

METRIC PROPERTIES OF MANIFOLDS BIMEROMORPHIC TO COMPACT KÄHLER SPACES

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Introduction

A goal of this paper is to prove that: “Every compact complex manifold M bimeromorphic to a compact Kähler manifold M' is balanced; that is, M has a hermitian metric with Kähler form ω such that $d\omega^{N-1} = 0$, $N = \dim M$ ” (Corollary 4.5). Of course, every Kähler manifold is balanced; the interest of the above result stems from the fact that we find out a metric property which transfers from M' to M , while it is well known that the Kähler property is not stable under bimeromorphic maps.

This introduction is mainly devoted to outline the background.

Let M and \widetilde{M} be compact complex manifolds and $f: \widetilde{M} \rightarrow M$ be a modification. It is well known that:

- (1) If f is a blow-up of M with smooth center and M is Kähler, then \widetilde{M} is Kähler too [4],

however

- (2) in general, if f is a modification and M is Kähler, \widetilde{M} fails to be Kähler.

A counterexample is given in [12, p. 505] by a compact non-Kähler threefold X and a modification $f: X \rightarrow \mathbf{P}_3$. In order to illustrate Chow's lemma, Hironaka builds up also a projective threefold Y and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & \searrow h & \downarrow f \\ & & \mathbf{P}_3 \end{array}$$

where g and h are obtained as a finite sequence of blow-ups with smooth centers.

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Let us consider for a moment the threefold X . Since a compact Kähler manifold cannot contain any complex curve homologous to zero, but X contains such a curve (see [17, Chapter VIII,3.3.] or [9, p. 444]), it is not Kähler. On the other hand, a compact balanced manifold contains no hypersurfaces homologous to zero and neither does X by construction: so X could be balanced. This property of compact balanced manifolds has a weak converse, if you look at hypersurfaces as positive currents of degree (1,1). Indeed Michelson proved that a compact complex manifold is balanced if and only if it carries no positive currents of degree (1, 1) which are components of a boundary [16, Proposition 4.5]. This result suggested that X is balanced, as we proved in [1]; yet it is only a particular case of the following general statement:

(3) If M is Kähler, then \widetilde{M} is balanced [2].

Balanced manifolds have been studied from a differential point of view in [6]; other results and examples can be found of course in [16]. (3) shows how balanced manifolds can be “produced” in a very natural way by using modifications. Besides, we can also “pull back” the property of having a balanced metric (not the balanced metric itself, in general!) as is shown in [3]:

(3') If M is balanced, \widetilde{M} is balanced too.

(3') proves that going from M to \widetilde{M} via f no new hypersurface (and also positive current) which is the component of a boundary can appear (on the contrary, new curves may appear as $f: X \rightarrow P_3$ shows).

Thus the following question arises naturally: Is the class of compact balanced manifolds invariant by modifications? In other words, can statement (3') be reversed: If \widetilde{M} is balanced, is M balanced too? The problem looks interesting even in the simplest case because, as the modification $g: Y \rightarrow X$ shows,

(4) even if f is a blow-up of M with smooth center and \widetilde{M} is Kähler, in general M fails to be Kähler.

In this paper we prove that for a generic modification $f: \widetilde{M} \rightarrow M$

(5) if \widetilde{M} is Kähler, then M is balanced.

The proof of (5) depends heavily on the quoted result of [16] and on our Theorem 3.9: “Suppose M and \widetilde{M} are complex manifolds (not necessarily compact) and $f: \widetilde{M} \rightarrow M$ is a proper modification. If T is a positive

$\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$, then there exists a positive $\partial\bar{\partial}$ -closed current \tilde{T} on \tilde{M} of degree $(1, 1)$ such that $f_*\tilde{T} = T$. Moreover if \tilde{M} is compact, such a current is unique.”

Since locally $\partial\bar{\partial}$ -closed currents are components of a boundary, it is convenient to translate the condition in [16] in terms of Aeppli cohomology groups (see (1.3)). To generalize (5), we shall introduce a cohomological condition, called (B), which holds in particular for compact Kähler manifolds. In Theorem 3.9, if \tilde{M} satisfies (B), the cohomology class $[\tilde{T}]$ of \tilde{T} is exactly $f^*[T]$ (see Proposition 3.10). So we get the following converse of (3'):

(5') (Main Theorem 4.2) If \tilde{M} is balanced and satisfies (B),
then M is balanced and satisfies (B) too.

Now Corollary 4.5 announced at the beginning is simply a consequence of (5) and of a theorem of Varouchas [22].

As one can see in the literature, the most interesting case is that of compact complex manifolds which are bimeromorphic to projective varieties, that is, Moishezon manifolds. Namely, let M be a Moishezon manifold: If \tilde{M} is projective and $f: \tilde{M} \rightarrow M$ is a modification, it is difficult to find smooth objects on M coming from \tilde{M} . For instance, if $\tilde{\omega}$ is a Kähler form on \tilde{M} , then $f_*\tilde{\omega}$ is not smooth: its coefficients are in L_{loc}^1 . Moreover, if L is a positive line bundle on \tilde{M} , although f_*L is a holomorphic line bundle on the whole of M [18], it is not, in general, positive (for a survey see [23]).

Therefore our techniques based on positive, $\partial\bar{\partial}$ -closed currents seem to be more appropriate and allow us to assert that every Moishezon manifold carries a balanced metric.

Finally notice that Michelson’s characterization theorem is not constructive, therefore if \tilde{M} and M are balanced, the results (3') and (5') do not give any information about the link between balanced metrics on \tilde{M} and on M . Nevertheless, we shall prove that: “For every balanced metric h on M with Kähler form ω there exists a balanced metric \tilde{h} on \tilde{M} with Kähler form $\tilde{\omega}$ such that $\omega^{N-1} - f_*\tilde{\omega}^{N-1}$ is a $\partial\bar{\partial}$ -exact current.” This is a corollary of Theorem 4.8: “Let M and \tilde{M} be p -Kähler manifolds, let $f: \tilde{M} \rightarrow M$ be a proper modification and call Y the degeneracy set, with $p > \dim Y$. For every p -Kähler form Ω on M , there exists a p -Kähler form $\tilde{\Omega}$ on \tilde{M} such that $\Omega - f_*\tilde{\Omega}$ is a $\partial\bar{\partial}$ -exact current.”

1. Preliminaries and notation

(1.1) Throughout the paper, whether explicitly stated or not, \widetilde{M} and M are assumed to be complex N -dimensional manifolds. A *proper modification* $f: \widetilde{M} \rightarrow M$ is a proper holomorphic map such that, for a suitable analytic set Y in M , $E := f^{-1}(Y)$ (the *exceptional set of the modification*) is a hypersurface and $\widetilde{M} - E \xrightarrow{f} M - Y$ is a biholomorphism. Moreover, Y has codimension ≥ 2 or f is a biholomorphism.

In particular, if Y is smooth and f is the blow-up of M along Y , suitable coordinates can be chosen in M as follows. (As usual, $B_k(z^\circ, r)$ denotes the euclidean open ball in \mathbf{C}^k with center z° and radius r . $B_k(0, r)$ is simply denoted by $B_k(r)$ and $B_k(1)$ is denoted by B_k . Take $B_0 := \{0\}$.) For every $y \in Y$ take an open neighborhood $U = B_m \times B_n$ ($m := \dim Y$ and $N := m + n$) such that $U \cap Y = B_m \times \{0\}$. Call $\widetilde{U} := f^{-1}(U)$: We can identify

$$f|_{\widetilde{U}}: \widetilde{U} \rightarrow U \quad \text{with the natural projection } \pi: B_m \times \widetilde{B}_n \rightarrow B_m \times B_n,$$

where \widetilde{B}_n denotes the blow-up of B_n at $\{0\}$. In the following we shall simply say: “identify locally f with π .”

Recall that if $(t_1, \dots, t_m) \in \mathbf{C}^m$ and $(z_1, \dots, z_n) \in \mathbf{C}^n$,

$$\widetilde{B}_n = \{(z, \xi) \in B_n \times \mathbf{P}^{n-1} \mid z_j \xi_k = z_k \xi_j, 1 \leq j, k \leq n\}$$

so that the natural Kähler form on \widetilde{U} is given by

$$\tilde{\omega} := \frac{i}{2} \partial \bar{\partial} \|t\|^2 + \frac{i}{2} \partial \bar{\partial} \|z\|^2 + \frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2;$$

therefore (see [23, p. 37])

$$\pi_* \tilde{\omega} = \frac{i}{2} \partial \bar{\partial} \|t\|^2 + \frac{i}{2} \partial \bar{\partial} \|z\|^2 + \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2.$$

(1.2) As usual, $\mathcal{E}^{k,k}(M)_{\mathbf{R}}$ (resp. $\mathcal{D}^{k,k}(M)_{\mathbf{R}}$) denotes the space of smooth real (k, k) -forms (resp. smooth real (k, k) -forms with compact support) on M . Their duals are spaces of real currents of bidimension (k, k) (or of degree $(N-k, N-k)$), i.e., real $(N-k, N-k)$ -forms with distribution coefficients.

Let $\varphi \in \mathcal{E}^{k,k}(M)_{\mathbf{R}}$. In local coordinates, we shall often write

$$\begin{aligned} \varphi &= \frac{ik^2}{2^k} \sum_{1 \leq \alpha_1 < \dots < \alpha_k \leq N, 1 \leq \beta_1 < \dots < \beta_k \leq N} \varphi_{\alpha_1, \dots, \alpha_k \bar{\beta}_1, \dots, \bar{\beta}_k}(z) dz_{\alpha_1} \\ &\quad \wedge \dots \wedge dz_{\alpha_k} \wedge d\bar{z}_{\beta_1} \wedge \dots \wedge d\bar{z}_{\beta_k} \\ &= \sigma_k \sum'_{|A|=|B|=k} \varphi_{A\bar{B}} dz_A \wedge d\bar{z}_B, \end{aligned}$$

where \sum' denotes the sum on strictly increasing multi-indices.

A real current T on M of bidimension (k, k) is called *positive* (in the sense of Lelong [15]) if, for every choice of $\varphi_1, \dots, \varphi_k \in \mathcal{D}^{1,0}(M)_{\mathbb{R}}$, $T(\sigma_k \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_k) \geq 0$. Moreover T is said to be *strictly positive* if $\varphi_1 \wedge \cdots \wedge \varphi_k \neq 0$ implies $T(\sigma_k \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_k) > 0$. It is well known that a positive current is of order zero. We shall denote by $\|T\|$ the mass of T . A smooth form $\psi \in \mathcal{E}^{N-k, N-k}(M)_{\mathbb{R}}$ is *positive* (resp. *strictly positive*) if the associated current T_ψ , defined as

$$T_\psi(\varphi) = \int \varphi \wedge \psi \quad \forall \varphi \in \mathcal{D}^{k,k}(M)_{\mathbb{R}},$$

is a positive (resp. strictly positive) current.

If X is a p -dimensional irreducible analytic subset of M , we shall denote by $[X]$ the positive current defined as

$$[X](\varphi) := \int_X \varphi \quad \forall \varphi \in \mathcal{D}^{p,p}(M)_{\mathbb{R}}.$$

It is well known that $[X]$ is closed; moreover, if $u: X \rightarrow \mathbb{R}$ is a pluriharmonic function, $u[X]$ is a $\partial\bar{\partial}$ -closed current.

(1.3) As regards the statement of the Main Theorem, we recall here the definition of balanced manifold and define condition (B).

1.1. Definition. Let M be a complex N -dimensional manifold. M is said to be *balanced* (or semi-Kähler) if there exists a hermitian metric h on M , called the *balanced metric*, such that its Kähler form ω satisfies $d\omega^{N-1} = 0$.

This class of manifolds obviously includes that of Kähler manifolds (for $N = 2$ they coincide) but also many important classes of non-Kähler manifolds, such as the complex solvmanifolds, twistor spaces of oriented riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [16]), hermitian compact manifolds which are locally flat [6], manifolds obtained as modifications of compact Kähler manifolds [2]. As well as in the Kähler case (see [11]), there exists an intrinsic characterization of compact balanced manifolds by means of positive currents.

1.2. Theorem [16, Theorem 4.5]. *Suppose M is a compact complex manifold. The following conditions are equivalent:*

(i) M is balanced.

(ii) If T is a positive current on M of degree $(1, 1)$ which is the component of a boundary (i.e., there exists a current S such that $T = \bar{\partial}S + \partial\bar{S}$), then $T = 0$.

Let us say a few words on currents which are components of boundaries. If T is a current of bidimension $(N-1, N-1)$ which is the component of

a boundary, or, more generally, which is a (weak) limit of currents which are components of boundaries, then $\partial\bar{\partial}T = 0$ and moreover $T(\varphi) = 0$ for every closed $\varphi \in \mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}}$; that is, if we consider the operator

$$d: \mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}} \rightarrow (\mathcal{D}^{N, N-1}(M) \oplus \mathcal{D}^{N-1, N}(M))_{\mathbf{R}}$$

and its dual

$$(\partial \oplus \bar{\partial}): (\mathcal{D}^{N, N-1}(M) \oplus \mathcal{D}^{N-1, N}(M))_{\mathbf{R}}' \rightarrow (\mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}})',$$

then $(\text{Ker } d)^{\perp} = \overline{\text{Im}(\partial \oplus \bar{\partial})}$.

In [16, Lemma 4.8] it is proved that, if M is compact, $\text{Im}(\partial \oplus \bar{\partial})$ is a closed subspace of $\text{Ker } i\partial\bar{\partial}$; hence every current which is the limit of currents which are components of boundaries is the component of a boundary itself. Nevertheless, we shall work mainly in the noncompact case.

The *real* $(1, 1)$ -Aeppli groups are defined as follows:

$$V^{1,1}(M)_{\mathbf{R}} = \frac{\text{Ker}(i\partial\bar{\partial}: \mathcal{E}^{1,1}(M)_{\mathbf{R}} \rightarrow \mathcal{E}^{2,2}(M)_{\mathbf{R}})}{(\partial\mathcal{E}^{0,1}(M) + \bar{\partial}\mathcal{E}^{1,0}(M))_{\mathbf{R}}}$$

and

$$\Lambda^{1,1}(M)_{\mathbf{R}} = \frac{\text{Ker}(d: \mathcal{E}^{1,1}(M)_{\mathbf{R}} \rightarrow (\mathcal{E}^{2,1}(M) + \mathcal{E}^{1,2}(M))_{\mathbf{R}})}{i\partial\bar{\partial}\mathcal{E}^{0,0}(M)_{\mathbf{R}}}.$$

As usual, we shall denote by $H^{1,1}(M, \mathbf{R})$ the set of classes in $H^2(M, \mathbf{R})$ which have a $(1, 1)$ -representative. It is well known that all these groups can be defined also by means of real currents of degree $(1, 1)$. Thus a $\partial\bar{\partial}$ -closed current T is the component of a boundary if and only if its class in $V^{1,1}(M)_{\mathbf{R}}$ is zero.

A class of $V^{1,1}(M)_{\mathbf{R}}$ is said to be *positive* if it can be represented by a positive current: hence Theorem 1.2 can be written as follows: “ M is balanced if and only if every nonzero positive $\partial\bar{\partial}$ -closed current of degree $(1, 1)$ represents a nonzero class in $V^{1,1}(M)_{\mathbf{R}}$.” Finally, let us consider the natural maps:

$$\begin{aligned} \alpha: \Lambda^{1,1}(M)_{\mathbf{R}} &\rightarrow H^{1,1}(M, \mathbf{R}), \\ \beta: H^{1,1}(M, \mathbf{R}) &\rightarrow V^{1,1}(M)_{\mathbf{R}} \end{aligned}$$

and the following condition:

(B) β is injective and $\text{Im } \beta$ contains all positive elements of $V^{1,1}(M)_{\mathbf{R}}$.

1.3. Proposition. *If $\beta \circ \alpha: \Lambda^{1,1}(M, \mathbb{R})_{\mathbb{R}} \rightarrow V^{1,1}(M)_{\mathbb{R}}$ is an isomorphism, then α and β are isomorphisms. In particular, every compact Kähler manifold satisfies (B).*

Proof. It is enough to notice that α is always surjective. Moreover, if M is regular (in particular, if it is Kähler or Moishezon or in the class \mathcal{C} of Fujiki), then $\beta \circ \alpha$ is an isomorphism (see [21] for the definition of regular manifold and its cohomological properties).

2. $\partial\bar{\partial}$ -closed currents and pluriharmonic functions

We study in this section the behaviour of real $\partial\bar{\partial}$ -closed currents of degree $(1, 1)$ and of order zero, whose support is contained in the exceptional set of a proper modification. If the support is “too small” (that is, it is contained in an analytic subset of dimension $< N - 1$), the current vanishes (see Theorem 2.1; if T is also positive see [2, Theorem 1.5]). On the other hand, if the modification $f: \widetilde{M} \rightarrow M$ is obtained as a finite sequence of blow-ups with smooth centers, we get a characterization of the set of currents described above: “Every real $\partial\bar{\partial}$ -closed current T of degree $(1, 1)$ and order zero supported in the exceptional set E of f is of the form $\sum_{\alpha} u_{\alpha}[E_{\alpha}]$, where $\{E_{\alpha}\}$ is the set of irreducible components of E and u_{α} is a pluriharmonic function on E_{α} ” (Proposition 2.5). These results are well known for locally flat currents, but we are not in this case, as Remark 2.4 shows.

We get moreover that if such a current is limit of currents which are components of boundaries, then it vanishes. This holds also in a weaker form if f is a generic proper modification (Proposition 2.7).

2.1. Theorem. *Let Ω be an open set in \mathbb{C}^N , and suppose T is a real current of bidimension (p, p) and of order zero on Ω such that $\partial\bar{\partial}T = 0$. If the support of T is contained in an analytic subset Y of Ω of dimension $q < p$, then $T = 0$.*

Proof. Let $x \in \text{Reg } Y$; in a neighborhood U of x choose coordinates (z_1, \dots, z_N) such that

$$Y \cap U = \{z \in U \mid z_j = 0 \text{ for } j = q + 1, \dots, N\}.$$

Call $z' = (z_1, \dots, z_q)$ and $z'' = (z_{q+1}, \dots, z_N)$. In U ,

$$T = \sigma_{N-p} \sum'_{|A|=|B|=N-p} T_{A\bar{B}} dz_A \wedge d\bar{z}_B$$

where the measures $t_{A\bar{B}}$ can be written as

$$t_{A\bar{B}}(z) = r_{A\bar{B}}(z') \otimes \delta(z'')$$

because $\text{supp } T \subset Y$ (see for instance [14], p. 47).

Call $I = \{q+1, \dots, N\}$: Since $q < p$, $A \supsetneq I$ and $B \supsetneq I$ for all strictly increasing $(N-p)$ -indices A and B . Choose $\alpha \in I \setminus A$, $\beta \in I \setminus B$ and let $A' = A \cup \{\alpha\}$, $B' = B \cup \{\beta\}$ (arrange indices in increasing order). Compute $i\partial\bar{\partial}T$, and notice that, in the coefficient of $dz_{A'} \wedge d\bar{z}_{B'}$, the only addendum containing $\partial_{\alpha\bar{\beta}}^2 \delta$ is

$$r_{AB}(z') \otimes \partial_{\alpha\bar{\beta}}^2 \delta(z'').$$

As $\partial\bar{\partial}T = 0$, we get $r_{AB} = 0$. Therefore $\text{supp } T \subseteq \text{Sing } Y$, and we get the result by induction on the dimension of Y . q.e.d.

The next result is a vanishing lemma based on the Kodaira-Nakano Vanishing Theorem.

2.2. Lemma. *Let $f: \tilde{M} \rightarrow M$ be the blow-up of M along a submanifold Y . Then $H^0(E, \Omega_E^1(N_{E|\tilde{M}})) = 0$, where $N_{E|\tilde{M}}$ is the normal bundle of the exceptional set E .*

Proof. Identify f locally with the blow-up π as said in (1.1). Take $y \in U \cap Y = B_m$ and identify the singular fibre $\pi^{-1}(y)$ with \mathbf{P}_{n-1} . Call $E_U := E \cap U$. Let us recall some easy facts about normal and conormal bundles:

(i) The conormal bundle N^* is defined by the following exact sequence of vector spaces, for $x \in \mathbf{P}_{n-1}$:

$$0 \rightarrow N_{\mathbf{P}_{n-1}|E_{U,x}}^* \rightarrow T_{E_{U,x}}'^* \rightarrow T_{\mathbf{P}_{n-1,x}}'^* \rightarrow 0.$$

(ii) Since $E_U = B_m \times \mathbf{P}_{n-1}$, the conormal bundle $N_{\mathbf{P}_{n-1}|E_U}^*$ is trivial.

(iii) $N_{\mathbf{P}_{n-1}|\tilde{B}_n}^* = [\mathbf{P}_{n-1}]|_{\mathbf{P}_{n-1}} = [-H]$ (notations are the standard ones, see, e.g., [8]).

From (i) we get

$$(2.1) \quad \begin{aligned} 0 \rightarrow \mathcal{O}(N_{\mathbf{P}_{n-1}|E_U}^*) \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) &\rightarrow \Omega_{E_U|_{\mathbf{P}_{n-1}}}^1 \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) \\ &\rightarrow \Omega_{\mathbf{P}_{n-1}}^1 \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) \rightarrow 0 \end{aligned}$$

and from (ii) and (iii) we infer that

$$\mathcal{O}(N_{\mathbf{P}_{n-1}|E_U}^*) \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) = \mathcal{O}(-1).$$

Hence the long exact sequence of cohomology groups of (2.1) starts with

$$\begin{aligned} 0 \rightarrow H^0(\mathbf{P}_{n-1}, \mathcal{O}(-1)) &\rightarrow H^0(\mathbf{P}_{n-1}, \Omega_{E_U|_{\mathbf{P}_{n-1}}}^1 \otimes \mathcal{O}(-1)) \\ &\rightarrow H^0(\mathbf{P}_{n-1}, \Omega_{\mathbf{P}_{n-1}}^1(-1)) \rightarrow \dots \end{aligned}$$

From the Kodaira-Nakano Vanishing Theorem (and some easy facts about Riemann surfaces for $n = 2$), we get

$$H^0(\mathbf{P}_{n-1}, \Omega^1(-1)) = H^0(\mathbf{P}_{n-1}, \mathcal{O}(-1)) = 0;$$

thus $H^0(\mathbf{P}_{n-1}, \Omega_{E_U}^1|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(-1)) = 0$.

Let $h \in H^0(E, \Omega_{E_U}^1(N_{E_U|\widetilde{U}}))$. Since

$$N_{E_U|\widetilde{U}}|_{\mathbf{P}_{n-1}} = N_{\mathbf{P}_{n-1}|\widetilde{B}_n},$$

$h|_{\mathbf{P}_{n-1}}$ is a section of $\Omega_{E_U}^1|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) = \Omega_{E_U}^1|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(-1)$ (by iii)). Thus $h|_{\mathbf{P}_{n-1}} = 0$; i.e. h , restricted to a generic fibre $\pi^{-1}(y)$, is zero. This achieves the proof. q.e.d.

Now we are ready to prove the following result.

2.3. Theorem. *Let $f: \widetilde{M} \rightarrow M$ be the blow-up of M along a submanifold Y . If T is a real $\partial\bar{\partial}$ -closed current on \widetilde{M} of order zero and degree $(1, 1)$ whose support is contained in the exceptional set E , then there exists a pluriharmonic function $h: Y \rightarrow \mathbf{R}$ such that*

$$T = (h \circ f)[E].$$

Moreover, if T is a (weak) limit of currents which are components of boundaries, then $T = 0$.

Proof. Let us fix a coordinate neighborhood $(V, v_1, \dots, v_N) = (V, (v', v_N))$ in \widetilde{M} such that $V \cap E = \{v_N = 0\}$. In V , T has the following expression:

$$T = \frac{i}{2} \sum_{\alpha, \beta=1}^N t_{\alpha\bar{\beta}}(v) dv_\alpha \wedge d\bar{v}_\beta.$$

As $\text{supp } T \subseteq E$,

$$t_{\alpha\bar{\beta}}(v) = r_{\alpha\bar{\beta}}(v') \otimes \delta(v_N)$$

where $r_{\alpha\bar{\beta}}$ is a measure and $r_{\bar{\beta}\alpha} = \overline{r_{\alpha\bar{\beta}}}$.

Fix $\alpha, \beta < N$ and compute $i\partial\bar{\partial}T$. The coefficient of $dv_\alpha \wedge dv_N \wedge d\bar{v}_\beta \wedge d\bar{v}_N$, which has to vanish, is given by

$$\begin{aligned} & -r_{\alpha\bar{\beta}}(v') \otimes \partial_{N\bar{N}}^2 \delta(v_N) + \partial_\alpha r_{N\bar{\beta}}(v') \otimes \partial_{\bar{N}} \delta(v_N) + \partial_{\bar{\beta}} r_{\alpha\bar{N}}(v') \otimes \partial_N \delta(v_N) \\ & - \partial_{\alpha\bar{\beta}}^2 r_{N\bar{N}}(v') \otimes \delta(v_N). \end{aligned}$$

Hence we conclude that

$$(2.2) \quad \begin{cases} r_{\alpha\bar{\beta}} = 0 & \text{for } 1 \leq \alpha, \beta < N, \\ r_{\alpha\bar{N}} \text{ is holomorphic} & \text{for } 1 \leq \alpha < N, \\ r_{N\bar{N}} \text{ is pluriharmonic.} & \end{cases}$$

Let us now check what happens in another chart. Choose another coordinate neighborhood $(W, w_1, \dots, w_N) = (W, w', w_N)$ with $W \cap V \neq \emptyset$ and $W \cap E = \{w_N = 0\}$. Assume

$$T = \frac{i}{2} \sum_{\lambda, \mu=1}^N s_{\lambda\bar{\mu}}(w') \otimes \delta(w_N) dw_\lambda \wedge d\bar{w}_\mu \quad \text{in } W,$$

by (2.2) and similar results for $\{s_{\lambda\bar{\mu}}\}$, and using the fact that $\partial v_N / \partial w_\lambda = \partial w_N / \partial v_\alpha = 0$ on E for $\alpha, \lambda < N$, we obtain the following relations:

$$r_{\alpha\bar{N}}(v') = \sum_{\lambda=1}^{N-1} s_{\lambda\bar{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_\alpha} \quad \text{if } \alpha < N,$$

and

$$\begin{aligned} r_{N\bar{N}}(v') &= \sum_{\lambda=1}^{N-1} s_{\lambda\bar{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_N} \\ &\quad + \sum_{\mu=1}^{N-1} \overline{s_{\mu\bar{N}}(w'(v'))} \frac{\overline{\partial v_N}}{\partial w_N} \frac{\overline{\partial w_\mu}}{\partial v_N} + s_{N\bar{N}}(w'(v')). \end{aligned}$$

But now it is a matter of course that, if we cover \widetilde{M} by charts of type (V, v', v_N) , then $\{r_{1\bar{N}}, \dots, r_{n-1\bar{N}}\}$ is nothing but a global section of $\Omega_E^1(N_E|_{\widetilde{M}})$ on E : Indeed, the cocycles of $N_E|_{\widetilde{M}}$ are given by $\partial v_N / \partial w_N$. By Lemma 2.2 we get $r_{\alpha\bar{N}} = s_{\lambda\bar{N}} = 0$, and $r_{N\bar{N}} = s_{N\bar{N}} =: h$ is a pluriharmonic map on \widetilde{M} . Since the fibers of f are compact, h depends only on the coordinates of Y .

To get the second part of the statement, by (1.3) we need only to prove that if $h: Y \rightarrow \mathbb{R}$ is not identically zero, then there exists a closed form $\Theta \in \mathcal{D}^{N-1, N-1}(\widetilde{M})_{\mathbb{R}}$ such that

$$T(\Theta) = \int_E (h \circ f) \Theta \neq 0.$$

Choose $y \in Y$ such that $h(y) \neq 0$ (suppose $h(y) > 0$); choose an open neighborhood U of y , biholomorphic to $B_m \times B_n$ and such that $U \cap Y \cong B_m \times \{0\}$ and $h > 0$ in $U \cap Y$; then identify $f|_{f^{-1}(U)}$ with the blow-up π . Take real functions u and v as follows:

$$\begin{aligned} u &\in \mathcal{C}_0^\infty(B_m), \quad u \neq 0, \quad u(t) \geq 0, \\ v &\in \mathcal{C}_0^\infty(B_n), \quad v(z) = 1 \text{ near the origin.} \end{aligned}$$

Then assume

$$\begin{aligned}\psi(z) &= \frac{i}{2\pi} \partial \bar{\partial}((1 - v(z)) \log \|z\|^2), \\ \Theta &= \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 - \pi^* \psi \right)^{n-1} \wedge u(t) \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m.\end{aligned}$$

It is straightforward to verify that Θ satisfies what is required. In particular, if $i: E \rightarrow f^{-1}(U)$ is the inclusion, $i^* \Theta = (\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2)^{n-1} \wedge u(t) (\frac{i}{2} \partial \bar{\partial} \|t\|^2)^m$; hence

$$(h \circ f)[E](\Theta) = \int_E (h \circ f \circ i)(i^* \Theta) = c \int_{B_m} h(t) u(t) \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m > 0.$$

2.4. Remark. An example of a $\partial \bar{\partial}$ -closed current T of order zero which is not locally flat is given here. Let us use the same notation of the previous theorem, and define

$$T = \frac{i}{2} c \delta(v_N)(dv_1 \wedge d\bar{v}_N + dv_N \wedge d\bar{v}_1), \quad c \in \mathbb{R}.$$

T is a real $\partial \bar{\partial}$ -closed current of degree $(1, 1)$ and order zero, and $\text{supp } T \subseteq \{v_N = 0\}$. By the Support Theorem (see [10, Theorem 1.7]), if T were locally flat, there would exist a pluriharmonic function $h: E \rightarrow \mathbb{R}$ such that $T = h[E]$, but this is not the case, if $c \neq 0$.

2.5. Proposition. *Let $f: \widetilde{M} \rightarrow M$ be a proper modification which is obtained as a finite sequence of blow-ups with smooth centers. Call $\{E_\alpha\}$ the set of the irreducible components of the exceptional set E . Then:*

(i) *Every real $\partial \bar{\partial}$ -closed current T of order zero and degree $(1, 1)$ on \widetilde{M} supported in E is of the form $\sum_\alpha u_\alpha [E_\alpha]$, where u_α is a pluriharmonic function on E_α .*

(ii) *Moreover, T is a (weak) limit of currents which are components of boundaries if and only if every u_α vanishes.*

Proof. By our assumption there is a finite sequence

$$f_j: V_{j+1} \rightarrow V_j, \quad 0 \leq j < r,$$

of blow-ups with smooth centers $Y_j \subseteq V_j$ and exceptional sets $E'_{j+1} \subseteq V_{j+1}$ such that $V_0 = M$, $V_r = \widetilde{M}$, $f = f_0 \circ \cdots \circ f_{r-1}$. By Theorem 2.3 we get

$$(f_1 \circ \cdots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1] \quad \text{with } u_1: Y_0 \rightarrow \mathbb{R} \text{ pluriharmonic.}$$

Let \tilde{E}'_1 be the strict transform of E'_1 under f_1 ; $(u_1 \circ f_0 \circ f_1)[\tilde{E}'_1]$ is $\partial \bar{\partial}$ -closed. Therefore we can apply Theorem 2.3 again to obtain

$$(2.3) \quad (f_2 \circ \cdots \circ f_{r-1})_* T - (u_1 \circ f_0 \circ f_1)[\tilde{E}'_1] = (u_2 \circ f_1)[E'_2]$$

with $u_2: Y_1 \rightarrow \mathbf{R}$ pluriharmonic. Eventually we get

$$T = \sum_{j=1}^r (u_j \circ f_{j-1} \circ \cdots \circ f_{r-1})[\tilde{E}'_j],$$

where \tilde{E}'_j is the strict transform of E'_j via $f_{j-1} \circ \cdots \circ f_{r-1}$, $1 \leq j < r$, $\tilde{E}'_r = E'_r$ and $u_j: Y_{j-1} \rightarrow \mathbf{R}$ is pluriharmonic.

(ii) Suppose T is a limit of components of boundaries. Then

$$(f_1 \circ \cdots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1]$$

is limit of components of boundaries too so that by Theorem 2.3 $u_1 = 0$. From (2.3) and Theorem 2.3 we infer $u_2 = 0$, and so on. q.e.d.

Let us consider now a generic proper modification $f: \tilde{M} \rightarrow M$. The following lemma is essentially contained in [13] to which we refer step by step.

2.6. Lemma. *Let $f: \tilde{M} \rightarrow M$ be a proper modification; for every $x \in M$ there exist an open neighborhood V of x in M , a complex manifold Z and holomorphic maps $g: Z \rightarrow \tilde{M}$, $h: Z \rightarrow V$ such that $h = f \circ g$. Moreover $g: Z \rightarrow f^{-1}(V)$ is a blow-up, and $h: Z \rightarrow V$ is obtained as a finite sequence of blow-ups with smooth centers.*

Proof. Locally, f is dominated by a blow-up; that is, [13, Lemma 8, p. 321] for every $x \in M$ there exist an open neighborhood V of x in M and a complex subspace $(D, \tilde{\mathcal{O}}_D)$ of V such that, if $h': V' \rightarrow V$ is the blow-up with center $(D, \tilde{\mathcal{O}}_D)$, then there exists a holomorphic map $g': V' \rightarrow \tilde{M}$ with $h' = f \circ g'$. Let \mathcal{I} be the coherent ideal sheaf in \mathcal{O}_V which defines the complex space $(D, \tilde{\mathcal{O}}_D)$; by applying Lemma 7 [13, p. 320] to V and \mathcal{I} we get a suitable finite sequence

$$h_j: V_{j+1} \rightarrow V_j, \quad 0 \leq j < r,$$

of blow-ups with smooth centers such that $Z := V_r$ is smooth, and if $h := h_0 \circ \cdots \circ h_{r-1}$, $h^{-1}(\mathcal{I})$ is invertible (see the remark after Lemma 7 in [13]). Shrinking V we can also suppose $V_0 = V$. Therefore, by means of the universal property of blow-ups [13, Definition 1, p. 315], we get a holomorphic map $g'': Z \rightarrow V'$ such that $h = h' \circ g''$. If $g := g' \circ g'': Z \rightarrow \tilde{M}$, then $h = f \circ g$ and $g: Z \rightarrow f^{-1}(V)$ becomes a blow-up since $h: Z \rightarrow V$ is obtained as a finite sequence of blow-ups and $f: f^{-1}(V) \rightarrow V$ is a proper modification (see Corollary 1, p. 320 and Lemma 4, p. 318 of [13]). q.e.d.

We can prove now the last result of this section.

2.7. Proposition. *Let $f: \widetilde{M} \rightarrow M$ be a proper modification and let $\{E_\alpha\}$ be the set of irreducible components of the exceptional set E . If $T = \sum_\alpha c_\alpha [E_\alpha]$, $c_\alpha \in \mathbf{R}$, and for every $x \in M$ there exists an open neighborhood V of x such that*

(2.4) *$T|_{f^{-1}(V)}$ is a (weak) limit of currents which are components of boundaries,*

then $c_\alpha = 0 \forall \alpha$.

Proof. Fix α° and choose $x \in f(E_{\alpha^\circ})$. For a suitable open neighborhood V of x in M we get holomorphic maps $g: Z \rightarrow f^{-1}(V)$ and $h: Z \rightarrow V$ as in the previous lemma. Now $T|_{f^{-1}(V)} = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ where $\{E'_{\alpha,j}\}$ is the set of connected components of $E_\alpha \cap f^{-1}(V)$. Let $\{F_\beta\}$ be the set of irreducible components of the exceptional set of $g: Z \rightarrow f^{-1}(V)$, and denote the strict transform of $E'_{\alpha,j}$ under g by $\tilde{E}'_{\alpha,j}$. Thus $\{F_\beta\} \cup \{\tilde{E}'_{\alpha,j}\}$ is the set of irreducible components of the exceptional set of $h: Z \rightarrow V$, and therefore the total transform \hat{T} of $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ under g is of the form $\hat{T} = \sum_{\alpha,j} c_\alpha [\tilde{E}'_{\alpha,j}] + \sum_\beta c'_\beta [F_\beta]$. By Proposition 2.5 we need only to prove that $\hat{T}|_{h^{-1}(V)}$ satisfies (2.4).

Let $\varphi \in \mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$ be a representative of the fundamental class of $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ in $H^2(f^{-1}(V), \mathbf{R})$; i.e., $\varphi = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] + dQ$ for a suitable current Q in $f^{-1}(V)$. Then $g^*\varphi$ represents the fundamental class of the total transform \hat{T} ; i.e.,

$$(2.5) \quad g^*\varphi = \hat{T} + dQ'$$

for a suitable current Q' in Z .

The hypothesis (2.4) provides a sequence $\{R_\mu\}$ of $(1,0)$ -currents in $f^{-1}(V)$ such that

$$\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] = \lim_\mu (\bar{\partial} R_\mu + \partial \bar{R}_\mu) \quad (\text{weakly}).$$

By smoothing R_μ and Q we get

$$\varphi = \lim_\mu (\bar{\partial} \rho_\mu + \partial \bar{\rho}_\mu) \quad (\text{weakly})$$

where ρ_μ are smooth $(1,0)$ -forms in $f^{-1}(V)$.

Let S be a closed current of degree $(N-1, N-1)$ with compact support in $f^{-1}(V)$ and let $\psi \in \mathcal{D}^{N-1, N-1}(f^{-1}(V))_{\mathbf{R}}$ such that $S = \psi + i\partial\bar{\partial}u$

for a suitable current u with compact support in $f^{-1}(V)$. Now,

$$S(\varphi) = \int \varphi \wedge \psi + i\partial\bar{\partial}u(\varphi) = \lim_{\mu} \int (\bar{\partial}\rho_{\mu} + \partial\bar{\rho}_{\mu}) \wedge \psi + i\bar{\partial}u(\partial\varphi) = 0.$$

Finally, consider the operator

$$(\partial \oplus \bar{\partial}): (\mathcal{E}^{0,1}(f^{-1}(V)) \oplus \mathcal{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}} \rightarrow \mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$$

and its dual

$$d: (\mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}})' \rightarrow (\mathcal{E}^{0,1}(f^{-1}(V)) \oplus \mathcal{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}}';$$

we have just proved that $\varphi \in (\text{Ker } d)^{\perp} = \overline{\text{Im}(\partial \oplus \bar{\partial})}$. Thus φ is limit of components of boundaries in the strong sense, and the same holds for $g^*\varphi$; so, by (2.5), the same holds for \tilde{T} in a weak sense.

3. Positive $\partial\bar{\partial}$ -closed currents have a pullback to \tilde{M}

In this section we start with the following data: $f: \tilde{M} \rightarrow M$ is a proper modification and T is a positive $\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$, and we try to find a “nice pullback,” say \tilde{T} , of T to \tilde{M} . If E is the exceptional set of f and $Y := f(E)$, then $\tilde{M} - E \xrightarrow{f} M - Y$ is a biholomorphism. Therefore such a pullback must extend $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$ from $\tilde{M} - E$ to the whole of \tilde{M} . What we are looking for is a positive $\partial\bar{\partial}$ -closed extension \tilde{T} on \tilde{M} , which also satisfies

- (3.1) $\forall x \in M$, there exists an open neighborhood W of x such that $\tilde{T}|_{f^{-1}(W)}$ is a (weak) limit of currents which are components of boundaries.

But $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$ has an extension of order zero to \tilde{M} if and only if it has locally finite mass across E (see [15, p. 10]); i.e., $\forall x \in E$, there is a neighborhood V of x in \tilde{M} such that

$$(3.2) \quad \int_{V-E} ((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y}) \wedge \theta^{N-1} < \infty,$$

where θ is a smooth strictly positive $(1, 1)$ -form on \tilde{M} .

Since (3.2) is a local statement, we shall carry on the computations in coordinates, starting with the case of a blow-up with smooth center. After having proved that $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$ admits extensions of order zero, we construct an extension \tilde{T} which satisfies the previous demands (see Theorem 3.9) and such that, if condition (B) holds, its class $[\tilde{T}]$ in

the Aeppli group $V^{1,1}(\widetilde{M})_{\mathbb{R}}$ coincides with $f^*([T])$ (Proposition 3.10). These properties are not enjoyed, in general, by the simple extension \widetilde{T}° , as an example shows.

To prove the following result, we shall follow [19] (page 129 and ff. for the case $k = 1$); nevertheless, since Siu works with d -closed currents, we shall give here a sufficiently detailed proof in order to check that his arguments also work in the $\partial\bar{\partial}$ -closed case and for the sake of completeness.

3.1. Proposition. *Let π be the blow-up of $U := B_m \times B_n$ with center $Y = B_m \times \{0\}$, and let $\tilde{\omega}$ be the Kähler form for $B_m \times B_n$ defined in (1.1). Suppose $\{T_\epsilon\}$ is a family of $\partial\bar{\partial}$ -closed smooth positive $(1,1)$ -forms on U , such that there is a current T on U , $T = \lim_\epsilon T_\epsilon$ (weakly). Then $\forall t^\circ \in B_m$, there exists a neighborhood V of $(t^\circ, 0)$ in U such that*

$$(3.3) \quad \sup_\epsilon \int_{\pi^{-1}(V)} \pi^* T_\epsilon \wedge \tilde{\omega}^{N-1} < \infty.$$

Proof. Choose a unitary linear coordinates system $w = w(t, z) = (w_1, \dots, w_N)$ of \mathbb{C}^N such that $(w_I, z) := (w_{i_1}, \dots, w_{i_m}, z_1, \dots, z_n)$ form a coordinates system of \mathbb{C}^N for every $I = (i_1, \dots, i_m)$ with $1 \leq i_1 < \dots < i_m \leq N$. Look at the $(1,1)$ -form $(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2)$. Its matrix is positive semidefinite; more precisely, at $z \neq 0$, it has 0 as simple eigenvalue, with eigendirection z , and $1/\pi \|z\|^2$ as eigenvalue of multiplicity $(n-1)$, with eigenspace $(z)^\perp$. Hence $(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2)^h = 0$ if $h \geq n$; this implies that there exists a constant $c > 0$ such that

$$(\pi_* \tilde{\omega})^{m+n-1} \leq c \sum_{k=m}^{m+n-1} \binom{m+n-1}{k} \sum_I' \left(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \wedge \left(\frac{i}{2} \partial\bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left(\frac{i}{2} \partial\bar{\partial} \|w_I\|^2 \right)^m.$$

Let $t^\circ \in B_m$ and let $\rho_I: \mathbb{C}^N \rightarrow \mathbb{C}^n$ be defined by $\rho_I(t, z) = w_I(t, z)$. There exist an open ball A_I with center $w_I(t^\circ, 0)$ in \mathbb{C}^m and $r_I > 0$ such that

$$X_I := \rho_I^{-1}(A_I) \cap (\mathbb{C}^m \times B_n(r_I)) \Subset U.$$

Thus, if we take $V := \bigcap_I X_I$, to check (3.3) we have only to prove that $\forall I, \forall k, m \leq k \leq m+n-1$,

$$(3.4) \quad \sup_\epsilon \int_{X_I} T_\epsilon \wedge \left(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \wedge \left(\frac{i}{2} \partial\bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left(\frac{i}{2} \partial\bar{\partial} \|w_I\|^2 \right)^m < \infty.$$

The form which is integrated in (3.4) is smaller than or equal to

$$\frac{1}{(\pi \|z\|^2)^{N-1-k}} T_\epsilon \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m,$$

which is in $L^1_{\text{loc}}(U)$; this implies that in (3.4) we can ignore the singularity of $\partial \bar{\partial} \log \|z\|^2$. Since $i \partial \bar{\partial} T_\epsilon = 0$, there exist $(1, 0)$ -forms S_ϵ on $U = B_m \times B_n$ such that $T_\epsilon = \bar{\partial} S_\epsilon + \partial \bar{S}_\epsilon$. Thus denoting the (topological) boundary of X_I by bX_I

$$\begin{aligned} & \int_{X_I} T_\epsilon \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \int_{bX_I} (S_\epsilon + \bar{S}_\epsilon) \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \\ &\quad \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{bX_I} (S_\epsilon + \bar{S}_\epsilon) \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{X_I} T_\epsilon \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m, \end{aligned}$$

where the reason for the second equality is the following:

$$bX_I := [\rho_I^{-1}(bA_I) \cap (\mathbf{C}^m \times B_n(r_I))] \cup [\rho_I^{-1}(A_I) \cap (\mathbf{C}^m \times bB_n(r_I))] = Y_1 \cup Y_2;$$

integration on Y_1 gives no contribution because $(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2)^m$ is a $2m$ -form on the manifold bA_I of real dimension $2m-1$. On the other hand, on Y_2 we have

$$\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 = \frac{i}{2\pi r_I^2} \partial \bar{\partial} \|z\|^2.$$

Now we need the following result [19, p. 66].

3.2. Lemma. *Suppose $G_1 \Subset G_2 \Subset U$ are relatively compact open subsets of \mathbf{C}^N , φ is a product of $(N-k)$ smooth positive $(1, 1)$ -forms and $\{T_\epsilon\}$ is a sequence of positive currents on U of degree (k, k) converging (weakly) to a current T on U . Then*

$$\limsup_\epsilon \int_{G_1} T_\epsilon \wedge \varphi \leq \int_{G_2} T \wedge \varphi \quad \text{and} \quad \int_{G_1} T \wedge \varphi \leq \liminf_\epsilon \int_{G_2} T_\epsilon \wedge \varphi.$$

If we choose G such that $X_I \Subset G \Subset U$, by virtue of (3.5) and Lemma 3.2 we get

$$\begin{aligned} & \limsup_{\epsilon} \int_{X_I} T_\epsilon \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \\ & \quad \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ & \leq \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_G T \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m < \infty. \end{aligned}$$

Thus also

$$\begin{aligned} & \sup_{\epsilon} \int_{X_I} T_\epsilon \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \\ & \quad \wedge \left(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m < \infty. \text{ q.e.d.} \end{aligned}$$

We would like to mention the following easy consequence of the above lemma, which is used in what follows:

(3.6) If $\{T_\epsilon\}$ and φ are as in Lemma 3.2, and L is a Borel set, $L \Subset U$, then

$$\lim_{\epsilon} \int_L T_\epsilon \wedge \varphi = \int_L T \wedge \varphi \text{ if } \|T\|(bL) = 0.$$

3.3. Proposition. Let $f: \widetilde{M} \rightarrow M$ be the blow-up of M along a submanifold Y . If T is a positive $\partial \bar{\partial}$ -closed current on M of degree $(1, 1)$, then the current $((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y})$ has locally finite mass across E , and hence extends to a current of order zero.

Proof. Identify f locally with the blow-up π and let $y = (t^\circ, 0) \in Y$. By smoothing T by convolutions in a suitable open neighborhood U of y in M , we get a family $\{T_\epsilon\}$ as in Proposition 3.1. Choose $r > 0$ and a sequence $\{r_j\}$ of positive real numbers such that $r_j \downarrow 0$, $V_0 := B_m(t^\circ, r) \times B_n(r_0) \Subset U$ and, for every $j \geq 0$,

$$(3.7) \quad \|T\|(bV_j) = 0, \quad \text{where } V_j := B_m(t^\circ, r) \times B_n(r_j).$$

Since

$$\lim_j \chi_{\pi^{-1}(V_0 - V_j)} = \chi_{\pi^{-1}(V_0 - Y)},$$

where, as usual, χ_L is the characteristic function of L , we infer that

$$\begin{aligned} \lim_j \int_{\pi^{-1}(V_0 - V_j)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1} \\ = \int_{\pi^{-1}(V_0 - Y)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1}, \end{aligned}$$

where $\widetilde{U} := f^{-1}(U)$; but

$$\int_{\pi^{-1}(V_0 - V_j)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1} = \int_{V_0 - V_j} T \wedge \pi_* \tilde{\omega}^{N-1}.$$

By (3.7) and (3.6), we have

$$\begin{aligned} \int_{V_0 - V_j} T \wedge \pi_* \tilde{\omega}^{N-1} &= \lim_\epsilon \int_{V_0 - V_j} T_\epsilon \wedge \pi_* \tilde{\omega}^{N-1} \\ &\leq \sup_\epsilon \int_{V_0} T_\epsilon \wedge \pi_* \tilde{\omega}^{N-1} < \infty. \end{aligned}$$

Thus

$$\int_{\pi^{-1}(V_0 - Y)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1} < \infty.$$

Remark. One of the possible extensions of order zero is the “simple extension” [15], which is defined as follows:

$$\widetilde{T}^\circ(\varphi) = \int_{\widetilde{M}-E} ((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y}) \wedge \varphi$$

for every $\varphi \in \mathcal{D}^{N-1, N-1}(\widetilde{M})_{\mathbf{R}}$. \widetilde{T}° is called also “extension by zero” since $\|\widetilde{T}^\circ\|(E) = 0$. Nevertheless we are interested in an extension \widetilde{T} which is also $\partial\bar{\partial}$ -closed and satisfies (3.1), therefore we go on otherwise.

First we recall a lemma [19, p. 69].

3.4. Lemma. Suppose Ω is an open subset of \mathbf{C}^N and θ a strictly positive $(1, 1)$ -form on Ω . Suppose $\{T_\lambda\}$ is a sequence of smooth positive (k, k) -forms on Ω which satisfy

$$\sup_\lambda \int_K T_\lambda \wedge \theta^{N-k} < \infty$$

for every compact K of Ω . Then there exists a subsequence $\{T_{\lambda_\mu}\}$ of $\{T_\lambda\}$ which converges (weakly) on Ω .

Using this result, we are able to complete Proposition 3.1 as follows.

3.5. Corollary. In the hypotheses of Proposition 3.1, there exists a sequence $\{\varepsilon_\mu\}$, $\varepsilon_\mu \rightarrow 0$, such that $\pi^* T_{\varepsilon_\mu}$ converges (weakly) on \widetilde{U} to a

current \tilde{T}_U . This current does depend not on the sequence ε_μ but only on T .

Proof. We can take coordinates open sets Ω_j such that $\tilde{U} = \bigcup_{j=1}^n \Omega_j$. By Proposition 3.1, Ω_j , $\pi^* T_e|_{\Omega_j}$, and $\theta = \tilde{\omega}|_{\Omega_j}$ satisfy the hypothesis of Lemma 3.4. Therefore, if we consider subsequently $\Omega_1, \dots, \Omega_n$, we find a sequence $\{\varepsilon_\mu\}$ such that $\pi^* T_{\varepsilon_\mu}$ converges on each Ω_j , hence on \tilde{U} to a current \tilde{T}_U . Let $\{T'_{\varepsilon_\mu}\}$ be another sequence, with the same properties of $\{T_{\varepsilon_\mu}\}$, such that

$$\lim_{\mu} \pi^* T'_{\varepsilon_\mu} = \tilde{T}'_U.$$

Since $U \cong B_m \times B_n$, T'_{ε_μ} and T_{ε_μ} are components of boundaries; therefore we can get $\tilde{T}'_U = \tilde{T}_U$ by applying Theorem 2.3 to $\tilde{T}'_U - \tilde{T}_U$.

3.6. Remark. Another proof of the second part of the previous corollary is given here, to emphasize the link between $\|\tilde{T}_U\|(E)$ and a kind of “Lelong number of T along Y ”.

Proof. Let $\{T'_{\varepsilon_\mu}\}$ be another sequence, with the same properties of $\{T_{\varepsilon_\mu}\}$, such that

$$\lim_{\mu} \pi^* T'_{\varepsilon_\mu} = \tilde{T}'_U.$$

Take an open ball $A \Subset B_m$ and $B_n(r) \Subset B_n$ such that

$$(3.8) \quad \|\tilde{T}_U\|(b\pi^{-1}(A \times B_n(r))) = \|\tilde{T}'_U\|(b\pi^{-1}(A \times B_n(r))) = 0.$$

Since on $bB_n(r)$

$$\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 = \frac{i}{2\pi r_I^2} \partial \bar{\partial} \|z\|^2,$$

we get as in the proof of Proposition 3.1

$$(3.9) \quad \begin{aligned} & \int_{\pi^{-1}(A \times B_n(r))} \pi^* T_{\varepsilon_\mu} \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \int_{A \times B_n(r)} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \frac{1}{(\pi r^2)^{n-1}} \int_{A \times B_n(r)} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m. \end{aligned}$$

Notice that (3.8) implies $\|T\|(b(A \times B_n(r))) = 0$; by letting $\mu \rightarrow \infty$ in

(3.9) we conclude that

$$\begin{aligned} & \int_{\pi^{-1}(A \times B_n(r))} \tilde{T}_U \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \frac{1}{(\pi r^2)^{n-1}} \int_{A \times B_n(r)} T \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m, \end{aligned}$$

and the same holds for \tilde{T}'_U .

By applying Theorem 2.3 to the current $\tilde{T}_U - \tilde{T}'_U$ we get

$$\begin{aligned} 0 &= \int_{\pi^{-1}(A \times B_n(r))} (\tilde{T}_U - \tilde{T}'_U) \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \int_A h \left(\frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m, \end{aligned}$$

and since A is arbitrary, $h = 0$; i.e., $\tilde{T}_U = \tilde{T}'_U$. q.e.d.

We would like also to compare the previous proof with the analogous situation occurring in [19, p. 129]. The author has a closed current \tilde{T}_U ; hence he gets

$$\chi_{E_U} \tilde{T}_U = c[E_U]$$

by deep results based on Bombieri-Hörmander estimates (see [19, Chapter 5]). Next,

$$\chi_{E_U} \tilde{T}'_U = c'[E_U],$$

for another \tilde{T}'_U implies $\tilde{T}_U - \tilde{T}'_U = (c - c')[E_U]$. But the class of $[E_U]$ does not vanish in the cohomology ring of \tilde{U} , while the class of $\tilde{T}_U - \tilde{T}'_U$ is zero, because \tilde{T}_U and \tilde{T}'_U are limits of sequences of boundaries; this implies $c = c'$.

3.7. Remark. Let $f: \widetilde{M} \rightarrow M$ be the blow-up of M along a submanifold Y , and let T be a positive $\partial \bar{\partial}$ -closed current on M of degree $(1, 1)$. For every $x \in M$, there exists an open neighborhood U of x such that, smoothing T by convolutions, we can apply Corollary 3.5 to get a positive extension \tilde{T}_U of $((f|_{\widetilde{U}-E})^{-1})_* (T|_{U-Y})$ in $f^{-1}(U)$. By construction, such an extension is the limit of currents which are components of boundaries; hence \tilde{T}_U is $\partial \bar{\partial}$ -closed. But in general $\tilde{T}^\circ \neq \tilde{T}_U$, as the following example shows.

Example. Let us take $Y = \{0\}$ (that is, $m = 0$ and $N = n$), and let us consider a sequence $\{T_\varepsilon\}$ as in Proposition 3.1. For $0 < r_1 < r_2 < 1$,

$$\begin{aligned} & \int_{r_1 < \|z\| < r_2} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \int_{r_1 < \|z\| < r_2} T_\varepsilon \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^h \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1-h} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_\varepsilon \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1} \\ &\quad - \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_1^2)^h} \int_{\|z\| < r_1} T_\varepsilon \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}. \end{aligned}$$

For the second equality, see [20, p. 364, Remark 1] or also our (3.5). For $r_1 \rightarrow 0$,

$$\int_{\|z\| < r_2} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} = \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_\varepsilon \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}.$$

Now choose the subsequence ε_μ of Corollary 3.5 and r_2 such that $\|T\|(bB_n(r_2)) = 0$. For $\mu \rightarrow \infty$

$$\begin{aligned} \|\tilde{T}_U\|(\pi^{-1}(B_n(r_2))) &= \int_{\pi^{-1}(\|z\| < r_2)} \tilde{T}_U \wedge \tilde{\omega}^{N-1} \\ &= \lim_{\mu} \int_{\pi^{-1}(\|z\| < r_2)} \pi^* T_{\varepsilon_\mu} \wedge \tilde{\omega}^{N-1} = \lim_{\mu} \int_{\|z\| < r_2} T_{\varepsilon_\mu} \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \lim_{\mu} \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}; \end{aligned}$$

thus

$$(3.10) \quad \|\tilde{T}_U\|(E \cap U) = \lim_{r_2 \rightarrow 0} \|\tilde{T}_U\|(\pi^{-1}(B_n(r_2))) = n(T, 0).$$

Hence, if the Lelong number $n(T, 0) \neq 0$, $\tilde{T}^\circ \neq \tilde{T}_U$.

It is perhaps interesting to check the difference between \tilde{T}° and \tilde{T}_U . Take $T = [H]$, where $H = \{z_n = 0\}$; we get easily that \tilde{T}° is nothing else than the strict transform of H under π . Hence \tilde{T}° is closed, and therefore $\tilde{T} - \tilde{T}^\circ$ is $\partial \bar{\partial}$ -closed; obviously

$$\tilde{T} - \tilde{T}^\circ \geq 0 \quad \text{and} \quad \text{supp}(\tilde{T} - \tilde{T}^\circ) \subseteq E = \mathbf{P}_{n-1}.$$

By Theorem 1.5 in [2], there exists a constant $k \geq 0$ such that $\tilde{T} - \tilde{T}^\circ = k[E]$. This implies

$$\|\tilde{T}\|(E) = k \text{ vol}(\mathbf{P}_{n-1}) = k,$$

but from (3.10)

$$\|\tilde{T}\|(E) = n([H], 0) = 1.$$

Thus $\tilde{T} = \tilde{T}^\circ + [E]$.

This example reflects a more general situation, which is discussed in Proposition 3.10: that is, if T is a divisor, its total transform is \tilde{T} and not, in general, \tilde{T}° .

Let us collect our results in the following theorem.

3.8. Theorem. *Let $f: \tilde{M} \rightarrow M$ be the blow-up of M along a submanifold Y . If T is a positive $\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$, then $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$ can be extended to \tilde{M} , and there exists an extension \tilde{T} which is positive and $\partial\bar{\partial}$ -closed, and satisfies the following condition:*

- (3.1) $\forall x \in M$, there exists an open neighborhood W of x such that $\tilde{T}|_{f^{-1}(W)}$ is a (weak) limit of currents which are components of boundaries.

Now we consider the general case of a proper modification.

3.9. Theorem. *Let $f: \tilde{M} \rightarrow M$ be a proper modification and let T be a positive $\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$. Then the following hold:*

(i) *There exists a positive $\partial\bar{\partial}$ -closed current \tilde{T} on \tilde{M} of degree $(1, 1)$ such that $f_*\tilde{T} = T$ and (3.1) holds.*

(ii) *If M is compact, such a current \tilde{T} is unique.*

Proof. (i) Let $x \in M$ and choose an open neighborhood V of x and maps $g: Z \rightarrow f^{-1}(V)$ and $h: Z \rightarrow V$ as in Lemma 2.6. Since h is obtained as a finite sequence of blow-ups with smooth centers, by Theorem 3.8 we get a positive $\partial\bar{\partial}$ -closed current \hat{T} on Z such that $h_*\hat{T} = T$. Shrinking V , we can suppose that it is contained in a coordinate chart and is biholomorphic to an open ball, and that there exists a sequence $\{T_\epsilon\}$ of components of boundaries, $T_\epsilon \rightarrow T$ (e.g., by smoothing T by convolutions). By construction, $\hat{T} = \lim_\epsilon h^*T_\epsilon$ and it does not depend on $\{T_\epsilon\}$. Define

$$\tilde{T}_V := g_*\hat{T} = \lim_\epsilon g_*h^*T_\epsilon = \lim_\epsilon (f|_{f^{-1}(V)})^*T_\epsilon.$$

Since \tilde{T}_V does not depend on $\{T_\epsilon\}$ nor on the factorization $h = f \circ g$, $\tilde{T}|_{f^{-1}(V)} := \tilde{T}_V$ defines the required current.

(ii) Let \tilde{T} and \tilde{T}' be currents on \tilde{M} which satisfy (i). Let $x \in M$ and Z, g, h be as above; using (i) for the map g , we get positive $\partial\bar{\partial}$ -closed currents \hat{T} and \hat{T}' on Z such that $g_*\hat{T} = \tilde{T}$ and $g_*\hat{T}' = \tilde{T}'$ on

$f^{-1}(V)$; hence $h_*\widehat{T} = h_*\widehat{T}' = T$ on V . Let $\{E'_\gamma\}$ be the set of irreducible components of $E \cap f^{-1}(V)$, and $\{F_\beta\}$ be the set of irreducible components of the exceptional set F of $g: Z \rightarrow f^{-1}(V)$, and denote by \widetilde{E}'_γ the strict transform of E'_γ under g . Thus $\{F_\beta\} \cup \{\widetilde{E}'_\gamma\}$ is the set of irreducible components of the exceptional set of $h: Z \rightarrow V$, and by Proposition 2.5 we get

$$\widehat{T} - \widehat{T}' = \sum_\gamma u_\gamma[\widetilde{E}'_\gamma] + \sum_\beta u_\beta[F_\beta],$$

where u_γ and u_β are pluriharmonic functions. Thus $\widetilde{T} - \widetilde{T}' = g_*(\widehat{T} - \widehat{T}') = \sum_\gamma u'_\gamma[E'_\gamma]$ on $f^{-1}(V)$, where u'_γ is a well-defined pluriharmonic function on $E'_\gamma - g(\widetilde{E}'_\gamma \cap F)$, locally bounded on E'_γ . Hence u'_γ extends to E'_γ so that we get on \widetilde{M}

$$\widetilde{T} - \widetilde{T}' = \sum_\alpha u_\alpha[E_\alpha],$$

where $\{E_\alpha\}$ is the set of irreducible components of E , and u_α is pluriharmonic on E_α . But each E_α is compact, so each u_α is constant. The thesis follows from Proposition 2.7. q.e.d.

Let us consider now the class in $V^{1,1}(\widetilde{M})_{\mathbb{R}}$ of the current \widetilde{T} given by the previous theorem.

3.10. Proposition. *Let M , \widetilde{M} be complex manifolds which satisfy condition (B), and let $f: \widetilde{M} \rightarrow M$ be a proper modification. Let T be a positive $\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$, and \widetilde{T} be a current on \widetilde{M} satisfying Theorem 3.9 (i). Let $f^*: V^{1,1}(M)_{\mathbb{R}} \rightarrow V^{1,1}(\widetilde{M})_{\mathbb{R}}$ be the natural map. Then $f^*([T]) = [\widetilde{T}]$.*

Proof. By condition (B), there exist d -closed smooth real $(1, 1)$ -forms φ and $\tilde{\varphi}$ such that $\beta([\varphi]) = [T]$ and $\tilde{\beta}([\tilde{\varphi}]) = [\widetilde{T}]$; i.e., $T = \varphi + \bar{\partial}S + \partial\bar{S}$ and $\widetilde{T} = \tilde{\varphi} + \bar{\partial}R + \partial\bar{R}$ for suitable currents S and R . Therefore

$$f_*\tilde{\varphi} - \varphi = \bar{\partial}(S - f_*R) + \partial(\bar{S} - f_*\bar{R}),$$

but β is injective; hence $f_*\tilde{\varphi} - \varphi$ is d -exact. Using the results on the link between the cohomology rings of M , \widetilde{M} , E and Y (see, e.g., [7, p. 285]) we get

$$\tilde{\varphi} - f^*\varphi = \sum_\alpha c_\alpha[E_\alpha] + dQ$$

for a suitable current Q .

Recall that there exist open sets V such that $\widetilde{T}|_{f^{-1}(V)}$ is the limit of components of boundaries; hence the same holds for $\tilde{\varphi}$. Moreover

we can choose V such that $H^2(V, \mathbf{R}) = 0$, so that $\varphi|_V = d\psi$. Thus $\sum_\alpha c_\alpha [E_\alpha] = \tilde{\varphi} - f^*\varphi - dQ$ is the limit of components of boundaries in $f^{-1}(V)$; by Proposition 2.7, $c_\alpha = 0 \ \forall \alpha$, so that

$$f^*([T]) = f^*\beta([\varphi]) = \tilde{\beta}f^*([\varphi]) = \tilde{\beta}f^*([\varphi]) = \tilde{\beta}([\tilde{\varphi}]) = \tilde{T}.$$

4. The main theorem

In this section we use the machinery developed in the previous sections to get some metric results. We start with a lemma concerning condition (B).

4.1. Lemma. *Let $f: \widetilde{M} \rightarrow M$ be a proper modification. If \widetilde{M} satisfies condition (B) (that is, $\beta: H^{1,1}(M, \mathbf{R}) \rightarrow V^{1,1}(M)_\mathbf{R}$ is injective and $\text{Im } \beta$ contains all positive elements of $V^{1,1}(M)_\mathbf{R}$), then M also satisfies (B).*

Proof. Let us consider the following commutative diagram (see (1.3)):

$$\begin{array}{ccc} H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_\mathbf{R} \\ \downarrow f^* & & \downarrow f^* \\ H^{1,1}(\widetilde{M}, \mathbf{R}) & \xrightarrow{\tilde{\beta}} & V^{1,1}(\widetilde{M})_\mathbf{R} \\ \downarrow f_* & & \downarrow f_* \\ H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_\mathbf{R} \end{array}$$

where $f_* \circ f^*$ is the identity. Denote by $[]$ the classes in all groups that appear in the diagram. By hypothesis, $\tilde{\beta}$ is injective; hence β is injective too. Let T be a positive $\partial\bar{\partial}$ -closed current on M of degree $(1, 1)$, and \tilde{T} be a positive $\partial\bar{\partial}$ -closed current on \widetilde{M} of degree $(1, 1)$ given by Theorem 3.9. We know that there exists a d -closed form $\psi \in \mathcal{E}^{1,1}(\widetilde{M})_\mathbf{R}$ such that $\tilde{\beta}([\psi]) = [\tilde{T}]$. Therefore

$$\beta f_*([\psi]) = f_*\tilde{\beta}([\psi]) = [f_*\tilde{T}] = [T]. \quad \text{q.e.d.}$$

Now we can state and prove the Main Theorem. Here the manifolds are supposed to be compact, since Theorem 1.2 is needed.

4.2. Main Theorem. *Let M, \widetilde{M} be compact complex manifolds, and $f: \widetilde{M} \rightarrow M$ be a modification. If \widetilde{M} is balanced and satisfies (B) (in particular, if it is Kähler), then M is balanced and satisfies (B) too.*

Proof. (B) holds for M by Lemma 4.1.

Let $T = \bar{\partial}S + \partial\bar{S}$ be a positive current of degree $(1, 1)$ on M . If we prove that $T = 0$, we get the thesis by Theorem 1.2. Let \tilde{T} be given

by Theorem 3.9; then $f^*([T]) = [\tilde{T}]$ by Proposition 3.10. Hence \tilde{T} is a positive component of a boundary on a balanced manifold. This implies $\tilde{T} = 0$. Thus $\text{supp } T \subseteq Y$, but the codimension of Y is greater than one; hence by Theorem 2.1, $T = 0$.

Remark. In the proof of the previous theorem, problems arising from changing charts are avoided. As a matter of fact, one may hope to prove the Main Theroem directly. Starting from a strictly positive $(1, 1)$ -form $\tilde{\omega}$ on \tilde{M} with $d\tilde{\omega}^{N-1} = 0$, try to construct an analogous form ω on M . But obviously we cannot hope that $\omega|_{M-Y} = f_*\tilde{\omega}|_{M-Y}$, because $f_*\tilde{\omega}$ “blows up” near Y . So we should modify $f_*\tilde{\omega}$ on coordinate open sets which meet Y , and then glue together these currents. This seems to be much more complicated than our procedure, which consists basically in extending the current T from $\tilde{M} - E$ to \tilde{M} .

Theorem 4.2 has some interesting corollaries; to state them let us recall the definition of the class \mathcal{C} of Fujiki [5, p. 34–35].

4.3. Definition. A reduced (compact) complex analytic space X belongs to \mathcal{C} if it is a meromorphic image of a compact Kähler space.

Varouchas proved that \mathcal{C} is nothing but the class of spaces bimeromorphic to some compact Kähler manifold:

4.4. Theorem [22, Theorem 3]. *If $X \in \mathcal{C}$, then there exist a compact Kähler manifold K and a modification $f: K \rightarrow X$.*

So we get by Theorem 4.2 a result about “nice” hermitian metrics on manifolds in the class \mathcal{C} .

4.5. Corollary. *Every manifold in the class \mathcal{C} is balanced.*

And in particular we have

4.6. Corollary. *Moishezon manifolds are balanced.*

Notice that there exist compact balanced manifolds not in the class \mathcal{C} , e.g., the Iwasawa manifold I_3 .

In order to study metrics in connection with modifications of balanced manifolds, let us start from a more general situation and introduce the following definition.

4.7. Definition. A complex manifold M is called p -Kähler if it carries a strictly weakly positive smooth closed (p, p) -form, called a p -Kähler form.

For more details about this subject see [2], here we may only point out that 1-Kähler is equivalent to Kähler and $(N - 1)$ -Kähler is equivalent to balanced.

4.8. Theorem. *Let M and \tilde{M} be compact p -Kähler manifolds, let $f: \tilde{M} \rightarrow M$ be a proper modification, and call Y the degeneracy set, with $p > \dim Y$. For every p -Kähler form Ω on M , there exists a*

p -Kähler form $\tilde{\Omega}$ on \tilde{M} such that $[\Omega] = [f_* \tilde{\Omega}]$ in the (p, p) -Aeppli group $\Lambda^{p,p}(M)_{\mathbf{R}}$.

Proof. Let Ω and Ω' be p -Kähler forms on M and \tilde{M} respectively. Since $p > \dim Y$, arguing as in [22, pp. 251–252], we get an open neighborhood U of Y in M and a real current R in U such that

$$f_* \Omega' = i\partial\bar{\partial}R.$$

Since $f_* \Omega'$ is smooth and $\partial\bar{\partial}$ -exact in $U - Y$, there exists a smooth real $(p-1, p-1)$ -form β in $U - Y$ such that

$$f_* \Omega' |_{U-Y} = i\partial\bar{\partial}\beta.$$

Moreover, $R - \beta = \gamma + \bar{\partial}C + \partial\bar{C}$ in $U - Y$, for a suitable smooth real $\partial\bar{\partial}$ -closed $(p-1, p-1)$ -form γ and a current C .

Now choose an open set W such that $Y \subset W \Subset U$ and a real function $g \in \mathcal{C}_0^\infty(U)$, with $g = 1$ in W ; take

$$D := g(\beta + \gamma) + \bar{\partial}(gC) + \partial(g\bar{C}).$$

Since $i\partial\bar{\partial}D$ is smooth in $M - Y$, $\chi := (f|_{\tilde{M}-E})^* i\partial\bar{\partial}D$ is a smooth (p, p) -form in $\tilde{M} - E$ which coincides with Ω' in $f^{-1}(W) - E$; therefore χ can be extended to a real smooth (p, p) -form on the whole of \tilde{M} , which is supported in $f^{-1}(U)$ and strictly weakly positive in $f^{-1}(W)$. Choose $\varepsilon > 0$ such that

$$\tilde{\Omega} := f^* \Omega + \varepsilon \chi$$

is strictly weakly positive on \tilde{M} , so we get $\Omega - f_* \tilde{\Omega} = i\partial\bar{\partial}\varepsilon D$. q.e.d.

In [2] we studied a kind of modification for which the hypotheses of Theorem 4.8 hold, so that we have now some new information about the link between p -Kähler forms on M and \tilde{M} . Moreover for $p = N - 1$ we can give a metric interpretation:

4.9. Corollary. *Let M and \tilde{M} be compact balanced manifolds and $f: \tilde{M} \rightarrow M$ a modification. For every balanced metric h on M with Kähler form ω there exists a balanced metric \tilde{h} on \tilde{M} with Kähler form $\tilde{\omega}$ such that $\omega^{N-1} - f_* \tilde{\omega}^{N-1}$ is a $\partial\bar{\partial}$ -exact current.*

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