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METRIC DIOPHANTINE APPROXIMATION IN JULIA SETS OF EXPANDING RATIONAL MAPS

by RICHARD HILL *and* SANJU L. VELANI

In loving memory of Janaki Divani

ABSTRACT

Let $T : J \to J$ be an expanding rational map of the Riemann sphere acting on its Julia set J and $f : J \to \mathbf{R}$ denote a Hölder continuous function satisfying $f(x) \geqslant \log | T'(x) |$ for all x in J. Then for any point z_0 in J define the set $D_{z_0}(f)$ of " well-approximable " points to be the set of points in J which lie in the Euclidean ball

$$B(y, \exp(-\textstyle\sum_{i=0}^{n-1} f(T^i x)))$$

for infinitely many pairs (y, n) satisfying $T^n(y) = z_0$. We prove that the Hausdorff dimension of $D_{z_0}(f)$ is the unique positive number $s(f)$ satisfying the equation $P(T, -s(f).f) = 0$, where P is the pressure on the Julia set. This result is then shown to have consequences for the limsups of ergodic averages of Hölder continuous functions. We also obtain local counting results which are analogous to the orbital counting results in the theory of Kleinian groups.

1. INTRODUCTION

1.1. Part of number theory is concerned with finding rational numbers p/q which are good approximations to a real number x. For any x one can find infinitely many p/q whose distance from x is less than q^{-2}. If one can find infinitely many p/q whose distance from x is less than $q^{-\tau}$ with $\tau > 2$ then x is said to be a τ-well approximable number. In this article we shall associate to a dynamical system $T : X \to X$ various sets of well approximable points in X in analogy with the classical theory of well approximable real numbers. We shall calculate (Theorem 1) the Hausdorff dimensions of these sets. As a consequence we obtain in 1.5 results on the distribution of ergodic averages for Hölder continuous functions $f: X \to \mathbf{R}$, and we also solve in 1.3 the " shrinking target problem " introduced in [9]. Furthermore our method shows a link between conformal measures and local counting results, which we describe in 3.

We shall restrict our attention to the case where $T : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is an expanding rational map of degree $d \geqslant 2$ of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{ \infty \}$ and $J = J(T)$ is its associated Julia set (see [1]). It is known that J is non-empty, perfect and fully invariant, which means that

$$T^{-1}(J) = J = T(J).$$

By the definition of expanding (see Chap. 9 of [1]), there exists a $\lambda > 1$ and an integer $m \geqslant 1$ such that

$$|(T^m)'(z)| \geqslant \lambda \quad \text{for all } z \text{ in } J(T),$$

where T' is the derivative of T. For expanding maps, J is not the whole of $\overline{\mathbf{C}}$ (see Theorem 4 of [21]) and so we may and will assume throughout that $\infty \notin J$. Thus we can think of J as a metric space with the usual metric on $\mathbf{C}$.

1.2. *The main result.* — We shall now introduce a class of "well approximable" subsets of J for which we shall calculate the Hausdorff dimensions. The notation $B(x, r)$ will mean a ball with centre $x \in \mathbf{C}$ and radius $r > 0$ (with respect to the usual metric on $\mathbf{C}$), and δ will denote the Hausdorff dimension of J.

Let $f: J \to \mathbf{R}$ be a Hölder continuous function satisfying $f(x) \geqslant \log |T'(x)|$ for all x in J and write f_n for the n-th ergodic sum of f. This means that

$$f_n(x) := \sum_{i=0}^{n-1} f(T^i x).$$

All logarithms in this article will be to the base e. For any point z_0 in J, let $I = I(z_0)$ be the set of pairs (y, n) $(n \in \mathbf{N})$ such that $T^n(y) = z_0$. We now define the following subset of J

$$D_{z_0}(f) := \{ x \in J : x \in B(y, \exp(-f_n(y))) \text{ for infinitely many pairs } (y, n) \in I \}.$$

In § 4 of this article we shall prove the following:

Theorem 1. — *The set* $D_{z_0}(f)$ *has Hausdorff dimension* $s(f)$, *where* $s(f)$ *is the unique solution to the pressure equation*

$$P(T, -s f) = 0.$$

For the definitions of pressure, Hölder continuity and Hausdorff dimension the reader is referred to section 2. The Hausdorff dimension of $D_{z_0}(f)$ is independent of our choice of z_0, thus in future we shall simply write $D(f)$ for the set $D_{z_0}(f)$. Also we shall only be working with one rational map T at any one time and so we will leave T out of the pressure notation, i.e. write $P(-s f)$ for $P(T, -sf)$. Theorem 1 generalizes the results and conjectures of [9], which we shall presently describe. The theorem can also be viewed as an extension of the Bowen-Manning-McCluskey formula which states that $P(-\delta \log |T'|) = 0$. Furthermore Theorem 1 can be thought of as an analogy of the Jarník-Besicovitch theorem, see 1.4.

For $f(x) = \log |T'(x)|$ one can easily deduce that $\dim D(f) = \dim J$. Thus for $f < \log |T'|$, we have $D(f) \supset D(\log |T'|)$ and there is nothing to investigate.

1.3. *Connections with previous work.* — Let z_0 be a point of J and let τ be a real, positive number. In [9] we considered the following two sets of " well approximable " points in J,

$$W(\tau) := \{ x \in J : T^n(x) \in B(z_0, \exp(-\tau n)) \text{ for infinitely many } n \in \mathbf{N} \}.$$

$$W^{\bullet}(\tau) := \{ x \in J : T^n(x) \in B(z_0, |(T^n)'(x)|^{-\tau}) \text{ for infinitely many } n \in \mathbf{N} \}.$$

Note that $W(\tau)$ is the set of points whose forward orbits enter a shrinking target (centred at z_0) infinitely often. For this reason we refer to the problem of describing the set $W(\tau)$ as the " shrinking target problem " (see [9, 10, 11]). It was shown in [9] that for $\tau \geqslant 0$, $W^{\bullet}(\tau)$ has Hausdorff dimension $\delta/(1+\tau)$ and that the Hausdorff dimension of $W(\tau)$ satisfies the bounds

$$\frac{\delta\chi}{\chi+\tau} \leqslant \dim(W(\tau)) \leqslant s(\tau),$$

where $s(\tau) \in \mathbf{R}^{\geqslant 0}$ is the unique solution to the pressure equation

$$P(-s \log |T'|) = s\tau$$

and χ is a positive constant. It was conjectured that $\dim(W(\tau)) = s(\tau)$, and we shall now show that this is true. We shall show that these results and the conjecture are in fact consequences of Theorem 1.

We first note the relations between the sets $D(f)$ (f as above) and the sets $W(\tau)$ and $W^{\bullet}(\tau)$.

Proposition 1. — *If* $f(x) = (1+\tau)\log|T'(x)|$ *then there is an* $N \in \mathbf{N}$ *such that*

$$T^{2N} D(f) \subset T^N W^{\bullet}(\tau) \subset D(f).$$

Proposition 2. — *If* $f(x) = \log|T'(x)| + \tau$ *then there is an* $N \in \mathbf{N}$ *such that*

$$T^{2N} D(f) \subset T^N W(\tau) \subset D(f).$$

These propositions follow from the Köbe Distortion Theorem and the fact that all analytic inverse branches of T^n are well defined on balls in a neighbourhood U of J (see 2.2). We sketch a proof of Proposition 1 to illustrate how these propositions can be proved. In fact they follow from Propositions 2 and 3 of [9]. Suppose $x \in W^{\bullet}(\tau)$. This implies that for infinitely many natural numbers n one has

$$T^n(x) \in B(z_0, |(T^n)'(x)|^{-\tau}).$$

This, in turn, implies that

$$x \in T^{-n}(B(z_0, |(T^n)'(x)|^{-\tau})).$$

Thus for some inverse branch T_i^{-n} of T^n one has

$$x \in T_i^{-n}(B(z_0, |(T^n)'(x)|^{-\tau})).$$

If $y = T_i^{-n}(z_0)$, then by using the Köbe Distortion Theorem one can show that there is a constant $C > 1$ such that

$$B(y, C^{-1}|(T^n)'(x)|^{-\tau-1}) \subset T_i^{-n}(B(z_0, |(T^n)'(x)|^{-\tau})) \subset B(y, C|(T^n)'(x)|^{-\tau-1}).$$

Using the chain rule we have that

$$|(T^n)'(x)|^{-\tau-1} = \prod_{i=0}^{n-1} |T'(T^i x)|^{-\tau-1} = \exp(-\sum_{i=0}^{n-1} (\tau+1)\log|T'(T^i x)|)$$
$$= \exp(-f_n(x)).$$

We therefore have that

$$B(y, C^{-1}\exp(-f_n(x))) \subset T_i^{-n}(B(z_0, |(T^n)'(x)|^{-\tau})) \subset B(y, C\exp(-f_n(x))).$$

Choose N large enough so that for all $n \geqslant N$, $x \in J$ one has $\exp(f_n(x)) \geqslant C$. The result now follows.

We now show how Theorem 1 combined with Propositions 1 and 2 give the results and the conjecture of [9]. We first treat the case of $W^*(\tau)$. We have the relations

$$T^{2N} D((1+\tau)\log|T'(x)|) \subset T^N W^*(\tau) \subset D((1+\tau)\log|T'(x)|),$$

from which it follows that

$$\dim W^*(\tau) = \dim D((1+\tau)\log|T'(x)|).$$

Therefore by Theorem 1 we have

$$\dim W^*(\tau) = s(f),$$

where $s(f)$ is the unique solution to the pressure equation

$$P(-s(f).(1+\tau)\log|T'|) = 0.$$

However the Hausdorff dimension δ of J is characterised as the unique solution to the equation $P(-s\log|T'|) = 0$ (this is the Bowen-Manning-McCluskey formula). We therefore have that $\dim W^*(\tau) = s(f) = \delta/(1+\tau)$ as required.

Now consider the set $W(\tau)$. It follows as above from Proposition 2 and Theorem 1 that $\dim W(\tau) = s(f)$ where $s(f)$ is the solution to

$$P(-s(f).(\log|T'| + \tau)) = 0.$$

From this we have that

$$P(-s(f).\log|T'|) = s(f)\,\tau,$$

which is the result forseen in [9].

1.4. *Classical Diophantine approximation.* — We will now describe a connection between our theory and an aspect of the classical theory of Diophantine approximation, which we briefly recall.

Let x be any real number. It was shown by Dirichlet that there are infinitely many rational approximations p/q to x, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

This result is the best possible of its kind, in that if one replaces the q^{-2} on the right by $q^{-\tau}$ with $\tau > 2$ then the set $C(\tau)$ of x for which there are infinitely many approximations p/q with

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\tau}$$

has zero Lebesgue measure. The set $C(\tau)$ is the classical set of τ-well approximable numbers. The Jarník-Besicovitch Theorem [2, 14] states that for $\tau > 2$ the Hausdorff dimension of $C(\tau)$ is equal to $2/\tau$. We shall now reinterpret this result.

For the moment let J be the closed interval [0, 1] instead of a Julia set, and let $T : J \to J$ be the Gauss map, which is given by

$$T(x) = x^{-1} \bmod 1 \quad \text{if } x \neq 0.$$

For convenience we shall define $T(0) = \sqrt{2} - 1$. The point of this is to ensure that zero is not a periodic point. Furthermore we shall set $z_0 = 0 \in J$.

With this notation in mind, define as before the set

$$D(f) := \left\{ x \in J : \begin{array}{r} x \in B(y, \exp(-f_n(y))) \text{ for infinitely many pairs } (y, n) \\ \text{satisfying } T^n(y) = z_0 \end{array} \right\},$$

where f_n is an ergodic sum of a Hölder continuous function $f : J \to \mathbf{R}$. Letting $f(x) = \frac{\tau}{2} \log | T'(x) |$ we have

Proposition 3. — *The equality* $D\left(\frac{\tau}{2} \log | T' |\right) = C(\tau)$ *holds true.*

This is an immediate consequence of the following observation.

Lemma. — *If p/q is a rational number (we are assuming that p and q are coprime) then there is a unique n such that $T^n(p/q) = 0$ and one has*

$$| (T^n)'(p/q) | = q^2.$$

The lemma can be proved by induction on q; the uniqueness of n follows from our choice of T(0).

If one could prove an analogue of Theorem 1 for the Gauss map then this would imply by Proposition 3 the result $\dim \mathrm{C}(\tau) = 2/\tau$. At present we have not been able to prove such an analogue. However using the techniques of [9] it is possible to show (see [10] when it is completed) that in the special case when $f(x) = \frac{\tau}{2} \log | \mathrm{T}'(x) |$, one still has for the Gauss Map and similar transformations $\mathrm{P}(-\dim \mathrm{D}(f) . f(x)) = 0$. This implies the classical result.

1.5. *Exceptional sets for ergodic averages.* — We will now point out some consequences of Theorem 1 concerning ergodic averages of Hölder continuous functions. We return to the rational map setting, in which T is an expanding rational map and J is its Julia set.

Recall that if one has a T-invariant, ergodic probability measure μ on J, then for μ-almost all x in J

$$\frac{1}{n} f_n(x) \to \int_{\mathrm{J}} f(x)\, d\mu(x)$$

as $n \to \infty$. However this need not be true for all x in J. To study those points x for which this fails to hold we define for $\chi \in \mathbf{R}$ the sets

$$\mathrm{Ex}_f^+(\chi) := \left\{ x \in \mathrm{J} : \limsup_{n \to \infty} \frac{1}{n} f_n(x) \geqslant \chi \right\},$$

$$\mathrm{Ex}_f^-(\chi) := \left\{ x \in \mathrm{J} : \liminf_{n \to \infty} \frac{1}{n} f_n(x) \leqslant \chi \right\}.$$

Note the trivial relations:

$$\mathrm{Ex}^+_{\mathrm{A}.f+\mathrm{B}}(\mathrm{A}\chi + \mathrm{B}) = \mathrm{Ex}_f^{\mathrm{sign}(\mathrm{A})}(\chi) \quad \text{for } \mathrm{A}, \mathrm{B} \in \mathbf{R},\ \mathrm{A} \neq 0, \tag{1}$$

hence results concerning $\mathrm{Ex}_f^-(\chi)$ are equivalent to results on $\mathrm{Ex}_f^+(\chi)$. As before let d denote the degree of T. We shall prove the following

Theorem 2. — Let $f : \mathrm{J} \to \mathbf{R}$ *be a Hölder continuous function and let* $\chi \in \mathbf{R}$. *Then*

$$\dim \mathrm{Ex}_f^-(\chi) \geqslant \sup \left\{ \frac{s(f+\mathrm{B})}{\mathrm{A}} \right\},$$

where the sup *is over real numbers* $\mathrm{A} > 0$ *and* B *such that the following inequalities are satisfied*

$$\mathrm{A}(f(x) + \mathrm{B}) \geqslant \log | \mathrm{T}'(x) | \quad \textit{for all } x \textit{ in } \mathrm{J},$$

$$\mathrm{A}(\chi + \mathrm{B}) \geqslant \sup_{x \in \mathrm{J}} \log | \mathrm{T}'(x) |$$

and

$$\mathrm{P}\left(-\log d \frac{f + \mathrm{B}}{\chi + \mathrm{B}} \right) \leqslant 0.$$

This can be reduced to the following weaker result.

Proposition 4. — *Let* $f: \mathrm{J} \to \mathbf{R}$ *be as in* 1.2 *and* $\chi \geqslant \sup_{x \in \mathrm{J}} \log |\mathrm{T}'(x)|$. *If* $\chi > \log d/\dim \mathrm{D}(f)$ *then*

$$\dim \mathrm{Ex}_f^-(\chi) \geqslant \dim \mathrm{D}(f).$$

To prove Proposition 4 we require the following

Lemma 1. — *With* $f: \mathrm{J} \to \mathbf{R}$ *as in* 1.2 *one has* $\mathrm{D}(f) \backslash \mathrm{D}(\chi) \subset \mathrm{Ex}_f^-(\chi)$ *and* $\mathrm{D}(\chi) \backslash \mathrm{D}(f) \subset \mathrm{Ex}_f^+(\chi)$.

We first prove the lemma. Suppose $x \in \mathrm{D}(f) \backslash \mathrm{D}(\chi)$. Then $x \in \mathrm{B}(y, \exp(-f_n(y)))$ for infinitely many $(y, n) \in \mathrm{I}$, but there are at most finitely many $(y, n) \in \mathrm{I}$ for which we have $x \in \mathrm{B}(y, \exp(-n\chi))$. There are therefore infinitely many $(y, n) \in \mathrm{I}$ for which one has $x \in \mathrm{B}(y, \exp(-f_n(y))) \backslash \mathrm{B}(y, \exp(-n\chi))$. Now by the Hölder continuity of f and the fact that $f \geqslant \log |\mathrm{T}'|$ there is a positive constant $\mathrm{C}(f)$ such that

$$x \in \mathrm{B}(y, \mathrm{C}(f) \exp(-f_n(x))) \backslash \mathrm{B}(y, \exp(-n\chi))$$

for infinitely many n. This implies that

$$\mathrm{C}(f) \exp(-f_n(x)) > \exp(-n\chi),$$

for infinitely many n and therefore

$$\liminf_{n \to \infty} \frac{1}{n} f_n(x) \leqslant \chi.$$

This shows that $\mathrm{D}(f) \backslash \mathrm{D}(\chi) \subset \mathrm{Ex}_f^-(\chi)$. A similar argument proves the other assertion made in the lemma.

We shall now prove Proposition 4. Suppose that $\dim \mathrm{D}(f) > \dim \mathrm{D}(\chi)$. Then it follows that $\dim(\mathrm{D}(f) \backslash \mathrm{D}(\chi)) = \dim \mathrm{D}(f)$. By the lemma this implies that $\dim \mathrm{Ex}_f^-(\chi) \geqslant \dim \mathrm{D}(f)$ whenever $\dim \mathrm{D}(f) > \dim \mathrm{D}(\chi)$. We now calculate $\dim \mathrm{D}(\chi)$. By Theorem 1 we have that $\mathrm{P}(-\dim(\mathrm{D}(\chi))\,\chi) = 0$. On the other hand by the variational principle for pressure (see section 2.3) we have that

$$\begin{aligned} 0 = \mathrm{P}(-\dim(\mathrm{D}(\chi))\,\chi) &= \sup_\sigma \left\{ h_\sigma + \int_\mathrm{J} -\dim(\mathrm{D}(\chi))\,\chi\, d\sigma(x) \right\} \\ &= \sup_\sigma \{ h_\sigma - \dim(\mathrm{D}(\chi))\,\chi \} \\ &= h_{\mathrm{top}} - \dim(\mathrm{D}(\chi))\,\chi, \end{aligned}$$

where h_{top} is the topological entropy of $\mathrm{T}: \mathrm{J} \to \mathrm{J}$. This in combination with the well known fact [15] that for rational maps $h_{\mathrm{top}} = \log d$ (d is the degree of the rational map) implies

$$\dim \mathrm{D}(\chi) = \log d/\chi.$$

We therefore have that $\dim \mathrm{Ex}_f^-(\chi) \geqslant \dim \mathrm{D}(f)$ whenever $\dim \mathrm{D}(f) > \log d/\chi$ and this completes the proof of the proposition.

We now prove Theorem 2 using Proposition 4 and the relations (1). Let f be any Hölder continuous function and $\chi \in \mathbf{R}$, choose $A > 0$, $B \in \mathbf{R}$ so that for all $x \in J$, $A(f(x) + B) > \log | T'(x) |$ and so that

$$A(\chi + B) \geqslant \sup_{x \in J} \log | T'(x) |$$

and

$$A(\chi + B) > (\log d)/s(A(f + B)).$$

Then by Proposition 4, we have

$$\dim \mathrm{Ex}_f^-(\chi) = \dim \mathrm{Ex}_{A(f+B)}^-(A(\chi + B)) \geqslant s(A(f + B)).$$

Since $s(A(f + B)) = A^{-1} s(f + B)$, the condition $A(\chi + B) > (\log d)/s(A(f + B))$ reduces to $\chi + B > (\log d)/s(f + B)$. On reformulating these conditions we obtain the theorem.

1.6. *On the proof of Theorem 1 and counting results.* — The proof of Theorem 1, the main result of this paper, follows by obtaining the upper and lower bounds for $\dim D(f)$ separately. The set $D(f)$ is a lim sup set and the upper bound for $\dim D(f)$ follows by considering its natural cover (see 4.1). The proof of the lower bound result is based on the classical approach of constructing a " Cantor-type " subset of $D(f)$ on which a probability measure satisfying a certain mass distribution principle is constructed (see 4.2-4.4). The construction relies heavily on the existence of the Denker-Urbański conformal measures supported on the Julia set of a rational map (see 2.4), which are a generalization of the more standard δ-conformal measures initially constructed by Patterson [18] on the limit sets of Kleinian groups and later extended to the rational map setting by Sullivan [21]. The Denker-Urbański conformal measures combined with a well controlled covering of the Julia set allow us to obtain local pre-image counting results (see section 3, in particular Theorem 4 and the " Key Lemma "). For example we obtain the following result as a special case of Theorem 4. Let B be a ball centred on J. Then for any real number $X \geqslant X_0(B)$

$$\sum_{\substack{(y,n) \in I: \\ y \in B \text{ and } |(T^n)'(y)| \leqslant X}} 1 \asymp \nu(B)\, X^{\delta},$$

where ν is the δ-conformal measure supported on J which for expanding maps is a constant multiple of δ-dimensional Hausdorff measure. Such counting results are central to our particular Cantor construction.

Notation. — To simplify notation the symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable. If f is a differentiable function we shall denote by f' the derivative of f. The set of non-negative real numbers will be written $\mathbf{R}^{\geqslant 0}$. We shall use the following convention for constants. Constants which arise during a proof will be called $c_1, c_2, \ldots$, whereas those constants appearing in the statements of lemmas will be called C_3, C_4, ... We shall treat the symbols c_1, etc. as reusable constants, so that c_1 will have a different meaning in the proof of Lemma 5 from that in the proof of Proposition 7. The constants in capital letters will on the other hand have a fixed meaning throughout the paper. Finally, we mention that the number 2 appears a lot in this paper. The only aspect of the number 2 which we shall be interested in is the fact that it is bigger than 1; it could (almost) be replaced throughout the paper by for example the number 3.7.

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2. MATERIAL REQUIRED FOR THE PROOF

The proof of Theorem 1 makes essential use of the conformal measures defined and constructed by Denker and Urbański [7], and the concept of pressure, introduced by Ruelle [17, 19]. We shall also require some geometric results on J which we give first.

2.1. Hausdorff measure and dimension

The Hausdorff dimension of a non-empty subset X of the k-dimensional Euclidean space $\mathbf{R}^k$ is an aspect of the size of X which can discriminate between sets of Lebesgue measure zero. The upper bounds on the Hausdorff dimensions of the sets $D(f)$ of well approximable points will follow from the definition of this dimension, which we include in order to establish notation.

The diameter $\sup\{ |\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in V \}$ of a non-empty subset V of $\mathbf{R}^k$ will be denoted by $d(V)$. A collection $\{ V_i \}$ such that $0 < d(V_i) \leqslant \rho$ for each i and $X \subset \bigcup_i V$ is called a ρ-cover of X.

Let s be a non-negative number and for any positive ρ define,

$$\mathscr{H}^s_\rho(X) = \inf\{ \sum_{i=1}^{\infty} d(V_i)^s : \{ V_i \} \text{ is a countable } \rho\text{-cover of } X \}.$$

The *s-dimensional Hausdorff measure* $\mathscr{H}^s(X)$ of X is defined by

$$\mathscr{H}^s(X) = \lim_{\rho \to 0} \mathscr{H}^s_\rho(X) = \sup_{\rho > 0} \mathscr{H}^s_\rho(X)$$

and the *Hausdorff dimension* $\dim X$ of X by

$$\dim X = \inf\{ s : \mathscr{H}^s(X) = 0 \} = \sup\{ s : \mathscr{H}^s(X) = \infty \}.$$

Further details and alternative definitions of Hausdorff measure and dimension can be found in [8].

A general and classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set X in $\mathbf{R}^k$ is the following mass distribution principle [8].

Lemma 2 (Mass Distribution Principle). — *Let* D *be a metric space with a Borel probability measure* μ. *Suppose there are constants* r_0, s, $C > 0$ *such that for all* $x \in D$, $0 < r < r_0$ *one has*

$$\mu(B(x, r)) < C . r^s.$$

Then the following holds

$$\dim D \geqslant s.$$

Proof. — Suppose one has a ρ-cover $\{ V_i \}$ of D with $\rho < r_0$. Then one has $\Sigma_i\, d(V_i)^s \geqslant \Sigma_i\, 2C^{-1}\, \mu(V_i) \geqslant 2C^{-1}\, \mu(W)$. Therefore $\mathscr{H}^s(W) \geqslant 2C^{-1}\, \mu(W) > 0$, which implies the lower bound on the dimension. □

2.2. Geometry of the Julia set

We need the following powerful result from complex analysis (see [13]).

Köbe Distortion Theorem. — *Let* $\Delta \subset \overline{\mathbf{C}}$ *be a topological disc with boundary containing at least two points and let* $V \subset \Delta$ *be compact. Then there exists a constant* $K(\Delta, V)$ *such that for any univalent holomorphic function* $f : \Delta \to \mathbf{C}$ *the following inequality is satisfied,*

$$\sup_{x, y \in V} \frac{|f'(x)|}{|f'(y)|} \leqslant K(\Delta, V).$$

One useful consequence of this theorem is the following:

Bounded Distortion Property. — *There is an absolute constant* K *such that if* f *is a univalent holomorphic function defined on a disc* $B(z, 2r)$ *in* $\mathbf{C}$ *then*

$$B(f(z), K^{-1} r\, |f'(z)|) \subset f(B(z, r)) \subset B(f(z), Kr\, |f'(z)|).$$

Throughout this article K *will denote the constant arising in the statement of the Bounded Distortion Property.*

The following lemma is also well known, and can be found (amongst other places) in [9].

Lemma 3. — *Let* T *be an expanding rational map with Julia set* J. *Then there is a neighbourhood* U *of* J *such that* $T^{-1}(U) \subset U$ *and for any ball* $B \subset U$, *all inverse branches of iterates of* T *are defined on* B.

Covering Lemma. — *Let* z_0 *be a point in* J. *Then there exist positive constants* C_1, C_2 *and a positive integer* n_0 *with the following properties: for all* $n \geqslant n_0$,

$$J \subset \bigcup_{y : T^n(y) = z_0} B(y, C_1\, |(T^n)'(y)|^{-1});$$

the following union

$$\mathop{\overset{\circ}{\bigcup}}_{y : T^n(y) = z_0} B(y, C_2\, |(T^n)'(y)|^{-1})$$

is disjoint.

Let U be the neighbourhood of J in $\mathbf{C}$ constructed in Lemma 3. For a ball $B = B(x, r)$ we shall sometimes write 2B for $B(x, 2r)$. If $B \subset \mathbf{C}$ is a ball centred on a point of J, then we define

$$n_0(B) := \sup\{ n \in \mathbf{N} : T^n(2B) \subset U \text{ and } T^n|_{2B} \text{ is injective} \},$$
$$n_1(B) := \inf\{ n \in \mathbf{N} : T^n(B) \supset J \}.$$

Note that one always has the inequality $n_1(B) \geqslant n_0(B)$. Also note that $n_1(B)$ always exists and is finite since $T : J \to J$ is an open map and is topologically exact.

Lemma 4. — There is a constant C_3 *independent of* $B = B(x, r)$ *such that* $n_1(B) - n_0(B) \leqslant C_3$. *Furthermore one has*

$$| (T^{n_0(B)})'(x) | \asymp | (T^{n_1(B)})'(x) | \asymp r^{-1}.$$

Proof. — Let

$$c := \frac{1}{2} \inf_{x \in J, y \in \mathbf{C} \setminus U} | x - y |.$$

Thus any ball centred on J and of radius $2c$ must be contained in U. For any $n \in \mathbf{N}$ let $T_x^{-n} : B(T^n x, 2c) \to U$ be the inverse branch of T^n which takes $T^n x$ back to x. The existence of this inverse branch is guaranteed by Lemma 3. Note that inverse branches are automatically injective. Applying the Bounded Distortion Property to this map, we obtain for any $n \in \mathbf{N}$,

$$B(x, cK^{-1} | (T^n)'(x) |^{-1}) \subset T_x^{-n}(B(T^n x, c)) \subset B(x, cK | (T^n)'(x) |^{-1}).$$

Choose natural numbers p, q to satisfy

$$cK^{-1} | (T^{p+1})'(x) |^{-1} \leqslant 2r \leqslant cK^{-1} | (T^p)'(x) |^{-1},$$
$$cK | (T^q)'(x) |^{-1} \leqslant r \leqslant cK | (T^{q-1})'(x) |^{-1}.$$

Since T is expanding, it follows that $| p - q |$ is bounded independently of x, r. Note that T^p maps $T_x^{-p}(B(T^p x, c))$ injectively to U. Therefore by the above inclusion, T^p maps $B(x, cK^{-1} | (T^p)'(x) |^{-1})$ injectively to U. However by choice of p, we have $2B \subset B(x, cK^{-1} | (T^p)'(x) |^{-1})$. This shows that $n_0(B) \geqslant p$. On the other hand, by choice of q and using the other inclusion above, one can show that $T^q(B) \supset B(T^q(x), c)$. This implies that $n_1(B) \leqslant q + n_1(B(T^q(x), c))$. Let

$$c_2 := \sup\{ n_1(B(z, c)) : z \in J \}.$$

By the Covering Lemma and the fact that T is expanding, one knows that c_2 is finite. Therefore $n_1(B) \leqslant q + c_2$. It follows that $n_1(B) = n_0(B) + O(1) = p + O(1)$, and the other equation follows from the choice of p. □

An essentially equivalent statement to part of Lemma 4 is the following (which we shall occasionally find more appropriate).

Lemma 5. — *Let* $x \in \mathrm{J}$ *and* $n \in \mathbf{N}$. *Then* $n_0(\mathrm{B}(x, |(\mathrm{T}^n)'(x)|^{-1})) = n + \mathrm{O}(1)$.

2.3. Pressure

As before, let T be an expanding rational map with Julia set J. At this point it is worth mentioning that since T is expanding, the dynamical system (J, T) has a finite Markov partition (see section 7.29 of [19]). Using this fact we may apply results proved for shift spaces (for example in [17]) to our case. We shall define the pressure of a Hölder continuous function. For a comprehensive account of the concepts introduced in this subsection, the reader is referred to [3, 6, 17, 19].

Recall that a function $f : \mathrm{J} \to \mathbf{R}$ is said to be Hölder continuous if there is a constant $\mathrm{C}(f)$ satisfying the following condition: for any ball $\mathrm{B} = \mathrm{B}(x, r) \subset \mathrm{U}$ with $x \in \mathrm{J}$ and any n in $\mathbf{N}$ such that T^n is injective on B, one has for all y in $\mathrm{B} \cap \mathrm{J}$

$$|f_n(x) - f_n(y)| \leqslant \mathrm{C}(f).$$

This definition of Hölder continuity is the standard definition given for functions on shift spaces [17]. It is in fact equivalent to the form more usually used in complex dynamics, which states that there are constants r_0, $\alpha > 0$ such that if $x, y \in \mathrm{J}$ satisfy $|x - y| < r_0$ then one has $|f(x) - f(y)| \leqslant |x - y|^\alpha$. We now give a sketch of how one can obtain the above property from the more standard definition. By the Hölder continuity of f,

$$|f_n(x) - f_n(y)| \leqslant \sum_{i=0}^{n-1} |\mathrm{T}^i(x) - \mathrm{T}^i(y)|^\alpha.$$

Now T^n is injective on B, thus by the Bounded Distortion Property the right hand side of the above inequality is less than or equal to the quantity

$$|\mathrm{T}^{n-1}(x) - \mathrm{T}^{n-1}(y)|^\alpha (1 + \mathrm{K}^\alpha \sum_{i=1}^{n-1} |(\mathrm{T}^i)'(x)|^{-\alpha}).$$

Since T is expanding, one can find a bound $\mathrm{C}(f)$ on this expression which depends only on T, α and K.

Let $f : \mathrm{J} \to \mathbf{R}$ be a Hölder continuous function. For any positive ε one defines an (n, ε)-separated set to be a set $\mathrm{F}_n(\varepsilon)$ of points of J such that for any two distinct points $x, y \in \mathrm{F}_n(\varepsilon)$ one has

$$|\mathrm{T}^k x - \mathrm{T}^k y| > \varepsilon \quad \text{for some } k < n \text{ which may depend on } x, y.$$

We shall write

$$\mathrm{P}(\mathrm{T}, f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \sup \{ \sum_{z \in \mathrm{F}_n(\varepsilon)} \exp(f_n(z)) \}$$

where the supremum is over all maximal (n, ε)-separated sets $F_n(\varepsilon)$ of J. The *pressure* $P(T, f)$ of f is defined as

$$P(T, f) = \lim_{\varepsilon \to 0} P(T, f, \varepsilon).$$

The existence of this limit is a consequence of the Hölder continuity of f. We shall usually write $P(f)$ instead of $P(T, f)$.

An important property of pressure is the following variational principle (see [22]),

$$P(f) = \sup_{\sigma} \left\{ h_\sigma + \int_J f(x)\, d\sigma(x) \right\}$$

where the supremum extends over all ergodic T-invariant Borel probability measures σ on J and h_σ denotes the measure-theoretical entropy of T with respect to σ.

In the case $f = 0$, the pressure $P(0)$ reduces to the topological entropy h_{top} of T. This is always equal to $\log d$ [15], where d is the degree of the map T.

For $s \geqslant 0$, consider the function $s \mapsto P(-s f)$. This is a strictly decreasing, convex function which vanishes at exactly one point $s =: s(f)$. For $f(x) = \log |T'(x)|$ one has that $s(f) = \delta$ $(:= \dim(J))$. This equality is known as the Bowen-Manning-McCluskey formula. We remark again that Theorem 1 can be viewed as an elaborate generalization of this formula.

2.4. Conformal measures

Let $T : J \to J$ be as before. Furthermore let $h : J \to \mathbf{R}$ be a Hölder continuous function. A measure ν_h on J is said to be h-conformal (with respect to T) if

$$\nu_h(T(A)) = \int_A \exp(h(x))\, d\nu_h(x)$$

for every Borel subset A of J such that T restricted to A is injective. Suppose that ν is h-conformal with respect to T, and suppose that T^n is injective on some $A \subset J$. Then iterating the above relation n times (each time approximating h uniformly by a step function) one obtains

$$\nu_h(T^n(A)) = \int_A \exp(h_n(x))\, d\nu_h(x),$$

where h_n is the n-th ergodic sum of h. This means that ν is h_n-conformal with respect to T^n.

We shall require the following theorem of Denker and Urbański (see the first part of the theorems on p. 104 and p. 125 of [7]) which guarantees the existence of h-conformal measures.

Theorem. — *Let* $h : J \to \mathbf{R}$ *be a Hölder continuous function satisfying* $P(-h) = 0$ *and* $h(x) > 0$ *for all* $x \in J$. *Then there is a unique non-atomic h-conformal probability measure on* J.

In fact we require the following slight generalization of this, a proof of which we sketch.

Theorem 3. — Let $h : J \to \mathbf{R}$ be a Hölder continuous function satisfying $P(-h) = 0$, *and suppose that for some $n \in \mathbf{N}$, $h_n(x) > 0$ for all $x \in J$ (here h_n is the n-th ergodic sum of h). Then there is a unique non-atomic h-conformal probability measure on* J.

Proof. — Let n be chosen so that $h_n(x) > 0$ and $h_{n+1}(x) > 0$ for all $x \in J$. Then by the previous theorem, there exists a measure ν_n on J which is h_n-conformal with respect to the transformation T^n. Similarly, there is a measure ν_{n+1} which is h_{n+1}-conformal with respect to the transformation T^{n+1}. However, by iterating the conformality relation $n+1$ times one may deduce that ν_n is $h_{n(n+1)}$-conformal with respect to $T^{n(n+1)}$. Similarly one shows that ν_{n+1} is $h_{n(n+1)}$-conformal with respect to $T^{n(n+1)}$. Therefore by the uniqueness part of the previous theorem, we have $\nu_n = \nu_{n+1}$. Now using the fact that ν_n is both h_n-conformal with respect to T^n and h_{n+1}-conformal with respect to T^{n+1}, one deduces that it is h-conformal with respect to T. We have thus proved the existence part. For uniqueness, suppose ν is any h-conformal measure. Iterating the conformality relation n times, one shows that ν is h_n-conformal with respect to T^n, and thus by the uniqueness part of the previous theorem we have $\nu = \nu_n$. □

Remark. — Another way of deducing Theorem 3 from the theorem of Denker and Urbański would be to observe that h_n is cohomologous to h (via a Hölder coboundary). The conclusions of Theorem 3 remain unchanged if h is replaced by a cohomologous function.

As in 1.2, let $f : J \to \mathbf{R}$ be a Hölder continuous function satisfying $f(x) \geqslant \log |T'(x)|$ for all x in J. We shall apply the above theorem to the function $h := s(f).f : x \mapsto s(f).f(x)$ where $s(f)$ is the unique solution to $P(-s\,f) = 0$. Since T is expanding and $f \geqslant \log |T'|$, there is an $n \in \mathbf{N}$ such that $f_n(x) > 0$ for all $x \in J$. Thus $h = s(f).f$ satisfies the conditions of the theorem. Therefore there is a unique non-atomic $s(f).f$-conformal probability measure supported on J, which we shall denote by ν. The conformality condition means that, for every Borel subset $A \subset J$ on which T is injective,

$$\nu(T(A)) = \int_A \exp(s(f)\, f(x))\, d\nu(x). \tag{2}$$

Lemma 6. — There are constants C_4, C_5 such that for any ball B *centred on a point of* J *and for any $x \in B$ one has*

$$C_4 \exp(-s(f)\, f_{n_0(B)}(x)) \leqslant \nu(B) \leqslant C_5 \exp(-s(f)\, f_{n_0(B)}(x)).$$

Proof. — By the Hölder continuity of f there is a constant $C(f)$ such that for $x, y \in B$ one has $|f_{n_0(B)}(x) - f_{n_0(B)}(y)| \leqslant C(f)$. Iterating the relation (2) we obtain

$$\nu(T^{n_0(B)}(B)) = \int_B \exp(s(f)\, f_{n_0(B)}(z))\, d\nu(z),$$

$$1 \leqslant \int_B \exp(s(f)\, f_{n_1(B)}(z))\, d\nu(z).$$

Therefore, using Lemma 4 we obtain the inequalities

$$e^{-s(f)C(f)} \exp(s(f) f_{n_0(B)}(x))\, \nu(B) \leqslant \nu(T^{n_0(B)}(B)),$$
$$1 \leqslant e^{s(f)\, C(f)} \exp(s(f)(f_{n_0(B)}(x) + C_3 \sup_{x \in J} (f(x))))\, \nu(B).$$

The lemma follows by setting $C_4 := \exp(-s(f)\,(C(f) + C_3 \sup_{x \in J}(f(x))))$ and $C_5 := \exp(s(f)\, C(f))$. □

Lemma 7. — *There is a positive constant* C_6 *such that for any ball* B *centred on* J *one has*

$$\nu(2B) \leqslant C_6\, \nu(B).$$

Proof. — This follows from the previous lemma when one knows that $n_0(B) - n_0(2B)$ is bounded from above. Such a bound follows from Lemma 5 using the fact that T is expanding. □

3. LOCAL COUNTING RESULTS

We now depart from our main aim to describe a simple application of our methods. Let J, T, f be as before. *Counting results* are results which describe the number of periodic points or pre-images or preperiodic points, etc. that there are in a dynamical system. However there are usually infinitely many of them, and so one tries to answer the question how many there are of a given " size ". In order to obtain asymptotic estimates for such numbers as the " size " increases it is usual practise to define a dynamical zeta function and to obtain some kind of analytic continuation. These methods are based on techniques from analytic number theory, in which one counts the number of prime numbers of a given size. We give an example of this, counting pre-images of a given " size ". Let I be the set of all pairs $(y, n) \in J \times \mathbf{N}$ such that $T^n(y) = z_0$. Define

$$\pi(X) := \#\{ (y, n) \in I : f_n(y) \leqslant X \}.$$

In order to estimate $\pi(X)$ one defines the dynamical zeta function

$$Z(s) := \sum_{(y, n) \in I} \exp(-s f_n(y)).$$

It is known that $Z(s)$ converges in the right half plane $P(-\Re(s).f) < 0$, and that under certain conditions on f one may obtain an analytic continuation of Z to a larger domain. Then using a Tauberian Theorem one deduces that

$$\pi(X) \sim e^{s(f) X},$$

where $s(f)$ is the unique solution to the equation $P(-s.f) = 0$. This was by way of an introduction; we shall not use these techniques here. As usual, the notation $\pi(X) \sim e^{s(f)X}$ means that $\pi(X)\, e^{-s(f)X} \to 1$ as X tends to infinity.

We shall be interested in counting pre-images which lie in a given ball in J. Let $B = B(x, r)$ be a ball centred on a point $x \in J$, and define

$$\pi(B, X) := \#\{ (y, n) \in I : y \in B \text{ and } f_n(y) \leqslant X \}.$$

We shall prove the following.

Theorem 4. — *Let* $s(f)$ *be the solution to the equation* $P(-s.f) = 0$ *and let* ν *be the unique* $s(f).f$*-conformal measure. If* B *is a ball centred on a point of* J *then there is a constant* $X_0(B)$, *such that for* $X > X_0(B)$ *one has*

$$\pi(B, X) \asymp \nu(B)\, e^{s(f)X},$$

where the implied constants are independent of B.

This result is analogous to those for the orbital counting function in the discrete group setting first found in [16]. To prove Theorem 4 we need the following lemma, which is in turn proved using the Covering Lemma of 2.2.

Lemma 8 (Covering Result). — *There are constants* C_7, C_8, C_9, C_{10} *with the following property. For any* $x \in J$, $X \in \mathbf{R}^{\geqslant 0}$ *there is a pair* $(y, n) \in I$ *such that*

$$x \in B(y, C_7 \mid (T^n)'(y) \mid^{-1}), \quad \textit{and} \quad f_n(y) - C_8 \leqslant X \leqslant f_{n+1}(y) + C_8.$$

Furthermore, there are no more than C_9 *pairs* $(y, n) \in I$ *with* $f_n(y) - C_8 \leqslant X \leqslant f_{n+1}(y) + C_8$, *such that*

$$x \in B(y, C_{10} \mid (T^n)'(y) \mid^{-1}).$$

Proof. — Let x, X be given. Then choose n so that

$$f_n(x) < X \leqslant f_{n+1}(x).$$

By the Covering Lemma of 2.2, there is a point $y \in J$ satisfying $T^n(y) = z_0$ and with $x \in B(y, C_7 \mid (T^n)'(y) \mid^{-1})$. Since f is Hölder continuous there is a constant $C(f)$ such that $|f_n(y) - f_n(x)| \leqslant C(f)$. We therefore have

$$f_n(y) - C(f) \leqslant X \leqslant f_{n+1}(y) + C(f).$$

The first part of the lemma follows by setting $C_8 = C(f)$. Now suppose that $(y_1, n_1) \in I$ also has these properties. Since some finite ergodic sum of f is bounded away from zero it follows that $\mid n - n_1 \mid$ can take on only a bounded number of values. By setting C_{10} to be the C_2 of the Covering Lemma of 2.2, we have that for any n_1 there is at most one y_1 satisfying $x \in B(y_1, C_{10} \mid (T^n)'(y) \mid^{-1})$. Thus there are only a bounded number of pairs (y_1, n_1) satisfying the conditions of the lemma. We would like to thank Manfred Denker for suggesting the proof of this lemma. □

Proof of Theorem 4. — Let $I(X)$ be the set of all $(y, n) \in I$ such that

$$f_n(y) - C_8 \leqslant X \leqslant f_{n+1}(y) + C_8.$$

Then by the previous lemma one has

$$J \subset \bigcup_{(y,n)\in I(X)} B(y, C_7 \mid (T^n)'(y) \mid^{-1}),$$

and the multiplicity of $\bigcup_{(y,n)\in I(X)} B(y, C_{10} \mid (T^n)'(y) \mid^{-1})$ at any point is $\leqslant C_9$. Therefore for any measure m on J one has

$$\frac{1}{C_9} \sum_{(y,n)\in I(X)} m(B(y, C_{10} \mid (T^n)'(y) \mid^{-1})) \leqslant m(J)$$
$$\leqslant \sum_{(y,n)\in I(X)} m(B(y, C_7 \mid (T^n)'(y) \mid^{-1})).$$

Similarly we have for any ball B,

$$\frac{1}{C_9} \sum_{\substack{(y,n)\in I(X):\\ B(y, C_{10}\mid(T^n)'(y)\mid^{-1}) \subset B}} m(B(y, C_{10} \mid (T^n)'(y) \mid^{-1})) \leqslant m(B)$$
$$\leqslant \sum_{\substack{(y,n)\in I(X):\\ B(y, C_3\mid(T^n)'(y)\mid^{-1}) \cap B \neq \varnothing}} m(B(y, C_7 \mid (T^n)'(y) \mid^{-1})).$$

Let X be large enough so that for $(y, n) \in I(X)$ one has $C_7 \mid (T^n)'(y) \mid^{-1} < r/2$. We then have

$$\frac{1}{C_9} \sum_{\substack{(y,n)\in I(X):\\ y\in B(x, r/2)}} m(B(y, C_{10} \mid (T^n)'(y) \mid^{-1})) \leqslant m(B)$$
$$\leqslant \sum_{\substack{(y,n)\in I(X):\\ y\in B(x, 2r)}} m(B(y, C_7 \mid (T^n)'(y) \mid^{-1})).$$

Now let ν be the $s(f).f$-conformal measure. We have for $(y, n) \in I(X)$ by Lemmas 6, 7 and 5,

$$\nu(B(y, C_7 \mid (T^n)'(y) \mid^{-1})) \asymp \nu(B(y, C_{10} \mid (T^n)'(y) \mid^{-1}))$$
$$\asymp \exp(-s(f) f_n(y)) \asymp e^{-s(f)X}.$$

We therefore have

$$\pi(B(x, r/2), X + C_8) - \pi(B(x, r/2), X - C_8) \leqslant \nu(B)\, e^{s(f)X}$$

and $$\nu(B)\, e^{s(f)X} \leqslant \pi(B(x, 2r), X + C_8) - \pi(B(x, 2r), X - C_8).$$

Thus by Lemma 7 we have

$$\pi(B(x, r), X + C_8) - \pi(B(x, r), X - C_8) \asymp \nu(B)\, e^{s(f)X}.$$

This is equivalent to the theorem. □

We shall now prove by a similar method (but using the Covering Lemma of 2.2 instead of Lemma 8) a result which we need in § 4. The number $\Sigma(B, n)$ for $n \in \mathbf{N}$ will be defined as follows:

$$\Sigma(B, n) := \sum_{y:T^n y = z_0,\ B(y, \exp(-f_n(y))) \subset B} \exp(-s(f) . f_n(y)). \tag{3}$$

Key Lemma. — *There are constants* $C_{11}, C_{12}, C_{13} > 0$ *depending only on* T, J, f *such that if* B *is a ball and* $n > n_0(B) + C_{13}$ *then*

$$C_{11}\, \Sigma(B, n) < \nu(B) < C_{12}\, \Sigma(B, n).$$

Proof. — This lemma is a straightforward application of the transformation formula (2) for ν to the Covering Lemma of 2.2. We have a covering of B:

$$B \subset \bigcup_{y:T^n y = z_0,\ B(y, C_1 |(T^n)'(y)|^{-1}) \cap B \neq \emptyset} B(y, C_1 \,|\, (T^n)'(y) \,|^{-1}). \tag{4}$$

From this follows

$$\nu(B) \leqslant \sum_{y:T^n y = z_0,\ B(y, C_1 |(T^n)'(y)|^{-1}) \cap B \neq \emptyset} \nu(B(y, C_1 \,|\, (T^n)'(y) \,|^{-1})). \tag{5}$$

By Lemmas 7, 6 and 5 we have $\nu(B(y, C_1 \,|\, (T^n)'(y) \,|^{-1})) \asymp \exp(-s(f) f_n(y))$. Together with (5) this gives us

$$\nu(B) \ll \sum_{y:T^n y = z_0,\ B(y, \exp(-f_n(y))) \cap B \neq \emptyset} \exp(-s(f) f_n(y)). \tag{6}$$

This is almost one half of the lemma. To obtain the other half one notes that

$$B \supset \mathring{\bigcup_{y:T^n y = z_0,\ B(y, \exp(-f_n(y))) \subset B}} B(y, C_2 \,|\, (T^n)'(y) \,|^{-1}).$$

This gives us in the same way (Lemmas 7, 6 and 5) that

$$\nu(B) \gg \sum_{y:T^n y = z_0,\ B(y, \exp(-f_n(y))) \subset B} \exp(-s(f) f_n(y)). \tag{7}$$

Now let r be the radius of B. Choose C_{13} large enough so that for all $x \in J$, $n \geqslant C_{13}$ one has $|\, (T^n)'(x) \,| \geqslant 2K^{-1}$. Since $n > n_0(B) + C_{13}$ it follows by Lemma 4, the chain rule and the Köbe Distortion Theorem that

$$|\, (T^n)'(y) \,|^{-1} \leqslant r/2,$$

so the condition $B(y, \exp(-f_n(y))) \cap B \neq \emptyset$ implies $B(y, \exp(-f_n(y))) \subset 2B$. We have therefore shown in (6) and (7) that

$$\Sigma(2B, n) \gg \nu(B) \gg \Sigma(B, n).$$

Thus by Lemma 7 we have

$$\nu(B) \asymp \Sigma(B, n).$$

This finishes the proof of the lemma. □

The following corollary is used in the upper bound on $\dim D(f)$ in 4.1.

Corollary 1. — *One has*

$$\sum_{y:T^n y = z_0} \exp(-s(f).f_n(y)) \asymp 1.$$

4. PROOF OF THEOREM 1

We now prove Theorem 1. The proof consists in obtaining an upper bound and a lower bound for the Hausdorff dimension of $D(f)$ separately. The upper bound is easy and we shall prove that first. Let f be as in 1.2 and let $s(f)$ be the unique solution to $P(-s.f) = 0$.

4.1. *The upper bound.* — The set $D(f)$ is a lim sup set, that is

$$D(f) := \bigcap_{q=1}^{\infty} \bigcup_{n=q}^{\infty} \bigcup_{y:T^n(y)=z_0} B(y, \exp(-f_n(y))).$$

Thus for any natural number q there is a " natural " cover of $D(f)$:

$$D(f) \subset \bigcup_{n=q}^{\infty} \bigcup_{y:T^n(y)=z_0} B(y, \exp(-f_n(y))).$$

Since T is expanding, there exists a $\lambda > 1$ and an integer $m \geqslant 1$ such that $|(T^m)'(x)| \geqslant \lambda$ for all x in J. Let $c_1 = \min\{|(T^n)'(x)| : x \in J, 1 \leqslant n \leqslant m\}$. Then for any $q \geqslant 1$

$$|(T^q)'(x)| \geqslant \lambda^{\left[\frac{q}{m}\right]} c_1 \quad \text{for all } x \text{ in J},$$

where $\left[\frac{q}{m}\right]$ denotes the integer part of $\frac{q}{m}$. Clearly, $\lambda^{\left[\frac{q}{m}\right]} c_1 \to \infty$ as $q \to \infty$. We have that $f \geqslant \log |T'|$, hence for all x in J

$$\exp(-f_q(y)) \leqslant |(T^q)'(x)|^{-1} \leqslant \lambda^{-\left[\frac{q}{m}\right]} c_1^{-m}. \tag{8}$$

Fix a positive ε. By the definition of the Hausdorff measure

$$\mathcal{H}_\rho^{s(f)+\varepsilon}(D(f)) \leqslant \sum_{n=q}^{\infty} \sum_{y:T^n(y)=z_0} \exp(-(s(f)+\varepsilon)f_n(y)),$$

where $\rho \geqslant 2c_1\lambda^{-\left[\frac{q}{m}\right]}$. From inequality (8) we see that $\exp(-\varepsilon f_n(y)) \ll \lambda^{-\left[\frac{n}{m}\right]\varepsilon}$. This fact combined with Corollary 1 implies that

$$\mathcal{H}_\rho^{s(f)+\varepsilon}(D(f)) \leqslant \sum_{n=q}^{\infty} \lambda^{-\left[\frac{n}{m}\right]\varepsilon}.$$

Since $\lambda > 1$, the above sum tends to zero as $q \to \infty$. Thus $\mathcal{H}^{s(f)+\varepsilon}(D(f))$ is zero and on letting ε tend to zero we obtain

$$\dim D(f) \leqslant s(f),$$

as required. □

4.2. *The lower bound.* — We have given the upper bound on $\dim \mathrm{D}(f)$; we now prove the lower bound. In what follows the word "measure" will mean "Borel probability measure". We shall use a classical method of constructing a Cantor-like subset K of $\mathrm{D}(f)$ and a measure μ supported on this subset. The measure will satisfy the condition

$$\mu(\mathrm{B}(x, r)) \leqslant r^{s(f)-\varepsilon} \quad \text{for all } r < r_0(\varepsilon),\ x \in \mathrm{K}. \tag{9}$$

This implies by the Mass Distribution Principle (cf. 2.1, Lemma 2) that K has dimension $\geqslant s(f) - \varepsilon$. Letting ε tend to zero and observing that $\mathrm{K} \subset \mathrm{D}(f)$ we obtain that $\dim \mathrm{D}(f) \geqslant s(f)$. This together with the upper bound obtained in 4.1 completes the proof of Theorem 1.

4.3. *The Cantor set.* — We begin by constructing the Cantor-like set $\mathrm{K} \subset \mathrm{D}(f)$. Let $\mathrm{N}(l)$ for $l \in \mathbf{N}$ be a rapidly increasing sequence of natural numbers. We shall use the notation

$$r(y, l) := \mathrm{C}_2 \exp(-f_{\mathrm{N}(l)}(y))$$

and

$$\mathbf{B}(y, l) := \mathrm{B}(y, r(y, l)).$$

Note that the constant C_2 (from the Covering Lemma) forces the various $\mathbf{B}(y, l)$ with the same level l to be disjoint. We define sets $\mathrm{K}(l)$ for $l \in \mathbf{N}$ recursively as follows:

$$\mathrm{K}(1) := \mathrm{J},$$
$$\mathrm{K}(l+1) := \bigcup \mathbf{B}(y, l+1),$$

the union being taken over all pairs $(y, \mathrm{N}(l+1)) \in \mathbf{I}$ such that $\mathbf{B}(y, l+1) \subset \mathrm{K}(l)$. The set K is defined to be the intersection

$$\mathrm{K} = \bigcap_{l=1}^{\infty} \mathrm{K}(l).$$

We must show that K is a subset of $\mathrm{D}(f)$. Let $x \in \mathrm{K}$. Then $x \in \mathrm{K}(l)$ for every $l \in \mathbf{N}$. Therefore $x \in \mathrm{B}(y, \exp(-f_{\mathrm{N}(l)}(y)))$ for the infinite sequence $\mathrm{N}(l)$. This implies that x is in $\mathrm{D}(f)$. We therefore have that

$$\mathrm{K} \subset \mathrm{D}(f).$$

Generalizing the $\mathbf{B}(y, l)$ notation, for any $x \in \mathrm{K}(l)$ we shall write $\mathbf{B}(x, l)$ to mean the unique ball $\mathbf{B}(y, l)$ containing x.

4.4. *A measure on* K. — We now construct a measure μ on K. This will be defined to be the limit of a sequence of measures μ_l, where μ_l is a measure supported on $\mathrm{K}(l)$. This sequence of measures will be defined recursively. We define μ_1 to be any probability measure on $\mathrm{K}(1) := \mathrm{J}$. Suppose that the measure μ_l on $\mathrm{K}(l)$ is defined. The set $\mathrm{K}(l)$ is a union of balls $\mathbf{B}(y, l)$. We shall use the notation

$$\mu(y, l) := \mu_l(\mathbf{B}(y, l))$$

for each of the balls $\mathbf{B}(y, l)$ which constitute $\mathrm{K}(l)$.

Now consider one of the balls $\mathbf{B}(y, l)$ which make up $K(l)$. It contains a finite number of balls $\mathbf{B}(z, l+1)$ which make up its intersection with $K(l+1)$. We define $\mu_{l+1}(\mathbf{B}(z, l+1)) := \mu(z, l+1)$, where the numbers $\mu(z, l+1)$ are given by the recursive formula

$$\mu(z, l+1) := \frac{r(z, l+1)^{s(f)}}{C_2\,\Sigma(\mathbf{B}(y, l), N(l+1))}\,\mu(y, l), \tag{10}$$

where, as in 3,

$$\Sigma(B, n) := \sum_{(y, n) \in I : B(y, \exp(-f_n(y))) \subset B} \exp(-s(f) f_n(y)).$$

It follows from the way we have constructed the measure that $\mu_{l+1}(\mathbf{B}(y, l)) = \mu_l(\mathbf{B}(y, l))$. Since the sets $\mathbf{B}(y, l) \cap K$ generate the σ-algebra of Borel subsets of K it follows that the measures μ_l tend to a limit μ. Since each μ_l is supported on $K(l)$ and $K(l) \supset K(l+1) \supset \ldots$ it follows that μ is a measure on K, see Proposition 1.7 of [8]. It remains to prove the estimate (9) on μ.

4.5. *An estimate on the numbers* $\mu(y, l)$. — By the Key Lemma and (10) we have that

$$\mu(y, l) \asymp \frac{r(y, l)^{s(f)}}{\nu(\mathbf{B}(y, l-1))}\,\mu(y, l-1).$$

Iterating this relation we get

$$\mu(y, l) = r(y, l)^{s(f)} \times \prod_{i=1}^{l-1} \frac{r(y, i)^{s(f)}}{\nu(\mathbf{B}(y, i))} \times \exp(O(l)). \tag{11}$$

The terms in the product in (11) are of the form $r(B)^{s(f)}/\nu(B)$ for a ball B. We now prove a general estimate on such terms. To state this it will be convenient to use the notation $g(x) = f(x) - \log |T'(x)|$. We shall also write g_n for the ergodic sum:

$$g_n(x) = \sum_{i=0}^{n-1} g(T^i x).$$

Lemma 9. — *Let* $B = B(x, r)$, $x \in J$. *Then for any* $y \in B \cap J$ *one has*

$$\frac{r^{s(f)}}{\nu(B)} \asymp \exp(s(f)\, g_{n_0(B)}(y)).$$

Proof. — Lemma 6 tells us that

$$\nu(B) \asymp \exp(-s(f) f_{n_0(B)}(y)) = \exp(-s(f)\, g_{n_0(B)}(y))\, |(T^{n_0(B)})'(y)|^{-s(f)}.$$

On the other hand, we have, by Lemma 4,

$$r \asymp |(T^{n_0(B)})'(y)|^{-1}.$$

The lemma follows from the two formulae. □

Using Lemma 9 we now have from (11)

$$\mu(y, l) = r(y, l)^{s(f)} \times \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(y,i))}(y)) \times \exp(\mathrm{O}(l)). \tag{12}$$

4.6. *The measure of an arbitrary ball.* — We shall now estimate $\mu(\mathrm{B})$ for a general ball B. Given a ball B centred on a point x of K we choose l such that $\mathrm{N}(l-1) \leqslant n_0(2\mathrm{B}) + \mathrm{C}_{13} \leqslant \mathrm{N}(l)$. From this condition it follows that B intersects $\ll 1$ of the balls $\mathbf{B}(y, l-1)$ in the construction on $\mathrm{K}(l-1)$. It is thus sufficient from the point of view of obtaining the bound (9), to estimate $\mu(\mathrm{B} \cap \mathbf{B}(y, l-1))$ for each $\mathbf{B}(y, l-1)$ in $\mathrm{K}(l-1)$ separately, and we might as well assume that B intersects only one $\mathbf{B}(y, l-1)$. We may also assume (from the point of view of obtaining an upper bound on $\mu(\mathrm{B} \cap \mathbf{B}(y, l-1)))$ that $\mathrm{B} \subset \mathbf{B}(y, l-1)$. This implies the inequality $n_0(\mathrm{B}) \geqslant n_0(\mathbf{B}(x, l-1))$, which we shall use below.

We have

$$\mu(\mathrm{B}) \leqslant \sum_{(y, \mathrm{N}(l)) \in \mathrm{I} : \mathbf{B}(y, l) \cap \mathrm{B} \neq \emptyset} \mu(y, l).$$

By (12) we get

$$\mu(\mathrm{B}) \leqslant \sum_{(y, \mathrm{N}(l)) \in \mathrm{I} : \mathbf{B}(y, l) \cap \mathrm{B} \neq \emptyset} r(y, l)^{s(f)} \times \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(y,i))}(y)) \times \exp(\mathrm{O}(l)).$$

Since $\mathrm{B} \subset \mathbf{B}(x, l-1)$, we have for each term in this product $\mathbf{B}(y, i) = \mathbf{B}(x, i)$. Therefore

$$\mu(\mathrm{B}) \leqslant \sum_{(y, \mathrm{N}(l)) \in \mathrm{I} : \mathbf{B}(y, l) \cap \mathrm{B} \neq \emptyset} r(y, l)^{s(f)} \times \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(y)) \times \exp(\mathrm{O}(l)).$$

Since g is Hölder continuous and $y \in \mathbf{B}(x, i)$ for $i \leqslant l-1$, this gives us

$$\mu(\mathrm{B}) \leqslant \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(x)) \times \sum_{(y, \mathrm{N}(l)) \in \mathrm{I} : \mathbf{B}(y, l) \cap \mathrm{B} \neq \emptyset} r(y, l)^{s(f)} \times \exp(\mathrm{O}(l)).$$

Now from the condition on l we deduce that $r(y, l) \leqslant r/2$ for every y appearing in the above sum. Therefore the condition $\mathbf{B}(y, l) \cap \mathrm{B} \neq \emptyset$ implies $\mathbf{B}(y, l) \subset 2\mathrm{B}$, and we have

$$\mu(\mathrm{B}) \leqslant \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(x)) \times \Sigma(2\mathrm{B}, \mathrm{N}(l)) \times \exp(\mathrm{O}(l)).$$

Applying the Key Lemma (§ 3) we obtain

$$\mu(\mathrm{B}) \leqslant \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(x)) \times \nu(2\mathrm{B}) \times \exp(\mathrm{O}(l)).$$

Now from Lemmas 7 and 9 we get

$$\mu(\mathrm{B}) \leqslant \prod_{i=1}^{l-1} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(x)) \times r(\mathrm{B})^{s(f)} \times \exp(-\, s(f)\, g_{n_0(\mathrm{B})}(x)) \times \exp(\mathrm{O}(l)),$$

and the inequality $n_0(\mathrm{B}) \geqslant n_0(\mathbf{B}(x, l-1))$ gives us

$$\mu(\mathrm{B}) \leqslant \prod_{i=1}^{l-2} \exp(s(f)\, g_{n_0(\mathbf{B}(x,i))}(x)) \times r(\mathrm{B})^{s(f)} \times \exp(\mathrm{O}(l)).$$

Let $\varepsilon > 0$. We now choose the sequence $\mathrm{N}(l)$ to grow quickly enough so that for all $x \in \mathrm{K}$, $l \in \mathbf{N}$,

$$r(y, l-1)^{-\varepsilon} > \prod_{i=1}^{l-2} \exp(s(f)\, g_{n_0(\mathbf{B}(y,i))}(y)) \times \exp(\mathrm{O}(l)).$$

We then have

$$\mu(\mathrm{B}) \leqslant r(\mathrm{B})^{s(f)-\varepsilon}.$$

This finishes the proof. □

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