1}{4}(J^2[\xi, Y] - J[\xi, JY]) \\ &= \frac{1}{4}J(J[\xi, Y] - [\xi, JY]) \\ &= -\frac{1}{4}J\phi Y. \end{aligned} \tag{4.31}$$

Putting together the above remarks and (4.30) and (4.31), we obtain the claimed formula. Using this, we obtain for the torsion of ∇^{TW}:

$$\begin{aligned} T^{TW} &= \frac{1}{4}((J\phi \wedge \eta) - \eta \otimes d\eta) + \eta \otimes d\eta + \frac{1}{4}((J\phi - J)\eta + \eta \wedge d\eta) \\ &= \frac{1}{2}(J\phi \wedge \eta) + \frac{3}{4}\eta \otimes d\eta - \frac{1}{4}J \wedge \eta + \frac{1}{4}\eta \wedge d\eta. \end{aligned}$$

Thus, for $X, Y, Z \in \Gamma(\mathcal{C})$, we have

$$\begin{aligned} T^{TW}(X; Y, Z) &= 0 \text{ because } \eta(\mathcal{C}) = 0, \\ T^{TW}(\xi; X, Y) &= \frac{3}{4} d\eta(X, Y) + \frac{1}{4}\eta \wedge d\eta(\xi, X, Y) \\ &= d\eta(X, Y), \\ T^{TW}(X; \xi, Y) &= \frac{1}{2}((J\phi X)^\flat \wedge \eta)(\xi, Y) + \frac{1}{4}\eta \wedge d\eta(X, \xi, Y) - \frac{1}{4}((JX)^\flat \wedge \eta)(\xi, Y) \\ &= -\frac{1}{2}g(J\phi X, Y) - \frac{1}{4}d\eta(X, Y) + \frac{1}{4}g(JX, Y) \\ &= -\frac{1}{2}g(X, J\phi Y) - \frac{1}{4}d\eta(X, Y) + \frac{1}{4}d\eta(X, Y) = -\frac{1}{2}g(X, J\phi Y), \\ T^{TW}(\xi; \xi, Y) &= 0. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
T^{TW}(X,Y) &= d\eta(X,Y)\xi \\
&= -L_\eta(X,JY)\xi \\
&= L_\eta(JX,Y), \\
T^{TW}(\xi,X) &= -\frac{1}{2}J\phi X \\
&= -\frac{1}{2}J([\xi,JX]-J[\xi,X]) \\
&= -\frac{1}{2}(J[\xi,JX]+[\xi,X]-\eta([\xi,X])\xi) \\
&= -\frac{1}{2}(J[\xi,JX]+[\xi,X]+d\eta(\xi,X)\xi) \\
&= -\frac{1}{2}(J[\xi,JX]+[\xi,X]).
\end{aligned}$$

Hence, ∇^{TW} is the Tanaka-Webster connection in this case. Putting the above results together, we obtain the following result

4.2.7 Theorem
Let (M,H,J_{CR}) a CR manifold and (g,η,J) a metric contact structure such that $\ker\eta = H$ *and $J|_H = J_{CR}$. Let furthermore $(\hat{M},\hat{J},\hat{g})$ given by $\hat{M}=\mathbb{R}\times M$, $\hat{J}\partial t=\xi, \hat{J}\xi=-\partial t$ and $\hat{J}|_H = J|_H$ and $\hat{g}=dt^2+g$. Then the Tanaka-Webster connection of M is uniquely determined as the restriction to M of the connection $\hat{\nabla}^b$ on the almost-hermitian manifold $(\hat{M},\hat{J},\hat{g})$ satisfying the following conditions:*

(i) ∇^b *is hermitian,*

(ii) ∇^b *is nice,*

(iii) ∇^b *is quasi-equivalent to* ∇^0*,*

(iv) $(T^b)^{1,1}_s = B$ *with B as described in (the proof of) lemma 4.2.6.*

We conclude this section by noting one further property of the torison of the Tanaka-Webster connection.

4.2.8 Definition Let (M,g,η,J) be a metric contact manifold that is CR. Then a contact connection on TM whose torsion satisfies

$$g(X,T(Y,Z)) = 0 \quad \text{for any } X,Y,Z\in\Gamma(\mathcal{C})$$

is called a *CR connection.*

It is immediate from the explicit description of the Tanka-Webster connection that it is CR.

In this chapter, we have provided an alternative description of the Tanaka-Webster connection. However, we still need to explicitly describe a part of its torsion. In the following chapter, we will give a characterization by means of the Dirac operator it induces, where there will be no more need for an explicit description of the torsion. The property that ∇^{TW} is a CR connection will be very useful for that, in general, many connections induce the same Dirac operator, but, as we will see, there is at most one amongst them that is CR.

5

The Hodge-Dolbeault operator and geometric Dirac operators

This section will be devoted to the study of the relationship between the Hodge-Dolbeault operator and certain geometric Dirac operators, in particular those induced by canonical and basic connections. In the first section, we study Dirac operators on almost-hermitian manifolds, showing in particular that the Hodge-Dolbeault operator is a geometric Dirac operator. In the following section, we then look at the operators induced on a metric contact manifold by those on the associated almost-hermitian manifold. In particular, we see that the Tanaka-Webster connection induces a Hodge-Dolbeault-like operator and is the only CR connection to do so.

5.1 Dirac operators on almost-hermitian manifolds

In this section, we will study Dirac operators on almost-hermitian manifolds. Recall that every almost-hermitian manifold has a canonical $Spin^c$ structure with spinor bundle $\mathbb{S}^c \simeq \Lambda^{0,*}(T^*M)$. On this bundle, we have a particular Dirac type operator:

5.1.1 Definition The *Hodge-Dolbeault operator* is the operator

$$\begin{aligned} \mathcal{H} : \Gamma(\mathbb{S}^c) \simeq \Omega^{0,*}(M) &\longrightarrow \Omega^{0,*}(M) \\ \omega &\longmapsto \sqrt{2}(\overline{\partial}\omega + \overline{\partial}^*\omega). \end{aligned}$$

We will compare this operator with the geometric Dirac operators induced by the various connections on an almost-hermitian manifold (Levi-Cività, canonical, basic). In particular, we will show that $\mathcal{H}$ is a geometric Dirac operator.

We begin by proving some auxiliary results on the covariant derivative of differential forms, linking it to the exterior differential d and the Dolbeault operator $\overline{\partial}$. Recall that every connection ∇ on TM induces one on $\Omega^*(M)$ by

$$(\nabla_X\omega)(X_1, ..., X_k) = X(\omega(X_1, ..., X_k)) - \sum_{i=1}^{k} \omega(X_1, ..., \nabla_X X_i, ..., X_k) \tag{5.1}$$

This covariant derivative is closely related to the exterior differential as the following lemma, stated in [Gau97, section 3.5], shows.

5.1.2 Lemma
Let ∇ be any metric connection on the tangent space TM of some almost-hermitian manifold (M, g, J) and T its torsion. Then the following equalities hold for the exterior differential d and co-differential δ of a differential form $\omega \in \Omega^k(M)$:

$$\begin{aligned} dw(X_0, ..., X_k) &= \sum_{j=0}^{k} (-1)^j (\nabla_{X_j}\omega)(X_0, ..., \hat{X}_j, ..., X_k) \\ &\quad + \sum_{\alpha<\beta} (-1)^{\alpha+\beta}\omega(T(X_\alpha, X_\beta), X_0, ..., \hat{X}_\alpha, ..., \hat{X}_\beta, ..., X_k) \\ \delta\omega(X_1, ..., X_{k-1}) &= -\sum_{j=1}^{n} (\nabla_{b_j}\omega)(b_j, X_1, ..., X_{k-1}) + \omega((\operatorname{tr} T)^\natural, X_1, ..., X_{k-1}) \\ &\quad - \sum_{j=1}^{k-1} (-1)^j g(T(X_j; \cdot, \cdot)), \omega(\cdot, \cdot, X_1, ..., \hat{X}_j, ..., X_{k-1})) \end{aligned}$$

where, as usual, $\hat{X}_j$ means that X_j does not appear in the formula, and where T with three arguments and $\operatorname{tr} T$ are to be understood in the sense introduced in section 1.2.2.

Proof: (1) The right hand side is equal to

$$\sum_{j=0}^{k}(-1)^j \Big(X_j(\omega(X_0,...,\hat{X}_j,...,X_k)) - \omega(\nabla_{X_j}X_0,...,\hat{X}_j,...,X_k) - ... - \omega(X_0,...,\hat{X}_j,...,\nabla_{X_j}X_k)\Big)$$
$$+\sum_{\alpha<\beta}(-1)^{\alpha+\beta}\omega(T(X_\alpha,X_\beta),X_0,...,\hat{X}_\alpha,...,\hat{X}_\beta,...,X_k).$$

Reordering the first sum and using $\nabla_{X_k}X_j - \nabla_{X_j}X_k = [X_k,X_j] + T(X_k,X_j)$, we obtain that the r.h.s. is equal to

$$\sum_{j=0}^{k}(-1)^j X_j(\omega(X_0,...,\hat{X}_j,...,X_k)) - \sum_{j<l}(-1)^{j+l}(\omega([X_j,X_l],X_0,...,\hat{X}_j,...,\hat{X}_l,...,X_k)$$
$$+\,\omega(T(X_i,X_j),X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_k))$$

$$\begin{aligned}&\sum_{\alpha<\beta}(-1)^{\alpha+\beta}\omega(T(X_\alpha,X_\beta),X_0,...,\hat{X}_\alpha,...,\hat{X}_\beta,...,X_k)\\ =&d\omega(X_0,...,X_k) - \sum_{i<j}(-1)^{i+j}\omega(T(X_i,X_j),X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_k),\end{aligned}$$

which proves the first identity.

(2) To prove this identity, we recall that the codifferential is defined by

$$\delta\omega = (-1)^{nk+1} * d * \omega$$

for any $\omega \in \Omega^k(M)$. Now, we write ω with respect to a local basis (b_j) with dual (b^j) as $\omega = \sum_I \omega_I b^I$ where for $I=(i_1,...,i_k)$, $b^I = b^{i_1}\wedge\cdots\wedge b^{i_k}$ and $\omega_I = \omega(b_{i_1},...,b_{i_k})$. We then have locally that

$$*\omega = \sum_I \omega_I * b^I = \sum_I \omega_I \operatorname{sgn}(I,J) b^J,$$

where J is the complement of I in $\{1,...,n\}$ and $\operatorname{sgn}(I,J)$ is the sign of the permutation $(1,...,n)\mapsto(I,J)$. Thus, we obtain

$$*\,\omega(b_{j_1},...,b_{j_{n-k}}) = \operatorname{sgn}(I,J)\omega_I. \tag{5.2}$$

We now first prove the second formula for the Levi-Cività-connection, which has no torsion. In that case, we have the following equalities:

$$\begin{aligned}\delta\omega(b_{i_1},...,b_{i_{k-1}}) &= (-1)^{nk+1} * d * \omega(b_{i_1},...,b_{i_{k-1}})\\ &= (-1)^{nk+1}\operatorname{sgn}(\alpha,I)(d*\omega)(b_{\alpha_1},...,b_{\alpha_{n-k+1}}) \quad (\alpha=\{1,...,n\}\setminus I)\\ &\overset{(1)}{=} (-1)^{nk+1}\operatorname{sgn}(\alpha,I)\sum_{j=1}^{n-k+1}(-1)^j(\nabla^g_{b_{\alpha_j}} *\omega)(b_{\alpha_1},...,\widehat{b_{\alpha_j}},...,b_{\alpha_{n-k+1}}).\end{aligned}$$

We consider this sum in some more detail. We have that $*\nabla = \nabla *$ and thus, the sum is equal to

$$
\begin{aligned}
&(-1)^{nk+1}\operatorname{sgn}(\alpha, I)\sum_{j=1}^{n-k+1}(-1)^j(*\nabla^g_{b_{\alpha_j}}\omega)(b_{\alpha_1},...,\widehat{b_{\alpha_j}},...,b_{\alpha_{n-k+1}})\\
\overset{(5.2)}{=}&(-1)^{nk+1}\operatorname{sgn}(\alpha, I)\sum_{j=1}^{n-k+1}(-1)^j\operatorname{sgn}(\alpha_j, I, (\alpha\setminus\{\alpha_j\}))(\nabla^g_{b_{\alpha_j}}\omega)(b_{\alpha_j}, b_{i_1},...,b_{i_{k-1}})\\
=&\sum_{j=1}^{n-k+1}(-1)^j(-1)^{j-1}(\nabla^g_{b_{\alpha_j}}\omega)(b_{\alpha_j}, b_{i_1},...,b_{i_{k-1}})\\
=&-\sum_{j=1}^{n-k+1}(\nabla^g_{b_{\alpha_j}}\omega)(b_{\alpha_j}, b_{i_1},...,b_{i_{k-1}})\\
=&-\sum_{j=1}^{n}(\nabla^g_{b_j}\omega)(b_j, b_{i_1},...,b_{i_{k-1}}),
\end{aligned}
$$

where the last identity is due to the fact that the terms we have added are zero because $(\nabla_{b_j}\omega)(b_j, b_{i_1},...,b_{i_{k-1}}) = 0$ if $j \in I$.
Next, we prove the claim for any metric connection ∇. We begin by comparing the covariant derivatives ∇ and ∇^g induce on forms. Using that $\nabla = \nabla^g + A$, we deduce

$$
\begin{aligned}
\nabla_X\omega(X_1,...,X_k) &= X(\omega(X_1,...,X_k)) - \sum_{j=1}^{k}\omega(X_1,...,\nabla_X X_j,...,X_k)\\
&= X(\omega(X_1,...,X_k)) - \sum_{j=1}^{k}\omega(X_1,...,\nabla^g_X X_j + A_X X_j,...,X_k)\\
&= \nabla^g_X\omega(X_1,...,X_k) - \sum_{j=1}^{k}\omega(X_1,...,A_X X_j,...,X_k).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\delta\omega(X_1,...,X_{k-1}) &= -\sum_{\mu=1}^{n}(\nabla^g_{b_\mu}\omega)(b_\mu, X_1,...,X_{k-1})\\
&= -\sum_{\mu=1}^{n}(\nabla_{b_\mu}\omega)(b_\mu, X_1,...,X_{k-1}) - \sum_{\mu=1}^{n}\omega(A_{b_\mu}b_\mu, X_1,...,X_{k-1})\\
&\quad -\sum_{\mu=1}^{n}\sum_{j=1}^{k-1}\omega(b_\mu, X_1,...,A_{b_\mu}X_j,...,X_k).
\end{aligned}
$$

Because

$$A_{b_\mu}b_\mu = \sum_{\nu=1}^{n} g(A_{b_\mu}b_\mu, b_\nu)b_\nu = \sum_{\nu=1}^{n} A(b_\mu; b_\mu, b_\nu)b_\nu,$$

we obtain that

$$\sum_{\mu=1}^{n} A_{b_\mu}b_\mu = (\operatorname{tr} A)^\natural = -(\operatorname{tr} T)^\natural.$$

This leaves us to consider the third sum. We have that

$$\sum_{\mu=1}^{n}\sum_{j=1}^{k-1}\omega(b_\mu, X_q, ..., A_{b_\mu}X_j, ..., X_k)$$
$$=\sum_{j=1}^{k-1}\sum_{\mu,\nu=1}^{n}(-1)^\mu g(A_{b_\mu}X_j, b_\nu)\omega(b_\nu, b_\mu, X_1, ..., \hat{X}_\mu, ..., X_{ik-1}).$$

Recalling that $A = -T + \frac{3}{2}\mathfrak{b}T$, we see that this is equal to

$$\sum_{j=1}^{k-1}\sum_{\mu,\nu=1}^{n}(-1)^j\left(-T(b_\mu; X_j, b_\nu) + \frac{1}{2}(T(b_\mu; X_j, b_\nu) + T(X_j; b_\nu, b_\mu) + T(b_\nu; b_\mu, X_j))\right)\cdot$$
$$\omega(b_\nu, b_\mu, X_1, ..., \hat{X}_\mu, ..., X_{k-1})$$

$$=\sum_{j=1}^{k-1}\sum_{\nu<\mu}(-1)^j T(X_j, b_\nu, b_\mu)\omega(b_\nu, b_\mu, X_1, ..., \hat{X}_\mu, ..., X_{k-1})$$
$$-\frac{1}{2}\sum_{j=1}^{k-1}(-1)^j\sum_{\mu,\nu=1}^{n}(T(b_\mu, X_j, b_\nu) - T(b_\nu, b_\mu, X_j))\omega(b_\nu, b_\mu, X_1, ..., \hat{X}_\mu, ..., X_{k-1}).$$

The first of these two sums is precisely equal to

$$\sum_{j=1}^{k-1}(-1)^j g(T(X_j; \cdot, \cdot)), \omega(\cdot, \cdot, X_1, ..., \hat{X}_j, ..., X_{k-1})).$$

The second sum, on the other hand, vanishes, because we can write it as

$$-\frac{1}{2}\sum_{j=1}^{k-1}(-1)^j\sum_{\mu,\nu=1}^{n}T(b_\mu, X_j, b_\nu)\omega(b_\nu, b_\mu, X_1, ..., \hat{X}_\mu, ..., X_{k-1})$$
$$+\frac{1}{2}\sum_{j=1}^{k-1}(-1)^j\sum_{\mu,\nu=1}^{n}T(b_\nu, X_j, b_\nu)\omega(b_\mu, b_\nu, X_1, ..., \hat{X}_\mu, ..., X_{k-1}),$$

which is zero. This yields the claim. □

This lemma has an extension to the operators $\overline{\partial}$ and $\overline{\partial}^*$.

5.1.3 Lemma (cf. [Gau97, Lemma 3.5])
Let (M, g, J) be an almost-hermitian manifold and let ∇ be a hermitian connection on TM with torsion T. Then, for $\omega \in \Omega^{0,k}$ and $Z_0, ..., Z_k \in TM^{0,1}$ we have that

$$\overline{\partial}\omega(Z_0, ..., Z_k) = \sum_{j=1}^{k}(-1)^j(\nabla_{Z_j}\omega)(Z_0, ..., \hat{Z}_j, ..., Z_k)$$
$$+\sum_{j<l}(-1)^{j+l}\omega(T^{2,0}(Z_j, Z_l), Z_0, ..., \hat{Z}_j, ..., \hat{Z}_l, ..., Z_k),$$
$$\overline{\partial}^*\omega(Z_1, ..., Z_{k-1}) = -\sum_{j=1}^{n}(\nabla_{b_j}\omega)(b_j, Z_1, ..., Z_{k-1}) + \omega((\operatorname{tr} T)^\natural, Z_1, ..., Z_{k-1})$$
$$-\sum_{j=1}^{k-1}g(g(Z_j, T^{2,0}(\cdot, \cdot)), \omega(\cdot, \cdot, Z_1, ..., \hat{Z}_j, ..., Z_{k-1})).$$

Proof: One uses lemma 5.1.2. For $Z_0, ..., Z_k \in TM^{0,1}$, we simply have $d\omega(Z_0, ..., Z_k) = \overline{\partial}\omega(Z_0, ..., Z_k)$. We can then replace T by $T^{2,0}$ for the following reason:

$$\begin{aligned} T(Z_j, Z_l) &= T(X_j, Z_l) + iT(JX_j, Z_l) \\ &= T(X_j, X_l) - T(JX_j, JX_l) + i(T(JX_j, X_l) + T(X_j, JX_l)). \end{aligned}$$

Hence, for $T^{1,1}$ we have $T^{1,1}(X_j, X_l) = T^{1,1}(JX_j, JX_l)$ and $T^{1,1}(JX_j, X_l) = -T^{1,1}(X_j, JX_l)$ and thus the (1,1)-part vanishes. For the (0,2)-part we obtain $T^{0,2}(X_j, X_l) = -T^{0,2}(JX_j, JX_l)$ and $T^{0,2}(JX_j, X_l) = -J(T^{0,2}(X_j, X_l)) = T^{0,2}(X_j, JX_l)$ and thus

$$T^{0,2}(Z_j, Z_l) = 2T^{0,2}(X_j, X_l) - 2iJT^{0,2}(X_j, X_l) \in TM^{1,0},$$

which vanishes when we take the scalar product with an element of $TM^{0,1}$. Therefore the only part that remains is the (2,0)-part (and this part is, by an analogous calculation, indeed in $TM^{0,1}$).
The second identity follows by the same arguments. □

We will now use these results to compare the Hodge-Dolbeault operator and geometric Dirac operators. Throughout this section, we assume that the manifold M is of even dimension $n = 2m$ and equipped with an almost-complex structure J and an almost-hermitian metric J. We will use the following local frames without further explanation: $e_1, f_1, ..., e_m, f_m$ denotes a J-adapted frame and given such a frame we set

$$\begin{aligned} z^j &:= \frac{1}{\sqrt{2}}(e^j + if^j) \quad (j = 1, \ldots, n), \\ \overline{z^j} &:= \frac{1}{\sqrt{2}}(e^j - if^j) \quad (j = 1, \ldots, n). \end{aligned}$$

Compare section 1.1 for more details on this. One easily deduces

$$\begin{aligned} (e_j)^{1,0} &= \frac{1}{\sqrt{2}} z_j, & (e_j)^{0,1} &= \frac{1}{\sqrt{2}} \overline{z_j}, \\ (f_j)^{1,0} &= \frac{i}{\sqrt{2}} z_j, & (f_j)^{0,1} &= -\frac{i}{\sqrt{2}} \overline{z_j}. \end{aligned}$$

Recall that in the case of a $Spin^c$ structure, to induce a connection on the spinor bundle, we not only need a connection on TM but also an auxiliary connection form Z on P_1. Throughout this section, we take the point of view that the connection form Z is the det-extension of some connection form A^z on $P_U(M)$ induced by a connection ∇^z as described in section 3.4. We will denote such a Dirac operator by $\mathcal{D}_c(\nabla, \nabla^z)$ with the conventions that we note the Riemannian Dirac operator $\mathcal{D}_c^g(\nabla^z) := \mathcal{D}_c(\nabla^g, \nabla^z)$ and the Dirac operators of the canonical and basic connections $\mathcal{D}_c^t(\nabla^z) = \mathcal{D}_c(\nabla^t, \nabla^z)$ and $\mathcal{D}_c^b(\nabla^z) = \mathcal{D}_c(\nabla^b, \nabla^z)$ respectively.

In [Gau97], Gauduchon considers Dirac operators of hermitian connections. However, he chooses another approach to the auxiliary connection, considering the $Spin^c$ spinor bundle as the tensor product of the spin spinor bundle with a square root L^{-1} of the anti-canonical bundle and then introducing a connection on this bundle as the product of a connection on the spinor bundle (for the spin structure) and a unitary connection on L^{-1}. Using this construction, he deduces the following relationship between the Hodge-Dolbeault operator and the geometric Dirac $\mathcal{D}_c^{g,1}$ operator induced by the Levi-Cività connection on TM and a connection on L that is induced by the second canonical connection (cf. [Gau97, section 3.6]):

$$\mathcal{H} = \mathcal{D}_c^{g,1} - \frac{1}{4}\left(c((d^cF)^+) - c((d^cF)^-)\right).$$

As we take another approach to the definition of a connection on $\mathbb{S}^c$, his results differ somewhat from ours.
The first result that we prove is the central one of this section, relating the Hodge-Dolbeault operator to a geometric Dirac operator, namely the geometric Dirac operator induced by a basic connection.

5.1.4 Theorem
Let (M, g, J) be an almost-hermitian manifold and $\mathbb{S}^c$ the spinor bundle of its canonical Spin^c structure. Then the Hodge-Dolbeault operator on $\mathbb{S}^c$ is the geometric Dirac operator induced by the basic connection ∇^b and the connection form Z^b which is also induced by the basic connection:

$$\mathcal{H} = \mathcal{D}^b_c(\nabla^b).$$

Proof: Recall the expressions for $\overline{\partial}$ and $\overline{\partial}^*$ from lemma 5.1.3. We will apply those for a basic connection. As ∇^b is nice, the trace of its torsion vanishes. Furthermore, comparing the explicit formulæ for the torsion of ∇^b and the first canonical connection ∇^0 (cf. the discussion in section 4.1)

$$\begin{aligned} T^b &= N - \frac{1}{4}((d^cF)^+ + \mathfrak{M}(d^cF)^+) + B \qquad\qquad (B \in \Omega^{1,1}_s(M, TM)),\\ T^t &= N - \frac{1}{4}((d^cF)^+ + \mathfrak{M}(d^cF)^+), \end{aligned}$$

we see that they only differ in their $(1,1)$-part. As we know that the (2,0)-part of the torsion of the first canonical connection vanishes, the respective part of the torsion of the basic connection must also vanish. Therefore, applying the results of lemma 5.1.3 to the basic connection, we obtain for any $Z_0, ..., Z_q \in \Gamma(TM^{0,1})$:

$$\begin{aligned} \overline{\partial}\omega(Z_0, ..., Z_q) &= \sum_{j=1}^{q} (-1)^j (\nabla^b_{Z_j}\omega)(Z_0, ..., \hat{Z}_j, ..., Z_q),\\ \overline{\partial}^*\omega(Z_1, ..., Z_{q-1}) &= -\sum_{j=1}^{n} (\nabla^b_{b_j}\omega)(b_j, Z_1, ..., Z_{q-1}). \end{aligned}$$

We will rewrite these two expressions in a slightly different way. Any $Z \in \Gamma(TM^{0,1})$ can locally be written as $Z = \sum_j \zeta^j \overline{z_j}$ and $\zeta^j = \overline{z^j}(Z)$. Therefore, we can write

$$\begin{aligned} \overline{\partial}\omega(Z_0, ..., Z_q) &= \sum_{j=0}^{q} (-1)^j (\nabla^b_{Z_j}\omega)(Z_0, ..., \hat{Z}_j, ..., Z_q)\\ &= \sum_{j,k}^{q} (-1)^j \overline{z^k}(Z_j)(\nabla^b_{\overline{z_k}}\omega)(Z_0, ..., \hat{Z}_j, ..., Z_k)\\ &= \sum_{k=1}^{m} \overline{z^k} \wedge (\nabla^b_{\overline{z_k}}\omega)(Z_0, ..., Z_q). \end{aligned}$$

Because all forms appearing above are of type $(0, *)$, both sides are equal to zero if an argument is of type (1,0). Therefore, we can use the above formula not only for arguments of type (0,1), but write generally

$$\overline{\partial}\omega = \sum_{k=1}^{m} \overline{z^k} \wedge (\nabla^b_{\overline{z_k}}\omega).$$

Concerning the formula for $\overline{\partial}^*$, we can write

$$\begin{aligned}\overline{\partial}^*\omega &= \sum_{j=1}^m e_j \lrcorner \nabla^b_{e_j}\omega + f_j \lrcorner \nabla^b_{f_j}\omega \\ &= \sum_{j=1}^m (e_j)^{0,1} \lrcorner \nabla^b_{e_j}\omega + (f_j)^{0,1} \lrcorner \nabla^b_{f_j}\omega.\end{aligned}$$

Thus, we obtain

$$\mathcal{H}\omega = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) = \sum_{j=1}^m \sqrt{2}\left[\overline{z^j} \wedge (\nabla^b_{\overline{z_j}}\omega) + (e_j)^{0,1} \lrcorner \nabla^b_{e_j}\omega + (f_j)^{0,1} \lrcorner \nabla^b_{f_j}\omega\right].$$

Using the definition of $\overline{z_j}$, we obtain

$$\begin{aligned}\mathcal{H}\omega = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) &= \sum_{j=1}^m \overline{z^j} \wedge (\nabla^b_{e_j}\omega) + i\overline{z^j} \wedge (\nabla^b_{f_j}\omega) + \sqrt{2}(e_j)^{0,1} \lrcorner \nabla^b_{e_j}\omega + \sqrt{2}(f_j)^{0,1} \lrcorner \nabla^b_{f_j}\omega \\ &= \sum_{j=1}^m \sqrt{2}\left[(e_j^{1,0})^\flat \wedge (\nabla^b_{e_k}\omega) + (f_j^{1,0})^\flat \wedge (\nabla^b_{f_k}\omega) + (e_j)^{0,1} \lrcorner \nabla^b_{e_j}\omega + (f_j)^{0,1} \lrcorner \nabla^b_{f_j}\omega\right] \\ &= \sum_{j=1}^m e_j.\nabla^b_{e_j}\omega + f_j.\nabla^b_{f_j}\omega.\end{aligned}$$

Using theorem 3.4.8, we deduce that the connection ∇^b on forms coincides with the spinor derivative induced by ∇^b and the auxiliary connection Z^b induced by ∇^b. This yields the claim. □

Using this result, we can now deduce results on the relationship between the Hodge-Dolbeault operator and any geometric operator, using lemma 3.3.13 (which compares two Dirac operators induced by two different connections on TM and the same auxiliary connection Z) and theorem 3.4.8 (which can be used to compare two Dirac operators induced by different connections Z). We recall the following definition for a form $\Omega \in \Omega^2(M, TM)$:

$$\operatorname{tr}_c \Omega(X) = i \sum_{j=1}^m \Omega(X; b_{2j}, b_{2j-1}),$$

where (s_j) is an adapted basis. We begin by giving a general formula which we will then apply to some special cases.

5.1.5 Proposition

Let ∇ be any metric connection and ∇^z a hermitian connection on the tangent bundle of an almost-hermitian manifold (M, g, J). Let A and A^z note their potentials. Then the following relationship between the Dirac operator $\mathcal{D}_c(\nabla, \nabla^z)$ and the Hodge-Dolbeault operator holds:

$$\mathcal{D}_c(\nabla, \nabla^z) = \mathcal{H} - \frac{1}{2}c(\operatorname{tr}(A - A^b)) + \frac{1}{2}c(\mathfrak{b}(A - A^b)) - \frac{1}{2}c(\operatorname{tr}_c(A^z - A^b)).$$

Proof: From the main theorem, we have that

$$\mathcal{H} = \sum_{j=1}^m e_j.\nabla^b_{e_j}\omega + f_j.\nabla^b_{f_j}\omega.$$

For the spinor connection induced by ∇^b and ∇^z we have by theorem 3.4.8 that

$$\nabla^b_X\omega = \widetilde{\nabla}^Z_X\omega + \frac{1}{2}\operatorname{tr}_c(A^z - A^b)(X)\omega.$$

Thus, we obtain the following equality:

$$\begin{aligned}\mathcal{H} &= \sum_{j=1}^{m} e_j.\widetilde{\nabla}^Z_{e_j}\omega + f_j.\widetilde{\nabla}^Z_{f_j}\omega + e_j.\frac{1}{2}\operatorname{tr}_c(A^z - A^b)(e_j)\omega + f_j.\frac{1}{2}\operatorname{tr}_c(A^z - A^b)(f_j)\omega \\ &= \mathcal{D}_c(\nabla^b, \nabla^z) + \frac{1}{2}\operatorname{tr}_c(A^z - A^b).\omega. \qquad (*)\end{aligned}$$

Furthermore, by lemma 3.3.13, we have that

$$\begin{aligned}\mathcal{D}_c(\nabla, \nabla^z) &= \mathcal{D}^g_c(\nabla^z) - \frac{1}{2}c(\operatorname{tr}(A)) + \frac{1}{2}c(\mathfrak{b}A), \\ \mathcal{D}_c(\nabla^b, \nabla^z) &= \mathcal{D}^g_c(\nabla^z) - \frac{1}{2}c(\operatorname{tr}(A^b)) + \frac{1}{2}c(\mathfrak{b}A^b).\end{aligned}$$

Combining these two with (*) yields the claim. □

Note that $\operatorname{tr}_c(\cdot)$ gives an *imaginary* one-form, which (as opposed to a real-valued one-form) acts symetrically on spinors, thus not interfering with the formal self-adjointness of $\mathcal{D}(\nabla, \nabla^z)$. As a first application, we compare the Riemannian Dirac operator with the Hodge-Dolbeault operator:

5.1.6 Corollary
For the Riemannian Dirac operator, the following identity is satisfied:

$$\mathcal{D}^g_c(\nabla^b) = \mathcal{H} + \frac{1}{12}c((d^cF)^+ - (d^cF)^-).$$

Proof: By the above proposition, we have

$$\mathcal{D}^g_c(\nabla^b) = \mathcal{H} + \frac{1}{2}c(\operatorname{tr} A^b) - \frac{1}{2}c(\mathfrak{b}A^b).$$

We will therefore need to calculate A^b. Recall that $A^b = -T^b + \frac{3}{2}\mathfrak{b}T^b$. Therefore, $\operatorname{tr} A^b = -\operatorname{tr} T^b = 0$. Furthermore, $\mathfrak{b}A^b = \frac{1}{2}\mathfrak{b}T^b$ and we have

$$T^b = N - \frac{1}{4}\left((d^cF)^+ + \mathfrak{M}(d^cF)^+\right) + B, .$$

and therefore, using $3\mathfrak{b}\mathfrak{M}(d^cF)^+ = (d^cF)^+$, we obtain

$$\mathfrak{b}T^b = \mathfrak{b}N - \frac{1}{4}\left((d^cF)^+ + \frac{1}{3}(d^cF)^+\right) + \mathfrak{b}B.$$

Because $B \in \Omega^{1,1}_s(M, TM)$, we have that $\mathfrak{b}B = 0$. Furthermore, we know from theorem 1.3.3 that $\mathfrak{b}N = \frac{1}{3}(d^cF)^-$. We obtain

$$\mathfrak{b}T^b = \frac{1}{3}(d^cF)^- - \frac{1}{3}(d^cF)^+$$

and therefore

$$\mathfrak{b}A^b = \frac{1}{6}\left((d^cF)^- - (d^cF)^+\right).$$

This yields the claimed equation. □

As another application, we consider the geometric Dirac operators induced by the canonical connections.

5.1.7 Corollary
For the geometric Dirac operators of the canonical connections, we have the following equality:

$$\mathcal{D}_c^g(\nabla^t, \nabla^b) = \mathcal{H} - \frac{t+1}{4}c(\theta) + \frac{t}{6}c((d^cF)^+).$$

Proof: From proposition 5.1.5, we have that

$$\mathcal{D}_c(\nabla^t, \nabla^s) = \mathcal{H} - \frac{1}{2}c(\mathrm{tr}(A^t - A^b)) + \frac{1}{2}c(\mathfrak{b}(A^t - A^b)).$$

We know that

$$T^t = N + \frac{3t-1}{4}(d^cF)^+ - \frac{t+1}{4}\mathfrak{M}(d^cF)^+$$

and therefore, as $\mathrm{tr}\, N = 0$,

$$\begin{aligned}\mathrm{tr}\, T^t &= -\frac{t+1}{4}\,\mathrm{tr}\,\mathfrak{M}(d^cF)^+ \\ &= -\frac{t+1}{2}\theta.\end{aligned}$$

Therefore

$$\mathrm{tr}\, A^t = \frac{t+1}{2}\theta$$

and using that $\mathrm{tr}\, A^b = 0$, we obtain $\mathrm{tr}(A^b - A^t) = -\frac{t+1}{2}\theta$. It remains to calculate $\mathfrak{b}(A^t - A^b)$. We have already calculated that

$$\mathfrak{b}A^b = \frac{1}{6}\left((d^cF)^- - (d^cF)^+\right).$$

Using again the formula for the torsion of ∇^t, we deduce

$$\begin{aligned}\mathfrak{b}T^t &= \mathfrak{b}N + \frac{3t-1}{4}(d^cF)^+ - \frac{t+1}{12}(d^cF)^+ \\ &= \frac{1}{3}(d^cF)^- + \frac{2t-1}{3}(d^cF)^+\end{aligned}$$

and thus

$$\begin{aligned}\mathfrak{b}(A^t - A^b) &= \frac{1}{6}(d^cF)^- + \frac{2t-1}{6}(d^cF)^+ - \frac{1}{6}\left((d^cF)^- - (d^cF)^+\right) \\ &= \frac{t}{3}(d^cF)^+.\end{aligned}$$

Combining all these results, we obtain the claimed formula. □

One might also use proposition 5.1.5 to deduce formulæ for the Dirac operators $\mathcal{D}(\nabla^g, \nabla^t)$ and $\mathcal{D}(\nabla^t, \nabla^s)$ but we omit these as they would lead to quite tedious calculations without any direct interest to us.
This concludes our discussion of Dirac operators on almost-hermitian manifolds. The most important result is that the Hodge-Dolbeault operator is induced by the basic connection. This will be used in the following section to describe the Dirac operator of the Tanaka-Webster connection.

5.2 Dirac operators on contact manifolds

As we have done for connections, we will now use the theory of Dirac operators on almost-hermitian manifolds developed in the previous section to describe Dirac operators on a metric contact manifold with particular focus on the CR case. In this section, let (M^{2m+1}, g, η, J) be a metric contact and $(\hat{M}, \hat{g}, \hat{J})$ the associated almost-hermitian manifold (for more details on the manifold $\hat{M}$, compare section 4.2). We denote $\mathbb{S}^c$ and $\hat{\mathbb{S}^c}$ the spinor bundles of their respective canonical $Spin^c$ structures. Using the relationship between Hodge-Dolbeault operator and geometric Dirac operators on $\hat{M}$ and the relationship between the $Spin^c$-spinor bundles of $\hat{M}$ and M, we describe a Hodge-Dolbeault-like operator that is the geometric Dirac operator induced by the generalized Tanaka-Webster connection.
We know from the preceding section that the Hodge-Dolbeault operator $\hat{\mathcal{H}}$ on $\hat{M}$ coincides with the geometric Dirac operator induced by a basic connection ∇^b and the connection form Z^b on P_1 which is also induced by ∇^b. In our case, we will always choose the basic connection $\hat{\nabla}^b$ as defined after lemma 4.2.6.
As we are interested in Dirac operators and connections on M, we want to study how $\hat{\mathcal{H}}$ acts on M, i.e. on $\mathbb{S}^c$. In order to do so, we will take a closer look at the relationship between $\hat{\mathbb{S}^c}$ and $\mathbb{S}^c$ and the Clifford multiplications on the two bundles. The calculations we are about to present roughly follow [Nic05, section 3.3]. Recall that because $\mathcal{C}$ admits an almost-hermitian structure, its complexification admits a splitting into the $\pm i$-eigenspaces of J, which we denote $\mathcal{C}^{1,0}$ and $\mathcal{C}^{0,1}$. Taking the duals of each subbundle, this splitting extends to the dual bundle $\mathcal{C}_c^*$. Taking exterior powers, we obtain the bundle of (p,q)-forms

$$\Lambda^{p,q}(\mathcal{C}^*) = \Lambda^p((\mathcal{C}^{1,0})^*) \wedge \Lambda^q((\mathcal{C}^{0,1})^*).$$

Using this bundle, we can describe the spinor bundles as follows (cf. propositions 3.4.2 and 3.4.9):

$$\begin{aligned}\hat{\mathbb{S}^c} &\simeq \Lambda^{0,*}(T^*\hat{M}) = \Lambda^*(T^*\hat{M}^{0,1}),\\ \mathbb{S}^c &\simeq \Lambda^{0,*}(\mathcal{C}^*) = \Lambda^*((\mathcal{C}^*)^{0,1}).\end{aligned}$$

To compare the two, we first make some remarks on local bases of $\hat{M}$. We will always use a $\hat{J}$-adapted local basis $(e_0, f_0, e_1, f_1, ..., e_m, f_m)$, where $e_0 = \partial t$, $f_0 = \xi$ and $(e_1, f_1, ..., e_m, f_m)$ is a J-adapted basis of $\mathcal{C}$. From this basis, we can deduce bases (z_j) of $TM^{1,0}$ and $(\overline{z_j})$ of $TM^{0,1}$ with duals (z^j), $(\overline{z^j})$ in the usual way (cf. the beginning of section 5.1). Of particular importance are the following elements:

$$\begin{aligned}z_0 &= \frac{1}{\sqrt{2}}(e_0 - if_0) = \frac{1}{\sqrt{2}}(\partial t - i\xi),\\ \overline{z_0} &= \frac{1}{\sqrt{2}}(e_0 + if_0) = \frac{1}{\sqrt{2}}(\partial t + i\xi),\\ z^0 &= \frac{1}{\sqrt{2}}(dt + i\eta),\\ \overline{z^0} &= \frac{1}{\sqrt{2}}(dt - i\eta).\end{aligned}$$

Then, we note that

$$\begin{aligned}T^*\hat{M} &= \mathbb{R}dt \oplus \mathbb{R}\eta \oplus \mathcal{C}^*\\ T^*\hat{M}_c &= \mathbb{C}z^0 \oplus \mathbb{C}\overline{z^0} \oplus \mathcal{C}_c^*.\end{aligned}$$

and therefore

$$T^*\hat{M}^{1,0} = \mathbb{C}z^0 \oplus (\mathcal{C}^*)^{1,0} \quad \text{and} \quad T^*\hat{M}^{0,1} = \mathbb{C}\overline{z^0} \oplus (\mathcal{C}^*)^{0,1},$$

which implies

$$\Lambda^{0,p}(T^*\hat{M}) = \mathbb{C}\overline{z^0} \wedge \Lambda^{0,p-1}(\mathcal{C}^*) \oplus \Lambda^{0,p}(\mathcal{C}^*).$$

In particular, we have

$$\Lambda^{0,even}(T^*\hat{M}) = \mathbb{C}\overline{z^0} \wedge \Lambda^{0,odd}(\mathcal{C}^*) \oplus \Lambda^{0,even}(\mathcal{C}^*)$$

and

$$\Lambda^{0,odd}(T^*\hat{M}) = \mathbb{C}\overline{z^0} \wedge \Lambda^{0,even}(\mathcal{C}^*) \oplus \Lambda^{0,odd}(\mathcal{C}^*).$$

Now, $\hat{\mathbb{S}^c}$ splits as $\hat{\mathbb{S}^c} = \hat{\mathbb{S}^c_+} \oplus \hat{\mathbb{S}^c_-}$ with $\hat{\mathbb{S}^c_\pm} = \Lambda^{0,even/odd}(T^*\hat{M})$. We then have an identification

$$\begin{aligned} \chi\colon \hat{\mathbb{S}^c_+} \simeq \Lambda^{0,even}(T^*\hat{M}) &\xrightarrow{\sim} \mathbb{S}^c \simeq \Lambda^{0,*}(\mathcal{C}^*) \\ \overline{z^0} \wedge \omega_0 + \omega_1 &\longmapsto \omega_0 + \omega_1 \end{aligned}$$

with the inverse mapping given by $\chi^{-1}(\omega) = \omega$ for $\omega \in \Lambda^{0,even}(\mathcal{C}^*)$ and $\chi^{-1}\omega = \overline{z^0} \wedge \omega$ for $\omega \in \Lambda^{0,odd}(\mathcal{C}^*)$. Now, we set

$$\mathcal{J} = \hat{c}(dt) = \frac{1}{\sqrt{2}}(\hat{c}(z^0) + \hat{c}(\overline{z^0})).$$

By definition of the Clifford multiplication, we have that

$$\begin{aligned} \hat{c}(z^0) &= \sqrt{2}\left((z^0)^{0,1} \wedge - \left((z^0)^{1,0}\right)^\natural \lrcorner\right) = -\sqrt{2}\,\overline{z_0}\lrcorner, \\ \hat{c}(\overline{z^0}) &= \sqrt{2}\left(\overline{z^0}^{0,1} \wedge - \left(\overline{z^0}^{1,0}\right)^\natural \lrcorner\right) = \sqrt{2}\,\overline{z^0} \wedge . \end{aligned}$$

Because elements of $\mathbb{S}^c$ are differential forms on $\mathcal{C}$, we have that $\overline{z_0}\lrcorner\psi = 0$ for any $\psi \in \mathbb{S}^c$. Thus, $\mathcal{J}|_{\mathbb{S}^c} = \sqrt{2}\,\overline{z^0}\wedge$ and for $\psi = \overline{z^0} \wedge \omega_0 + \omega_1 \in \hat{\mathbb{S}^c_+}|_M$, we have

$$\mathcal{J}\psi = \omega_0 + \overline{z^0} \wedge \omega_1 \in \hat{\mathbb{S}^c_-}|_M$$

and every element of $\hat{\mathbb{S}^c_-}|_M$ can be written in this form. Hence, we have $\mathcal{J}\hat{\mathbb{S}^c_+} = \hat{\mathbb{S}^c_-}|_M$ and obtain an identification $-\chi\mathcal{J}\colon \mathbb{S}^c_-|_M \to \mathbb{S}^c$. Therefore, along M, we can write $\hat{\mathbb{S}^c}|_M \simeq \mathbb{S}^c \oplus \mathbb{S}^c$. Using this decomposition, we can write forms $\psi \in \hat{\mathbb{S}^c} \simeq \mathbb{S}^c \oplus \mathbb{S}^c$ as column vectors by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \overset{\sim}{\mapsto} \chi^{-1}\psi_1 + \mathcal{J}\chi^{-1}\psi_2 \tag{5.3}$$

with $\psi_1, \psi_2 \in \mathbb{S}^c$. With respect to this decomposition, we can write $\mathcal{J}$ as

$$\mathcal{J} = \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix} \tag{5.4}$$

and because $\mathcal{J}^2 = c(dt)^2 = -\|dt\|^2 = -1$, we have that $K_1K_2 = -1$, $K_2K_1 = -1$.
We now compare the Clifford multiplication on $\hat{\mathbb{S}^c}$ and $\mathbb{S}^c$.

5.2.1 Lemma
The Clifford multiplication $\hat{c}$ of $\hat{\mathbb{S}^c}$ and c of $\mathbb{S}^c$ are related by

$$c(X)(\psi) = \chi\left(\mathcal{J}\hat{c}(X)(\chi^{-1}(\psi))\right)$$

for any $X \in TM, \psi \in \mathbb{S}^c$.

Proof: By propositions 3.4.2 and 3.4.9 the two Clifford multiplications on a $(0,q)$-form are given by

$$\hat{c}(X) = \sqrt{2}\left((X^{1,0})^\flat \wedge - X^{0,1}\lrcorner\right),$$
$$c(X) = \sqrt{2}((X^{1,0}_{\mathcal{C}})^\flat \wedge - X^{0,1}_{\mathcal{C}}\lrcorner) + (-1)^{q+1} i\eta(X).$$

Note that for $X \in \Gamma(\mathcal{C})$, the (1,0)+(0,1)-splitting of $T^*\hat{M}$ and $\mathcal{C}$ agree because $X = X^{1,0}_{\mathcal{C}} + X^{0,1}_{\mathcal{C}}$ and because $(\mathcal{C}^*)^{1,0} \subset T^*M^{1,0}$ we have $X^{1,0}_{\mathcal{C}} \in T^*M^{1,0}$ and analogously for $X^{0,1}_{\mathcal{C}}$. We now check that the claimed relationship holds on a basis of TM.
We begin with ξ. We have that $c(\xi)\psi = i(-1)^{q+1}\psi$. On the other hand, $\xi^{1,0} = \frac{i}{\sqrt{2}} z_0$ and $\xi^{0,1} = -\frac{i}{\sqrt{2}}\overline{z_0}$. Therefore, for $\psi \in \Omega^{0,q=2k}(\mathcal{C})$

$$\begin{aligned}\hat{c}(\xi)(\chi^{-1}\psi) &= \sqrt{2}(\frac{i}{\sqrt{2}}(z_0)^\flat \wedge \psi + \frac{i}{\sqrt{2}}\underbrace{\overline{z_0}\lrcorner\psi}_{=0}) \\ &= i\overline{z^0} \wedge \psi\end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{J}\hat{c}(\xi)(\chi^{-1}\psi) &= i\mathcal{J}(\overline{z^0}\wedge\psi) \\ &= i\overline{z^0} \wedge \overline{z^0} \wedge \psi - i\overline{z_0}\lrcorner(\overline{z^0}\wedge\psi) \\ &= -i\psi + i\overline{z^0} \wedge (\overline{z_0}\lrcorner\psi) \\ &= -i\psi = i(-1)^{2k+1}\psi.\end{aligned}$$

Then, we obtain the following equality:

$$\chi\left(\mathcal{J}\hat{c}(\xi)(\chi^{-1}\psi)\right) = i(-1)^{2k+1}\psi.$$

Analogously, for $\psi \in \Omega^{0,q=2k+1}(\mathcal{C})$, one uses that $\hat{c}(\xi)(\chi^{-1}\psi) = \hat{c}(\xi)(\overline{z^0}\wedge\psi)$ and obtains the claimed equality as well.
Next up, we consider an element e_j of a J-adapted basis (e_j, f_j) of $\mathcal{C}$. We have that $e_j^{1,0} = \frac{1}{\sqrt{2}} z_j$ and $e_j^{0,1} = \frac{1}{\sqrt{2}}\overline{z_j}$. Thus, we have

$$c(e_j)\psi = \overline{z^j} \wedge \psi - \overline{z_j}\lrcorner\psi$$

and for $\psi \in \Omega^{0,2k}(\mathcal{C})$, we have

$$\hat{c}(e_j)(\chi^{-1}\psi) = \overline{z^j}\wedge\psi - \overline{z_j}\lrcorner\psi.$$

Hence, we obtain

$$\begin{aligned}\mathcal{J}\hat{c}(e_j)(\chi^{-1}\psi) &= \overline{z^0}\wedge\overline{z^j}\wedge\psi - \overline{z_0}\lrcorner\overline{z^j}\wedge\psi - \overline{z^0}\wedge\overline{z_j}\lrcorner\psi + \overline{z_0}\lrcorner\overline{z_j}\lrcorner\psi \\ &= \overline{z^0}\wedge(\overline{z^j}\wedge\psi - \overline{z_j}\lrcorner\psi), \\ \chi\left(\mathcal{J}\hat{c}(e_j)(\chi^{-1}\psi)\right) &= \overline{z^j}\wedge\psi - \overline{z_j}\lrcorner\psi.\end{aligned}$$

For $\psi \in \Omega^{0,2k+1}(\mathcal{C})$, we have

$$\begin{aligned}
\hat{c}(e_j)(\chi^{-1}\psi) &= \hat{c}(e_j)(\overline{z^0} \wedge \psi) \\
&= \overline{z^j} \wedge \overline{z^0} \wedge \psi - \overline{z_j} \lrcorner (\overline{z^0} \wedge \psi) \\
&= \overline{z^j} \wedge \overline{z^0} \wedge \psi + \overline{z^0} \wedge (\overline{z_j} \lrcorner \psi), \\
\mathcal{J}\hat{c}(e_j)(\chi^{-1}\psi) &= \overline{z^0} \wedge \overline{z^j} \wedge \overline{z^0} \wedge \psi + \overline{z^0} \wedge \overline{z^0} \wedge (\overline{z_j} \lrcorner \psi) - \overline{z_0} \lrcorner (\overline{z^j} \wedge \overline{z^0} \wedge \psi) - \overline{z_0} \lrcorner (\overline{z^0} \wedge (\overline{z_j} \lrcorner \psi)) \\
&= -\overline{z_0} \lrcorner \overline{z^j} \wedge \overline{z^0} \wedge \psi - \overline{z_j} \lrcorner \psi \\
&= \overline{z^j} \wedge \psi - \overline{z_j} \lrcorner \psi.
\end{aligned}$$

This yields

$$\chi\left(\mathcal{J}\hat{c}(e_j)(\chi^{-1}\psi)\right) = \overline{z^j} \wedge \psi - \overline{z_j} \lrcorner \psi.$$

An analogous argument for f_j then yields the claim. $\square$

We now conclude our discussion of the relationship between the two spinor bundles and apply the results. We want to consider the geometric Dirac operator induced by ∇^{TW}. To this end, we make some remarks about connections induced on the spinor bundle $\mathbb{S}^c$ associated to the canonical $Spin^c$ structure on M. As in the almost-hermitian case, a connection on $\mathbb{S}^c$ is defined by a metric connection ∇ on TM and a connection form Z on P_1 which can be induced by a hermitian connection on TM (see section 3.4 for more details). Here, we will consider the geometric Dirac operator $\mathcal{H} = \mathcal{D}_c(\nabla^{TW}, \nabla^{TW})$ (for the notation, see section 5.1). For reasons which will become clear later, we will call it the *contact Hodge-Dolbeault operator*. To compare it with the Hodge-Dolbeault operator on $\hat{\mathbb{S}^c}$, we will first need to compare the spinor connections on the two bundles. We know that ∇^{TW} is the restriction of $\hat{\nabla}^b$ to TM. Thus, the covariant derivatives induced by the two connections on the bundles of forms coincide on $\Lambda^{0,*}(\mathcal{C}^*) \subset \Lambda^{0,*}(T^*\hat{M})$. By theorems 3.4.8 and 3.4.12 we then deduce that the spinor connections coincide on $\Lambda^{0,*}(\mathcal{C}^*)$.
Then, from the local formulæ of the geometric Dirac operators of $\hat{\nabla}^b$ and ∇^{TW}, we obtain the following expressions, where $\psi \in \Gamma(\mathbb{S}^c)$:

$$\hat{\mathcal{H}}(\chi^{-1}\psi) = \hat{c}(\partial t)\hat{\nabla}^b{}_{\partial t}(\chi^{-1}\psi) + \hat{c}(\xi)\hat{\nabla}^b{}_\xi(\chi^{-1}\psi) + \sum_{i=1}^m \left(\hat{c}(e_j)\hat{\nabla}^b{}_{e_j} + \hat{c}(f_j)\hat{\nabla}^b{}_{f_j}\right)(\chi^{-1}\psi), \quad (5.5)$$

$$\mathcal{H}\psi = c(\xi)\hat{\nabla}^b{}_\xi\psi + \sum_{i=1}^m \left(c(e_j)\hat{\nabla}^b{}_{e_j} + c(f_j)\hat{\nabla}^b{}_{f_j}\right)\psi.$$

Now, using the relationship between Clifford multiplication on $\hat{\mathbb{S}^c}$ and $\mathbb{S}^c$ above, we deduce

$$\mathcal{H}\psi = \chi\mathcal{J}\hat{c}(\xi)\chi^{-1}\nabla^{TW}_\xi\psi + \sum_{i=1}^m \chi\mathcal{J}\hat{c}(e_j)\chi^{-1}\nabla^{TW}_{e_j}\psi + \chi\mathcal{J}\hat{c}(f_j)\chi^{-1}\nabla^{TW}_{f_j}\psi. \quad (5.6)$$

Furthermore, $\hat{\nabla}^b$ and χ commute: Let $\varphi = \overline{z^0} \wedge \psi_0 + \psi_1 \in \Gamma(\hat{S^c_+})$, $\psi_0, \psi_1 \in \Gamma(\mathbb{S}^c)$. Then,

$$\begin{aligned}
\chi(\hat{\nabla}^b{}_X\varphi) &= \chi\left(\hat{\nabla}^b\overline{z^0} \wedge \psi_0 + \overline{z^0} \wedge \hat{\nabla}^b{}_X\psi_0 + \hat{\nabla}^b{}_X\psi_1\right) \\
&= \chi\left(\hat{\nabla}^b\overline{z^0} \wedge \psi_0\right) + \hat{\nabla}^b{}_X\psi_0 + \hat{\nabla}^b{}_X\psi_1 \\
&= \chi\left(\hat{\nabla}^b\overline{z^0} \wedge \psi_0\right) + \hat{\nabla}^b{}_X(\chi\varphi).
\end{aligned}$$

Yet, $\hat{\nabla}^b\overline{z^0} = 0$ for the following reason: We have

$$\hat{\nabla}^b{}_X\overline{z^0} = \frac{1}{\sqrt{2}}(\hat{\nabla}^b{}_X dt - i\hat{\nabla}^b{}_X\eta).$$

Now, from equation (4.29) in the discussion of contact connections, we know that $\hat{\nabla}^b\partial t = 0$. Furthermore, the restriction of $\hat{\nabla}^b$ to TM is contact and thus ξ is $\hat{\nabla}^b$-parallel. Then, by corollary 3.4.6, $\hat{\nabla}^b dt = \hat{\nabla}^b\eta = 0$ and thus $\hat{\nabla}^b\overline{z^0} = 0$. This proves the claimed relationship.
Hence, we deduce the following formula from (5.6):

$$\mathcal{H}\psi = \chi\mathcal{J}\hat{c}(\xi)\hat{\nabla}^b{}_\xi(\chi^{-1}\psi) + \sum_{i=1}^{m}\chi\mathcal{J}\hat{c}(e_j)\hat{\nabla}^b{}_{e_j}(\chi^{-1}\psi) + \chi\mathcal{J}\hat{c}(f_j)\hat{\nabla}^b{}_{f_j}(\chi^{-1}\psi).$$

Using (5.5), we obtain that

$$\hat{\mathcal{H}}(\chi^{-1}\psi) = \hat{c}(\partial t)\chi^{-1}\hat{\nabla}^b{}_{\partial t}\psi - \mathcal{J}\chi^{-1}\mathcal{H}\psi.$$

A similar result can be obtained using the isomorphism between $\mathbb{S}^c_-|_M$ and $\mathbb{S}^c$ instead of χ. Writing this in the block form defined in (5.3) with respect to the splitting $\hat{\mathbb{S}^c} = \mathbb{S}^c \oplus \mathcal{J}\mathbb{S}^c$, we have

$$\hat{\mathcal{H}} = \mathcal{J}\left(\hat{\nabla}^b{}_{\partial t} - \begin{pmatrix}\mathcal{H} & 0\\ 0 & K_1\mathcal{H}K_2\end{pmatrix}\right). \tag{5.7}$$

We go back to the splitting $\Omega^{0,*}(\hat{M}) = \overline{z^0}\wedge\Omega^{0,*}(\mathcal{C})\oplus\Omega^{0,*}(\mathcal{C})$ and will give an alternative description. In order to do so, we remark that by the definition of z^0 and $\overline{z^0}$, we have $\eta^{1,0} = -\frac{i}{\sqrt{2}}z^0$ and $\eta^{0,1} = \frac{i}{\sqrt{2}}\overline{z^0}$ and thus $\hat{c}(\eta) = i\overline{z^0}\wedge + i\overline{z_0}\lrcorner$. Hence, for $\omega_1,\omega_2\in\Omega^{0,*}(\mathcal{C})$ we have

$$\begin{aligned}\mathcal{J}\hat{c}(i\eta)(\overline{z^0}\wedge\omega_1+\omega_2) &= -\mathcal{J}(\overline{z^0}\wedge + \overline{z_0}\lrcorner)(\overline{z^0}\wedge\omega_1+\omega_2)\\ &= \mathcal{J}(-\omega_1 - \overline{z^0}\wedge\omega_2)\\ &= (-\overline{z^0}\wedge\omega_1+\omega_2).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}(1-\mathcal{J}\hat{c}(i\eta))(\overline{z^0}\wedge\omega_1+\omega_2) &= \overline{z^0}\wedge\omega_1,\\ \frac{1}{2}(1+\mathcal{J}\hat{c}(i\eta))(\overline{z^0}\wedge\omega_1+\omega_2) &= \omega_2,\end{aligned}$$

i.e. we have that

$$\begin{aligned}\frac{1}{2}(1-\mathcal{J}\hat{c}(i\eta))\hat{\mathbb{S}^c}|_M &= \overline{z^0}\wedge\Lambda^{0,*}(\mathcal{C}^*),\\ \frac{1}{2}(1+\mathcal{J}\hat{c}(i\eta))\hat{\mathbb{S}^c}|_M &= \Lambda^{0,*}(\mathcal{C}^*).\end{aligned}$$

In particular, we have that

$$\begin{aligned}\frac{1}{2}(1-\mathcal{J}\hat{c}(i\eta))\hat{\mathbb{S}^c_+}|_M &= \overline{z^0}\wedge\Lambda^{0,odd}(\mathcal{C}^*),\\ \frac{1}{2}(1+\mathcal{J}\hat{c}(i\eta))\hat{\mathbb{S}^c_+}|_M &= \Lambda^{0,even}(\mathcal{C}^*).\end{aligned}$$

We define

$$\mathbb{S}^c_+ = \Lambda^{0,even}(\mathcal{C}^*) \quad\text{and}\quad \mathbb{S}^c_- = \Lambda^{0,odd}(\mathcal{C}^*).$$

By the above result and the known relationship between Clifford multiplication on $\hat{\mathbb{S}^c}$ and $\mathbb{S}^c$, we have that $\mathbb{S}^c_\pm$ are the eigenspaces of $c(i\eta)$ to the eigenvalue ± 1.
Now, let $\omega \in \Omega^{0,q}(\mathcal{C})$. Then, $\overline{\partial}\omega \in \Omega^{0,q+1}(\hat{M})$ and therefore, it admits a splitting in the above sense, i.e. there exist $(\overline{\partial}\omega)_0 \in \Omega^{0,q}(\mathcal{C})$ and $(\overline{\partial}\omega)_1 \in \Omega^{0,q+1}(\mathcal{C})$ such that

$$\begin{aligned}\overline{\partial}\omega &= \overline{z^0} \wedge (\overline{\partial}\omega)_0 + (\overline{\partial}\omega)_1\\ &= \frac{1}{2}(1 - \mathcal{J}\hat{c}(i\eta))\overline{\partial}\omega + \frac{1}{2}(1 + \mathcal{J}\hat{c}(i\eta))\overline{\partial}\omega.\end{aligned}$$

Thus we obtain operators on $\mathbb{S}^c$ as follows:

5.2.2 Definition We define the following operators:

$$\begin{aligned}\overline{\partial}_0 \colon \mathbb{S}^c &\longrightarrow \mathbb{S}^c\\ \omega &\longmapsto (\overline{\partial}\omega)_0,\\ \overline{\partial}_{\mathcal{C}} \colon \mathbb{S}^c &\longrightarrow \mathbb{S}^c\\ \omega &\longmapsto (\overline{\partial}\omega)_1 = \frac{1}{2}(1 + \mathcal{J}\hat{c}(i\eta))\overline{\partial}\omega\end{aligned}$$

with notation as above.

By construction, we have that $\overline{\partial}_0(\mathbb{S}^c_\pm) \subset \mathbb{S}^c_\pm$ and $\overline{\partial}_{\mathcal{C}}(\mathbb{S}^c_\pm) \subset \mathbb{S}^c_\mp$. Furthermore, one has that

$$\begin{aligned}\overline{z^0} \wedge \overline{\partial}_0\omega &= \frac{1}{2}(1 - \mathcal{J}\hat{c}(i\eta))\overline{\partial}\omega\\ &= \frac{1}{2}\overline{\partial}\omega + \frac{1}{2}\mathcal{J}(\overline{z^0} \wedge \overline{\partial}\omega + \overline{z_0} \lrcorner \overline{\partial}\omega)\\ &= \frac{1}{2}\overline{\partial}\omega + \frac{1}{2}(-\overline{z_0} \lrcorner (\overline{z^0} \wedge \overline{\partial}\omega) + \overline{z^0} \wedge (\overline{z_0} \lrcorner \overline{\partial}\omega))\\ &= \overline{z^0} \wedge (\overline{z_0} \lrcorner \overline{\partial}\omega)\end{aligned}$$

and thus, we have that $\overline{\partial}_0\omega = \overline{z_0} \lrcorner \overline{\partial}\omega$. We keep this in mind for later use.
Now, let $\varphi \in \Gamma(\hat{\mathbb{S}^c_+})$ be t-independent. Then, using (5.7), we have that

$$\begin{aligned}\hat{\mathcal{H}}\psi &= -\mathcal{J}\begin{pmatrix}\mathcal{H} & 0\\ 0 & K_2\mathcal{H}K_1\end{pmatrix}\begin{pmatrix}\chi\varphi\\ 0\end{pmatrix}\\ &= -\begin{pmatrix}0 & K_1\\ K_2 & 0\end{pmatrix}\begin{pmatrix}\mathcal{H} & 0\\ 0 & K_2\mathcal{H}K_1\end{pmatrix}\begin{pmatrix}\chi\varphi\\ 0\end{pmatrix}\\ &= \begin{pmatrix}0 & \mathcal{H}K_1\\ -K_2\mathcal{H} & 0\end{pmatrix}\begin{pmatrix}\chi\varphi\\ 0\end{pmatrix},\end{aligned}$$

i.e. we have that $\hat{\mathcal{H}}\psi = -\chi^{-1}\mathcal{J}\mathcal{H}\chi\psi$, which implies

$$\chi^{-1}\mathcal{H}\chi\varphi = \sqrt{2}\mathcal{J}(\overline{\partial} + \overline{\partial}^*)\varphi$$

for any t-independent $\varphi \in \Gamma(\mathbb{S}^c_+)$, or, equivalently,

$$\mathcal{H}\psi = \sqrt{2}\chi\mathcal{J}(\overline{\partial} + \overline{\partial}^*)\chi^{-1}\psi$$

for any $\psi \in \Gamma(\mathbb{S}^c)$.

Now, let $\psi \in \Gamma(\mathbb{S}^c)$. Then $\psi = \psi_- + \psi_-$ with $\psi_\pm \in \Gamma(\mathbb{S}^c_\pm)$ and we obtain that

$$\begin{aligned}(\overline{\partial} + \overline{\partial}^*)(\chi^{-1}\psi) &= (\overline{\partial} + \overline{\partial}^*)(\overline{z^0} \wedge \psi_- + \psi_+) \\ &= (\overline{\partial}\overline{z^0}) \wedge \psi_- - \overline{z^0} \wedge (\overline{\partial}\psi_-) + \overline{\partial}\psi_+ + \overline{\partial}^*(\overline{z^0} \wedge \psi) + \overline{\partial}^*\psi \\ &= -\overline{z^0} \wedge (\overline{\partial}_{\mathcal{C}}\psi_-) + \overline{z^0} \wedge \overline{\partial}_0\psi_+ + \overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}^*(\overline{z^0} \wedge \psi_-) + (\overline{z^0} \wedge \overline{\partial}_o + \overline{\partial}_{\mathcal{C}})^*\psi_+ \\ &= \overline{z^0} \wedge (\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_-) + \overline{\partial}_{\mathcal{C}}\psi_+ + (\overline{z^0} \wedge \overline{\partial}_0)^*\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}^*(\overline{z^0} \wedge \psi_-).\end{aligned}$$

We know that $(\overline{z^0}\wedge)^* = \overline{z_0}\lrcorner$ ie $(\overline{z^0} \wedge \overline{\partial}_0)^* = \overline{\partial}_0^*\overline{z_0}\lrcorner$ and because $\overline{z_0}\lrcorner\psi_+ = 0$, we obtain

$$(\overline{\partial} + \overline{\partial}^*)(\overline{z^0} \wedge \psi_- + \psi_+) = \overline{z^0} \wedge (\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_-) + \overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}^*(\overline{z^0} \wedge \psi_-). \tag{5.8}$$

The term that still remains somewhat unclear is $\overline{\partial}^*(\overline{z^0} \wedge \psi_-)$ and we now want to study it in some more detail: We have that

$$\begin{aligned}(\alpha, \overline{\partial}^*(\overline{z^0} \wedge \psi_-))_{L^2} &= (\overline{\partial}\alpha, \overline{z^0} \wedge \psi_-)_{L^2} \\ &= (\overline{z^0} \wedge \overline{\partial}_0\alpha + \overline{\partial}_{\mathcal{C}}\alpha, \overline{z^0} \wedge \psi_-)_{L^2} \\ &= (\overline{\partial}_0\alpha, \overline{z_0}\lrcorner(\overline{z^0} \wedge \psi_-))_{L^2} + (\overline{\partial}_{\mathcal{C}}\alpha, \overline{z^0} \wedge \psi_-)_{L^2} \\ &= (\alpha, \overline{\partial}_0^*\psi_-)_{L^2} + (\overline{\partial}_{\mathcal{C}}\alpha, \overline{z^0} \wedge \psi_-)_{L^2}.\end{aligned}$$

Now, α admits a splitting $\alpha = \alpha_- + \overline{z^0} \wedge \alpha_+$. Using this, we obtain

$$\begin{aligned}(\alpha, \overline{\partial}^*(\overline{z^0} \wedge \psi_-))_{L^2} &= (\overline{\partial}_0\alpha, \overline{\partial}_0^*\psi_-)_{L^2} + (\overline{\partial}_{\mathcal{C}}\alpha_- + \overline{\partial}_{\mathcal{C}}(\overline{z^0} \wedge \alpha_+), \overline{z^0} \wedge \psi_-)_{L^2} \\ &= (\alpha, \overline{\partial}_0^*\psi_-)_{L^2} + (\overline{\partial}_{\mathcal{C}}\alpha_-, \overline{z^0} \wedge \psi_-)_{L^2} - (\overline{z^0} \wedge (\overline{\partial}_{\mathcal{C}}\alpha_+), \overline{z^0} \wedge \psi_-)_{L^2} \\ &= (\alpha, \overline{\partial}_0^*\psi_-)_{L^2} + (\underbrace{\overline{z_0}\lrcorner\overline{\partial}_{\mathcal{C}}\alpha_-}_{=0}, \psi_-)_{L^2} - (\alpha_+, \overline{\partial}_{\mathcal{C}}^*\psi_-)_{L^2} \\ &= (\alpha, \overline{\partial}_0^*\psi_-)_{L^2} - (\overline{z^0} \wedge \alpha_+, \overline{z^0} \wedge \overline{\partial}_{\mathcal{C}}^*\psi_-)_{L^2} - (\underbrace{\overline{z_0}\lrcorner\alpha_-}_{=0}, \overline{\partial}_{\mathcal{C}}^*\psi_-)_{L^2} \\ &= (\alpha, \overline{\partial}_0^*\psi_-)_{L^2} - (\alpha, \overline{z^0} \wedge \overline{\partial}_{\mathcal{C}}^*\psi_-)_{L^2}.\end{aligned}$$

Therefore, we conclude

$$\overline{\partial}^*(\overline{z^0} \wedge \psi_-) = \overline{\partial}_0^*\psi_- - \overline{z^0} \wedge \overline{\partial}_{\mathcal{C}}^*\psi_-.$$

Combining this with (5.8) we obtain that

$$\begin{aligned}(\overline{\partial} + \overline{\partial}^*)(\overline{z^0} \wedge \psi_- + \psi_+) &= \overline{z^0} \wedge (\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_-) + \overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}_0^*\psi_- - \overline{z^0} \wedge \overline{\partial}_{\mathcal{C}}^*\psi_- \\ &= \overline{z^0} \wedge (\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_- - \overline{\partial}_{\mathcal{C}}^*\psi_-) + \overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}_0^*\psi_-.\end{aligned}$$

Using (5.2), we then obtain that

$$\begin{aligned}\mathcal{H}(\overline{z^0} \wedge \psi_- + \psi_+) &= \sqrt{2}\chi^{-1}\left(\mathcal{J}(\overline{\partial} + \overline{\partial}^*)(\overline{z^0} \wedge \psi_- + \psi_+)\right) \\ &= \sqrt{2}\chi^{-1}\Big((\overline{z^0} \wedge -\overline{z_0}\lrcorner)(\overline{z^0} \wedge (\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_- - \overline{\partial}_{\mathcal{C}}^*\psi_-) \\ &\qquad\qquad + \overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}_0^*\psi_-)\Big) \\ &= -\sqrt{2}\chi^{-1}\left((\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_- - \overline{\partial}_{\mathcal{C}}^*\psi_-) - \overline{z^0} \wedge (\overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}_0^*\psi_-)\right) \\ &= -\sqrt{2}(\overline{\partial}_0\psi_+ - \overline{\partial}_{\mathcal{C}}\psi_- - \overline{\partial}_{\mathcal{C}}^*\psi_-) + \sqrt{2}(\overline{\partial}_{\mathcal{C}}\psi_+ + \overline{\partial}_{\mathcal{C}}^*\psi_+ + \overline{\partial}_0^*\psi_-).\end{aligned}$$

Writing this in block form, i.e. writing $\psi_+ + \psi_- = (\psi_+, \psi_-)^T$, we obtain

$$\mathcal{H}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \sqrt{2}\begin{pmatrix} -\overline{\partial}_0 & (\overline{\partial}_{\mathcal{C}} + \overline{\partial}_{\mathcal{C}}^*) \\ (\overline{\partial}_{\mathcal{C}} + \overline{\partial}_{\mathcal{C}}^*) & \overline{\partial}_0^* \end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \tag{5.9}$$

Therefore, we can characterize the Tanaka-Webster connection of M as a connection inducing the Hodge-Dolbeault-like operator defined by (5.9). Yet, as we know, there may be many connections inducing the same Dirac operator. We do, however, have the following uniqueness result:

5.2.3 Theorem (cf. [Nic05, Proposition 3.11])
Let (M, g, η, J) be a metric contact manifold that is CR. Then each class of Dirac equivalent connections contains at most one nice CR connection.

Proof: In order to prove this theorem, we extend ∇ to a connection $\hat{\nabla}$ on the almost-hermitian manifold $\hat{M}$ via the following formulæ:

$$\hat{\nabla}.\partial t = 0 \quad \text{and} \quad (\hat{\nabla}_U V)(t,\cdot) = \nabla_{U(t,\cdot)} V(t,\cdot) \quad \text{and} \quad \hat{\nabla}_{\partial t} U = \frac{\partial}{\partial t} U,$$

where $U, V \in \Gamma(\hat{M}, TM)$ are interpreted as elements of $\mathfrak{X}(M)$ parametrized by t, i.e. $U(t,\cdot) \in \mathfrak{X}(M)$. It is obvious that the operator thus defined is a connection. We now prove that it is hermitian, i.e. that $\hat{\nabla}\hat{J} = 0$. As $\hat{\nabla}\hat{J}$ is tensorial in both arguments, it is enough to test it on vector fields which form a pointwise basis, i.e. on ∂t and vector fields $X, Y \in \mathfrak{X}(M)$ that are independent of t. For these, we obtain

$$\begin{aligned}
\hat{\nabla}_{\partial t}\hat{J}(\partial t) &= \hat{\nabla}_{\partial t}(\hat{J}\partial t) - \hat{J}(\hat{\nabla}_{\partial t}\partial t)\\
&= \frac{\partial}{\partial t}\xi = 0,\\
\hat{\nabla}_X\hat{J}(\partial t) &= \hat{\nabla}_X(\hat{J}\partial t) - \hat{J}(\hat{\nabla}_X\partial t)\\
&= \nabla_X \xi = 0,\\
\hat{\nabla}_{\partial t}\hat{J}(X) &= \hat{\nabla}_{\partial t}(\hat{J}X) - \hat{J}(\hat{\nabla}_{\partial t}X)\\
&= \hat{\nabla}_{\partial t}(JX) - \hat{\nabla}_{\partial t}\eta(X)\partial t - \hat{J}(\frac{\partial}{\partial t}X)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\hat{\nabla}_X\hat{J}(Y) &= \hat{\nabla}_X\hat{J}Y - \hat{J}(\hat{\nabla}_X Y)\\
&= \nabla_X JY - J\nabla_X Y + \hat{\nabla}_X(\eta(Y)\partial t) + \eta(\nabla_X Y)\partial t\\
&= (\nabla_X J)Y + (-X(\eta(Y)) + \eta(\nabla_X Y))\partial t\\
&= 0,
\end{aligned}$$

where the last equality follows, because ∇ is contact and because

$$X(\eta(Y)) = X(g(Y,\xi)) = g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi) = \eta(\nabla_X Y).$$

Having established that $\hat{\nabla}$ is indeed hemitian, we now consider its torsion $\hat{T}$. For $u, v \in T_x M$ and a t-independent vector field $U \in \mathfrak{X}(M)$ such that $U(x) = u$, we have

$$\begin{aligned}
\hat{T}(u,v) &= T(u,v),\\
\hat{T}(u,\partial t) &= \hat{\nabla}_u \partial t - \hat{\nabla}_{dt} U - [U, \partial t] = 0,
\end{aligned}$$

which completely determines $\hat{T}$ because it is tensorial. Thus, we can write

$$\hat{T} = T,$$

where we set $\partial t \lrcorner T = 0$. Then, by theorem 4.1.2, we can write

$$\hat{T} = \widehat{N} + \frac{1}{8}(\hat{d}^c\widehat{F})^+ - \frac{3}{8}\mathfrak{M}(\hat{d}^c\widehat{F})^+ + \frac{9}{8}\hat{\omega}^+ - \frac{3}{8}\mathfrak{M}\hat{\omega}^+ + \hat{B}, \tag{5.10}$$

where $\hat{\omega}^+ \in \Omega^+(\hat{M})$ and $\hat{B} \in \Omega^{1,1}_s(\hat{M}, T\hat{M})$. On the other hand, we have for $\hat{\nabla}^b$

$$T^b = \widehat{N} + \frac{1}{8}(\hat{d}^c\widehat{F})^+ - \frac{3}{8}\mathfrak{M}(\hat{d}^c\widehat{F})^+ + \frac{9}{8}\omega_b^+ - \frac{3}{8}\mathfrak{M}\omega_b^+ + B_b,$$

where again $\omega_b \in \Omega^3(\hat{M})$ and $B_b \in \Omega^{1,1}_s(\hat{M}, T\hat{M})$. Thus, we can write

$$\begin{aligned} \hat{T} &= T^b + (\hat{T} - T^b) \\ &= T^b + \frac{9}{8}\omega^+ - \frac{3}{8}\mathfrak{M}\omega^+ + B, \end{aligned} \tag{5.11}$$

where $\omega^+ = \hat{\omega}^+ - \omega_b^+$ and $B = \hat{B} - B_b$. In particular, this implies

$$\begin{aligned} \mathfrak{b}T = \mathfrak{b}\hat{T} &= \mathfrak{b}T^b + \frac{9}{8}\omega^+ - \frac{3}{8}\mathfrak{b}\mathfrak{M}\omega^+ \\ &= \mathfrak{b}T^b + \omega^+, \end{aligned}$$

where we made use of the fact that $\mathfrak{b}\mathfrak{M}\omega^+ = \frac{1}{3}\omega^+$. Therefore, we obtain

$$\omega^+ = \mathfrak{b}T - \mathfrak{b}T^b.$$

Because $\mathfrak{b}T$ does not depend on ∇ but only on its Dirac equivalence class (cf. corollary 3.3.9), the Dirac equivalence class of ∇ completely determines ω^+ and thus $\hat{\omega}^+$. What is more, we know that $\omega^+ = \mathfrak{b}T - \frac{1}{3}\eta \wedge d\eta$ and thus that $\partial t \lrcorner \omega^+ = 0$.
To show that ∇ is uniquely determined, we still need to show that B is also completely determined by the Dirac equivalence class of ∇. Then, by (5.10), $\hat{T}$ would be completely determined and so qould be $\hat{\nabla}$ and thus ∇. We begin by noting that we know

$$T(\partial t; \cdot, \cdot) = 0, \qquad T^b(\partial t; \cdot, \cdot) = 0 \quad \text{and} \tag{5.12}$$

$$T(\cdot; \partial t, \cdot) = 0, \qquad T^b(\cdot, \partial t, \cdot) = 0. \tag{5.13}$$

Because $\partial t \lrcorner \omega^+ = 0$, (5.12) and (5.11) imply that

$$B(\partial t; \cdot, \cdot) = 0.$$

Furthermore, by (5.11) and (5.13), we obtain that

$$B(\cdot; \partial t, \cdot) = \frac{3}{8}\mathfrak{M}\omega^+(\cdot; \partial t, \cdot)$$

and thus, B is completely determined if one of the arguments is ∂t. Furthermore, because $B \in \Omega^{1,1}(\hat{M}, T\hat{M})$, we have that $B(X; \xi, Y) = B(X; \partial t, Y)$ which is known. Furthermore, because $\mathfrak{b}B$ is zero, we have

$$0 = \mathfrak{b}B(X; \xi, Y) = \underbrace{B(X; \xi, Y)}_{\text{known}} + B(\xi; Y, X) + \underbrace{B(Y; X, \xi)}_{\text{known}}$$

for any vector fields X and Y and thus, B is completely determined if one of the arguments is ξ. Finally, we use the property that ∇ and $\nabla^{TW} = \hat{\nabla}^b|_M$ are CR connections to deduce that

$$\begin{aligned} 0 = T(X;Y,Y) &= T^b(X;Y,Z) + \frac{9}{8}\omega^+(X;Y,Z) - \frac{3}{8}\mathfrak{M}\omega^+(X;Y,Z) + B(X;Y,Z) \\ &= \frac{9}{8}\omega^+(X;Y,Z) - \frac{3}{8}\mathfrak{M}\omega^+(X;Y,Z) + B(X;Y,Z) \end{aligned}$$

for any $X, Y, Z \in \Gamma(\mathcal{C})$, which completely determines B. This yields the claim. □

In particular, this implies that the Tanaka-Webster connection of a strictly pseudoconvex CR manifold is the unique nice CR connection that induces the operator $\mathcal{H}$ as defined in (5.9). We can summarize our findings on the (generalized) Tanaka-Webster connection as follows:

5.2.4 Theorem
Let (M, g, η, J) be a metric contact manifold. Then the generalized Tanaka-Webster connection induces the contact Hodge-Dolbeault operator $\mathcal{H}$ defined for any $\psi = \psi_+ + \psi_- \in \Gamma(\mathbb{S}^c)$ by

$$\mathcal{H}\psi = \sqrt{2}\chi\mathcal{J}(\overline{\partial} + \overline{\partial}^*)(\overline{z^0} \wedge \psi_- + \psi_+)$$

or, equivalently by

$$\mathcal{H}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \sqrt{2}\begin{pmatrix} -\overline{\partial}_0 & (\overline{\partial}_{\mathcal{C}} + \overline{\partial}^*_{\mathcal{C}}) \\ (\overline{\partial}_{\mathcal{C}} + \overline{\partial}^*_{\mathcal{C}}) & \overline{\partial}^*_0 \end{pmatrix}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

as its geometric Dirac operator.
If the manifold is CR, the Tanaka-Webster connection is the only nice CR connection to induce this operator as its geometric Dirac operator.

Appendix

Connections on principal bundles

A.1 Connections on principal bundles

The way we induce connections on a spinor bundle is based on the more general concept of connections on principal bundles and the connections (covariant derivatives) they induce on associated vector bundles. It is this theory that we review in this appendix. Because it serves mainly to list well-known facts for the convenience of the reader, we omit all proofs and refer the reader to [Bau09, chapter 3] which we used in writing this appendix. We begin by defining what we understand by a connection on a principal G-bundle $P \xrightarrow{\pi} M$.

A.1 Definition The *vertical tangent space* of P is the subbundle $TvP \subset TP$ given at each point by $Tv_pP = T_p(P_{\pi p})$.
A *horizontal tangent space* is a vector space complement (at each point) of the vertical tangent space.
A *connection* Th on P is a smooth and right-invariant choice of a horizontal tangent space, i.e. $Th_pP \subset T_pP$, $Th_pP \oplus Tv_pP = T_pP$ at any point $p \in M$ and $dR_g(Th_pP) = Th_{pg}P$, where R_g denotes right multiplication by g.

A.2 Remark The vertical tangent space is given by the kernel of the projection: $Tv_pP = \ker(\pi_p)$ and is furthermore isomorphic to all fundamental vector fields evaluated at p, where the fundamental vector fields are given by

$$\widetilde{X}(p) = \frac{d}{dt}(\exp(tX)p)|_{t=0} \text{ where } X \in \mathfrak{g}.$$

Another way to describe a connection is through a one-form with values in $\mathfrak{g}$, the Lie-algebra of G:

A.3 Definition A *connection one-form* is a form $C \in \Omega^1(P, \mathfrak{g})$ such that

(i) $R_g^*C = Ad(g^{-1}) \circ C$, where R_g stands for right multiplication,

(ii) $C(\widetilde{X}) = X$ for all $X \in \mathfrak{g}$.

These two definitions are equivalent in the following way:

A.4 Proposition (cf. [Bau09, Satz 3.2])
Connections and connection one-forms are in a 1-to-1 correspondence defined as follows:

(1) If a connection Th is given, define C by

$$C_p(\widetilde{X}(p) + Y_h) = X \quad \textit{for all } X \in \mathfrak{g}, Y_h \in TH_pP.$$

(2) Let C be given. Then define Th as the kernel of C.

Yet another way to describe a connection is by local connection forms: Given a local section $s \in \Gamma(U, P)$ and a connection one-form C, we define

$$C^s = C \circ ds \in \Omega^1(U, \mathfrak{g}).$$

These local forms then fulfil a certain transformation rule which we shall describe now. In order to do so, we establish certain functions: Let $s_i \in \Gamma(U_i, P), s_j \in \Gamma(U_i, P)$ be two local sections. Then there exists a transition function $g_{ij} \in C^\infty(U_i \cap U_j, G)$ such that

$$s_i(x) = s_j(x)g_{ij}(x)$$

for any $x \in U_i \cap U_j$. Furthermore, we consider the *Maurer Cartan form* $\mu_G \in \Omega^1(G, \mathfrak{g})$ defined by $(\mu_G)_g = dL_{g^{-1}}$. We then define the functions

$$\mu_{ij} = g_{ij}^* \mu_G \quad \text{i.e.} \quad \mu_{ij}(X) = dL_{g_{ij}(x)^{-1}}(dg_{ij}(X)) \text{ for} X \in T_x(U_i \cap U_j).$$

Using these, we can now state the transformation rule:

A.5 Proposition (cf. [Bau09, Satz 3.3])

(1) Let C be a connection one-form and s_i, s_j local sections of P. Then the local forms C^{s_i}, C^{s_j} satisfy the following transformation rule:

$$C^{s_i} = Ad(g_{ij}^{-1}) \circ C^{s_j} + \mu_{ij}.$$

(2) Let $\{(U_i, s_i)\}$ be an open cover of M with local sections of P and let $\{C_i \in \Omega^1(U_i, \mathfrak{g})\}$ fulfil

$$C_i = Ad(g_{ij}^{-1}) \circ C_j + \mu_{ij}.$$

Then there exists a connection one-form C such that $C^{s_i} = C_i$.

Having reviewed the various ways to define connections on a principal bundle, we now move on to discussing how a connection on a principal G-bundle P defines a connection on an associated vector bundle $E = P \times_\rho V$ where $\rho \colon G \to GL(V)$ is a representation of the Lie group G. The first step is to define an absolute differential $D^C \colon \Omega^k(M, E) \to \Omega^{k+1}(M, E)$. To define this operator, we note that $\Omega^k(M, E) \simeq \Omega^k(P, V)^{(G,\rho)}$, where

$$\Omega^k(P, V)^{(G,\rho)} = \{\omega \in \Omega^k(P, V) | X \lrcorner \omega = 0 \text{ for all vertical } X \text{ and } R_a^* \omega = \rho(a^{-1}) \circ \omega \quad \forall a \in G\}.$$

We then define

$$\begin{aligned} D^C \colon \Omega^k(P, V) &\longrightarrow \Omega^{k+1}(P, V) \\ \omega &\longmapsto d\omega(\text{proj}_{ThP} \cdot, ..., \text{proj}\, ThP \cdot). \end{aligned} \tag{A.14}$$

It can be shown that $D^C(\Omega^k(P, V)^{(G,\rho)}) \subset \Omega^{k+1}(P, V)^{(G,\rho)}$ and therefore D^C induces an operator on $\Omega^k(M, E)$.

A.6 Definition The operator $d^C \colon \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$ induced by the operator defined by (A.14) is called the *absolute differential* induced by the connection C.

For this absolute differential we have the following result:

A.7 Proposition (cf. [Bau09, Satz 3.11])

The absolute differential d^C satisfies the following identity for any $\sigma \in \Omega^k(M, E), \omega \in \Omega^l(M, E)$:

$$d^C(\sigma \wedge \omega) = (d^C \sigma) \wedge \omega + (-1)^k \sigma \wedge (d^C \omega).$$

Now, note that we have $\Omega^0(M, E) \simeq \Gamma(E)$ and $\Omega^1(M, E) \simeq \Gamma(T^*M \otimes E)$ and by the preceding proposition $d^C \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a covariant derivative.

A.8 Definition The differential operator

$$\nabla^C := d^C|_{\Omega^0(M,E)} : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

is called the covariant derivative induced by C.

We have the following local formula for ∇^C:

A.9 Proposition (cf. [Bau09, Satz 3.12])

Let C be a connection one-form on P and $E = P \times_\rho V$ a vector bundle associated to P. Let furthermore $e \in \Gamma(E)$ with local representation $e|_U = [s, v]$ where $s \in \Gamma(U, P)$ and $v \in C^\infty(U, V)$. Then the following formula holds:

$$(\nabla_X^C e)(x) = [s(x), d_x v(X(x)) + \rho_*(C^s(X(x)))v(x)].$$

A.2 Reductions and extensions of principal bundles

Reductions and extensions of principal bundles are important tools for changing the structure group of a principal bundle. In this appendix, we first introduce the two notions and then describe how connections behave with respect to extensions and reductions.

A.10 Definition Let (P, π, M, G) be a principal bundle and $\lambda\colon H \to G$ a morphism of Lie groups (i.e. smooth and compatible with the multiplication). Then a λ-*reduction* of P is a principal H-bundle (Q, π_Q, M, H) together with smooth map $f\colon Q \to P$ such that

(i) $\pi \circ f = \pi_Q$,

(ii) $f(qh) = f(q)\lambda(h)$ for all $q \in Q$ and $h \in H$,

or, equivalently, such that the following diagram commutes:

$$\begin{array}{ccccc} Q \times H & \longrightarrow & Q & & \\ \Big\downarrow{\scriptstyle f\times\lambda} & & \Big\downarrow{\scriptstyle f} & \searrow^{\pi_Q} & \\ P \times G & \longrightarrow & P & \xrightarrow{\pi} & M. \end{array}$$

If $\iota : H \hookrightarrow G$ is the inclusion map of the subgroup $H < G$, we simply call a ι-reduction an H-reduction.

Under a reduction, associated vector bundles behave as follows:

A.11 Proposition (cf. [Bau09, Satz 2.17])
Let (Q, f) be a λ-reduction of P and $\rho\colon G \to GL(V)$ a representation. Then there is a vector bundle isomorphism (i.e. a bijective vector bundle morphism) of the associated vector bundles

$$\begin{aligned} Q \times_{\rho\lambda} V &\xrightarrow{\sim} P \times_\rho V \\ [q, v] &\longmapsto [f(q), v] \end{aligned}$$

The operation of reducing the structure group of a principal bundles has an inverse of sorts, the extension of a principal bundle:

A.12 Definition Let Q be a principal H-bundle and let $\lambda\colon H \to G$ be a Lie group morphism. Then H acts on G by $h \cdot g = \lambda(h) \cdot g$ and

$$P := Q \times_\lambda G$$

is called a λ-*extension* of Q.

The following theorem explains how extensions and reductions relate to each other:

A.13 Proposition (cf. [Bau09, Satz 2.18])
Let $(Q, \pi_Q, M; H)$ be a principal H-bundle and $\lambda\colon H \to G$ a Lie group morphism. Then the following holds:

(1) The λ-extension of Q is a principal G-bundle over M.

(2) Define

$$\begin{aligned} f\colon Q &\longrightarrow P = Q \times_\lambda G \\ q &\mapsto [q, 1]. \end{aligned}$$

Then (Q, f) is a λ-reduction of P.

(3) Let (Q, f) be a λ-reduction of a principal G-bundle P. Then P is isomorphic to the λ-extension of Q.

Now, given a connection on a principal bundle, we ask ourselves whether this connection induces one on the extension (reduction) of the bundle. The answer is given by the following proposition:

A.14 Proposition (cf. [Bau09, Satz 4.1])
Let (P, π_P, M, G) be a principal G-bundle, $\lambda\colon H \to G$ and $((Q, \pi_Q, M, H), f)$ a λ-reduction of P. Let furthermore C be a connection form on Q. Then there exists exactly one connection form $\overline{C}$ on P such that

$$df_q(Th^C_q(Q)) = Th^{\overline{C}}_{f(q)}P, \tag{A.15}$$

or, equivalently,

$$f^*\overline{C} = \lambda_* \circ C. \tag{A.16}$$

A.15 Definition In the situation as described above, C is called a λ-reduction of $\overline{C}$ and $\overline{C}$ is called a λ-extension of C.

Note that by the above proposition, extensions of connections always exist. Reductions, however, do not exist in general.

A.3 The case of frame bundles

With any manifold, we have a particular principal bundle, the *frame bundle*. The (general) frame bundle $P_{GL}(M)$ is the bundle of all frames or bases, i.e.

$$\begin{aligned}(P_{GL}(M))_x &= \{s = (s_1, ..., s_n) | s \text{ is a basis of } T_xM\},\\ P_{GL}(M) &= \coprod_{x\in M} (P_{GL}(M))_x\end{aligned}$$

If M is n-dimensional, it is a principal GL_n-bundle, with the action of $A = (A_{ij}) \in GL_n$ given by

$$(s_1, ..., s_n) \cdot A = \left(\sum_{j=1}^n s_j A_{j1}, ..., \sum_{j=1}^n s_j A_{jn}\right).$$

This bundle is closely related to the tangent bundle and to its dual and exterior powers: The tangent bundle can be realized as an associated vector bundle as follows: Let $\rho\colon GL_n \to GL(\mathbb{R}^n)$ be the standard matrix action (left multiplication, with $\mathbb{R}^n$ considered as a space of column vectors). Then the tangent bundle is isomorphic to the vector bundle associated to $P_{GL}(M)$ by ρ:

$$TM \simeq P_{GL}(M) \times_\rho \mathbb{R}^n.$$

The isomorphism is given as follows: Denote $e_1, ..., e_n$ the standard basis of $\mathbb{R}^n$ and fix a basis $s = (s_1, ..., s_n)$ of T_xM and let $X \in T_xM$. Then, X can be written as $X = \sum_j X_j s_j$ and is then mapped to $[s, \sum_j X_j e_j]$. One easily verifies that this mapping is well-defined and is indeed an isomorphsim.
Furthermore, we consider the dual representation $\rho^*\colon GL_n \to GL((\mathbb{R}^n)^*)$. It is given by $\rho^*(A)(\alpha)(X) = \alpha(\rho(A^{-1})X)$ for any $A \in GL_n, \alpha \in (\mathbb{R}^n)^*$ and $X \in \mathbb{R}^n$. Thus, writing α as a line vector, $\rho^*(A)\alpha = \alpha \cdot A^{-1}$. Then we have that

$$T^*M \simeq P_{GL}(M) \times_{\rho^*} (\mathbb{R}^n)^*,$$

where the isomorphism is constructed analogously to the one for TM.
The space of exterior powers of T^*M can also be constructed as a vector bundle associated to the frame bundle in a similar way. To this end, we extend the representation ρ to a representation ρ^k with image in $GL(\Lambda^k(\mathbb{R}^n)^*)$ as follows: The endomorphism $\rho^k(A)$ is defined on elements of type $\alpha_1 \wedge ... \wedge \alpha_k$ as follows:

$$\rho^k(A)(\alpha_1 \wedge ... \wedge \alpha_k) = (\rho^*(A)\alpha_1) \wedge ... \wedge (\rho^*(A)\alpha_k)$$

and extended linearly. We then have an isomorphism

$$\begin{aligned} P_{GL}(M) \times_{\rho^k} \Lambda^k((\mathbb{R}^n)^*) &\xrightarrow{\sim} \Lambda^k(T^*M) \\ [(s_1, ..., s_n), e^{i_1} \wedge ... \wedge e^{i_k}] &\longmapsto s^{i_1} \wedge ... \wedge s^{i_k}. \end{aligned}$$

If M has additional structure, we can reduce the structure group of the frame bundle: Let (M, g) be a Riemannian manifold. We can then form the bundle of orthonormal bases $P_O(M)$ which is an O-reduction with the reduction map simply given by the inclusion map. If (M, g) is additionally oriented, one can form the bundle of oriented, orthonormal frames $P_{SO}(M)$ which is a reduction of the structure group to the special orthogonal group. The structure group can be further reduced to the unitary group when M admits an almost-hermitian structure; this is discussed in section 1.1.

Because the frame bundles are so closely related to the tangent bundle, every covariant derivative ∇ on TM induces a connection on $P_{GL}(M)$, defined by local connection forms as follows: Given a local basis $(s_1, ..., s_n)$ over $U \subset M$, ∇ can be locally expressed by $\nabla s_i = \sum_{k=1}^n \omega_{kj} \otimes s_k$ where $\omega_{kj} \in \Omega^1(U)$. Then a connection form on the frame bundle $C \in \Omega^1(P_{GL}(M), \mathfrak{gl}_n)$ is defined locally by

$$C^s = \sum_{i,j=1}^n \omega_{ij} B_{ij},$$

where $B_{ij} \in \mathbb{R}^{n\times n}$ is defined by $(B_{ij})_{kl} = \delta_{ik}\delta_{jl}$. In other words, $C^s(X)$ is simply the matrix $(\omega_i j(X))_{i,j=1}^n$.
Now, if (M, g) is Riemannian and ∇ is metric, it induces a connection on the bundle of orthogonal frames $P_O(M)$ (on $P_{SO}(M)$ if (M, g) is oriented) and the local formula can be written as follows:

$$(C^s)(X) = \frac{1}{2}\sum_{i<j} g(\nabla_X s_i, s_j) E_{ij},$$

where $E_{ij} = -B_{ij} + B_{ji}$.
The connection induced on $P_G(M)$ (we note the structure group simply G, the following result holds for all possible structure groups) then induces a covariant derivative on the associated vector bundle TM again. We shall note this covariant derivative ∇^C. Then ∇^C and ∇ coincide: By proposition A.9 we have

$$\begin{aligned} \nabla^C_X s_k &= [s, X(e_k) + \rho_*((C\nabla)^s(X))e_k] \\ &= [s, (\omega_{\mu\nu}(X)) \cdot e_k] \\ &= \left[s, \sum_\mu \omega_{\mu k}(X) e_\mu\right] \\ &= \sum_\mu \omega_{\mu k}(X) s_\mu = \nabla_X s_k, \end{aligned}$$

where we used $s_k = [s, e_k]$.

References

[Bau99] Helga Baum. Lorentzian twistor spinors and CR-geometry. *Differential Geometry and its Applications*, 11:69–96, 1999.

[Bau09] Helga Baum. *Eichfeldtheorie*. Springer, Berlin Heidelberg, 2009.

[Bej] Aurel Bejancu. CR-manifold. in: Springer Online Reference Works, `http://eom.springer.de/c/c110470.htm#c110470_00ca3`, retreived on August 21^{st}, 2011.

[Bis99] Jean-Michel Bismut. A local index theorem for non Kähler manifolds. *Mathematische Annalen*, 284:681–699, 1999.

[BJ10] Helga Baum and Andreas Juhl. *Conformal Differential Geometry*, volume 40 of *Oberwolfach Seminars*. Birkhäuser, Basel Boston Berlin, 2010.

[Bla02] David E. Blair. *Riemannian Geometry of Contact and Symplectic Manifolds*. Birkhäuser, Boston, 2002.

[CM74] S.S. Chern and J. Moser. Real hypersurfaces in complex manifolds. *Acta Mathematica*, 133, 1974.

[DT06] Sorin Dragomir and Giuseppe Tomanini. *Differential Geometry and Analysis on CR Manifolds*, volume 246 of *Progress in Mathematics*. Birkhäuser, Basel Boston Berlin, 2006.

[Fri00] Thomas Friedrich. *Dirac Operators in Riemannian Geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providece, RI, 2000.

[Gau97] Paul Gauduchon. Hermitian connections and Dirac operators. *Bolletino dell'Unione Mathematica Italiana*, 11-B:257–288, 1997. Suppl.fasc.2.

[GH80] Alfred Gray and Luis M. Hervella. The sixteen classes of almost hermitian manifolds and their linear invariants. *Annali di Matematica Pura ed Applicata*, 123:35–58, 1980.

[Hit74] Nigel Hitchin. Harmonic spinors. *Advances in Mathematics*, 14:1–55, 1974.

[Hör66] Lars Hörmander. *An Introduction to Complex Analysis in Several Variables*. Van Nostrand Reinhold Company, New York, 1966.

[Jac90] Howard Jacobowitz. *An Introduction to CR Structures*, volume 32 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[KN69] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume II. Interscience Publishers, New York, 1969.

[Lib54] Pauline Libermann. Sur les connexions hermitiennes. *Comptes rendus de l'Académie des Sciences*, 239:1579–1581, 1954.

[Lic55] André Lichnerowicz. *Théorie globale des connexions et des groupes d'holonomie*. Ed. Cremonese, Rome, 1955.

[Lic76] André Lichnerowicz. *Global theory of connections and holonomy groups*. Noordhoff, Leyden, 1976.

[LM89] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry.* Princeton University Press, Princeton, NJ, 1989.

[Mor96] John W. Morgan. *The Seiberg-Witten Equations and Applications to the Topology of Four-Manifolds*, volume 44 of *Mathematical Notes.* Princeton University Press, 1996.

[Nic05] Liviu Nicolaescu. Geometric connections and geometric Dirac operators on contact manifolds. *Differential Geometry and its Applications*, 22:355–378, 2005.

[NN57] Newlander and Louis Nirenberg. Complex analytic coordinates in almost complex manifolds. *The Annals of Mathematics*, 65(3):391–404, 1957.

[Pet05] Robert Petit. $Spin^c$-structures and Dirac operators on contact manifolds. *Differential Geometry and its Applications*, 22:229–252, 2005.

[Poi07] Henri Poincaré. Les fonctions analytiques de deux variables et la représentation conforme. *Rendiconti del Circolo Matematico di Palermo*, 23:185–220, 1907.

[Tan75] Noboru Tanaka. *A differential geometric study on strongly pseudo-convex manifolds.* Kinokuniya Book Store, Kyoto, 1975.

[Tan89] Shukichi Tanno. Variational problems on contact Riemannian manifolds. *Transactions of the American Mathematical Society*, 314:349–379, 1989.

[Web78] S.M. Webster. Pseudo-hermitian structures on a real hypersurface. *Journal of Differential Geometry*, 13:25–41, 1978.

[Wel08] Raymond O. Wells, Jr. *Differential Analysis on Complex Manifolds.* Springer, third edition, 2008.