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METRIC FOLIATIONS ON HYPERBOLIC SPACES

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Abstract. On the hyperbolic space D^n, codimension-one totally geodesic foliations of class C^k are classified. Except for the unique parabolic homogeneous foliation, the set of all such foliations is in one-one correspondence (up to isometry) with the set of all functions $z:\ [0,\pi]\to S^{n-1}$ of class C^{k-1} with $z(0)=e_1=z(\pi)$ satisfying

$$|z'(r)|\leq 1$$

for all r, modulo an isometric action by $O(n-1)\times\mathbb{R}\times\mathbb{Z}_2$.

Since 1-dimensional metric foliations on D^n are always either homogeneous or flat (that is, their orthogonal distributions are integrable), this classifies all 1-dimensional metric foliations as well.

Equations of leaves for a non-trivial family of metric foliations on D^2 (called "fifth-line") are found.

1. Introduction

Let $\mathfrak{F}$ be a foliation on a Riemannian manifold (M,g). The tangent vector field (vertical) and the complementary vector field (horizontal) of $\mathfrak{F}$ are denoted by $\mathcal{V}$ and $\mathcal{H}$, respectively. The foliation $\mathfrak{F}$ is said to be *metric* if $\nabla^v:\mathcal{H}\times\mathcal{H}\to\mathcal{V}$ is skew-symmetric, or equivalently, the leaves of $\mathfrak{F}$ are equi-distant locally.

Such a foliation is *flat* if the orthogonal distribution is integrable (and hence forms a totally geodesic foliation); is *homogeneous* if it consists of the orbits of a free action of a subgroup of the isometry group. Gromoll-Grove ([4]) showed that 1-dimensional metric foliations on constant curvature spaces are either flat or homogeneous. As a consequence, the only 1-dimensional metric foliations of Euclidean spheres are the Hopf fibrations $S^{2n+1}\to\mathbb{C}P^n$.

On the other hand, on the Euclidean spaces E^n, Gromoll-Walschap ([6]) proved the only metric foliations (of any dimension) are homogeneous (the orbits of a free isometric group action by generalized glide rotations).

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Not too much seems to be known for hyperbolic spaces even though the above statement (flat or homogeneous) holds true. In [3], there is a "classification" of metric foliations on $\mathbb{H}^2$, where it is stated that there are 3 kinds of 1-dimensional metric foliations. However, we show that there are infinitely many metric foliations which are all characteristically different.

Our model space is the open ball D^n in the Euclidean space with the hyperbolic metric given by, for $X, Y \in T_u(D^n)$,

$$\langle\!\langle X, Y \rangle\!\rangle_u = \frac{4\langle X, Y\rangle}{(1-|u|^2)^2},$$

where $\langle , \rangle$ is the Euclidean inner product. The aim of this paper is to find and *classify all codimension-one totally geodesic foliations* on D^n. The main result is the following.

Main Theorem. *Except for the unique parabolic homogeneous foliation, the set of all codimension-one totally geodesic foliations of class C^k on the hyperbolic space D^n is in one-one correspondence (up to isometry) with the set of all functions $z: [0,\pi] \to S^{n-1}$ of class C^{k-1} with $z(0) = e_1 = z(\pi)$ satisfying*

$$|z'(r)| \leq 1$$

for all r, modulo the action of $O(n-1) \times \mathbb{R} \times \mathbb{Z}_2$, where $O(n-1)$ and $\mathbb{R}$ are the elliptic and hyperbolic isometries associated with the geodesic axis joining $\pm e_1 = (\pm 1, 0, \ldots, 0)$ and $\mathbb{Z}_2$ is a reflection interchanging e_1 and $-e_1$.

We also obtained an equation of the leaves for a 1-dimensional metric foliation on D^2 which has the property that every leaf intersects one fixed geodesic curve at a constant angle.

Here are some well-known facts.

Proposition 1.1 ([10, Theorem 5.19])**.** *Let $\mathfrak{F}$ be a foliation with associated vector field V (V is a vector field tangent to the leaves of $\mathfrak{F}$). Then $\mathfrak{F}$ is a metric foliation if and only if*

$$\langle\!\langle [V, X], X \rangle\!\rangle = 0 \tag{1.1}$$

holds for every vector field X such that $\langle\!\langle V, X \rangle\!\rangle = 0$ and $\langle\!\langle X, X \rangle\!\rangle = 1$.

Proposition 1.2 ([10, Theorem 5.23 and Theorem 5.19])**.** *Let $\mathfrak{F}$ be a foliation on a Riemannian manifold, and $\mathfrak{X}$ its complementary distribution. Suppose $\mathfrak{X}$ is integrable. Then $\mathfrak{F}$ is a metric foliation if and only if $\mathfrak{X}$ yields a totally geodesic foliation.*

By [4], every 1-dimensional metric foliation on a constant curvature manifold is either flat or homogeneous. But it is not hard to see that on the hyperbolic space D^n, except loxodromics, every 1-dimensional homogeneous foliation is, in fact, flat. Thus, by Proposition 1.2, the complementary distribution of such becomes a codimension-one totally geodesic foliation. Therefore, there is one-one correspondence between the class of all 1-dimensional metric foliations except loxodromics and the class of all flat codimension-one totally geodesic foliations on D^n.

Remark. The Lifschitz function in [13, Section 3] is closely related to our construction. Our work gives more precise result using the characterizing equation $\langle\!\langle [V, X], X \rangle\!\rangle = 0$ and presents many interesting new examples, including the equation of a "fifth line".

2. The arc-radius and arc-center functions $z(r)$

S^{n-1} will denote the boundary sphere of D^n. Every codimension-one totally geodesic submanifolds are spheres perpendicular to the boundary sphere S^{n-1}. Even though such submanifolds do not touch S^{n-1}, we shall extend them to the boundary (so that each leaf is a compact set). We shall talk about only *codimension-one totally geodesic foliations* on D^n.

Let a codimension-one totally geodesic foliation of class C^k be given. There exists a unique leaf that contains the center of the space D^n. It is a D^{n-1} (an $(n-1)$-dimensional ball passing through the center) which divides the space into two regions. On each region, the leaves eventually gets smaller and smaller (in Euclidean sense). By compactness, such geodesic spheres converge to a point on the boundary sphere. Therefore, there are at most two such limit points, and it is possible for these two limit points coincide and there is only one limit point. Such a point on the boundary of the ball will be called a *limit of shrinking geodesics*. We name them as A and B.

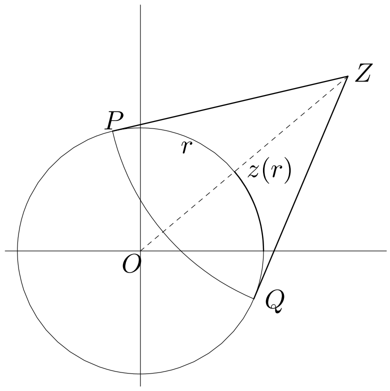
The set of all centroids of the fibers, $\mathfrak{C}$, is a 1-dimensional embedded submanifold of D^n, connecting A and B. For each $c \in \mathfrak{C}$, there is a unique leaf L_c, an $(n-1)$-dimensional Euclidean sphere perpendicular to the boundary sphere. Furthermore, $D^n = \cup\{L_c : c \in \mathfrak{C}\}$. Consider the (Euclidean) line connecting c and the center of D^n (There is one case where c equals the center of D^n. In that case, the line should be the line perpendicular to the fiber L_c which is a hyperplane passing through the center of D^n). These lines intersect the boundary sphere S^{n-1} at two points. Choose one point from these two points in a continuous fashion so that when c moves from A to B along $\mathfrak{C}$, the resulting points draw a curve on the boundary sphere joining A and B. This defines a map $z : \mathfrak{C} \to S^{n-1}$. Also determined is the (spherical) arc-length from $z(c)$ to the boundary of the fiber L_c, which will be called the *arc-radius* of L_c. The point $z(c)$ is called the *arc-center* of L_c. Summarizing, we have a bundle map

$$r : D^n \longrightarrow (0, \pi) \qquad \text{(arc-radius)}$$

which maps each leaf to its arc-length. This bundle is a product bundle (since the base space is contractible). Thus we have $(0, \pi) \longrightarrow \mathfrak{C} \hookrightarrow D^n$, a cross-section to the bundle. Composing the map $(0, \pi) \longrightarrow \mathfrak{C}$ with $\mathfrak{C} \xrightarrow{\text{proj}} S^{n-1}$, we get a map

$$z : (0, \pi) \longrightarrow S^{n-1} \qquad \text{(arc-center)}$$

which we call by z again (abuse of notation).



We use $u = (u_1, \ldots, u_n)$ for our coordinate system for the space D^n. For $r < \pi/2$, the real radius and center of a geodesic are related to r and $z(r)$ as follows:

$$\overline{PZ} = \text{Radius} = \tan r, \tag{2.1}$$

$$Z = \text{Center} = (\sec r) z(r). \tag{2.2}$$

For $r > \pi/2$, the real center is $-Z$ (on the opposite side of Z).

Using the relation between the real center, the real radius with r, $z(r)$ in (2.1), the equation of a totally geodesic sphere is

$$|u - (\sec r)z(r)|^2 = \tan^2 r, \tag{2.3}$$

where the norm is the Euclidean norm. Then the gradient vector field

$$V = u - (\sec r)z(r)$$

will generate a metric foliation, and its complementary distribution

$$X = (u - (\sec r)z(r))^{\perp}$$

will be a codimension-one totally geodesic foliation whose leaves have the equation (2.3).

The vector filed V is of class C^{k-1} if and only if X is of class C^{k-1}. If that is the case, the two foliations associated to the distributions V and X are of class C^k.

Lemma 2.1. *$z(r)$ is of class C^{k-1} as a function of r if and only if V (and X) is class C^{k-1} as a function of u.*

Proof. Suppose $z(r)$ is a function of r of class C^{k-1}. Let

$$F(u, r) = |u - (\sec r)z(r)|^2 - \tan^2 r, \quad 0 < r < \tfrac{\pi}{2}.$$

Then

$$F(u,r) = u \cdot u - 2(\sec r)u \cdot z(r) + 1,$$

since $|z(r)| = 1$. We have

$$\frac{\partial F}{\partial r} = -2(\sec r)\left((\tan r)u \cdot z(r) + u \cdot z'(r)\right).$$

We claim that

$$|u \cdot z'(r)| < |(\tan r)u \cdot z(r)|$$

on the totally geodesic sphere $|u - (\sec r)z(r)|^2 = \tan^2 r$ (This will say then $\frac{\partial F}{\partial r} \neq 0$).

Let α be the angle between u and $z'(r)$. Since $z(r) \cdot z'(r) = 0$, we have $|\frac{\pi}{2} - \alpha| < r$ so that $\frac{\pi}{2} - r < \alpha < \frac{\pi}{2} + r$ for $0 < r < \frac{\pi}{2}$. This implies

$$\begin{aligned} |u \cdot z'(r)| &= |u| \cdot |z'(r)| \cos\alpha \\ &\leq |u| \cos\alpha \\ &< |u| \cos(\frac{\pi}{2} - r) \\ &= |u| \sin r. \end{aligned}$$

On the other hand, from $u \cdot u - 2(\sec r)u \cdot z(r) + 1 = 0$, we get

$$u \cdot z(r) = \frac{1}{2}(\cos r)(|u|^2 + 1).$$

So,

$$\begin{aligned} |(\tan r)u \cdot z(r)| - |u \cdot z'(r)| &> \frac{1}{2}\sin r(|u|^2 + 1) - |u| \sin r \\ &= \frac{1}{2}\sin r(|u| - 1)^2 \\ &\geq 0. \end{aligned}$$

Consequently, $\frac{\partial F}{\partial r} \neq 0$ for all $0 < r < \pi/2$. Therefore, $\frac{\partial F}{\partial r}$ is never 0 for any $u \in D^n$. If $z(r)$ is a C^{k-1}-function of r, then $F(u,r)$ is a C^{k-1}-function (of u and r). By the *Implicit Function Theorem*, r can be represented as a C^{k-1}-function of u on D^n. By interchanging the role of the two limit points (A and B) of shrinking geodesics, we conclude also that $\frac{\partial F}{\partial r} \neq 0$ for all $\pi/2 < r < \pi$ in the above argument, and get the same conclusion. For the smoothness at $r = \frac{\pi}{2}$, see Remark 5.3. Consequently, V is of class C^{k-1}.

Conversely, suppose V is a function of u of class C^{k-1}. Then

$$(\sec r)z(r) = u - V$$

is of class C^{k-1}. Since $|z(r)| = 1$, we have

$$\sec r = |u - V| \quad \text{(Euclidean norm)},$$

and we see both r and

$$z(r) = \frac{1}{|u - V|}(u - V)$$

are of class C^{k-1} as functions of u. Now consider the obvious map

$$\xi : (0,\pi) \longrightarrow D^n$$

whose image is the geodesic connecting A and B (assuming $A \neq B$). Then

$$z(r) = \frac{1}{|u-V|}(u-V) \circ \xi(r),$$

which shows that $z(r)$ is a function of r of class C^{k-1}. The case $A = B$ is trivial. □

Proposition 2.2. *Let r and $z(r)$ be the arc-radius and the arc-center function for a totally geodesic foliation of class C^k $(k \geq 1)$, respectively. Then $z(r)$ is a function of class C^{k-1}, and satisfies $|z'(r)| \leq 1$ for all $0 < r < \pi$.*

Proof. Let $r_1 < r_2$. Find the unique leaves L_1 and L_2 of arc-radii r_1 and r_2, respectively (Note that there are two leaves of "arc-radius" r_1, but according to our convention, one is r_1 and the other is $r_1 + \frac{\pi}{2}$). These two circles on S^{n-1} do not intersect if and only if

$$\text{arc-distance}(z(r_2), z(r_1)) \leq r_2 - r_1 \tag{2.4}$$

(In fact, when arc-distance$(z(r_2), z(r_1)) = r_2 - r_1$, they do intersect at one point. However the two geodesic spheres (without the artificial boundary that we added) do not intersect). Let $\theta =$ arc-distance$(z(r_2), z(r_1))$. This is the angle between the two rays from the center O of D^n to $z(r_1)$ and $z(r_2)$.

From the cosine law, we have

$$1 + 1 - |z(r_2) - z(r_1)|^2 = 2\cos\theta.$$

Thus,

$$\theta = \cos^{-1}\left(1 - \tfrac{1}{2}|z(r_2) - z(r_1)|^2\right)$$

so that the condition (2.4) becomes

$$\cos^{-1}\left(1 - \tfrac{1}{2}|z(r_2) - z(r_1)|^2\right) \leq r_2 - r_1.$$

Since $\cos\theta$ is a decreasing function in $0 \leq \theta \leq \pi$, this yields

$$|z(r_2) - z(r_1)|^2 \leq 2\left(1 - \cos(r_2 - r_1)\right).$$

Dividing by $(r_2 - r_1)^2$, we get

$$\left|\frac{z(r_2) - z(r_1)}{r_2 - r_1}\right|^2 \leq 2 \times \frac{1 - \cos(r_2 - r_1)}{(r_2 - r_1)^2}.$$

This implies

$$|z'(r)|^2 \leq 2 \lim_{(r_2 - r_1)\to 0} \frac{1 - \cos(r_2 - r_1)}{(r_2 - r_1)^2} = 1.$$

□

Proposition 2.3. *Suppose $z = z(r) : (0, \pi) \to S^{n-1}$ is a differentiable map satisfying*

$$|z'(r)| \leq 1$$

for all $0 < r < \pi$. Then the collection of all totally geodesic spheres of arc-center $z(r)$ with arc-radius r for $0 < r < \pi$ forms a codimension-one totally geodesic foliation on D^n. If z is of class C^{k-1}, then the foliation is of class C^k.

Proof. For each $0 < r < \pi$, pick the point $z(r) \in S^{n-1}$, and draw the geodesic sphere of arc-radius r. By tracing back the proof of Proposition 2.2, we find that the condition $|z'(r)| \leq 1$ guarantees no two geodesic spheres intersect. Therefore, as r grows from 0 to π, the geodesic spheres sweep whole space D^n.

We prove that our vector field V satisfies the necessary differential equation (1.1) of the metric foliation. Let

$$v = u - (\sec r)z = \sum_i (u_i - (\sec r)z_i)E_i,$$

$$x = x(u) = \sum_i x_i E_i$$

be tangent vectors at u (which will be vertial and horizontal later), where $E_i = \frac{\partial}{\partial u_i}$. Since $|z|^2 = 1$, the geodesic equation

$$(u - (\sec r)z) \cdot (u - (\sec r)z) = \tan^2 r$$

becomes

$$|u|^2 - 2(\sec r)(u \cdot z) + 1 = 0 \tag{2.5}$$

or, equivalently

$$(\sec r)(u \cdot z) = \tfrac{1}{2}(1 + |u|^2), \tag{2.6}$$

where $u \cdot z$ denotes the Euclidean inner product of u and z.

Recall that our hyperbolic metric on D^n is given by, for $X, Y \in T_u(D^n)$,

$$\langle\!\langle X, Y \rangle\!\rangle_u = \frac{4\langle X, Y\rangle}{(1 - |u|^2)^2}.$$

Conditions for the vector field x to satisfy are

$$\langle\!\langle v, x \rangle\!\rangle_{D^n} = 0, \ \langle\!\langle x, x \rangle\!\rangle_{D^n} = 1.$$

The former is $\frac{4\langle v, x\rangle}{(1-|u|^2)^2} = 0$. Thus,

$$u \cdot x = (\sec r)(z \cdot x). \tag{2.7}$$

The latter condition yields

$$|x|^2 = \tfrac{1}{4}(1 - |u|^2)^2. \tag{2.8}$$

Now we start calculation:

$\langle [v, x], x\rangle$ (Euclidean inner product is good enough)

$$= \Big\langle \Big[\sum_i \Big(u_i - (\sec r) z_i \Big) E_i, \sum_j x_j E_j \Big], \sum_k x_k E_k \Big\rangle$$
$$= \sum_i \Big(u_i - (\sec r) z_i \Big) \sum_j \left(\frac{\partial x_j}{\partial u_i} x_j \right) \sum_{i,j} x_j x_i \frac{\partial}{\partial u_j} \Big(u_i - (\sec r) z_i \Big).$$

From the equality (2.8), we have

$$\sum_j \left(\frac{\partial x_j}{\partial u_i} x_j \right) = \tfrac{1}{2} \frac{\partial}{\partial u_i} |x|^2 = \tfrac{1}{2} \frac{\partial}{\partial u_i} \Big(\tfrac{1}{4}(1 - |u|^2)^2 \Big) = \tfrac{1}{2}(|u|^2 - 1) u_i$$

so that the first term becomes

$$\begin{aligned} &\sum_i (u_i - (\sec r) z_i) \sum_j \left(\frac{\partial x_j}{\partial u_i} x_j \right) \\ &= \tfrac{1}{2}(|u|^2 - 1) \Big((u \cdot u) - (\sec r)(z \cdot u) \Big) \\ &= \tfrac{1}{2}(|u|^2 - 1) \Big(|u|^2 - \tfrac{1}{2}(1 + |u|^2) \Big) \quad \text{(from (2.5))} \\ &= \tfrac{1}{4}(|u|^2 - 1)^2. \end{aligned}$$

In the second term,

$$\begin{aligned} \frac{\partial}{\partial u_j} \Big(u_i - (\sec r) z_i \Big) &= \delta_{ij} - z_i \frac{\partial}{\partial u_j} (\sec r) - (\sec r) \frac{\partial}{\partial u_j} z_i \\ &= \delta_{ij} - z_i (\sec r)(\tan r) \frac{\partial r}{\partial u_j} - (\sec r) z_i' \frac{\partial r}{\partial u_j} \end{aligned}$$

so that the second term becomes

$$\begin{aligned} &\sum_{i,j} x_j x_i \frac{\partial}{\partial u_j} \Big(u_i - (\sec r) z_i \Big) \\ &= \sum_{i,j} x_j x_i \Big(\delta_{ij} - z_i (\sec r)(\tan r) \frac{\partial r}{\partial u_j} - (\sec r) z_i' \frac{\partial r}{\partial u_j} \Big) \\ &= |x|^2 - (\sec r)(\tan r)(x \cdot z)(x \cdot \nabla r) - (\sec r)(x \cdot z')(x \cdot \nabla r) \\ &= \tfrac{1}{4}(1 - |u|^2)^2 - (\sec r)(\tan r)(x \cdot z)(x \cdot \nabla r) - (\sec r)(x \cdot z')(x \cdot \nabla r). \end{aligned}$$

Altogether, we get

$$\begin{aligned} &\langle [v, x], x \rangle \\ &= \tfrac{1}{4}(|u|^2 - 1)^2 \\ &\quad - \big(\tfrac{1}{4}(1 - |u|^2)^2 - (\sec r)(\tan r)(x \cdot z)(x \cdot \nabla r) - (\sec r)(x \cdot z')(x \cdot \nabla r) \big) \\ &= (\sec r)(x \cdot \nabla r) \Big((\tan r)(x \cdot z) + (x \cdot z') \Big). \end{aligned}$$

Once the arc-radius function $z(r)$ is known, the variable r is defined implicitly by the geodesic equation (2.5). In other words, the geodesics are the level surfaces for the function $r = r(u)$. Therefore, its gradient ∇r is orthogonal to all of the vectors x perpendicular to v. Thus,

$$x \cdot \nabla r = 0.$$

Consequently $\langle [v, x], x\rangle = 0$. The Euclidean inner product vanishes, and so does hyperbolic inner product (One can prove $x \cdot \nabla r = 0$ by brute force calculations. From (2.5), take derivatives, and form the dot-product. With the help of equalities (2.7), (2.8) and $|x| = 1$ etc., one can finally see this vanishes).

Consequently, the vector field V yields a metric foliation, and its natural orthogonal foliation X yields a totally geodesic foliation of class C^k. □

3. Examples

In this section, we present some 2 and 3-dimensional examples. For D^2, we use the coordinates (u, v) instead of $u = (u_1, u_2)$. For a fixed r, the normalized gradient of the circle

$$\left(u - \frac{z_1(r)}{\cos r}\right)^2 + \left(v - \frac{z_2(r)}{\cos r}\right)^2 - \tan^2 r = 0$$

defines a global unit vector field

$$V(u, v) = \frac{1 - u^2 - v^2}{2}\left(\frac{u\cos r - z_1(r)}{\sin r}\frac{\partial}{\partial u} + \frac{v\cos r - z_2(r)}{\sin r}\frac{\partial}{\partial v}\right)$$

which satisfies the differential equation (1.1). Therefore, as far as the function $z(r)$ satisfies the condition $|z'(r)| \leq 1$, the vector field $V(u, v)$ gives rise to a metric foliation on D^2.

Getting an explicit formula of the flows for the vector field V, one needs to solve differential equation from the vector field. This is not always easy. Here are some examples.

Example 3.1. $z(r) = (1, 0)$, constant. The geodesic foliation will be just the "concentric" (only the arg of the center is fixed) circles with the same arc-center $(1, 0)$. Obviously, this is a homogeneous one generated by a group of hyperbolic isometries. The vector field V is given by

$$V(u, v) = \frac{(u\cos r - 1)}{\sin r}\frac{\partial}{\partial u} + \frac{(v\cos r - 0)}{\sin r}\frac{\partial}{\partial v}$$

whose integral curves are

$$(u\cos r - 1)^2 + (v\cos r)^2 = \sin^2 r$$

for $r > 0$.

Example 3.2. $z(r) = \exp\left(\frac{1}{k}\sin kr\right)i$, k a constant. The arc-centers of the geodesic circles lie in the arc-interval $-\frac{1}{k}$ and $\frac{1}{k}$ on the boundary of the unit circle.

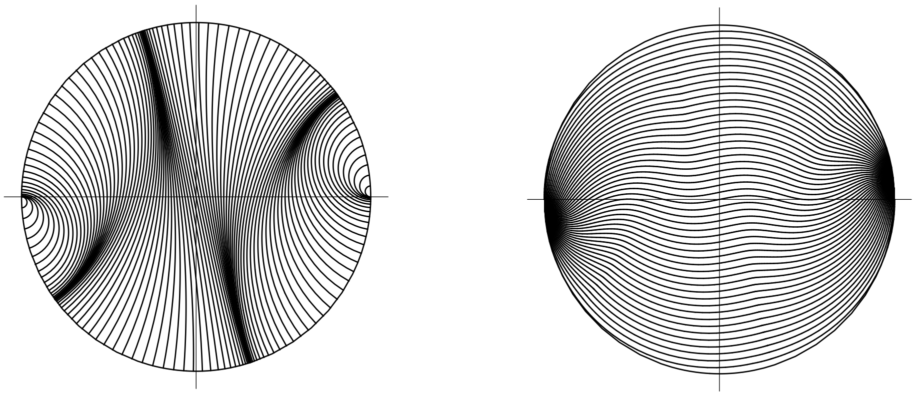


FIGURE 1. $z(r) = \exp\left(\frac{1}{k}\sin kr\right)i$

Example 3.3. If a piecewise-differentiable function $z(r)$ satisfies the condition $|z'(r)| \leq 1$ on each subinterval where $z(r)$ is differentiable, it still satisfies the condition $|z(r_2) - z(r_1)| \leq |r_2 - r_1|$ globally, and thus gives rise to a piecewise-smooth metric foliation. Since V involves $z(u, v)$ which is continuous, the flows will be of class C^1.

For example, $z(r)$ is given as a piecewise-smooth function:

$$\arg z(r) = \begin{cases} \frac{3\sqrt{3}}{4\pi} r, & 0 < r \leq \frac{2\pi}{3} \\ \sin r, & \frac{2\pi}{3} \leq r \leq \pi. \end{cases}$$

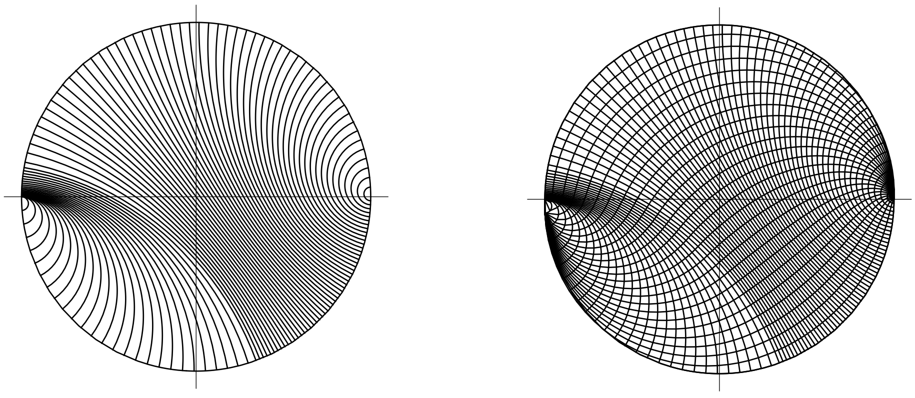


FIGURE 2. $z(r)$ is given as a piecewise-smooth function

Note that $z(r)$ is not smooth. The metric flows are not second-time differentiable. They are only of class C^1.

Example 3.4. The following totally geodesic foliation on D^3 is generated by the arc-center function $z : (0, \pi) \to S^2$ given by

$$z(r) = \left(\sin\tfrac{r}{2}\cos(\tfrac{1}{6}\sin(5r)), \sin\tfrac{r}{2}\sin(\tfrac{1}{6}\sin(5r)), \cos\tfrac{r}{2}\right).$$

Note that $|z'(r)| \leq 1$ is satisfied. The curve $z(r)$ starts at one limit of shrinking geodesics ($r = 0$) and ends at the antipodal point of the other limit of shrinking geodesics ($r = \pi$).

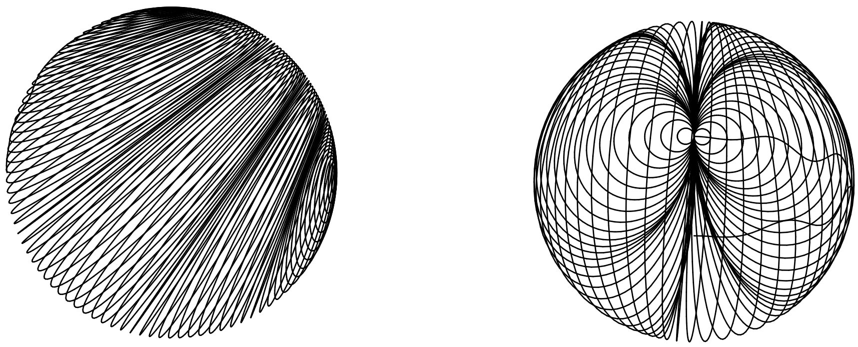


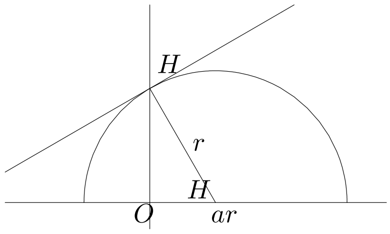
FIGURE 3. A codimension-one totally geodesic foliation on D^3 generated by $z(r)$, with the image of $z(r)$. Only the boundary S^2 are shown (from two different view points).

4. The "fifth line"

In this section, we study codimension-one totally geodesic foliations on D^n whose leaves have a same angle to a fixed geodesic curve. In dimension 2, the complementary foliation to such, that is, a metric foliation on D^2 whose leaves have a same angle to a fixed geodesic curve is called a *fifth line* in [3]. Such a condition gives rise to a certain differential equation. However, the differential equation on the disk model was exceptionally hard to solve. In fact, as far as the authors know, its complete solution seems to be unknown. See [3] for an integral form.

We change the equation to the upper half plane model, solve the differential equation, and translate to our disk model to get finally a solution on D^2.

We are in the upper half plane model with xy-coordinate. Without loss of generality, we may assume that the fixed geodesic is the y-axis. Suppose X is a geodesic foliation with the property that every leaf intersects the y-axis at a constant angle H.



From the geodesic equation $(x - ar)^2 + y^2 = r^2$, r can be solved in x and y. Thus the metric flow is

$$\begin{aligned} V(x,y) &= (x - ar, y) \\ &= \left(\frac{-x + a\sqrt{x^2 + (1-a^2)y^2}}{-1 + a^2}, y \right) \end{aligned}$$

and a non-unit speed geodesic flow is

$$X(x,y) = \left(-y, \frac{-x + a\sqrt{x^2 + (1-a^2)y^2}}{-1 + a^2} \right).$$

We know the integral curves of the geodesic flow already; namely, the solutions to

$$\begin{aligned} \frac{\partial x}{\partial t} &= -y, \\ \frac{\partial y}{\partial t} &= \frac{-x + a\sqrt{x^2 + (1-a^2)y^2}}{-1 + a^2} \end{aligned}$$

are the circles:

$$(x - ar)^2 + y^2 = r^2 \tag{4.1}$$

for varying r's.

To find the metric flow, we need to solve for $V(x,y)$. That is, we need to solve

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{-x + a\sqrt{x^2 + (1-a^2)y^2}}{-1 + a^2}, \\ \frac{\partial y}{\partial t} &= y. \end{aligned}$$

From the second, we get

$$y(t) = ce^t. \tag{4.2}$$

From

$$\frac{\partial x}{\partial t} = \frac{-x + a\sqrt{x^2 + (1-a^2)c^2 e^{2t}}}{-1 + a^2}$$

we do the change of variable. Let

$$x = \sqrt{1 - a^2}\, ce^t \tan s \tag{4.3}$$

and get a new equation in s:

$$(1-a^2)\frac{\partial s}{\partial t} = a(-1 + a\sin s)\cos s.$$

Solving this for s, we get

$$e^t = \frac{d(1-\sin s)^{\frac{1+a}{2a}}}{(1-a\sin s)(1+\sin s)^{\frac{1-a}{2a}}},$$

where d is a integral constant. Now the equations (4.3) and (4.2) yield

$$\begin{cases} x = \frac{(1-a)^2\lambda(1-\sin s)^{\frac{1+a}{2a}}}{(1-a\sin s)(1+\sin s)^{\frac{1-a}{2a}}}\tan s \\ y = \frac{\lambda(1-\sin s)^{\frac{1+a}{2a}}}{(1-a\sin s)(1+\sin s)^{\frac{1-a}{2a}}}, \end{cases} \tag{4.4}$$

where $\lambda = cd$. This is our equation for the leaves of the metric flow on the upper half plane model.

Proposition 4.1 (Equation of Fifth Line (on $\mathbb{H}^2$))**.** *The geodesic foliation with the property that every leaf intersects the y-axis at a constant angle H has integral curves given by* (4.4)*, where $a = \cos H$.*

The variable s is the curve parameter. For $s = 0$, $(x, y) = (0, \lambda)$ so that the variable λ gives continuous family of geodesic curves starting at points on the y-axis.

Proposition 4.2 (Equation of Fifth Line (on D^2))**.** *The 1-dimensional metric foliation in D^2 with the property that every leaf intersects the u-axis at a constant angle H has integral curves*:

$$\begin{cases} u(a,\eta,t) &= -q^{-1}\cdot d^2(a\sin t + 1)(1-\sin t)^{\frac{1}{a}} + (\sin\ t+1)^{\frac{1}{a}}(a\sin t - 1), \\ v(a,\eta,t) &= \quad q^{-1}\cdot 2\sqrt{1-a^2}d\cos^2 t^{\frac{a+1}{2a}}\tan t, \end{cases}$$

where

$$q(a,\eta,t) = 2d\cos^2 t^{\frac{a+1}{2a}} + (\sin\ t+1)^{\frac{1}{a}}(1-a\sin t) + d^2(1-\sin t)^{\frac{1}{a}}(a\ \sin t + 1)$$

and $d = \frac{\eta-1}{\eta+1}$.

The variable t is the curve parameter, and $a = \sin H$. For $t = 0$, $(u(a,\eta,0), v(a,\eta,0)) = (\eta, 0)$ so that the variable η gives continuous family of geodesic curves starting at points on the u-axis.

This was obtained by conjugating the previous function in (4.4) by following map $\mathbb{H}^2 \to D^2$ given by

$$(x,y) \ \mapsto \ \frac{1}{x^2+(1+y)^2}\,(1-x^2-y^2, 2x).$$

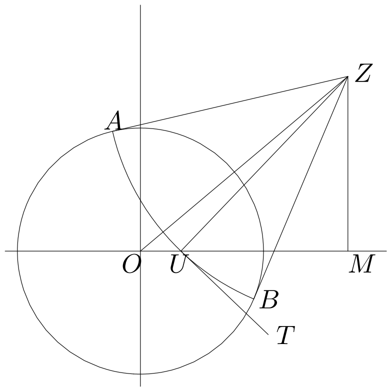
It is not hard to check that

$$\left.\frac{\frac{\partial v(a,\eta,t)}{\partial t}}{\frac{\partial u(a,\eta,t)}{\partial t}}\right|_{t=0} = \frac{a}{\sqrt{1-a^2}}.$$

Therefore, the angle of the curve with the u-axis is a constant, say H. Then $a = \sin H$.

Of course, such a foliation can be defined using our z-function. Consider the geodesic equation

$$\left(u - \frac{z_1(r)}{\cos r}\right)^2 + \left(v - \frac{z_2(r)}{\cos r}\right)^2 = \tan^2 r.$$



One can calculate

$$\frac{\overline{ZM}}{\overline{ZU}} = \frac{z_2(r)/\cos r}{\tan r} = \frac{z_2(r)}{\sin r}.$$

Set it to K and solve for z to get

$$z_1(r) = \sqrt{1 - K^2 \sin^2 r},$$
$$z_2(r) = K \sin r.$$

Recall that

$$K = \sin H = \cos J, \quad \text{where } J \text{ is the angle } \angle MUT.$$

By our procedure,

$$z(r) = (\sqrt{1 - K^2 \sin^2 r}, K \sin r), \quad 0 \leq K \leq 1,$$

yields a metric foliation whose leaves intersect the u-axis at the constant angle $J = \cos^{-1}(K)$.

One can verify that this function $z(r)$ satisfies the condition $|z'(r)| \leq 1$ since $|K| \leq 1$.

A similar construction as above works in any dimension D^n. It will yield codimension-one totally geodesic foliations of a constant angle with a fixed geodesic curve. Let

$$\Lambda = (\lambda_2, \ldots, \lambda_n) : (0, \pi) \longrightarrow S^{n-2} \subset \mathbb{R}^{n-1}$$

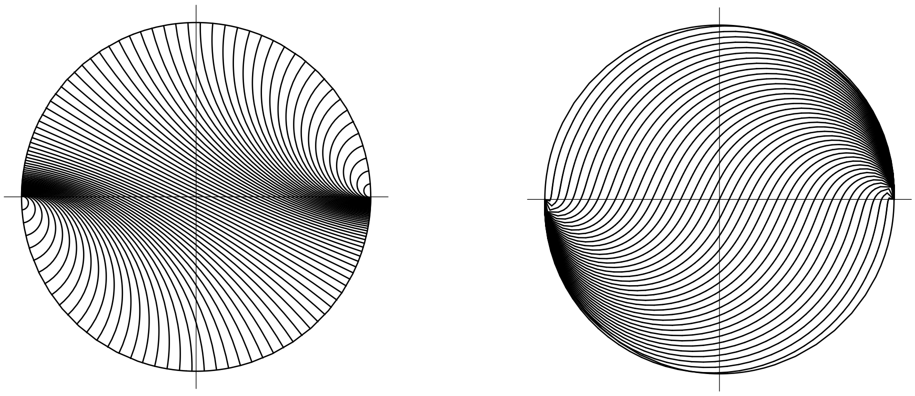


FIGURE 4. $z(r) = (\sqrt{1 - K^2 \sin^2 r}, K \sin r)$, (with $K = 0.9$) The leaves of the right-hand side foliation are the "fifth lines".

be a C^{k-1}-map. Then define an arc-center-function by

$$\begin{aligned} z_1(r) &= \sqrt{1 - K^2 \sin^2 r}, \\ z_2(r) &= K\lambda_2(r) \sin r, \\ &\cdots \\ z_n(r) &= K\lambda_n(r) \sin r. \end{aligned}$$

Then $\sum_{i=1}^{n} z_i(r)^2 = 1$ and K is the cosine of the angle of intersection with the u_1-axis. The arc-center function $z(r)$ of such foliations are simple closed curves, because $z(\pi) = z(0) = e_1$. Of course, z should satisfy the condition $|z'(r)| \leq 1$, which is equivalent to

$$\sum_{i=2}^{n} z_i'(r)^2 \leq \frac{1 - K^2}{K^2 \sin^2 r(1 - K^2 \sin^2 r)}.$$

Conversely, any codimension-one totally geodesic foliation which has a constant angle with the u_1-axis has the arc-center function as above.

Getting the equation of corresponding metric foliation seems to be very difficult in higher dimensions. Here is an example in dimension 3, with $\lambda_2(r) = \frac{r}{6K}$; $\lambda_3(r) = \sqrt{1 - (\frac{r}{6K})^2}$, where $K = 0.6$. Thus,

$$\begin{aligned} z_1(r) &= \sqrt{1 - K^2 \sin^2 r}, \\ z_2(r) &= \frac{r}{6} \sin r, \\ z_3(r) &= \sqrt{K^2 - (\tfrac{r}{6})^2} \sin r, \end{aligned}$$

where $K = 0.6$.

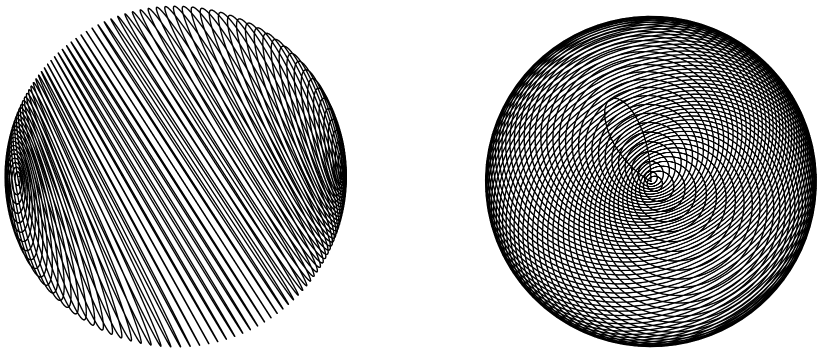


Figure 5. Codimension-one totally geodesic foliation of a constant angle with a fixed geodesic curve in D^3. Notice that $z(r)$ is a simple closed curve.

5. Classification

In Proposition 2.2 and Proposition 2.3, we showed that there is a correspondence between the totally geodesic foliations and the functions $z(r)$ satisfying $|z'(r)| \leq 1$.

$$\text{All codimension-one totally geodesic foliations} \\ \longleftrightarrow \{z : (0, \pi) \to S^{n-1} \ : \ |z'(r)| \leq 1\}.$$

Certainly, the group of isometries $\mathrm{Isom}(D^n) \cong O(1, n)$ acts on this functions space. We shall find a subset of the functions space which contains a fundamental domain of the action.

Recall that every codimension-one totally geodesic foliation has at most two *limit of shrinking geodesics* A (and B). This means that the arc-center function $z(r)$ has values near e_1 for small values of r. Observe that

$$\begin{aligned} z(0) &= \quad A(= e_1), \\ z(\pi) &= -B \end{aligned}$$

for any $z(r)$. Furthermore, we have:

Lemma 5.1. *The foliation satisfies $z(0) = -z(\pi)$ if and only if it is the unique 1-dimensional parabolic homogeneous foliation.*

Proof. Clearly $z(0) = -z(\pi)$ is equivalent to $A = B$. A 1-dimensional parabolic homogeneous foliation has a unique limit of shrinking geodesics. That is, $A = B$.

Conversely, suppose $z(0) = e_1 = -z(\pi)$. It is not too hard to see the following: If the spherical geodesic distance from $z(p)$ to $z(q)$ is equal to $|q-p|$, then

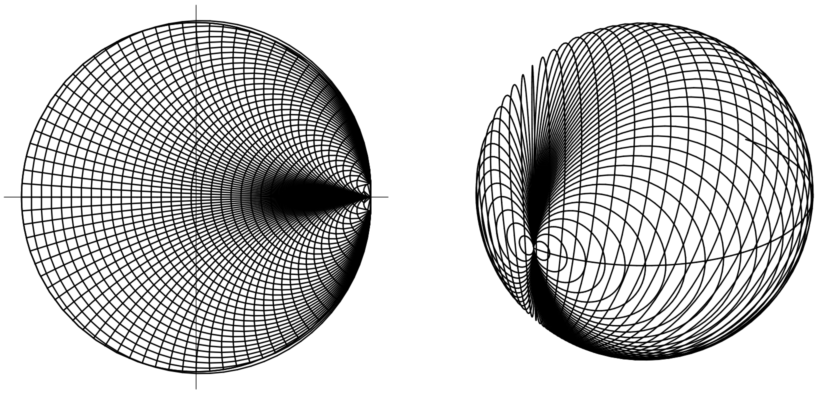


FIGURE 6. Parabolic homogeneous foliations: $z(r) = (\cos r)e_1 + (\sin r)e_2$.

$z(r)$ itself is the spherical geodesic curve from $z(p)$ to $z(q)$ for $p \le r \le q$. This fact forces $z(r)$ to be of the form

$$z(r) = (\cos r)e_1 + (\sin r)e$$

for $0 < r < \pi$, where e is a point on the equator $u_1 = 0$. We may assume that $e = e_2$ (by an isometric rotation). Thus, $\arg(z(r)) = r$. It is clear that this is the unique parabolic homogeneous foliation. More precisely, let

$$N = \begin{bmatrix} 0 & 0 & 1 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ 1 & -1 & 0 & . & 0 \\ . & . & . & 0 & . \\ 0 & 0 & 0 & . & 0 \end{bmatrix} \in \mathfrak{so}(1,n).$$

Then,

$$\varphi(t) = e^{tN} = \begin{bmatrix} \frac{t^2+2}{2} & -\frac{t^2}{2} & t & . & 0 \\ \frac{t^2}{2} & \frac{2-t^2}{2} & t & . & 0 \\ t & -t & 1 & . & 0 \\ . & . & . & 1 & . \\ 0 & 0 & 0 & . & 1 \end{bmatrix} \in \mathrm{SO}(1,n).$$

With the isometry $D^n \longrightarrow \mathbb{R}^{1,n}$ (the hyperboloid model) given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \mapsto \frac{1}{1 - |\mathbf{x}|^2} \left(1 + |\mathbf{x}|^2, 2x_1, 2x_2, \ldots, 2x_n\right),$$

which is a parabolic map, sending $\mathbf{x}$ to

$$\frac{1}{4+4x_2t+(1-2x_1+|\mathbf{x}|^2)t^2}\Big(4x_2te_1+(1-2x_1+|\mathbf{x}|^2)(t^2e_1+2te_2)+4\mathbf{x}\Big).$$

The group $\{\varphi(t):\ t\in\mathbb{R}\}$ on D^n is a 1-dimensional isometry group which act on D^n freely, yielding the above foliation. If we fix $\mathbf{x}$ and move t in the above expression, we get the leaves of the foliation. ☐

Theorem 5.2 (Classification Theorem)**.** *Except for the unique parabolic homogeneous foliation* (*in Lemma 5.1*)*, the set of all codimension-one totally geodesic foliations of class C^k on the hyperbolic space D^n is in one-one correspondence* (*up to isometry*) *with the set of all functions $z:\ [0,\pi]\to S^{n-1}$ of class C^{k-1} with $z(0)=e_1=z(\pi)$ satisfying*

$$|z'(r)|\leq 1$$

for all r, modulo the action of $O(n-1)\times\mathbb{R}\times\mathbb{Z}_2$, the elliptic and hyperbolic isometries associated with the geodesic axis joining $-e_1$ and $e_1=(1,0,\dots,0)$ and a reflection interchanging e_1 and $-e_1$.

The condition $|z'(r)|\leq 1$ forces the curve $z(r)$ to lie only on one side of a hemisphere of S^{n-1}.

Proof. Since our foliation is not parabolic one, we know $A\neq B$. There exists an isometry which fixes e_1, and maps B to $-e_1$. This means that

$$z(0)=e_1=z(\pi). \tag{5.1}$$

The subgroup of isometries which map the geodesic axis joining e_1 and $-e_1$ onto itself is

$$\mathrm{SO}(n-1)\times\mathbb{R}\times\mathbb{Z}_2.$$

This group acts on the space of functions $\{z:(0,\pi)\longrightarrow S^{n-1}\}$ satisfying the conditions $|z'(r)|\leq 1$ and (5.1).

The isometries in $\mathrm{SO}(n-1)$ are the elliptic rotations around the axis joining e_1 and $-e_1$, as $\left[\begin{smallmatrix}1&0\\0&K\end{smallmatrix}\right]\in\mathrm{SO}(n)$, which is also Euclidean isometries on D^n. By such an isometry, the arc-center functions are rotated around the axis. The isometries in $\mathbb{R}$ are the hyperbolic isometries with axis joining e_1 and $-e_1$. More precisely, let

$$\psi(t)=\begin{bmatrix}\cosh(t)&\sinh(t)&0&0&0\\ \sinh(t)&\cosh(t)&0&0&0\\ 0&0&\cdots&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{bmatrix}\in\mathrm{SO}(1,n).$$

With the isometry $D^n\longrightarrow\mathbb{R}^{1,n}$ (the hyperboloid model) given by

$$(x_1,x_2,\dots,x_n)\mapsto\frac{1}{1-|\mathbf{x}|^2}\left(1+|\mathbf{x}|^2,2x_1,2x_2,\dots,2x_n\right),$$

this is a hyperbolic map, sending $\mathbf{x}$ to

$$\varphi(t): \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \mapsto \frac{2}{\Delta} \begin{bmatrix} x_1 \cosh(t) + \frac{1}{2}(1+|\mathbf{x}|^2)\sinh(t) \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

where $\Delta = 1 - |\mathbf{x}|^2 + (1+|\mathbf{x}|^2)\cosh(t) + 2x_1\sinh(t)$.

The group $\mathbb{Z}_2$ is generated by the reflection about the hyperplane $u_1 = 0$,

$$(x_1, x_2, \ldots, x_n) \leftrightarrow (-x_1, x_2, \ldots, x_n).$$

This interchanges e_1 and $-e_1$. The arc-center function will be changed to $z(\pi - r)$.

Clearly, the actions by $O(n-1)$ and $\mathbb{R}$ commute each other. Also the reflection about $u_1 = 0$ commutes with $O(n-1)\times\mathbb{R}$, and altogether, they form a subgroup isometries $O(n-1)\times\mathbb{R}\times\mathbb{Z}_2$. □

The set of foliations described in the theorem contains only one homogeneous one: Its arc-center function is given by the constant function $z(r) = e_1$.

Remark 5.3. Let $\mathfrak{F}$ be the metric foliation with the arc-center function $z(r)$. The hyperbolic isometry $\varphi(t)$ in the proof of Theorem 5.2 maps the level set $|u - (\sec r)z(r)|^2 = \tan^2 r$ with $r = \frac{\pi}{2}$ to a level set with smaller than r if $t > 0$. Therefore, in the proof of Proposition 2.3, one can apply the argument for the case of $0 < r < \frac{\pi}{2}$ to the new foliation $\varphi(t)(\mathfrak{F})$, and conclude that r is a smooth function of u near the level set for $r = \frac{\pi}{2}$.

According to [4], every 1-dimensional metric foliation on a constant curvature space is either homogeneous or *flat* (that is, the orthogonal distribution is integrable and hence forms a totally geodesic foliation). The only 1-dimensional homogeneous foliations on D^n are the ones generated by a parabolic isometry or by a loxodromic isometry which we know well.

Corollary 5.4. *Every 1-dimensional metric foliation on D^n is homogeneous or orthogonal to the codimension-one totally geodesic foliation classified in Theorem 5.2.*

References

[1] A. Basmajian and G. Walschap, *Metric flows in space forms of nonpositive curvature*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3177–3181.

[2] G. Cairns and R. H. Escobales, Jr., *Note on a theorem of Gromoll-Grove*, Bull. Austral. Math. Soc. **55** (1997), no. 1, 1–5.

[3] V. A. Efremovič and E. M. Gorelik, *Metric fibrations of Lobachevsky-Bolyai space*, Differential geometry and its applications (Eger, 1989), 223–230, Colloq. Math. Soc. Janos Bolyai, 56, North-Holland, Amsterdam, 1992.

[4] D. Gromoll and K. Grove, *One-dimensional metric foliations in constant curvature spaces*, Differential geometry and complex analysis, 165–168, Springer, Berlin, 1985.

[5] D. Gromoll and K. Tapp, *Nonnegatively curved metrics on $S^2 \times \mathbb{R}^2$*, Geom. Dedicata **99** (2003), 127–136.

[6] D. Gromoll and G. Walschap, *The metric fibrations of Euclidean space*, J. Differential Geom. **57** (2001), no. 2, 233–238.
[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. II*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.
[8] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics, 91. Cambridge University Press, Cambridge, 2003.
[9] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
[10] P. Tondeur, *Foliations on Riemannian Manifolds*, Springer, New York, 1998.
[11] G. Walschap, *Geometric vector fields on Lie groups*, Differential Geom. Appl. **7** (1997), no. 3, 219–230.
[12] S. Wolfram, *Mathematica*, Wolfram Research, 1993.
[13] A. Zeghib, *Lipschitz regularity in some geometric problems*, Geom. Dedicata **107** (2004), 57–83.

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