TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 317, Number 2, February 1990

METRIC TRANSFORMS AND EUCLIDEAN EMBEDDINGS

M. DEZA AND H. MAEHARA

ABSTRACT. It is proved that if $0 \leq c \leq 0.72/n$ then for any n-point metric space (X, d), the metric space (X, d^c) is isometrically embeddable into a Euclidean space. For 6-point metric space, $c = \frac{1}{2}\log_2 \frac{3}{2}$ is the largest exponent that guarantees the existence of isometric embeddings into a Euclidean space. Such largest exponent is also determined for all n-point graphs with "truncated distance".

1. INTRODUCTION

One of the fundamental problems in distance geometry is the isometric embedding problem, that is, to determine conditions for metric spaces to be isometrically embeddable in a given class of spaces, say l_p-spaces (e.g. [1–5, 8, 11, 12, 18, 23]). For finite metric spaces, there also arises the minimum dimensional embedding problem, that is, to decide the minimum dimension $m = m_p(n)$ such that any n-point subset of an l_p-space can be isometrically embedded in l_p^m. Except for the case $p = 2$ (in this case, $m_2(n) = n - 1$), the problem is not trivial. For some bounds on $m_p(n)$, $p \neq 2$, see e.g. Ball [6], Witsenhausen [24].

In this paper, we consider deforming the distance function of a metric space so that the resulting space can be embedded in a certain space, say l_2. This problem has its origin in the 1930s (e.g. [7, 16, 19, 22]).

Throughout this paper, X stands for a finite metric space and X_n stands for an n-point metric space. The distance between two points x, y of a metric space is denoted by the juxtaposition xy. If X is isometric to a subset of a Euclidean space, then X is said to be *Euclidean*. Let $F(t)$ be a continuous, monotone increasing, concave function of $t \geq 0$ with $F(0) = 0$. Then, replacing the distance xy in X by $F(xy)$, we have another metric space with the same point set as X. This new metric space is called the *metric transform* of X by $F(t)$, and it is denoted by the symbol $F(X)$.

Blumenthal [8, p. 131] proved that if $0 < c \leq \frac{1}{2}$ then the metric transform $(X_4)^c$ by $F(t) = t^c$ is Euclidean. Further, he showed that $c = \frac{1}{2}$ is the largest exponent that guarantees $(X_4)^c$ to be Euclidean for all X_4. We are going to

Received by the editors May 12, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 51K05, 51M05; Secondary 05C10.

consider similar problems for X_n, $n > 4$, in connection with "hypermetric" spaces (e.g. [3, 11, 13]).

We will prove that if $0 < c \leq 0.7213\ldots/n$, then $(X_n)^c$ is always Euclidean. In particular, $c = \frac{1}{2}\log_2\frac{3}{2} = 0.2924\ldots$ is the largest exponent that guarantees $(X_6)^c$ to be Euclidean.

The vertices of a connected graph G constitute a metric space with the shortest path distance. We denote this metric space by the same letter G. Winkler [23] gave a characterization of the connected graphs G whose metric transforms $(G)^{1/2}$ are Euclidean.

For a graph G, define the *truncated distance* between two vertices x, y of G by

$$xy = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \text{ and } y \text{ are adjacent}, \\ 2 & \text{otherwise.} \end{cases}$$

The metric space consisting of the vertices of G with truncated distance is denoted by G_T. If G is a graph with diameter ≤ 2, then G and G_T are the same space. Isometric embeddings of G_T into Hamming hypercubes and l_1 were considered in [2, 3].

We will prove that c_n, the largest c such that $(G_T)^c$ are Euclidean for all n-point graphs G, is given by

$$c_n = \begin{cases} \frac{1}{2}\log_2(k/(k-1)) & \text{if } n = 2k, \\ \frac{1}{2}\log_2(2k(k+1)/(2k^2-1)) & \text{if } n = 2k+1. \end{cases}$$

2. Preliminaries

A space X is said to be *hypermetric* if

$$\sum_{x\in X} a_x = 1 \Rightarrow \sum_{x,y\in X} a_x a_y (xy) \leq 0,$$

where a_x are integers. A space X is of *negative type* if the above holds with 1 replaced by 0. Schoenberg [18] proved that $(X)^{1/2}$ is Euclidean if and only if X is of negative type. This characterization goes back to Cayley, but was first stated in this form by Schoenberg in the 1930s. A space X is said to be l_1-embeddable if there is an isometric embedding of X into R^n with the l_1-metric, i.e. the metric

$$d(\overline{x}, \overline{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

A metric space is called *ultrametric* (see e.g. [1]) if it satisfies $xz \leq \max(xy, yz)$ for any three points x, y, z. In Maehara [15], an ultrametric space X is called an *irreducible space*, because any metric transform of X requires $(|X| - 1)$-dimensional Euclidean space for an isometric embedding. In particular, [15, Theorem 7] implies that ultrametric spaces are Euclidean (see also [1, Theorem 6.7]).

There is the following hierarchy in finite metric spaces (see, e.g. [2–4, 10–13, 15, 17. 23]):

X is ultrametric

$\Rightarrow X$ is Euclidean

$\Rightarrow X$ is l_1-embeddable

$\Rightarrow X$ is hypermetric

$\Rightarrow X$ is of negative type

$\Rightarrow$ The distance matrix of X has only one positive eigenvalue.

The conditions for hypermetricity and for negative type can be described by the polygonal inequalities. A space X is said to be $(2N+\varepsilon)$-*gonal* if the $(2N+\varepsilon)$-*gonal inequality*

$$\sum_{i<j} x_i x_j + \sum_{i<j} y_i y_j \le \sum_{i,j} x_i y_j \tag{1}$$

holds for any $(2N+\varepsilon)$-sequence of points of X,

$$x_1, \dots, x_N, y_1, \dots, y_{N+\varepsilon}, \tag{2}$$

where $\varepsilon = 0$ or 1. These $2N+\varepsilon$ points are not necessarily different. The 3-gonal inequality is just the triangle inequality. Then a space X is *hypermetric* if and only if it is $(2N+1)$-gonal for all $N \ge 1$, while X is of *negative type* if and only if it is $2N$-gonal for all $N \ge 1$ (e.g. [3]).

In this paper the function $\log_2(1+1/s)$ appears frequently, so we denote this function by γ:

$$\gamma(s) = \log_2(1+1/s).$$

First we prove two lemmas.

Lemma 1. *Suppose that* $0 < c \le \gamma(s)$, $s > 1$. *Then, in the metric transform* $(X)^c$, $xy \le yz$ *implies that*

$$xz \le (xy)/s + yz.$$

Proof. Denote the distance in the space X by $\overline{xy}$. Then $xy = (\overline{xy})^c$ by definition. Since $\overline{xy} + \overline{yz} \ge \overline{xz}$, it is enough to show that

$$(\overline{xy})^c/s + (\overline{yz})^c - (\overline{xy}+\overline{yz})^c \ge 0.$$

Dividing the left-hand side by $(\overline{xy})^c$ and letting $t = (\overline{yz})/(\overline{xy})$, we have

$$f(t) := 1/s + t^c - (1+t)^c.$$

Thus it remains only to show that $f(t) \ge 0$ for $t \ge 1$. Now, since the condition $0 < c \le \gamma(s)$ implies that $(1+1/s) \ge 2^c$, we have $f(1) \ge 0$. And since $c < 1$, we have

$$f'(t) = ct^{c-1} - c(1+t)^{c-1} > 0 \quad \text{for } t \ge 1.$$

Therefore $f(t) \ge 0$ for $t \ge 1$. □

Let (2) be a $(2N+1)$-sequence of points of $(X)^c$. By changing suffixes if necessary, we may assume that

$$x_k y_k \leq x_i y_j \quad \text{for } i \geq k, \quad j \geq k, \quad k = 1, \ldots, N \tag{3}$$

holds in the sequence (2). If $i < k$ then by (3) and Lemma 1,

$$x_i x_k \leq (x_i y_i)/s + x_k y_i \leq (x_k y_k)/s + x_k y_i,$$
$$y_i y_k \leq (x_i y_i)/s + x_i y_k \leq (x_k y_k)/s + x_i y_k.$$

Hence we have the following lemma.

Lemma 2. *Suppose that* $0 < c \leq \gamma(s)$ *and* $k \leq N$. *If* $i < k < j$, *then*

$$x_i x_k \leq (x_k y_k)/s + x_k y_i,$$
$$x_j x_k \leq (x_k y_k)/s + x_j y_k,$$
$$y_i y_k \leq (x_k y_k)/s + x_i y_k,$$
$$y_j y_k \leq (x_k y_k)/s + x_k y_j.$$

Therefore, for any $i, k \leq N$,

$$x_i x_k + y_i y_k \leq 2(x_k y_k)/s + x_i y_k + x_k y_i. \quad \square \tag{4}$$

3. The polygonal inequalities

Theorem 1. *If* $0 < c \leq \gamma(N)$, *then the metric transform* $(X)^c$ *is M-gonal for all* $M \leq 2N+2$, *and if* $c > \gamma(N)$ *then there is a space* X *such that* $(X)^c$ *is not* $(2N+2)$*-gonal.*

Proof. Suppose that $0 < c \leq \gamma(N)$. It is known that if a metric space is $(2N+1)$-gonal, then it is M-gonal for all $M \leq 2N+2$ (see e.g. [11]). Hence we show that $(X)^c$ is $(2N+1)$-gonal.

Let (2) be a $(2N+1)$-sequence taken from $(X)^c$. We may suppose that (3) holds for this sequence. Fix $k < N+1$ and sum (4) over $i \neq k$ to obtain

$$\sum_{i=1}^{N}(x_i x_k + y_i y_k) \leq \sum_{i=1}^{N}(x_i y_k + x_k y_i) - 2(x_k y_k) + 2(N-1)(x_k y_k)/N.$$

Summing over $k < N+1$ gives

$$\sum_{k=1}^{N}\sum_{i=1}^{N}(x_i x_k + y_i y_k) \leq \sum_{k=1}^{N}\sum_{i=1}^{N}(x_i y_k + x_k y_i) - 2\sum_{k=1}^{N}(x_k y_k)/N.$$

Dividing this inequality by 2, and adding the inequality

$$\sum_{k=1}^{N} y_k y_{N+1} \leq \sum_{k=1}^{N}\{(x_k y_k)/N + x_k y_{N+1}\},$$

we have

$$\sum_{i<j} x_i x_j + \sum_{i<j} y_i y_j \leq \sum_{i=1}^{N}\sum_{k=1}^{N} x_i y_k + \sum_{k=1}^{N} x_k y_{N+1} = \sum_{i,j} x_i y_j.$$

This is (1), proving the first part of the theorem. For the second part, consider the complete bipartite graph $K(N+1, N+1)$ with vertex classes $\{x_1, \ldots, x_{N+1}\}$ and $\{y_1, \ldots, y_{N+1}\}$. Then all the vertices of $K(N+1, N+1)$ constitute a metric space X by the usual graph distance. Suppose $c > \gamma(N) = \log_2(1+1/N)$. Then $2^c > (N+1)/N$. Hence, in $(X)^c$,

$$\sum_{i<j} x_i x_j + \sum_{i<j} y_i y_j = N(N+1)2^c > (N+1)^2;$$

on the other hand

$$\sum_{i,j} x_i y_j = (N+1)^2.$$

Thus the $(2N+2)$-gonal inequality fails in $(X)^c$. □

Deza [11] proved that X_5 is l_1-embeddable if and only if it is pentagonal (=5-gonal). Hence we have the following corollary.

Corollary 1. *The metric transform* $(X_5)^c$ *is* l_1*-embeddable for* $0 < c \leq \gamma(2)$, *and* $\gamma(2)$ *is the largest exponent that guarantees* $(X_5)^c$ *to be* l_1*-embeddable for all* X_5. □

4. Repeating numbers

In a sequence of points, some points may appear repeatedly. Define the maximum *repeating number* of a sequence as the maximum value of the number of times a point appears in that sequence. For example, the maximum repeating number of the sequence

$$a, b, c, a, b, b, c, b, c$$

is 4 provided that a, b, c are all different.

Lemma 3. *Suppose that* $0 < c \leq \gamma(s)$ *and* $(X)^c$ *is* $(2N-1)$*-gonal. Then, in* $(X)^c$, *the* $(2N+1)$*-gonal inequality holds for any* $(2N+1)$*-sequence with maximum repeating number at least* $2N/(s+1)$.

Proof. Suppose that (2) is a $(2N+1)$-sequence with maximum repeating number $r \geq 2N/(s+1)$.

First, we consider the case $\{x_1, \ldots, x_N\} \cap \{y_1, \ldots y_{N+1}\} \neq \emptyset$. Suppose, say, $x_2 = y_3$. Then since the $(2N-1)$-gonal inequality is valid for

$$x_1, x_3, x_4, \ldots, x_N, y_1, y_2, y_4, \ldots, y_{N+1}$$

and since

$$\sum_{i\neq 2} x_i x_2 + \sum_{j\neq 3} y_j y_3 = \sum_{i\neq 2} x_i y_3 + \sum_{j\neq 3} x_2 y_j,$$

the $(2N+1)$-gonal inequality (1) is valid for the $(2N+1)$-sequence (2).

Now, suppose that $\{x_1, \dots, x_N\} \cap \{y_1, \dots, y_{N+1}\} = \varnothing$. Changing suffixes of x_i, y_j if necessary, we may assume that (3) holds in the sequence (2). Let k be the smallest suffix i such that the maximum repeating number r equals the repeating number of z_i, where $z_i = x_i$ or y_i. In the following we assume that $z_k = x_k$. (The case $z_k = y_k$ is similar, and is omitted.) Since $(2N-1)$-gonal inequality is valid in $(X)^c$, we can apply it to the sequence left when x_k and y_k are removed, to wit

$$\sum_{\substack{i<j \\ i,j\neq k}} x_i x_j + \sum_{\substack{i<j \\ i,j\neq k}} y_i y_j \le \sum_{i,j\neq k} x_i y_j .$$

Hence, in order to prove the $(2N+1)$-gonal inequality (1) for the sequence (2), it is enough to show that

$$\sum_{i\in I+J} x_i x_k + \sum_{j\neq k} y_j y_k \le \sum_{i} x_i y_k + \sum_{j\neq k} x_k y_j , \tag{5}$$

where

$$I = \{i : i < k,\ x_i \neq x_k\}, \qquad J = \{i : i > k,\ x_i \neq x_k\}.$$

Note that since $\{x_1, \dots, x_N\} \cap \{y_1, \dots, y_{N+1}\} = \varnothing$, it follows that $|I| + |J| = N - r$. Now applying Lemma 2, we have

$$\begin{aligned}\sum_{i\in I+J} x_i x_k + \sum_{j=1}^{N+1} y_j y_k &\le \sum_{i\in I}\{(x_k y_k)/s + x_k y_i\} + \sum_{i\in J}\{(x_k y_k)/s + x_i y_k\} \\ &\quad + \sum_{j=1}^{k-1}\{(x_k y_k)/s + x_j y_k\} + \sum_{j=k+1}^{N+1}\{(x_k y_k)/s + x_k y_j\}.\end{aligned}$$

Since k is the minimum value of i such that x_i appears r times in the sequence (2), it follows that

$$I = \{1, 2, \dots, k-1\}.$$

Hence the right-hand side of the above inequality is

$$\begin{aligned}(|I| + |J| + N)(x_k y_k)/s &+ \sum_{i\in I+J} x_i y_k + \sum_{j\neq k} x_k y_j \\ = (2N - r)(x_k y_k)/s &+ \sum_{i\in I+J} x_i y_k + \sum_{j\neq k} x_k y_j .\end{aligned}$$

Since

$$\sum_i x_i y_k = r(x_k y_k) + \sum_{i\in I+J} x_i y_k ,$$

inequality (5) follows if we show $(2N - r)/s \le r$. But this inequality is equivalent to the assumption $r \ge 2N/(s+1)$. □

Theorem 2. *The metric transform* $(X_{2n})^c$ *is* $(2n+1)$*-gonal for* $0 < c \le \gamma(n-1)$.

Proof. Suppose $0 < c \le \gamma(n-1)$. Then by Theorem 1, $(X_{2n})^c$ is $(2n-1)$-gonal. To prove that $(X_{2n})^c$ is $(2n+1)$-gonal, let $s = n-1$, $N = n$ in Lemma

3. Since the maximum repeating number of any $(2n+1)$-sequence from $(X_{2n})^c$ is at least $\lceil (2n+1)/(2n) \rceil = 2 = 2n/(s+1)$, the $(2n+1)$-gonal inequality holds in $(X_{2n})^c$ by Lemma 3. □

Thus $(X_6)^c$ is heptagonal (7-gonal) for $0 < c \leq \gamma(2)$.

Recently, Avis [5] proved that X_6 is l_1-embeddable if and only if it is heptagonal. Hence we have the following corollary.

Corollary 2. *The metric transform $(X_6)^c$ is l_1-embeddable for $0 < c \leq \gamma(2)$, and $\gamma(2)$ is the largest exponent that guarantees $(X_6)^c$ to be l_1-embeddable for all X_6.* □

Theorem 3. *The metric transform $(X_n)^c$ is hypermetric for $0 < c \leq \gamma(n-1)$.*

Proof. We show that $(X_n)^c$ is $(2N+1)$-gonal for all $N \geq 1$. This is done by induction on N. Since $(X_n)^c$ is a metric space, it is 3-gonal by definition. Suppose that $(X_n)^c$ is $(2N-1)$-gonal, $N \geq 2$. Then since the maximum repeating number of any $(2N+1)$-sequence from $(X_n)^c$ is at least $(2N+1)/n$, the $(2N+1)$-gonal inequality holds in $(X_n)^c$ by Lemma 3. □

5. Euclidean embeddings

Since a hypermetric space is of negative type, its "square root" is Euclidean. Hence, from Theorem 3, we have the following.

Corollary 3. *The metric transform $(X_n)^c$ is Euclidean for $0 < c \leq \gamma(n-1)/2$.* □

Let $e(n)$ be the supremum of c such that $(X_n)^c$ is Euclidean for every X_n. Since any n-point metric space can be isometrically embedded into an $(n+1)$-point metric space, we have $e(n+1) \leq e(n)$. By Blumenthal,

$$e(4) = 1/2 = \gamma(1)/2.$$

Using the inequality

$$\log_e(1 + 1/s) > 1/s - (1/s)^2/2 > 1/(s+1),$$

we have

$$\tfrac{1}{2}\log_2(1 + 1/(n-1)) > \tfrac{1}{2}(\log_2 e)(1/n) = 0.7213\ldots/n.$$

Thus

$$e(n) \geq 0.7213\ldots/n.$$

By Corollary 1, $e(5) \geq \gamma(2)/2$. But this is the exact value of $e(6)$.

Theorem 4. $e(6) = \gamma(2)/2 = 0.2924\ldots$.

Proof. By Corollary 2, $(X_6)^c$ is l_1-embeddable for $0 < c \leq \gamma(2)$. Hence $(X_6)^c$ is of negative type for that range of c, and hence, applying the result of Schoenberg, we have $e(6) \geq \gamma(2)/2$. On the other hand, if $c > \gamma(2)$ then, by Theorem 1, there exists a metric space X_6 whose metric transform $(X_6)^c$ is not hexagonal. Therefore, $e(6) \leq \gamma(2)/2$. □

TABLE 1. Values of $e(n)$ for small n

n	2	3	4	5	6	7
$e(n)$	∞	1	1/2	?	$\gamma(2)/2$	?

Conjecture.

$$e(2n) = \gamma(n-1)/2,$$
$$e(2n+1) = \gamma(2n(n+1)/(2n+1)-1)/2.$$

These are true for the known values of $e(n)$. Further, as will be seen in the next section (Theorem 6), if we restrict ourselves to graphs with "truncated distance", the conjecture is also true.

Denote by $i(n)$ the sup of c such that $(X_n)^c$ has property (i) for every X_n. Then it may be also an interesting problem to determine $i(n)$ for such properties as:

(a) $\leq n$-gonal,
(b) hypermetric,
(c) negative type,
(d) l_1-embeddable.

6. GRAPHS WITH TRUNCATED DISTANCE

Here we consider the following problem: Given a graph G, up to what value of c is the metric transform $(G_T)^c$ Euclidean?

Let $\phi(G;x)$ be the characteristic polynomial of a graph G, that is $\phi(G;x) = \det(xI - A(G))$, where $A(G)$ is the adjacency matrix of G. Define the polynomial $P(G;x)$ by

$$P(G;x) = \phi(G;-x) - (-1)^{|G|}\phi(\overline{G};x-1),$$

where $\overline{G}$ denote the complement of the graph G. Then the following result was proved in Maehara [14]:

A graph G can be embedded in Euclidean space in such a way that adjacent vertices have distance 1 and nonadjacent vertices have distance $t > 1$ if and only if

$$1 - 1/t^2 \leq 1/z_{\max}, \qquad (6)$$

where $z_{\max}$ is the maximum root of the polynomial $P(G;x)$.

From this result the next follows easily.

Theorem 5. *For a graph G, the metric transform $(G_T)^c$ is Euclidean if and only if $c \leq \frac{1}{2}\gamma(z_{\max}-1)$.*

Proof. The inequality (6) is equivalent to

$$t \leq (1 + 1/(z_{\max}-1))^{1/2}.$$

Since the truncated distances between nonadjacent vertices of G are 2, letting $t = 2^c$ and applying the above result, we have that $(G_T)^c$ is Euclidean if and only if

$$c \le \tfrac{1}{2}\log_2(1 + 1/(z_{\max} - 1)) = \gamma(z_{\max} - 1)/2. \quad \square$$

Corollary 4. *For a graph G, the truncated metric space G_T is of negative type if and only if $z_{\max} \le 2$.* $\square$

If G is a regular graph of degree d, then the polynomial $P(G)$ is given by

$$P(G; x) = (|G|/(x + d))\phi(G; -x)$$

(see [14]). Hence, for a regular graph G,

$$z_{\max} = -(\text{minimum eigenvalue of } G).$$

Corollary 5. *For a regular graph G, G_T is of negative type if and only if the minimum eigenvalue of G is ≥ -2.* $\square$

The graphs with least eigenvalue ≥ -2 were characterized by Cameron, Goethals, Seidel and Shult [9] in terms of root systems. In the same terms, Terwillinger and Deza [21] gave a characterization of more general class of connected finite distance spaces of negative type [21, Theorem 1] and of hypermetric spaces between them [21, Theorem 2].

Example 1. Let G be the Petersen graph. Then

$$\phi(G; x) = (x - 3)(x - 1)^5(x + 2)^4$$

(see e.g. [20]), and hence G_T is of negative type. Further, since the diameter of G is 2, we have $G = G_T$, and hence the Petersen graph is of negative type.

Example 2. Let $G = K(m, n)$, the complete bipartite graph. Then

$$\phi(G: x) = (x^2 - mn)x^{m+n-2},$$
$$\phi(\overline{G}; x) = (x - m + 1)(x + 1)^{m-1}(x - n + 1)(x + 1)^{n-1}$$

(see e.g. [20]), and hence

$$P(G; x) = (-x)^{m+n-2}\{(m + n)x - 2mn\}.$$

Therefore, $z_{\max} = 2mn/(m + n)$. Thus $(G)^c = (G_T)^c$ is Euclidean if and only if $c \le \gamma(2mn/(m + n) - 1)/2$.

Theorem 6. *The sup c_n of c such that $(G_T)^c$ are Euclidean for all n-point graphs G is given by*

$$c_n = \begin{cases} \gamma(k - 1)/2 & \text{if } n = 2k, \\ \gamma(2k(k + 1)/(2k + 1) - 1)/2 & \text{if } n = 2k + 1. \end{cases} \tag{7}$$

Proof. Consider the $2N$-gonal inequality (1) in the metric transform $(G_T)^c$. The left-hand side of (1) does not decrease when the nonzero terms are replaced by 2^c's, and the right-hand side of (1) does not increase when the nonzero terms are replaced by 1's. Hence the worst case (i.e. the most difficult case for $2N$-gonal inequality to hold) will be attained by a complete bipartite graph. However, by Example 2, it will be easy to see that the sup of c for n-point complete bipartite graphs is given by (7). Hence the theorem follows. □

Winkler [23] proved that the graph $G = K_2 + \overline{K}_n$ (= the complement of the disjoint union of $\overline{K}_2$ and K_n) is not of negative type for $n > 4$ (though G has only one positive eigenvalue). Applying Corollary 4, let us present a shorter proof of this result.

Example 3 (cf. Winkler [23]). Let $G = K_2+\overline{K}_n$. Applying Cvetkovic's theorem [10, p. 57], we have

$$\phi(K_2 + \overline{K}_n; x) = x^{n-1}(x^3 - (2n+1)x - 2n),$$

and hence

$$P(G;x) = (-x)^{n-1}\{-(n+2)x^2 + 2(2n+1)x - 3n\}.$$

Hence the maximum root is $z_{\max} = 3n/(n+2)$. Thus, if $n > 4$, then $z_{\max} > 2$, and hence G $(= G_T)$ is not of negative type.

The distance matrix D of G is

$$\begin{array}{cccccccc} 0 & 1 & 1 & 1 & 1 & \cdots & 1 & \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 & \\ \hline 1 & 1 & 0 & 2 & 2 & \cdots & 2 & \\ 1 & 1 & 2 & 0 & 2 & \cdots & 2 & \\ \cdot & & & & \cdots & & & \\ \cdot & & & & \cdots & & & \\ 1 & 1 & 2 & 2 & \cdots & 2 & 0 & \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} n$$

Using the spectral resolution

$$\begin{pmatrix} -x & k & k & \cdots & k \\ k & -x & k & \cdots & k \\ & \cdots & & & \\ & \cdots & & & \\ k & k & \cdots & k & -x \end{pmatrix} = (kn - k - x)(1/n)J - (x+k)(I - (1/n)J)$$

(where J is the matrix with all entries 1 and I is the identity matrix), and the formula

$$\det\left(\begin{array}{c|c} M & N \\ \hline P & Q \end{array}\right) = \det(M)\det(Q - PM^{-1}N),$$

the charateristic polynomial of D is calculated as

$$(-1)^n(x+1)(x+2)^{n-1}(x^2 - (2n-1)x - 2).$$

Thus G has one positive eigenvalue but is not of negative type if $n > 4$.

References

1. M. Aschbacher, P. Bald, E. B. Baum and R. M. Wilson, *Embedding of ultrametric spaces in finite dimensional structures*, SIAM J. Algebraic Discrete Methods **8** (1987), 564–577.
2. P. Assouad and M. Deza, *Espaces metriques plongeables dans un hypercube*, Ann. Discrete Math. **8** (1980), 197–210.
3. ——, *Metric subspaces of* L^1 , Publications Mathematiques d'Orsay, 82. 03, Universite de Paris-Sud, 1982.
4. D. Avis, *Hypermetric space and Hamming cone*, Canad. J. Math. **33** (1981), 795–802.
5. ——, *All the facets of the six point Hamming cone*, Technical Report No. SOCS-88.7 1988.
6. K. Ball, *Isometric embedding in* l_p*-spaces*, European J. Combin. (to appear).
7. L. M. Blumenthal, *Remark concerning the euclidean four-point property*, Ergebnisse eines Math. Koll. **7** (1936), 8–10.
8. ——, *Theory and applications of distance geometry*, Chelsea, New York, 1970.
9. P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult, *Line graphs, root systems and elliptic goemetry*, J. Algebra **43** (1976), 305–327.
10. D. Cvetkovic, M. Doob and H. Sachs, *Spectra of graphs*, Academic Press, New York, 1980.
11. M. Deza (as M. E. Tylkin), *On Hamming geometry of unitary cubes*, Dokl. Akad. Nauk SSR **134** (1960), 1037–1040.
12. R. L. Graham and P. M. Winkler, *On isometric embeddings of graphs*, Trans. Amer. Math. Soc. **288** (1985), 527–536.
13. J. B. Kelly, *Hypermetric spaces*, Lecture Notes in Math., Vol. 490, Springer-Verlag 1975, pp. 17–31.
14. H. Maehara, *Regular embeddings of a graph*, Pacific J. Math. **107** (1983), 393–402.
15. ——, *Metric transforms of finite spaces and connected graphs*, Discrete Math. **61** (1986), 235–246.
16. J. von Neumann and I. J. Schoenberg, *Fourier integrals and metric geometry*, Trans. Amer. Math. Soc. **50** (1941), 226–251.
17. R. P. Roth and P. M. Winkler, *Collapse of the metric hierarchy for bipartite graphs*, European J. Combin. **7** (1986), 371–375.
18. I. J. Schoenberg, *Remarks to Maurice Frechet's article*, Ann. of Math. (2) **36** (1935), 724–732.
19. ——, *Metric space and completely monotone functions*, Ann. of Math. (2) **39** (1938), 811–841.
20. A. J. Schewenk and R. J. Wilson, *On the eigenvalue of a graph*, Selected Topics in Graph Theory (L. W. Beineke and R. J. Wilson, eds.), Academic Press, New York, 1978.
21. P. Terwillinger and M. Deza, *The classification of finite connected hypermetric spaces*, Graphs and Combinatorics **3** (1987), 293–298.
22. W. A. Wilson, *On certain types of continuous transformations of metric spaces*, Amer. J. Math. **57** (1935), 62–68.
23. P. M. Winkler, *On graphs which are metric spaces of negative type*, Graph Theory with Application to Algorithms and Computer Science (Y. Alavi et. al., eds.), Wiley, New York, 1985, pp. 801–810.
24. H. S. Witsenhausen, *Minimum dimension embedding of finite metric spaces*, J. Combin. Theory Ser. A **42** (1986), 184–199.

CNRS, Paris, France

Department of Mathematics, Ryukyu University, Okinawa, Japan