Hindawi Publishing Corporation
Geometry
Volume 2013, Article ID 694169, 12 pages
http://dx.doi.org/10.1155/2013/694169

Research Article

Metric Ricci Curvature for *PL* Manifolds

David Xianfeng Gu[1] and Emil Saucan[2,3]

[1] *Department of Computer Science, SUNY at Stony Brook, NY 11794-4400, USA*
[2] *Department of Mathematics, Technion, 32000 Haifa, Israel*
[3] *Department of Mathematics and Computer Science, The Open University, 53537 Ra'anana, Israel*

Correspondence should be addressed to Emil Saucan; semil@tx.technion.ac.il

Received 30 April 2013; Accepted 28 July 2013

Academic Editor: Bennett Palmer

We introduce a metric notion of Ricci curvature for *PL* manifolds and study its convergence properties. We also prove a fitting version of the Bonnet-Myers theorem, for surfaces as well as for a large class of higher dimensional manifolds.

1. Introduction

Recent years have seen a great "revival" of Ricci curvature, mainly due to not only Perelman's celebrated work on the Ricci flow and the Poincare conjecture [1, 2], but also to its extension to a far larger class of geometric objects than merely smooth 3-manifolds (see [3] and the bibliography therein). In consequence, Ricci curvature has become an object of interest and study in graphics and imaging. The approaches range from the implementations of the combinatorial Ricci curvature of Chow and Luo [4]—see, for example, [5], through—the classical approximation methods of smooth differential operators [6, 7] to discrete, purely combinatorial methods [8, 9].

We have addressed the problem of Ricci curvature of *PL* surfaces and higher dimensional *PL* (piecewise flat) manifolds, from a metric point of fview, both as a tool in studying the combinatorial Ricci flow on surfaces [10, 11] and, in a more general context, in the approximation in secant of curvature measures of manifolds and their applications [12, 13]. Computational applications aside, these and related problems—see [14, 15]—make the study of a robust notion of Ricci curvature for *PL* spaces a subject of thriving interest in the geometry and topology of (mainly 3-dimensional) manifolds. This paper represents a continuation of both papers above.

In all fairness, other definitions of Ricci curvature for n-dimensional *PL* manifolds have been considered previously. The main contribution, from our point of view, is that of Stone [16, 17] that we will discuss in detail and adapt here. More recent contributions are due to Alsing et al. [18] Glickenstein [19] and Trout [20]. However, our approach is quite different and owns nothing to the mentioned works. The only one of the quoted papers that has a similar starting point with ours—namely, Stone's articles—is [20]. Moreover he—as we do (and as Stone originally did) seeks a discrete version of the Bonnet-Myers theorem. However, these facts represent the only common ground: his approach is (as Stone's was) purely combinatorial whereas ours is metric; his methods are also combinatorial in nature (even the Morse function employed is combinatorial) while, in contrast, our basic tool of study is Wald's metric curvature. We should also mention the fact that the convergence and related results for the combinatorial Ricci curvature for images introduced in [8] (and further developed in [9]) were proven by passing from the original cell complex associated to an image to the *PL* manifold, so they also follow the basic pattern of proof considered in this paper and represent, as such, an alternative approach to Ricci curvature for "weighted" *PL* 2- and 3-dimensional manifolds.

The paper is structured as follows. In Section 2, we address the main problem, namely, that of defining a computable, discrete metric Ricci curvature for *PL* (piecewise flat) manifolds. Here, we use methods similar to those in [12] to address a problem—or, rather, a particular case—considered therein. We also investigate the convergence properties of this newly introduced curvature. In Section 3, we address a rather more theoretical question, namely, if the newly introduced version of Ricci curvature satisfies—as indeed expected from a proper ("correct") notion of Ricci curvature—a Bonnet-Myers type of theorem. The methods in this part are, partially, those developed in [10]. We further generalize these results by making appeal to the theory of Alexandrov spaces. In Section 4, we bring an immediate generalization to *PL* manifolds of a comparison theorem. We conclude with a few brief remarks regarding possible further directions of study.

A note to the reader before we proceed to the main part of our paper is that our default source for geometric differential definitions and results is [21], and if no other source is specified the reader should consult, if needed, this encyclopedic source. Also, as a background material for *PL* topology we refer the reader to [22].

2. Definition and Convergence

To begin with, we have to be able to properly define Ricci curvature for *PL* manifolds. This is indeed possible, not just for *PL* manifolds but also for polyhedral ones—and in a quite natural manner—combining ideas of Stone [16, 17] and metric curvatures. For this, one regards Ricci curvature as the mean of sectional curvatures:

$$\operatorname{Ric}(e_1) = \operatorname{Ric}(e_1, e_1) = \sum_{i=2}^{n} K(e_1, e_i), \tag{1}$$

for any orthonormal basis $\{e_1, \ldots, e_n\}$, and where $K(e_1, e_j)$ denotes the sectional curvature of the 2 sections containing the directions e_1.

First, one has, of course, to be able to define (*variational*) *Jacobi fields* (see below). This is where we rely upon Stons's work. However, we do not need the whole force of this technical apparatus, only to determine the relevant two sections, and, of course, to decide what a direction at a vertex of a *PL* manifold is.

In fact, in Stone's work, combinatorial Ricci curvature is defined both for the given simplicial complex $\mathcal{T}$ and also for its *dual complex* $\mathcal{T}^*$. In the latter case, cells—playing here the role of the planes in the classical setting of which sectional curvatures are to be averaged—are considered. However, his approach for the given complex, where one computes the Ricci curvature $\operatorname{Ric}(\sigma, \tau_1 - \tau_2)$ of an n-simplex σ in the direction of two adjacent $(n-1)$-faces, τ_1, τ_2, is not natural in a geometric context (even if useful in his purely combinatorial one), except for the 2-dimensional case, where it coincides with the notion of Ricci curvature in a direction (i.e., in this case, an edge—see also Remark 9 below). Passing to the dual complex will not restrict us since $(\mathcal{T}^*)^* = \mathcal{T}$ and, moreover—and more importantly—considering *thick* triangulations enables us to compute the more natural metric curvature for the dual complex and use the fact that the dual of a thick triangulation is thick, as we will detail below. Working only with thick triangulations does not restrict us, however, at least in dimension ≤ 4 since any triangulation admits a "thickening"—see [23], (This holds, as already mentioned, for any *PL* manifold of dimension ≤ 4, and in all dimensions for smoothable *PL* manifolds as well for any manifold of class $\geq \mathcal{C}^1$. Since the proof of the main result of Section 3, regarding manifolds of dimension higher than 3 holds only for manifolds admitting smoothings, restricting ourselves only to such manifolds does not represent any further hindrance.)

First, let us recall the definition of thick triangulations.

Definition 1. Let $\tau \subset \mathbb{R}^n$; let $0 \leq k \leq n$ be a k-dimensional simplex. The thickness (or *fatness*) φ of τ is defined as being

$$\varphi(\tau) = \frac{\operatorname{dist}(b, \partial\sigma)}{\operatorname{diam}\sigma}, \tag{2}$$

where b denotes the barycenter of σ and $\partial\sigma$ represents the standard notation for the boundary of σ (i.e., the union of the $(n-1)$-dimensional faces of σ).

A simplex τ is φ_0-*thick*, for some $\varphi_0 > 0$, if $\varphi(\tau) \geq \varphi_0$. A triangulation (of a submanifold of $\mathbb{R}^n$) $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is φ_0-*thick* if all its simplices are φ_0-*thick*. A triangulation $\mathcal{T} = \{\sigma_i\}_{i\in\mathbf{I}}$ is *thick* if there exists $\varphi_0 \geq 0$ such that all its simplices are φ_0-*thick*.

Remark 2. This is Munkres' definition [24]. For a discussion of other equivalent definitions, their mutual interplay and relationship with certain aspects of differential geometry (mainly curvature approximation), see [12]. Note that this definition holds for more general simplices not necessarily Euclidean ones.

To be able to define and estimate the Ricci curvature of $\mathcal{T}$ and $\mathcal{T}^*$ and the connection between them, we have to make appeal in an essential manner to the fatness of the given complex. We begin by noting—by keeping in mind formula (2)—that, since the length of the edge l_{ij}^*, dual to the edge l_{ij} common to the faces f_i, f_j, equals $r_i + r_j$, the first barycentric subdivision (needed in the construction of the dual complex—see e.g., [22]) of a thick triangulation is thick. For planar triangulations, and also for higher dimensional complexes embedded in some $\mathbb{R}^N$, one can realize the dual complex (also in $\mathbb{R}^N$) by constructing the dual edges l_{ij}^* orthogonal to the middle of the respective l_{ij}-s. To show the thickness of the dual simplices, one has also to make appeal to the characterization of thickness in terms of dihedral angles (Conditions (1.15) of [25].) Note that the notion of thickness also makes sense for general cells.

Definition 3. Let $\mathfrak{c} = \mathfrak{c}^k$ be a k-dimensional cell. The *thickness* (or *fatness*) of $\mathfrak{c}$ is defined as

$$\varphi(\mathfrak{c}) = \min_{\mathfrak{b}} \frac{\operatorname{Vol}(\mathfrak{b})}{\operatorname{diam}^l(\mathfrak{b})}, \tag{3}$$

where the minimum is taken over all the l-dimensional faces of $\mathfrak{c}$, $0 \leq k$. (If $\dim\mathfrak{b} = 0$, then $\operatorname{Vol}(\mathfrak{b}) = 1$ by convention.)

Therefore, we can summarize the discussion above as follows.

Lemma 4. *The dual complex $\mathcal{T}^*$ of a thick (simplicial) complex $\mathcal{T}$ is thick.*

Moreover, the following (common) Gromov-Hausdorff convergence property also follows immediately.

Lemma 5. *Let $\mathcal{T}, \mathcal{T}^*$ be as above. Then*

$$\lim_{\delta(\mathcal{T}) \to 0} (\mathcal{T}) = \lim_{\delta(\mathcal{T}^*) \to 0} (\mathcal{T}^*), \tag{4}$$

where $\delta(\mathcal{T}), \delta(\mathcal{T}^*)$ denote the mesh of $\mathcal{T}, \mathcal{T}^*$, respectively.

Remark 6. It is important to stress here the crucial role of the thickness of the triangulation; as far as geometry is concerned, thickness ensures, by its definition, the fact that no degeneracy of the simplices occurs, and hence no collapse and degeneracy of the metric can take place. Moreover, in its absence, no uniform estimates for the edge lengths can be made, and hence convergence of (dual) meshes and, as we will see shortly, of their metric Ricci curvatures can not be guaranteed.

Returning to the definition of Ricci curvature for simplicial complexes, given a vertex v_0, in the dual of an n-dimensional simplicial complex, a *direction* at v_0 is just an oriented edge $e_1 = v_0v_1$. Since there exist precisely n 2-cells $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$, having e_1 as an edge and, moreover, these cells form a part of n relevant variational (Jacobi) fields (see [16]); the Ricci curvature at the vertex v, in the direction e_1, is simply

$$\mathrm{Ric}\,(v) = \sum_{i=1}^{n} K\,(\mathfrak{c}_i)\,. \tag{5}$$

Remark 7. Observe that the index "i" in the definition (5) above runs from 1, and not from 2, as expected judging from the classical (smooth) setting. This is due to the fact that we defined Ricci curvature by passing to the dual complex, with its simple but demanding (so to say) combinatorics. For the implications of this fact, see Theorem 40 and Remark 41 below.

Remark 8. Note that we followed [16] only in determining the variational fields but not in his definition of Ricci curvature. Indeed, he considers a direction at a vertex v_0 to be the union of two edges e_1 and e_2 in the dual complex, where $e_1 = (v_0, v_1)$, and $e_2 = (v_1, v_2)$, and where the direction is determined by the lexicographical order. Then (according to [16, pp. 16-17]), the relevant variational fields are given by the $2n$ distinct 2-cells $\mathfrak{c}_1, \ldots, \mathfrak{c}_{2n}$, containing the edges e_1 and e_2, but only $2n - 1$ relevant ones, since one of the cells is enumerated twice. Hence, the Ricci curvature at v in the direction e_1e_2 is to be taken as the total defect of these $2n - 1$ cells as follows:

$$\mathrm{Ric}^*\,(v_0, e_1 - e_2) = 8n - \sum_{j=1}^{2n-1} \left\{ \left| \partial \mathfrak{c}_j \right| \mid e_1 < \mathfrak{c}_j \text{ or } e_2 < \mathfrak{c}_j \right\}. \tag{6}$$

See Figure 1.

This approach is necessary in the combinatorial case. However, it is more difficult than our approach, and it would produce unnecessary complications in determining the relevant analogues of the $(n - 1)$ 2-sections of the classical, smooth case. Moreover, it is quite possible that, in any practical implementation, the advantages obtained by considering larger variational fields would be countermined by "noise" added by considering such order 2 (or larger) neighbourhoods of the given vertex. However, computing Ricci curvature according to this scheme is still possible, using our metric approach (but see also Remark 9).

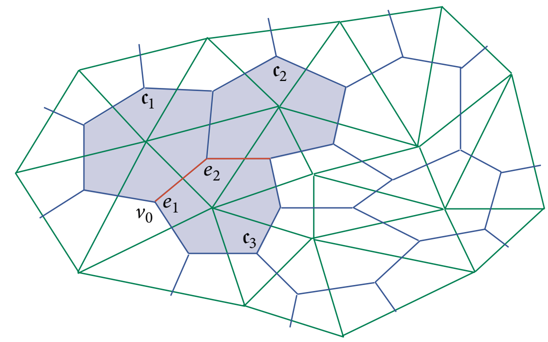

Figure 1: Part of the simplicial complex $\mathcal{T}$ and its dual cell complex $\mathcal{T}^*$. The (variational) Jacobi field given by $2n - 1$ cells ($n = 2$), in the direction $e_1 - e_2$, at the vertex v_0 is emphasized.

Remark 9. It is still possible (by dualization) to compute Ricci curvature according, more-or-less, to Stone's ideas, at least for the 2-dimensional case. Indeed, according to [17],

$$\begin{aligned} \mathrm{Ric}\,(\sigma, \tau_1 - \tau_2) &= 8n - \sum_{j=1}^{2n-1} \Big| \big\{ \beta_j \mid \beta_j < \tau_1 \text{ or } \beta_j < \tau_2; \\ &\qquad \dim \beta_j = n - 2 \big\} \Big| . \end{aligned} \tag{7}$$

This definition of Ricci curvature is a *combinatorial defect* one (presumably inspired by the classical definition of Gauss curvature as the angular defect at a vertex—see, e.g., [26]). This is evident from its expression but made more transparent by the 2-dimensional case. Indeed, in this case, the simplices β_j are 0-dimensional; that is, vertices and hence $N(\beta_j)$ represent just the number of 2-simplices having β_j as a common vertex; therefore $\mathrm{Ric}(\sigma, \tau_1 - \tau_2)$ represents nothing but the total combinatorial defect at these $2n - 1$ vertices (see also [16, p. 17.] for similar interpretation of Ric^*).

In consequence, using the approach of the original proof of Hilbert and Cohn-Vossen [26] (and following methods well established in graphics, etc.), we can consider, instead of the combinatorial defect, the angular defect of the cell $\mathfrak{c}_j$ dual to the vertex β_j. This, of course, applies both for our way and Stone's of determining a direction.

However, this approach to the definition of *PL* Ricci curvature is far less intuitive (and apparently has lesser geometric content, so to speak) in dimension ≥ 3. This is the reason why, for our present study, we have made use of the dual complex.

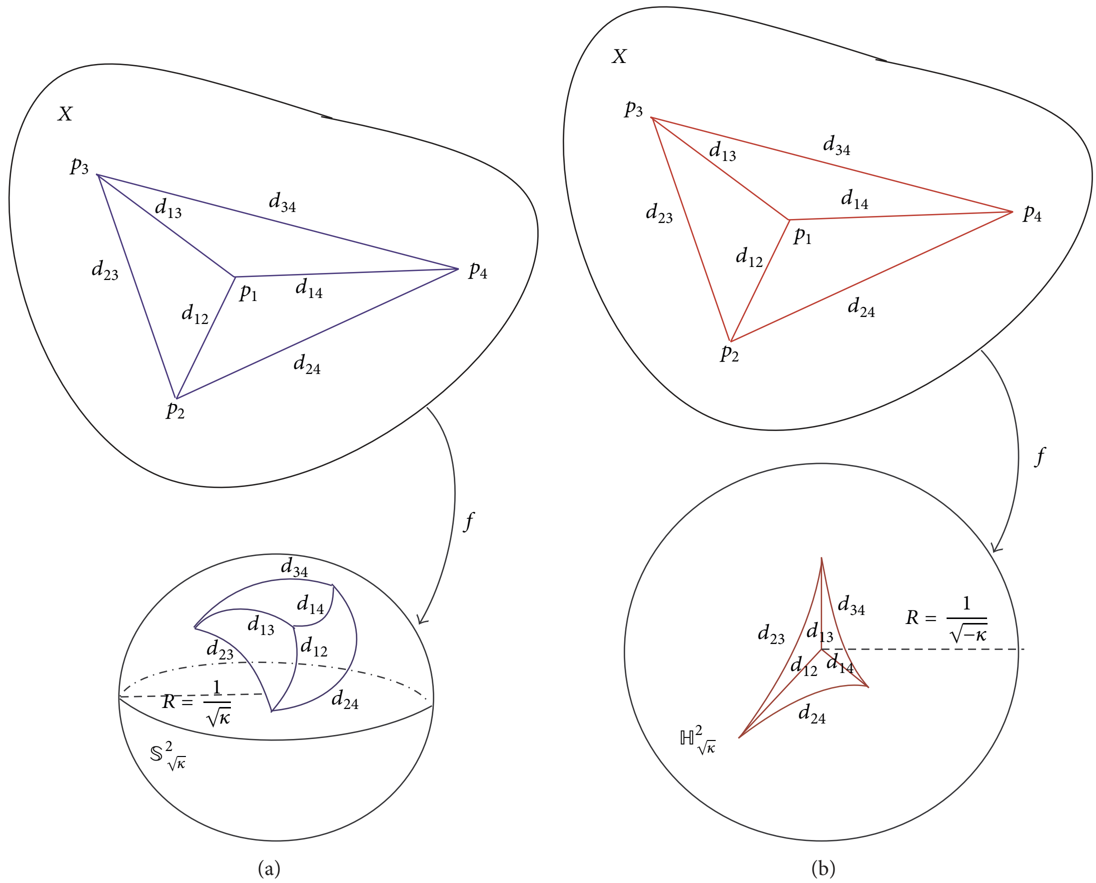

FIGURE 2: Isometric embedding of a metric quadruple in $\mathbb{S}^2_{\sqrt{\kappa}}$ (a) and $\mathbb{H}^2_{\sqrt{\kappa}}$ (b).

To determine—using solely metric considerations—the sectional curvatures $K(\mathfrak{c}_i)$ of the cells $\mathfrak{c}_i$, we shall employ the so-called (*modified*) *Wald curvature* K_W. At this point, we have to remind the reader of a number of definitions and results that, unfortunately, are perhaps (at least partly) forgotten. We begin with the following basic.

Definition 10. Let (M, d) be a metric space, and let $Q = \{p_1, \ldots, p_4\} \subset M$, together with the mutual distances: $d_{ij} = d_{ji} = d(p_i, p_j)$; $1 \le i, j \le 4$. The set Q together with the set of distances $\{d_{ij}\}_{1 \le i, j \le 4}$ is called a *metric quadruple*.

Remark 11. One can define metric quadruples in a somewhat more abstract manner, that is, without the aid of the ambient space. In this approach, a metric quadruple is defined as a 4-point metric space; that is, $Q = (\{p_1, \ldots, p_4\}, \{d_{ij}\})$, where the distances d_{ij} verify the axioms for a metric.

We next introduce some necessary notation. Let S_κ denote the complete, simply connected surface of constant Gauss curvature κ, that is; $S_\kappa \equiv \mathbb{R}^2$, if $\kappa = 0$; $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$, if $\kappa > 0$; and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$, if $\kappa < 0$. Here, $S_\kappa \equiv \mathbb{S}^2_{\sqrt{\kappa}}$ denotes the sphere of radius $R = 1/\sqrt{\kappa}$, and $S_\kappa \equiv \mathbb{H}^2_{\sqrt{-\kappa}}$ stands for the hyperbolic plane of curvature $\sqrt{-\kappa}$, as represented by the Poincaré model of the plane disk of radius $R = 1/\sqrt{-\kappa}$. Using this notation, we can next bring the following.

Definition 12. The embedding curvature $\kappa(Q)$ of the metric quadruple Q is defined to be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded (see Figure 2).

We are now able to bring the definition of *Wald curvature* [27, 28] (or rather of its modification due to Berestovskiĭ [29]).

Definition 13. Let (X, d) be a metric space. An open set $U \subset X$ is called a *region of curvature* $\ge \kappa$ if and only if any metric quadruple can be isometrically embedded in S_m, for some $m \ge k$. (While not needed in the remainder of the paper, we mention for the sake of completeness that a metric space (X, d) is said to have *Wald-Berestovskiĭ curvature* $\ge \kappa$ if and only if for any $x \in X$ is contained in a region U of curvature $\ge \kappa$).

Remark 14. Evidently, one can consider the Wald-Berestovskiĭ curvature at an accumulation point of a metric space, hence, on a smooth surface, by considering the limits of the curvatures of (nondegenerate) regions of diameter converging to 0.

Before we proceed further, let us make a certain modification of the notation in order to make it more uniform and more familiar to the reader working in classical differential geometry as well as in graphics. Henceforth, we shall denote by K_W the Wald curvature of a surface (*PL* or smooth), by analogy to its classical (Gauss) curvature K (of course, $K_W(p)$ will denote the Wald curvature of a point on the surface).

At this point, the question that naturally rises is whether it is possible to actually compute Wald curvature and, if possible, in what manner. First of all, the first basic step is to note that the role of the abstract open sets U in Definition 13 is naturally played by the cells $\mathfrak{c}_i$. We can state this as a formal definition, for the record.

Definition 15. Let $\mathfrak{c}$ be a cell with vertex set $V_{\mathfrak{c}} = \{v_1, \ldots, v_p\}$. The *embedding curvature* $K(\mathfrak{c})$ of $\mathfrak{c}$ is defined as

$$K(\mathfrak{c}) = \min_{\{i,j,k,l\} \subseteq \{1,\ldots,p\}} \kappa\left(v_i, v_j, v_k, v_l\right). \tag{8}$$

Remark 16. Evidently, the definition above presumes that the cells in the dual complex have at least 4 vertices. However, except for some utterly degenerate (planar) cases, this condition always holds. Moreover, it can be easily corrected by the truncation of the problematic vertices.

It is certainly worthwhile to note that it is possible to actually compute the Wald curvature of each of these cells, using the following formula for the embedding curvature $\kappa(Q)$ of a metric quadruple Q:

$$\kappa(Q) = \begin{cases} 0 & \text{if } \Gamma(Q) = 0, \\ \kappa,\ \kappa < 0 & \text{if } \det\left(\cosh\sqrt{-\kappa}\cdot d_{ij}\right) = 0, \\ \kappa,\ \kappa > 0 & \text{if } \det\left(\cos\sqrt{\kappa}\cdot d_{ij}\right),\ \sqrt{\kappa}\cdot d_{ij} \le \pi \\ & \text{and all the principal minors of order} \\ & 3 \text{ are} \ge 0, \end{cases} \tag{9}$$

where $d_{ij} = d(p_i, p_j)$, $1 \le i, j \le 4$, and $\Gamma(Q) = \Gamma(p_1, \ldots, p_4)$ denotes the Cayley-Menger determinant:

$$\Gamma(p_0, \ldots, p_3) = \begin{vmatrix} 0 & d_{01}^2 & \cdots & d_{13}^2 \\ d_{10}^2 & 0 & \cdots & d_{13}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{30}^2 & d_{31}^2 & \cdots & 0 \end{vmatrix}. \tag{10}$$

Remark 17. (1) For some first numerical results regarding the application of these formulas in a practical context, see [30, 31]. However, it should be noted that, apart from the Euclidean case, the equations involved are transcendental, and cannot be solved, in general, using elementary methods.

(2) We have also employed Wald curvature as a malleable tool in conjunction with Ricci curvature in a somewhat more theoretical context in [10]. We should remark here that, given its (metric) intrinsic nature, K_W "behaves well," so to speak, under Gromov-Hausdorff convergence (see [32, 33] and for the consequences of this fact for applications in graphics, imaging, etc., see [30, 31]). Moreover, since K_W or rather a somewhat modified version of it, K'_W identifies with Rinow curvature (see [34, 35]), it allows us to view the whole problem of defining and computing Ricci for *PL* (polyhedral) manifolds (in particular its applications in graphics, Regge calculus, etc.) in the larger context of Alexandrov spaces (see, e.g., [32, 33]).

Remark 18. Obviously, one can use the same method as above to compute the Ricci curvature (of $\mathcal{T}^*$), according to Stone's original approach for determining directions in cell complexes.

To return to the main problem of this section, from the definitions and results above, we obtain—first discretely, at finite scale bounded away from zero, and then passing to the limit—the following result connecting between the Ricci curvatures of a simplicial (polyhedral) complex and its dual.

Theorem 19. *Let $\mathcal{T}, \mathcal{T}^*$ be as in Lemma 5 above. Then,*

$$\lim_{mesh\,(\mathcal{T}) \to 0} \mathrm{Ric}(\sigma) = \lim_{mesh\,(\mathcal{T}^*) \to 0} C \cdot \mathrm{Ric}^*\left(\sigma^*\right), \tag{11}$$

where $\sigma \in \mathcal{T}$, and where $\sigma^* \in \mathcal{T}^*$ is (as suggested by the notation) the dual of σ.

Remark 20. This result is, admittedly, somewhat vague. However, to our defense, we can only underline the fact that the precise constant C is hard to determine. The thickness condition, that ensures a metric "quasiregularity" of the triangulation, supplies us only with weak estimates. To obtain stronger ones, one should be able to control the regularity of the combinatorical structure as well. This is evident, but it will become even clearer in the sequel. It should be noted in this context that, at least in graphics, mesh improvement techniques allow us to consider such "combinatorial almost regular" triangulations.

We now easily obtain the following theorem.

Theorem 21. *Let M^n be a (smooth) Riemannian manifold, and let $\mathcal{T}$ be a thick triangulation of M^n. Then,*

$$\mathrm{Ric}_{\mathcal{T}} \longrightarrow C_1 \cdot \mathrm{Ric}_{M^n}, \quad \text{as } mesh(\mathcal{T}) \longrightarrow 0, \tag{12}$$

where the convergence is the weak convergence (of measures).

For related results, see [12, 25] for the Lipschitz-Killing curvatures, [10, 36] for discrete (combinatorial, respective metric) Gaussian curvature, and [37], for the Einstein measures.

Proof. The theorem follows easily from Lemma 5; the fact that $\mathrm{Ric}(v)$ is defined in a purely metric, intrinsic manner is from the fact that intrinsic properties are preserved under Gromov-Hausdorff limits (see [33]) and also from Theorem 19. □

Remark 22. While the desired constant C_1 is, of course, $C_1 = 1$, and some first experimental results hint that, at least for certain "nice" triangulations, this is indeed the case, we cannot guarantee a better result—see the remark following the preceding theorem.

Remark 23. While we have adopted the Wald curvature as the metric curvatures for surfaces (and the Finsler-Haantjes one as a metric alternative for the computation of principal curvature) of our choice, for reasons detailed above, it would be interesting to explore the capabilities—both theoretical and practical—as far as *PL* differential geometry is concerned, of other metric curvatures (see [12] and the bibliography therein) and in particular of the *Menger curvature measure* [38]:

$$\mu(\mathcal{T}) = \mu_p(\mathcal{T}) = \sum_{T\in\mathcal{T}} \kappa_M^p(T)(\operatorname{diam} T)^2, \tag{13}$$

for some $p \geq 1$, where $\kappa_M(T)$ denotes the Menger curvature (of the simplex T).

3. The Bonnet-Myers Theorem

Having introduced a metric Ricci curvature for *PL* manifolds, one naturally wishes to verify that this represents, indeed, a proper notion of Ricci curvature and not just an approximation of the classical notion. According to the synthetic approach to differential geometry (see, e.g., [3, 33]), a proper notion of Ricci curvature should satisfy adapted versions of the main, essential theorems, that hold for the classical notions. Amongst such theorems the first and foremost is Myers' theorem (see, e.g., [21]). And, indeed, fitting versions for combinatorial cell complexes and weighted cell complexes were proven, respectively, by Stone [16, 17] and Forman [39]. Moreover, the Bonnet part of the Bonnet-Myers theorem, that is, the one appertaining to the sectional curvature, was also proven for *PL* manifolds, again by Stone—see [40, 41].

3.1. The 2-Dimensional Case. For the special—yet of main importance in applications (see [4, 5, 10])—case of 2-dimensional manifolds, such a result is easy to prove, given the fact that Ricci and sectional curvature essentially coincide. More precisely, we can formulate the following theorem.

Theorem 24 (Bonnet-Myers for *PL* 2-manifolds—combinatorial). *Let M^2_{PL} be a complete, connected 2-dimensional PL manifold such that*

(i) *there exists $d_0 > 0$, such that mesh $(M^2_{PL}) \leq d_0$, (where mesh (M^2_{PL}) denotes the mesh of the 1-skeleton of M^2_{PL}, i.e., the supremum of the edge lengths);*

(ii) $K_{Comb}(M^2_{PL}) \geq K_0 > 0$.

Then M^2_{PL} is compact, and moreover

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \begin{cases} 2\pi d_0, & k_0 \geq \left(2-\sqrt{2}\right)\pi, \\ \dfrac{4\pi^3 d_0}{\left[\left(2\pi - d_0\right)\left(4\pi k_0 - k_0^2\right)^{1/2}\right]} & \text{else}, \end{cases} \tag{14}$$

where K_{Comb} denotes the combinatorial Gauss curvature of M^2_{PL},

$$K_{Comb}(v_i) = 2\pi - \sum_{p=1}^{m_i} \alpha_p(v_i), \tag{15}$$

where $\alpha_1, \ldots, \alpha_{m_i}$ are the (interior) face angles adjacent to the vertex v_i.

Remark 25. Condition (1), that ensures that the set of vertices of the *PL* manifold is "fairly dense" (in Stone's formulation [41, p. 1062]), is nothing but the necessary and quite common density condition for good approximation both of distances and of curvature measures—see for example [12, 25] and the references therein. The mere existence of such a d_0 is evident for a compact manifold; however, it can't be apriorily supposed for a general manifold; hence, has to be postulated. Moreover, to ensure a good approximation of curvature, this density factor has to be properly chosen (see, e.g., [42]); thus, tighter estimates for the mesh of the triangulation can be obtained from (14) along with better curvature approximation. Not less importantly, an adequate choice of the vertices of the triangulation also ensures, via the thickness property, the nondegeneracy of the manifold (and of its curvature measures)—see [12].

Proof. The theorem follows readily from Theorem 3 of [40]. Indeed, in the two-dimensional case, the so-called *maximum* and *minimum curvatures*, k_+, respective k_- (see [40, p. 12], for the precise definitions), at the vertices of M^2_{PL} coincide with the combinatorial Gauss curvature. Moreover, due to the fact that here we are concerned solely with 2-dimensional simplicial complexes (*PL* manifolds), conditions (1) and (2) of Theorem 3 of [40] are equivalent, respectively, to our conditions (2) and (1) above. Therefore, the conditions in the statement of Theorem 3 of [40] are satisfied, and, by (ii) of the said result, the theorem follows immediately. □

Remark 26. It is easy to see that the theorem above extends to more general polyhedral surfaces. Indeed, by their very definition such surfaces admit simplicial subdivisions. However, during this subdivision process k_+ and k_- do not change since the only relevant contributions to these quantities occur at the vertices and depend only on the angles at these vertices, more precisely on the *normal geometry* (see [40, p. 12]), that suffer no change during the subdivision process.

Remark 27. The bound (14) is rather weak, as compared to the one for the classical case, but it is the only one supplied by Stone's result we made appeal to, namely, Theorem 3 of [40].

The proof above suffers from the disadvantage of making use of Stone's maximum and minimum curvatures even though in this context making appeal to them is rather natural. This drawback represents the main reason we did not feel compelled to detail here Stone's definitions, but rather to refer the reader to the original paper. We can, however, provide a different proof, independent of Stone's work but at the price of using some heavy (albeit classical) machinery, that, moreover, takes us away, so to say, from the discrete methods. On the other hand, smooth, analytical tools are far more familiar to a large research community in CAGD, imaging, and so forth.

Alternative Proof. The basic idea (which we first employed in [10]) is to consider a *smoothing* M^2 of M^2_{PL}. Since, by [24, Theorem 4.8], smoothings approximate arbitrarily well both distances and angles (more precisely, they are δ-approximation hence, for δ is small enough, also ε-approximations of M^2_{PL}—for details see [24], or, just for the minimal required facts, the Appendix of [10]) on M^2_{PL}, defects are also arbitrarily well approximated. Given that the combinatorial curvature of M^2_{PL} is bounded from below, it follows that so will be the sectional (i.e., Gauss) curvature of M^2.

Unfortunately, the Gaussian curvature of M^2 is positive only on isolated points (the set of vertices of M^2_{PL}), so we cannot apply the classical Bonnet theorem yet. However, we can ensure that M^2 is arbitrarily close to a smooth surface M^2_+, having curvature Gaussian curvature $K(M^2_+) > 0$. (This is easily seen by adding spherical "roofs" (of low curvature) over the faces and then slightly modifying the construction, to ensure that the curvature will be positive also on the "sutures" of the said roofs, corresponding to the edges of the original *PL* manifold). Therefore, the classical Bonnet theorem can be applied for M^2_+; hence, M^2_{PL} is compact and its diameter has the same upper bound (again using the same arguments as before (i.e., δ- and ε-approximations as that of M^2_+ (and M^2); namely,

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \frac{\pi}{\sqrt{K_0}}. \tag{16}$$

□

Remark 28. Apparently, the bound for diameter given by the proof above is tighter than the one obtained by Stone in [40]. Nevertheless, we should keep in mind that, in practice, one is more likely to encounter *PL* surfaces as approximations of smooth ones (and, obviously, *PL* surfaces are *PL* approximations of their own smoothings). However, the larger the mesh of the approximating surface (i.e., the "rougher" the approximation), the larger the deviation of the approximating triangles from the tangent planes is (at the vertices); hence, the more likely it is to obtain large combinatorial curvature. Hence, there is a correlation between the size of the simplices and curvature, even though it is not a straightforward one.

Since the leitmotif of the previous section was metric (Wald) curvature, it is natural to ask whether a fitting version of the Bonnet-Myers theorem exists for this type of curvature. The answer is—at least in dimension 2—positive: we can, indeed, state an analogue of Myers' theorem, in terms of the Wald curvature.

Theorem 29 (Bonnet-Myers for *PL* 2-manifolds—metric). *Let M^2_{PL} be a complete, connected 2-dimensional PL manifold such that*

(i′) *there exists $d_0 > 0$, such that mesh$(M^2_{PL}) \leq d_0$;*

(ii′) $K_W(M^2_{PL}) \geq K_0 > 0$.

Then M^2_{PL} is compact, and, moreover

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \frac{\pi}{\sqrt{K_0}}. \tag{17}$$

Proof. We employ again the basic argument first used in [10]. Since distances (and angles) are arbitrarily well approximated by smoothings, it follows that so are metric quadruples (including their angles); hence, so is Wald curvature. By [34] (see also [35], Theorems 11.2 and 11.3), the Wald curvature at any point of nontrivial geometry M^2, namely, at a vertex v, $K_W(v)$ equals the classical (Gauss) curvature $K(v)$ (and, of course, this is also true a fortiori at all the other points, where both the smooth and the *PL* manifolds are flat). Therefore; the Gauss curvature of M^2 approximates arbitrarily well the Wald curvature of M^2_{PL}; hence, we can apply the same argument as in the second proof of Theorem 24 to show that M^2_{PL} is, indeed, compact and, furthermore, satisfies the upper bound (16). □

Remark 30. Like the previous theorem, the result above can be extended to polyhedral manifolds, and even in a more direct fashion, since Wald curvature does not take into account the number of sides of the faces incident to a vertex but only their lengths.

Remark 31. This result, as well as its generalization to higher dimensions (see Theorem 33), is hardly surprising, given the fact that, by [43], Theorem 29, Myers' theorem holds for general Alexandrov spaces of curvature $\geq K_0 > 0$, and Wald-Berestovskii curvature is essentially equivalent to the Rinow curvature (see [34]), hence, to the Alexandrov curvature (see, e.g., [33, Chapter 1]). Rather, we give in the special case of *PL* surfaces (manifolds) a simpler, more intuitive proof of the Burago-Gromov-Perelman extension of Myers' theorem.

In higher dimension, none of the arguments applied in both proofs of Theorem 24 are applicable, at least not without imposing further conditions.

(1) Regarding the first proof, we have the following.

(i) In dimensions higher than 2, k_+ and k_- do not necessarily equal each other (see [40, Example 4, p. 14]), and, a fortiori, they fail to equal the combinatorial Gauss curvature. They do, however, according to Stone [40], resemble in their behaviour the minimum, respective maximum sectional curvature at a point common to 2-planes, that contain a given (fixed) tangent vector at the point in question.

An important proviso should be added, however: while for the general *PL* simplicial complexes, the equality between k_+ and k_- fails to hold, it is true for the most relevant—at least as far as our analysis is concerned-case, namely that of *PL* manifolds without boundary (see [40, Example 3, p. 13]). Therefore, in the light of the facts above, it follows that while for a fairly general and important setting the connection is straightforward, it is not clear how to compare, in the general case, our proposed metric discretization of Ricci curvature, with the maximal and minimal curvatures of Stone (hence to combinatorial curvature, whenever they equal it—and each other). A natural attempt would be to use straightforward extensions of k_+ and k_-—let us denote them, for convenience by, $\mathrm{Ric}_{\min}$ and $\mathrm{Ric}_{\max}$. However, it is not clear (at least at this point in time) how expressive these definitions would prove to be.

Remark 32. It is true that the lower bound on k_+, as considered in Theorem 3 of [40], has a simple expression, in any dimension, via a topological condition (cf. [40, Lemma 5.1]); namely, that the intersection of any (PL) geodesic segment of ends p and q with the 2-skeleton of M^2_{PL} is precisely the set $\{p, q\}$ (with the exception, of course, of the case when the segment is contained in a simplex. However, since the metric information contained in this new condition is void (or rather thoroughly encrypted, so to say), it has no apparent advantage for application in conjunction with metric curvature).

(ii) For an application of the Stone's methods in combination with the metric curvature approach to any dimension, one would have to make appeal to Jacobi fields, as defined in [16]. However, as discussed in the previous section, this would probably lead to numerical instability.

(2) Regarding the second proof we have the following.

> No smoothing of a PL manifold necessarily exists in a dimension higher than $n > 4$, and even if it exists, it is not necessarily unique, for $n \geq 4$ —see [44].
>
> However, if such a smoothing exists, then the second proof of Theorem 24 (and of Theorem 29) extends to any dimension, and we obtain the following PL (metric) versions of the classical results.

Theorem 33 (PL Bonnet—metric). *Let M^n_{PL} be a complete, n-dimensional PL, smoothable manifold without boundary, such that*

(i″) *there exists $d_0 > 0$, such that mesh $(M^n_{PL}) \leq d_0$;*

(ii″) $K_W(M^n_{PL}) \geq K_0 > 0$,

where $K_W(M^n_{PL})$ denotes the sectional curvature of the "combinatorial sections", that is, the cells c_i (see Section 1).

Then M^n_{PL} is compact, and, moreover,

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \frac{\pi}{\sqrt{K_0}}. \tag{18}$$

Proof. We should note in the beginning that the "rounding" argument of the second proof of Theorem 24 is not easy to extend directly—if at all—to higher dimension. Instead, a more subtle argument has to be devised. To this end, we make appeal again to Stone's paper [40], and we build the spherical simplicial complex $M^n_{\mathrm{Sph},\rho}$ associated with the given PL (or rather piecewise-flat) complex M^n_{PL}. This is constructed as follows: consider the sphere of radius $R = R(\sigma)$ and radius $O = O(\sigma)$, circumscribed to a given simplex σ, and its image $\sigma^* = \sigma^*(R^*)$ on a sphere of radius $R^* = R^*(\sigma), R^* \geq R$, via the central projection from O. We denote by $M^n_{\mathrm{Sph},\rho}$ the simplicial complex obtained by the remetrization of M^n_{PL} by the replacement of each σ by its spherical counterpart σ^*. Then, by Lemma 5.5 of [40], for large enough $R^* > R$, the following holds for any pair of points $p, q \in M^n_{PL}$: $\mathrm{dist}_{M^n_{PL}}(p, q) \leq C\mathrm{dist}_{M^n_{\mathrm{Sph},\rho}}(p^*, q^*)$, for a certain constant C, where p^*, q^* denote the spherical images of p, q. Since the curvature at each vertex of the spherical simplex obtained by central projection of the simplices of M^n_{PL} onto their circumscribed spheres is smaller than the corresponding one (at the same vertex) in the PL (piecewise-flat manifold), this holds a fortiori for $M^n_{\mathrm{Sph},\rho}$. It follows from the classical Bonnet theorem (after applying the necessary smoothing) that $\operatorname{diam}(M^n_{PL}) < \operatorname{diam}(M^n_{\mathrm{Sph},\rho})$. □

Remark 34. An approach similar to the one used in the proof above was also employed by Cheeger [45] in a rather similar context. We should stress here that, as a by product of the results in this paper, we also address—using our own methods—a problem posed by Cheeger in [45, Remark 3.5].

We should underline the fact that if we approach the problem of PL Ricci curvature from the viewpoint of the first part of the paper, that is, of PL (secant) approximations of smooth manifolds, then the situation changes dramatically. Indeed, even when such a smoothing M^n ($n \geq 3$) exists, it is not probable that its sections provided by M^n_{PL}, in the manner indicated in Section 2, suffice to approximate well enough—let alone reconstruct—the Ricci curvature of M^n. In simple words, "there are not enough directions" in M^n_{PL} to allow us to infer from the metric curvatures of a PL approximation, those of a given smooth manifold M^n (in fact, not even a good approximation); hence, we are faced again with a problem that we already mentioned in conjunction with the first proof, namely, that of insufficient "sampling of directions" in PL approximations. (On the other hand, the increasing of the number of directions, that is, of 2-dimensional sections (simplices), generates a decrease of the precision of the approximation due to the (possible) loss of thickness of the triangulation—a problem which we have discussed in some detail in [12].)

To sum up, all the considerations above show us that, unfortunately, in higher dimensional, no general analogue of Myers' theorem for PL manifolds can be obtained by applying solely smoothing arguments. It is true that *a* Ricci curvature of the smooth manifold M^n is obtained in terms of M^n_{PL}; however, it is not clear, in view of the paucity of sectional directions (i.e., possible 2-sections), how precisely this is connected to its discrete counterpart. Therefore, we can obtain, at best, an approximation result (with limits imposed by the thickness constraint—see discussion above).

We conclude with the following remarks. From the discussion above it is transparent that, unfortunately, at this point in time we can offer no proof for the general case, that is, for non-smoothable PL manifolds of dimension $n \geq 4$. To obtain such a proof for Bonnet's theorem, one should adapt Stone's methods, as developed in [40]; while for a comprehensive generalization of Myers' theorem, one has the apparently more difficult task of accordingly modifing, for the metric case, the purely combinatorial methods of [16]. A quite different approach, but one that would allow us to extend the metric approach to quite general weighted *CW* complexes, would be to adjust Forman's methods developed in [39] to our case. The essential step in this direction would be to find relevant geometric content (e.g., lengths, area, and volume) for Forman's "standard weights" associated to each cell.

3.2. Wald Curvature and Alexandrov Spaces. In this section, we bring a proof of a more general case, albeit at the price of using some extraneous and powerful techniques and results. For this, we first need some further preparations and background material, mainly the notion of *Alexandrov curvature*. Since this represents, by now, a quite classical and standard notion, and since introducing it formally here would take us too far afield, we will not bring here the technical definition and related material, but rather we refer the reader to, for example, [32]. However, we should mention that, in defining Alexandrov curvature, one makes appeal to *comparison triangles* in the model space (i.e., gauge surface S_κ), rather than quadrangles, as in the definition of Wald curvature.

It is more important to point out that Wald's curvature is essentially equivalent with the much more modern notion of Alexandrov curvature, at least for spaces in which there exist "sufficiently many" minimal geodesics (see, for instance, [46], Corollary 40), a condition that certainly is fulfilled in *PL* surfaces. For the practical consequences of the similarities and differences between the two approaches, see [11]. The reason we prefer working with the Wald curvature, is that it is computable, and, moreover, it has even simpler, more practical approximations—see [31]. For further theoretical relative advantages of the curvature types discussed above, see [46]. We should perhaps mention, that, in fact, we have first considered Wald's curvature—and the metric approach to curvature in general—as means of computing, in a direct and applicable manner, Alexandrov's curvature.

It is, however, important to notice that one has taken into account the "discrete" nature of the types of spaces considered, hence to compute solely the Wald curvature of the 1-star neighbourhood of a vertex, as already stressed above, and not to consider (ever) smaller neighbourhoods, as perhaps natural in other contexts. This, however, agrees with the method of computing discrete curvature as angular defect, as employed, for instance, in [36] and in the Chow-Luo discrete Ricci flow [4] (as well as in many other instances—see the bibliography for some of them). A positive consequence of this fact is that any such neighbourhood becomes a region having the same Alexandrov curvature bounded from below as the computed Wald one. Moreover, by the Alexandrov-Toponogo theorem (see, e.g., [46], Theorem 43 and its proof, pp. 837–840), the whole surface becomes a space of curvature (Wald or Alexandrov) bounded from below.

Moreover, taking into account only these "discrete" neighbourhoods is very important when equating the Wald and Alexandrov curvature, since it allows avoiding the blowup of Alexandrov curvature at the vertices during smoothing. However, if one still wishes to consider smaller-and-smaller neighbourhood of the vertices (motivated, perhaps, by other applications, then imaging and graphics, such as those in Regge calculus [25]), one can resort to the basic approach of Brehm and Kühnel, that is, "rounding" the edges by cylinders of radius ε (without any change in curvature) and replacing the polyhedral cones at the vertices by smooth "caps," up to a predetermined admissible error of, say, ε_1. Note that such a "filtration" of K_W by Gaussian curvature (of the approximating smooth surfaces) is in concordance with common practices in imaging, vision and, indeed, in many applicative fields. In addition, considering only this "discrete" neighbourhoods is very important when equating the Wald and Alexandrov curvature since it also allows us to avoid the blowup of Alexandrov curvature at the vertices during smoothing.

If one is willing to make appeal to the theory of Alexandrov spaces, then, by using this equivalence of Wald and Alexandrov curvatures with the above mentioned provisos, a result of the desired type follows immediately.

Theorem 35 (Bonnet-Myers-Alexandrov Spaces). *Let M^n_{PL} be a complete, connected PL manifold, such that $K_W(M^n_{PL}) \geq K_0 > 0$.*

Then M^n_{PL} is compact, and, moreover

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \frac{\pi}{\sqrt{K_0}}. \tag{19}$$

Proof. The theorem follows from [46, Corollary 47, p. 840] and from the fact that M^n_{PL} is locally compact. □

3.2.1. Thick Cell Complexes. Determining whether a general *PL* complex has Wald curvature bounded from below can be, in practice, quite difficult. However, in the special case of thick complexes (see definition in Section 1) one can determine a simple criterion as follows.

Lemma 36. *Let $M = M^n_{PL}$ be a complete, connected PL manifold thickly embedded in some $\mathbb{R}^N$, such that $K_W(M^2) \geq K_0 > 0$, where M^2 denotes the 2-skeleton of M. Then there exists $K_1 > 0$ such that $K_W(M^n_{PL}) \geq K_1 > 0$.*

Sketch of Proof. We indicate a proof only for the case $n = 3$; the general case follows by a simple inductive argument. Consider an edge e belonging to the 1-skeleton of M^n_{PL}. We have to show that $K_W(Q) \geq K_0 > 0$, for any quadruple incident to e. If Q is one of the quadruples determined by the original cells of M^2 (such as Q_1 in Figure 3), the condition is fulfilled trivially since $K_W(M^2) \geq K_0 > 0$. Otherwise, the edges of Q are either edges of the original cells (see Figure 3), or diagonals of such cells (e.g., d in Figure 3) or they connect vertices belonging to two different cells of the given complex (such as $\tilde{e}$ in Figure 3 connecting between vertices of the cells $\mathfrak{c}_2$ and $\mathfrak{c}_3$). But, it is quite standard to show that, by the fatness of the cells $\mathfrak{c}_i$, there exists a constant c_1 such that $(1/c_1)e \leq d \leq c_1 e$. In a similar manner, using the boundedness from below of the angles as an equivalent definition of thickness, one can show that the fatness of the embedding implies that there exists a c_2 such that $(1/c_2)e \leq \tilde{e} \leq c_2 e$.

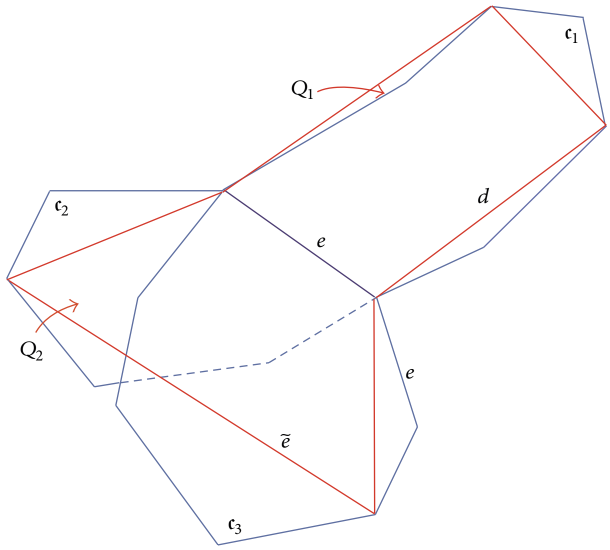

FIGURE 3: Thickness of metric quadruples adjacent to an edge in an embedded piecewise-flat 3-manifold.

The desired conclusion follows from the two double inequalities above and from the continuity of the determinant function that defines the Wald curvature.

The fitting version of Bonnet-Myers now follows as a direct corollary.

Theorem 37 (Bonnet-Myers—thick complexes). *Let $M = M^n_{PL}$ be a complete, connected PL manifold thickly embedded in some $\mathbb{R}^N$, such that $K_W(M^2) \geq K_0 > 0$, where M^2 denotes the 2-skeleton of M. Then M^n_{PL} is compact, and, moreover,*

$$\operatorname{diam}\left(M^2_{PL}\right) \leq \frac{\pi}{\sqrt{K_0}}. \tag{20}$$

Remark 38. We have formulated the theorem in terms of piecewise-flat manifolds since this is the case of the most interest, both for theoretical ends (see, e.g. [13, 23, 25]) and application oriented ones (see, for instance, [42]). The most natural and useful instance in which such manifolds arise is that of secant approximations to smooth manifolds, as emphasized in most of the papers mentioned above. However, the proof extends—mutatis mutandis—to the case of spaces whose simplices are modelled after spherical or hyperbolic spaces.

4. A Comparison Theorem

Up to this point, we have not yet defined the sectional curvature $K(\mathfrak{c})$ of a cell $\mathfrak{c}$. In light of our preceding discussion and results, the following definition is quite natural.

Definition 39. Let $M = M^n_{PL}$ be an n-dimensional *PL* manifold (without boundary). The *scalar metric curvature* scal_W of M is defined as

$$\mathrm{scal}_W(v) = \sum K_W(\mathfrak{c}), \tag{21}$$

the sum being taken over all the cells of M^* incident to the vertex v of M^*.

Using this definition and the results of Section 2, we immediately (and, in fact, quite trivially, since the result holds, regardless of the specific definition for the curvature of a cell) obtain, the following generalization of the classical curvature bounds comparison in Riemannian geometry (compare also with [47, Theorem 1]).

Theorem 40 (Comparison theorem). *Let $M = M^n_{PL}$ be an n-dimensional PL manifold (without boundary), such that $K_W(M) \geq K_0 > 0$; that is, $K(\mathfrak{c}) \geq K_0$, for any 2-cell of the dual manifold (cell complex) M^*. Then*

$$K_W \lesseqqgtr K_0 \Longrightarrow \mathrm{Ric}_W \lesseqqgtr nK_0. \tag{22}$$

Moreover,

$$K_W \lesseqqgtr K_0 \Longrightarrow \mathrm{scal}_W \lesseqqgtr n(n+1)K_0. \tag{23}$$

Remark 41. (1) Inequality (23) can be formulated in the seemingly weaker form

$$\mathrm{Ric}_W \lesseqqgtr nK_0 \Longrightarrow \mathrm{scal}_W \lesseqqgtr n(n+1)K_0. \tag{24}$$

(2) Note that in all the inequalities above, the dimension n appears, rather than $n-1$ as in the smooth, Riemannian case (hence, for instance, one has in (23) $n(n+1)K_0$ instead of $n(n-1)K_0$, (On the other hand, this holds even if $n = 3!\ldots$) as in the classical case). This is due to our definition (5) of Ricci (and scalar) curvature, via the dual complex of the given triangulation, hence, imposing standard and simple combinatorics, at the price of allowing for only such weaker bounds (without affecting the analogue of the Bonnet-Myers theorem—see Section 2 above).

5. Final Remarks

We conclude with a few short comments, regarding future study.

(i) The first and foremost concern would be to develop a proof for the general case, along the lines sketched above, and without making appeal to the theory of Alexandrov spaces.

(ii) Another less urgent task would be to provide full proofs and sharp, specific bounds in all the cases where these were only summarily sketched.

(iii) Note that the most common discretization of Gaussian curvature, namely, the angle deficit approach, corresponds in fact to curvature × area measure, so it is, essentially, a *curvature measure* (for a further development of this idea, see, e.g., [25]). In contrast, in our approach Wald's curvature is a (point-wise) *curvature function*. It is, therefore, only natural to try and define a curvature measure based on Wald's curvature. This would allow a better comparison with the previous works where the defect based discretization of curvature is used, not least the ones relating to the discrete Ricci flow [4, 5]. In addition, this would allow for a comparison with the other metric approaches to the curvature of *PL* manifolds (see Remark 23 above).

(iv) As a last—but certainly not least—open problem that we believe to be worth solving is to develop a fitting notion of "Einstein metric" associated to the metric Ricci curvature introduced in this paper. Such a metric will probably be related to the stationary points of the Regge functional.

Acknowledgments

The authors would like to thank the anonymous referee for his correction, constructive critique and stimulating questions, and for his relevant bibliographical references. In addition, they would like to thank the reviewer for bringing to their attention some new relevant bibliographical sources, as well as for suggesting the problem stated in Section 5(iv). Emil Saucan research is supported by Israel Science Foundation Grants 221/07 and 93/11 and by European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement no. [203134]. (The authors thank the anonymous reviewer for bringing to their attention this problem and its possible solution.).

References

[1] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," *http://arxiv.org/abs/math/0211159*. In press.

[2] G. Perelman, "Ricci flow with surgery on three-manifolds," http://arxiv.org/abs/math/0303109.

[3] C. Villani, *Optimal Transport, Old and New*, vol. 338 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 2009.

[4] B. Chow and F. Luo, "Combinatorial Ricci flows on surfaces," *Journal of Differential Geometry*, vol. 63, no. 1, pp. 97–129, 2003.

[5] X. D. Gu and S. T. Yau, *Computational Conformal Geometry*, Advanced Lectures in Mathematics no. 3, International Press, Somerville, Mass, USA, 2008.

[6] L. Astola, *Multi-scale riemann-finsler geometry [Ph.D. thesis]*, Eindhoven, The Netherlands, 2010.

[7] A. Fuster, L. Astola, and L. Florack, "A riemannian scalar measure for diffusion tensor images," in *Computer Analysis of Images and Patterns*, vol. 5702 of *Lecture Notes in Computer Science*, 419426, pp. 419–426, Springer, Berlin, Germany, 2009.

[8] E. Saucan, E. Appleboim, G. Wolansky, and Y. Y. Zeevi, "Combinatorial Ricci curvature and laplacians for image processing," in *Proceedings of the 2nd International Congress on Image and Signal Processing (CISP '09)*, vol. 2, pp. 992–997, Tianjin, China, 2009.

[9] E. Appleboim, E. Saucan, and Y. Y. Zeevi, "Ricci curvature and flow for image denoising and superesolution," in *Proceedings of the European Signal Processing Conference (EUSIPCO '12)*, pp. 2743–2747, Bucharest, Romania, 2012.

[10] E. Saucan, "On the properties of the combinatorial Ricci flow for surfaces," http://arxiv.org/abs/1104.2033.

[11] E. Saucan, "A metric Ricci flow for surfaces and its applications," *Imaging and Computing*, vol. 1, 2013.

[12] E. Saucan, "Fat triangulations and differential geometry," http://arxiv.org/abs/1108.3529 .

[13] E. Saucan and M. Katchalski, "Fat triangulations, curvature and quasiconformal mappings," *Axioms*, vol. 1, no. 2, pp. 99–110, 2012.

[14] J. H. Rubinstein, "Problems around 3-manifolds," vol. 12 of *Geometry and Topology Monographs*, Geometry & Topology Publications, 2007.

[15] A. Bernig, "On some aspects of curvature," Preprint, 2003.

[16] D. A. Stone, "A combinatorial analogue of a theorem of Myers," *Illinois Journal of Mathematics*, vol. 20, no. 1, pp. 12–21, 1976.

[17] D. A. Stone, "A combinatorial analogue of a theorem of Myers," *Illinois Journal of Mathematics*, vol. 20, pp. 551–554, 1920.

[18] P. M. Alsing, J. R. McDonald, and W. A. Miller, "The simplicial Ricci tensor," *Classical and Quantum Gravity*, vol. 28, no. 15, Article ID 155007, p. 17, 2011.

[19] D. Glickenstein, "Discrete conformal variations and scalar curvature on piecewise flat two- and three-dimensional manifolds," *Journal of Differential Geometry*, vol. 87, no. 2, pp. 201–237, 2011.

[20] A. Trout, "Positively curved combinatorial 3-manifolds," *Electronic Journal of Combinatorics*, vol. 17, no. 1, article R49, 23 pages, 2010.

[21] M. Berger, *A Panoramic View of Riemannian Geometry*, Springer, Berlin, Germany, 2003.

[22] J. F. Hudson, *Piecewise Linear Topology*, Lecture Notes in Mathematics, W. A. Benjamin, New York, NY, USA, 1969.

[23] E. Saucan, "Note on a theorem of Munkres," *Mediterranean Journal of Mathematics*, vol. 2, no. 2, pp. 215–229, 2005.

[24] J. R. Munkres, *Elementary Differential Topology*, Princeton University Press, Princeton, NJ, USA, 1966.

[25] J. Cheeger, W. Mäuller, and R. Schrader, "On the curvature of piecewise flat spaces," *Communications in Mathematical Physics*, vol. 92, no. 3, pp. 405–454, 1984.

[26] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, AMS Chelsea Publishing, American Mathematical Society, Chelsea, NY, USA, 1952.

[27] A. Wald, "Sur la courbure des surfaces," *Comptes Rendus de l'Académie des Sciences*, vol. 201, no. 21, pp. 918–920, 1935.

[28] A. Wald, "Begreudeung einer koordinatenlosen Differentialgeometrie der Fläachen," *Ergebnisse Eines Mathematischen Kolloquiums, First Series*, vol. 7, pp. 24–46, 1936.

[29] V. N. Berestovskiĭ, "Spaces with bounded curvature and distance geometry," *Siberian Mathematical Journal*, vol. 16, pp. 8–19, 1986.

[30] E. Saucan, "Surface triangulation—the metric approach," http://arxiv.org/abs/cs/0401023.

[31] E. Saucan and E. Appleboim, "Metric methods in surface triangulation," in *Mathematics of Surfaces XIII*, vol. 5654 of *Lecture Notes in Computer Science*, pp. 335–355, Springer, Berlin, Germany, 2009.

[32] D. Burago, Y. Burago, and S. Ivanov, *Course in Metric Geometry*, vol. 33 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2000.

[33] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, vol. 152 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 1999.

[34] L. M. Blumenthal, *Distance Geometry—Theory and Applications*, Clarendon Press, Oxford, UK, 1953.

[35] L. M. Blumenthal and K. Menger, *Studies in Geometry*, W. H. Freeman and Company, San Francisco, Calif, USA, 1970.

[36] U. Brehm and W. Kähnel, "Smooth approximation of polyhedral surfaces regarding curvatures," *Geometriae Dedicata*, vol. 12, no. 4, pp. 435–461, 1982.

[37] A. Bernig, "Curvature tensors of singular spaces," *Differential Geometry and its Applications*, vol. 24, no. 2, pp. 191–208, 2006.

[38] G. Lerman and J. T. Whitehouse, "On d-dimensional d-semimetrics and simplex-type inequalities for high-dimensional sine functions," *Journal of Approximation Theory*, vol. 156, no. 1, pp. 52–81, 2009.

[39] R. Forman, "Bochner's method for cell complexes and combinatorial Ricci curvature," *Discrete & Computational Geometry*, vol. 29, no. 3, pp. 323–374, 2003.

[40] D. A. Stone, "Geodesics in piecewise linear manifolds," *Transactions of the American Mathematical Society*, vol. 215, pp. 1–44, 1976.

[41] D. A. Stone, "Sectional curvatures in piecewise linear manifolds," *American Mathematical Society*, vol. 79, no. 5, pp. 1060–1063, 1973.

[42] E. Saucan, E. Appleboim, and Y. Y. Zeevi, "Geometric approach to sampling and communication," *Sampling Theory in Signal and Image Processing*, vol. 11, no. 1, pp. 1–24, 2012.

[43] Y. D. Burago, M. L. Gromov, and G. Y. Perelman, "A. D. Aleksandrov spaces with curvatures bounded below," *Russian Mathematical Surveys*, vol. 47, no. 2, p. 158, 1992.

[44] J. R. Munkres, "Obstructions to the smoothing of piecewise-differentiable homeomorphisms," *Annals of Mathematics*, vol. 72, no. 2, pp. 521–554, 1960.

[45] J. Cheeger, "A vanishing theorem for piecewise constant curvature spaces," in *Curvature and Topology of Riemannian Manifolds (Katata, 1985)*, vol. 1201 of *Lecture Notes in Mathematics*, pp. 33–40, Springer, Berlin, Germany, 1986.

[46] C. Plaut, "Metric spaces of curvature $\geq k$," in *Handbook of Geometric Topology*, R. J. Daverman and R. B. Sher, Eds., pp. 819–898, Elsevier, Amsterdam, The Netherlands, 2002.

[47] A. Bernig, "Curvature bounds of subanalytic spaces," Preprint, 2003.