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METRIC-VALUED MAPPINGS OF BOUNDED VARIATION

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1. Introduction

In the present paper, we construct a general theory of mappings of bounded variation that are defined on an arbitrary subset of the real line and take values in metric or linear normed spaces. Then this theory is applied to the proof of the existence of regular selections of multivalued mappings of bounded variation with compact graphs without the condition of convexity of their values. Below, we briefly outline the structure of the paper and the results obtained (see Sec. 2.1 for main definitions and notation).

First of all, in Sec. 2.2 we outline such characteristic properties of a functional of Φ-variation $\mathsf{V}_\Phi(\cdot,\cdot)$ as the monotonicity, minimality, additivity, sequential lower semicontinuity, regularity, and connection with a limit, and in Sec. 2.3, we prove the main relations between function spaces of Lipschitzian mappings, absolutely continuous mappings, and mappings of bounded Φ-variation, where $\Phi : [0,\infty) \to [0,\infty)$ is a continuous convex function such that $\Phi(\rho) = 0$ only for $\rho = 0$.

In Sec. 3, we prove a new structural theorem for all mappings of bounded Φ-variation $f : E \to X$ from E into X, where E is a nonempty subset of the real line $\mathbb{R}$ and X is a metric space. In the case where $E = [a, b]$ is a closed interval in $\mathbb{R}$ and $X = \mathbb{R}$, the following two criteria for functions $f : E \to X$ to be of bounded variation are well known: if $\Phi(\rho) = \rho$, which corresponds to the classical variation in the sense of C. Jordan, then f is a function of bounded variation if and only if f can be represented as the difference of two nondecreasing bounded functions (the Jordan decomposition); if $\Phi(\rho) = \rho^q$ for $q > 1$, which defines the "nonlinear" q-variation in the sense of F. Riesz, then f has a bounded q-variation if and only if f is absolutely continuous and the qth power of its derivative, which is defined almost everywhere on $[a, b]$, is Lebesgue integrable (the Riesz criterion). However, it is clear that none of these criteria is applicable if X is a metric space. In the most general case, our structural theorem (Theorem 3.1) asserts that $f : E \to X$ is a mapping of bounded Φ-variation if and only if it can be represented in the form of the composition $f = g \circ \varphi$, where $\varphi : E \to \mathbb{R}$ is a nondecreasing bounded function of bounded Φ-variation and g is a mapping acting from the image of φ into the metric space X and satisfying the Lipschitz condition with constant ≤ 1. The structural theorem of such a form was first proved in [6, 3.19] for continuous mappings of Jordan bounded variation, and then it was extended to various special classes of mappings in [7–11]. In addition, in Theorem 6.6(b), we generalize the Riesz criterion to the case where $\Phi \in \mathcal{N}$ and the mapping of bounded Φ-variation takes values in an arbitrary reflexive Banach space.

In the classical theory of real functions of Jordan bounded variation, an important role is played by the so-called E. Helly selection principle (see [28, Chapter 8, Sec. 4]), which is proved via the Jordan decomposition. On the basis of the structural theorem mentioned above, this principle was recently extended to the case of metric- and Banach-space-valued mappings [7–9, 11] of bounded variation from special classes. Here we present this theorem in a full generality (Theorem 4.2): an infinite family of continuous mappings on a closed interval of a real line of uniformly bounded Φ-variation with values in the compact subset of a metric space X contains a *pointwise* convergent sequence whose limit is a mapping of bounded Φ-variation. If X is a Banach space, then the condition of continuity of the family of mappings is superfluous, and if $\Phi \in \mathcal{N}$, then this principle can be strengthened up to the *uniform* convergence of a sequence that is chosen from the family (Theorem 4.1). The version of the selection principle presented

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above is substantially used in the proof of the existence of regular selections of multivalued mappings of bounded Φ-variation in Sec. 7.

In Sec. 5, we study the continuity properties of mappings of bounded Φ-variation on an arbitrary set $E \subset \mathbb{R}$ (Theorem 5.3), and we also prove the formulas for jumps of the functional of Φ-variation as well as the formulas for the Φ-variation of the mapping f on the set with a removed limit point (Theorem 5.6). Special cases of these formulas for mappings $f : E \to X$ of Jordan bounded variation are given in [6, 2.23] for $E = [a, b]$, where X is a Banach space, and in [8, Sec. 4] for $E \subset \mathbb{R}$ and for an arbitrary metric space X.

The case where the values of mappings of bounded Φ-variation lie in a linear normed or Banach space X is considered in Sec. 6. On the space of all such mappings, the norm (of Luxemburg type from the theory of Orlicz spaces [24, Chapter 2]) is introduced, its properties are studied, and it is proved that the space of mappings of bounded Φ-variation is a Banach space with this norm if X is a Banach space and Φ is a function that is moderately increasing at infinity (Sec. 6.1). In Sec. 6.2, we find an explicit formula (6.6) for the Φ-variation of continuously differentiable mappings without the assumption that the space X is complete (this formula is used in Sec. 7 for the proof of the existence of selections). In Sec. 6.3, we show that any mapping of bounded Φ-variation with values in a reflexive Banach space is weakly differentiable almost everywhere on $E = [a, b]$, and if $\Phi \in \mathcal{N}$, then it is almost everywhere strongly differentiable (Theorem 6.6). This circumstance allows one to generalize the Riesz criterion mentioned above. As a consequence of the structural theorem and the theorem of differentiation, we obtain that any absolutely continuous mapping on $E = [a, b]$ with values in an arbitrary *metric* space X is, in fact, a mapping of bounded Φ-variation with an appropriately chosen function Φ such that $\lim\limits_{\rho\to\infty} \Phi(\rho)/\rho = \infty$ (Theorem 6.7).

In Sec. 7, the problem of the existence of selections with prescribed properties (in particular, continuous selections) of multivalued mappings of bounded Φ-variation with respect to the Hausdorff metric is solved. By the Michael theorem [26], any lower semicontinuous multivalued mapping from a metric space into the space of closed convex subsets of a Banach space possesses a continuous selection. However, in the absence of the condition of convexity of values of a multivalued mapping, even Lipschitz-continuous mappings ([19, 29]) cannot have continuous selections if the domain of this multivalued mapping lies in a space of dimension greater than 1. The main result on the existence of selections in our case is Theorem 7.1: any multivalued mapping F from a connected interval $E \subset \mathbb{R}$ into the set of subspaces of the Banach space X with compact graph having the Φ-variation, which is bounded with respect to the Hausdorff metric, possesses a selection of bounded Φ-variation. In particular, if F is continuous or $\Phi \in \mathcal{N}$, then the selection is also continuous, and if $\Phi \in \mathcal{N}$ and X is reflexive, then the selection is a mapping that is almost everywhere strongly differentiable on E.

In Sec. 8, we propose a further generalization of the theory that is constructed up to the present, and, in particular, we present a generalization of Theorem 7.1 (Theorem 8.1).

Finally, in the Appendix, we cite the statements of auxiliary propositions in the form in which they are used in the main part of the paper; we also indicate the sources where the proofs of these propositions can be found.

The main results of the present paper were reported at the International Conference dedicated to the 90th anniversary of the birth of Academician L. S. Pontryagin, which was held in Moscow in 1998 from August 31 to September 6 [12].

2. Elementary Properties of the Variation

2.1. Notation and Definitions. The following notation is used in this paper:

- $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of positive integers;
- $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$ is the set of integers;
- $E \subset \mathbb{R}$ is a nonempty subset of the set of real numbers $\mathbb{R}$;
- $E_t^- = \{s \in E \mid s \le t\}$ and $E_t^+ = \{s \in E \mid t \le s\}$ if $t \in E$;

- $E_a^b = \{s \in E \mid a \leq s \leq b\}$ if $a,\, b \in E,\, a \leq b$;
- $[a, b] = \mathbb{R}_a^b$ is a closed interval of the real line $\mathbb{R}$ with endpoints $-\infty < a < b < \infty$;
- $\mathbb{R}^+ = \mathbb{R}_0^+ = \{s \in \mathbb{R} \mid s \geq 0\}$ is the set of nonnegative numbers;
- X is a fixed metric space with the metric $d(\cdot, \cdot)$ or a linear normed space (over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm $\|\cdot\|$;
- X^E is the set of all mappings $f : E \to X$ from E into the space X;
- $f(E) = \{f(t) \mid t \in E\} \subset X$ is the image of the mapping $f \in X^E$;
- $g \circ \varphi$ is the composition of two mappings $g : E \to X$ and $\varphi : E_1 \to E$, which acts from E_1 into X and is defined by the rule $(g \circ \varphi)(\tau) = g(\varphi(\tau))$ for all $\tau \in E_1$;
- $C(E; X)$ is the set of all continuous mappings from E into X;
- $\mathcal{F}_0$ is the set of all continuous strictly increasing functions Φ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\Phi(0) = 0$ and $\lim\limits_{\rho\to\infty} \Phi(\rho) = \infty$ (note that if $\Phi,\, \Psi \in \mathcal{F}_0$ and $c > 0$, then the functions $\Phi + \Psi$, $\Phi \cdot \Psi$, $c\Phi$, $\Phi \circ \Psi$, Φ^{-1}, $\min\{\Phi, \Psi\}$, and $\max\{\Phi, \Psi\}$ belong to $\mathcal{F}_0$, where Φ^{-1} stands for the inverse function to Φ);
- $\mathcal{M}$ is the set of all continuous convex functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Phi(\rho) = 0$ only for $\rho = 0$ (note that $\mathcal{M} \subset \mathcal{F}_0$, and if $\Phi,\, \Psi \in \mathcal{M}$ and $c > 0$, then $\Phi + \Psi$, $c\Phi$, and $\Phi \circ \Psi \in \mathcal{M}$); sometimes, functions Φ from $\mathcal{M}$ are called *$\mathcal{M}$-functions*;
- $\mathcal{N}$ is the set of all functions $\Phi \in \mathcal{M}$ such that $\lim\limits_{\rho\to\infty} \Phi(\rho)/\rho = \infty$ (note that $\mathcal{N} \subset \mathcal{M}$, and if $\Phi,\, \Psi \in \mathcal{N}$ and $c > 0$, then $\Phi + \Psi$, $c\Phi$, and $\Phi \circ \Psi \in \mathcal{N}$); functions Φ from the class $\mathcal{N}$ are called *$\mathcal{N}$-functions*;
- $P := Q$ or $Q =: P$; this notation means that the expression P is defined via the expression Q.

Everywhere in what follows in this section, unless otherwise stated, (X, d) is a fixed metric space.

A mapping $f : E \to X$ is said to be *Lipschitz continuous* (or, in abbreviated form, *Lipschitzian*) if the following quantity is finite:

$$\operatorname{Lip}(f) = \sup \left\{ \frac{d(f(t), f(s))}{|t-s|} \,\middle|\, t,\ s \in E, \quad t \neq s \right\};$$

this quantity is called the *Lipschitz constant* of the mapping f. The set of all Lipschitzian mappings from E into X is denoted by

$$C^{0,1}(E; X) = \{f : E \to X \mid \operatorname{Lip}(f) < \infty\}.$$

A mapping $f : E \to X$ is called *absolutely continuous* if there exists a function $\delta : (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0$ and any finite tuple $\{a_i, b_i\}_{i=1}^n \subset E$ of points such that $a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n$, the condition

$$\sum_{i=1}^n (b_i - a_i) \leq \delta(\varepsilon) \qquad \text{implies} \qquad \sum_{i=1}^n d(f(b_i), f(a_i)) \leq \varepsilon$$

holds. More precisely, such mappings f are called *$\delta(\cdot)$-absolutely continuous*, and since the function $\delta(\cdot)$ depends on f in general, we write $\delta(\cdot) = \delta_f(\cdot)$. The set of all absolutely continuous mappings from E into X is denoted by $AC(E; X)$.

Let

$$\mathcal{T}(E) = \{T = \{t_i\}_{i=0}^m \subset E \,|\, m \in \mathbb{N},\ t_{i-1} < t_i,\ i = 1, \ldots, m\} \tag{2.1}$$

be the set of all partitions of E into finite ordered tuples of points from E. For a function $\Phi \in \mathcal{F}_0$, a mapping $f : E \to X$, and a partition $T = \{t_i\}_{i=0}^m$ of the set E, we define

$$V_\Phi[f; T] \equiv V_{\Phi,d}[f; T] := \sum_{i=1}^m \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}} \right) \cdot (t_i - t_{i-1}), \tag{2.2}$$

which is called the *Φ-prevariation* of the mapping f corresponding to the partition T, and set

$$\mathsf{V}_\Phi(f, E) := \sup\{V_\Phi[f; T] \mid T \in \mathcal{T}(E)\}. \tag{2.3}$$

Formally, any partition from $\mathcal{T}(E)$ consists of at least two points; therefore, for the set E, which is empty or consists of one point, we explicitly set the following: $\mathsf{V}_\Phi(f,E)=V_\Phi[f;E]=0$ (and also $\mathcal{T}(\varnothing):=\varnothing$). The functional $\mathsf{V}_\Phi(\cdot,\cdot):X^E\times 2^E\to[0,\infty]$ is thus defined; it is called the *functional of* Φ-*variation* or simply the Φ-*variation*. The quantity $\mathsf{V}_\Phi(f,E)$ itself, finite or infinite, is called the (*total*) Φ-*variation* of the mapping f on the set E. If this quantity is finite, then f is called a *mapping of bounded* (or *finite*) Φ-*variation on* E (we also say that f *has* bounded Φ-variation on E). The set of all mappings from E into X of bounded Φ-variation is denoted by

$$BV_\Phi(E;X)=\{f:E\to X\mid \mathsf{V}_\Phi(f,E)<\infty\}.$$

If A is a nonempty subset of E, then we set $\mathsf{V}_\Phi(f,A)=\mathsf{V}_\Phi(f|_A,A)$, where $f|_A$ is the restriction of the mapping f to the set A, and the value $\mathsf{V}_\Phi(f,A)$ is called the Φ-*variation of the mapping* f *on the subset* A. In what follows, we assume that the set E is infinite.

If $\Phi(\rho)=\rho$, $\rho\in\mathbb{R}^+$ (so that $\Phi\in\mathcal{M}$), then definitions (2.1)–(2.3) give the classical notion of variation in the sense of C. Jordan [21] (see also [34, Chapter 4, Sec. 9]). Recently, this notion was studied in the authors's works [6–8] from the standpoint of describing the general properties of the variation and its application to the search for selections of multivalued mappings. We denote by $BV_1(E;X)$ the set of all mappings from E into X of Jordan bounded variation; the corresponding Φ-variation of a mapping $f:E\to X$ is denoted by $\mathsf{V}_1(f;E)$ and is simply called the *variation* of the mapping f or 1-*variation of* f *on* E.

If $\Phi(\rho)=\rho^q$, $\rho\in\mathbb{R}^+$, and $q>1$ (so that $\Phi\in\mathcal{N}$), then (2.1)–(2.3) define the notion of the q-variation in the sense of F. Riesz [32] (or [33, Chapter 2, Sec. 3.36]). In [9,10], the author proved that any multivalued mapping of bounded q-variation with a compact graph has selections of bounded q-variation. The set of all mappings of Riesz bounded q-variation is denoted by $BV_q(E;X)$, and the corresponding Φ-variation of the mapping f, which is called the q-*variation*, is denoted by $\mathsf{V}_q(f;E)$. The case where the function $\Phi\in\mathcal{N}$ and $E=[a,b]$ is a closed interval was studied by the author in [11]. In the present paper, we construct a general theory of mappings of bounded Φ-variation for functions $\Phi\in\mathcal{M}$ and then apply it to the search for selections of multivalued mappings of bounded Φ-variation (with respect to the Hausdorff metric). Note that, in contrast to the N-functions introduced by Krasnosel'skii and Rutitskii [24, Chapter 1], functions Φ from the class $\mathcal{N}$ do not satisfy the condition $\Phi'(0)=\lim\limits_{\rho\to 0}\Phi(\rho)/\rho=0$. Nevertheless, functions from the class $\mathcal{N}$ resemble N-functions, which play an important role in the construction of Orlicz spaces [24, Chapter 2]; therefore, the Φ-variation defined in (2.1)–(2.3) can be called the Φ-variation in the sense of Jordan–Riesz–Orlicz.

2.2. Main properties of Φ-variations. We begin with some elementary properties of mappings of bounded Φ-variation.

Proposition 2.1. *Let $\Phi\in\mathcal{M}$, and let $f:E\to X$. Then*

(a) *if $T\in\mathcal{T}(E)$ and $t\in E\setminus T$, we have $V_\Phi[f;T]\le V_\Phi[f;T\cup\{t\}]$;*
(b) *if T_1, $T_2\in\mathcal{T}(E)$ and $T_1\subset T_2$, we have $V_\Phi[f;T_1]\le V_\Phi[f;T_2]$;*
(c) *if $T\in\mathcal{T}(E)$, we have $\mathsf{V}_\Phi(f,T)=V_\Phi[f;T]$ (i.e., $\mathsf{V}_\Phi(f,\cdot)$ is a continuation of $V_\Phi[f;\cdot]$ from the set $\mathcal{T}(E)$ to the set 2^E of all subsets of the set E);*
(d) *the quantity $\mathsf{V}_\Phi(f,E)$ remains unchanged if when calculating the supremum in it, instead of all partitions of the set E, one restricts oneself to the consideration of only the $\{t_i\}_{i=0}^m\in\mathcal{T}(E)$ that have a finite number of points that are fixed in advance among the points $\{t_i\}_{i=0}^m$; in other words, if*

a partition $T_0 \in \mathcal{T}(E)$ *is fixed and* $\mathcal{T}_{T_0}(E) = \{T_0 \cup T \mid T \in \mathcal{T}(E)\}$, *then*

$$\mathsf{V}_\Phi(f, E) = \sup\{V_\Phi(f, T) \mid T \in \mathcal{T}_{T_0}(E)\}. \tag{2.4}$$

Proof. (a) Let $T = \{t_i\}_{i=0}^m$. For $i \in \{1, \ldots, m\}$, we set

$$V_i = \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}).$$

Since $t \notin T$, we have either $t < t_0$ or $t > t_m$, or otherwise, $t_{k-1} < t < t_k$ for some $k \in \{1, \ldots, m\}$. If $t < t_0$ or $t > t_m$, then inequality (a) is obvious. Now let $t_{k-1} < t < t_k$. Then we have

$$V_\Phi[f; T] = \left(\sum_{i=1}^{k-1} V_i\right) + V_k + \left(\sum_{i=k+1}^{m} V_i\right); \tag{2.5}$$

moreover, if $k = 1$, then the first sum does not appear in this relation, and if $k = m$, then the last sum does not appear in it. Using the triangle inequality for d and the fact that Φ is increasing and then applying the Jensen inequality for sums (A.1) for

$$\alpha_1 = t - t_{k-1}, \quad \alpha_2 = t_k - t, \quad x_1 = \frac{d(f(t), f(t_{k-1}))}{t - t_{k-1}}, \quad x_2 = \frac{d(f(t_k), f(t))}{t_k - t},$$

and observing that $\alpha_1 + \alpha_2 = t_k - t_{k-1}$, we find that

$$\begin{aligned} V_k &\leq \Phi\left(\frac{d(f(t), f(t_{k-1})) + d(f(t_k), f(t))}{(t - t_{k-1}) + (t_k - t)}\right) \cdot (t_k - t_{k-1}) \\ &\leq \Phi\left(\frac{d(f(t), f(t_{k-1}))}{t - t_{k-1}}\right) \cdot (t - t_{k-1}) + \Phi\left(\frac{d(f(t_k), f(t))}{t_k - t}\right) \cdot (t_k - t). \end{aligned} \tag{2.6}$$

Therefore, $V_\Phi[f; T] \leq V_\Phi[f; T \cup \{t\}]$, which was required.

(b) follows from (a) by induction.

(c) By virtue of (2.3), we have $\mathsf{V}_\Phi(f, T) \geq V_\Phi[f; T]$. On the other hand, if S is an arbitrary partition of the set T, then $S \subset T$; therefore, by (b), $V_\Phi[f; S] \leq V_\Phi[f; T]$, and, taking the supremum over all partitions S, we obtain $\mathsf{V}_\Phi(f, T) \leq V_\Phi[f; T]$, which implies the assertion.

(d) Since $\mathcal{T}_{T_0}(E) \subset \mathcal{T}(E)$, the left-hand side of (2.4) is not less than its right-hand side. On the other hand, if $T \in \mathcal{T}(E)$ is arbitrary, then $T \subset T_0 \cup T \in \mathcal{T}_{T_0}(E)$; therefore,

$$\mathsf{V}_\Phi(f, T) = V_\Phi[f; T] \leq V_\Phi[f; T_0 \cup T] \leq \sup\{V_\Phi[f; T] \mid T \in \mathcal{T}_{T_0}(E)\};$$

hence, the left-hand side of (2.4) is not more than its right-hand side and the equality is established. □

Proposition 2.1(c) implies that if $\Phi \in \mathcal{M}$ and $f \in X^E$, then, for any *finite* set $T \subset E$, we have $\mathsf{V}_\Phi(f, T) = V_\Phi[f; T]$; therefore, in what follows, instead of $V_\Phi[f; T]$, we will write $V_\Phi(f, T)$ in (2.2) and (2.3) (for uniformity).

Proposition 2.1(d) implies, in particular, that if $\Phi \in \mathcal{M}$, $E = [a, b]$ is a closed interval, $f : E \to X$, and

$$\mathcal{T}_a^b := \left\{T = \{t_i\}_{i=0}^m \subset [a, b] \mid m \in \mathbb{N}, \quad a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\right\}$$

is the set of all partitions of the closed interval $[a, b]$ containing two fixed points a and b, then

$$\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) := \sup\left\{V_\Phi(f, T) \mid T \in \mathcal{T}_a^b\right\} = \mathsf{V}_\Phi(f, [a, b]). \tag{2.7}$$

Proposition 2.2. *Let* $\Phi \in \mathcal{M}$ *and* $f : E \to X$. *We have*

(a) *if* $A \subset B \subset E$, *then* $\mathsf{V}_\Phi(f, A) \leq \mathsf{V}_\Phi(f, B)$ (*monotonicity*);

(b) *if* $t,\ s \in E$, *and* $s < t$, *then* $\Phi\Big(\frac{d(f(t),f(s))}{t-s}\Big) \le \frac{1}{t-s}\,\mathsf{V}_\Phi(f, E_s^t)$ (*equicontinuity and minimality*);

(c) *if* $t \in E$, *then* $\mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+) = \mathsf{V}_\Phi(f, E)$ (*additivity*);

(d) *if a sequence of mappings* $\{f_n\}_{n=1}^\infty \subset X^E$ *pointwise converges on* E *to* f (*i.e., if* $\lim\limits_{n\to\infty} d(f_n(t), f(t)) = 0$ *for all* $t \in E$), *then* $\mathsf{V}_\Phi(f, E) \le \liminf\limits_{n\to\infty} \mathsf{V}_\Phi(f_n, E)$ (*sequential lower semicontinuity*);

(e) $\mathsf{V}_\Phi(f, E) = \sup\{\mathsf{V}_\Phi(f, E_a^b) \mid a, b \in E,\ a < b\}$ (*regularity*);

(f) *if* $s = \sup E \in \mathbb{R} \cup \{\infty\}$ *and* $s \notin E$, *then* $\mathsf{V}_\Phi(f, E) = \lim\limits_{E \ni t \to s} \mathsf{V}_\Phi(f, E_t^-)$;

(g) *if* $i = \inf E \in \mathbb{R} \cup \{-\infty\}$ *and* $i \notin E$, *then* $\mathsf{V}_\Phi(f, E) = \lim\limits_{E \ni t \to i} \mathsf{V}_\Phi(f, E_t^+)$;

(h) *if* s *and* i *are such as in* (f) *and* (g), *and* $s \notin E$ *and* $i \notin E$, *then, in addition to* (*limit properties*) (f) *and* (g), *we have*

$$\mathsf{V}_\Phi(f, E) = \lim_{\substack{E \ni a \to i \\ E \ni b \to s}} \mathsf{V}_\Phi(f, E_a^b) = \lim_{E \ni b \to s} \lim_{E \ni a \to i} \mathsf{V}_\Phi(f, E_a^b) = \lim_{E \ni a \to i} \lim_{E \ni b \to s} \mathsf{V}_\Phi(f, E_a^b).$$

Proof. (a) If $T \in \mathcal{T}(A)$, then $T \in \mathcal{T}(B)$, and, therefore, $V_\Phi(f, T) \le \mathsf{V}_\Phi(f, B)$; it remains to take the supremum over all $T \in \mathcal{T}(A)$.

(b) Here it suffices to note that $\{s, t\}$ forms a partition of the set E_s^t and then use the definition of $\mathsf{V}_\Phi(f, E_s^t)$ from (2.3).

(c) For arbitrary partitions $T_1 \in \mathcal{T}(E_t^-)$ and $T_2 \in \mathcal{T}(E_t^+)$, we set

$$\widetilde{T}_i = \begin{cases} T_i & \text{if } t \in T_i, \\ T_i \cup \{t\} & \text{if } t \notin T_i, \end{cases} \qquad i = 1,\ 2.$$

Then $\widetilde{T}_1 \cup \widetilde{T}_2 \in \mathcal{T}(E)$, and we have

$$V_\Phi(f, T_1) + V_\Phi(f, T_2) \le V_\Phi(f, \widetilde{T}_1) + V_\Phi(f, \widetilde{T}_2) = V_\Phi(f, \widetilde{T}_1 \cup \widetilde{T}_2) \le \mathsf{V}_\Phi(f, E).$$

Taking the supremum over T_1 and T_2 indicated above, we find

$$\mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+) \le \mathsf{V}_\Phi(f, E). \tag{2.8}$$

We now prove the inequality converse to the last one (here we essentially use the convexity of Φ). Let $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$. If $t \in T$, then $t = t_k$ for some $k \in \{0, 1, \ldots, m\}$, and, therefore, we have

$$\begin{aligned} V_\Phi(f, T) &= V_\Phi(f, \{t_i\}_{i=0}^k) + V_\Phi(f, \{t_i\}_{i=k}^m) \\ &\le \mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+). \end{aligned} \tag{2.9}$$

On the other hand, if $t \notin T$, then $t < t_0$, or $t > t_m$, or otherwise, $t_{k-1} < t < t_k$ for some $k \in \{1, \ldots, m\}$. In the case where $t < t_0$ or $t > t_m$, it is obvious that

$$V_\Phi(f, T) \le V_\Phi(f, T \cup \{t\}) \le \mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+). \tag{2.10}$$

Now, if $t_{k-1} < t < t_k$, then, by (2.5) and (2.6), we have

$$V_\Phi(f, T) \le V_\Phi(f, \{t_i\}_{i=0}^{k-1} \cup \{t\}) + V_\Phi(f, \{t\} \cup \{t_i\}_{i=k}^m) \le \mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+).$$

Together with (2.9) and (2.10), this inequality means that

$$V_\Phi(f, T) \le \mathsf{V}_\Phi(f, E_t^-) + \mathsf{V}_\Phi(f, E_t^+) \qquad \forall\, T \in \mathcal{T}(E),$$

which just implies the inequality converse to (2.8), and, therefore, it implies (c) as well.

(d) Let $T=\{t_i\}_{i=0}^m$ be an arbitrary partition of E. By the definition of $\mathsf{V}_\Phi(f_n,E)$, we have

$$V_\Phi(f_n,T)\le \mathsf{V}_\Phi(f_n,E)\qquad \forall\, n\in\mathbb{N}. \tag{2.11}$$

We set $\Delta t_i=t_i-t_{i-1}$, $\rho_{i,n}=d(f_n(t_i),f_n(t_{i-1}))/\Delta t_i$, and $\rho_i=d(f(t_i),f(t_{i-1}))/\Delta t_i$. Then we find from (2.2) that

$$V_\Phi(f_n,T)-V_\Phi(f,T)=\sum_{i=1}^m\bigl(\Phi(\rho_{i,n})-\Phi(\rho_i)\bigr)\Delta t_i.$$

The continuity of the metric $d(\cdot,\cdot)$ and the pointwise convergence of f_n to f imply $\rho_{i,n}\to\rho_i$ as $n\to\infty$ for all $i=1,\dots,m$; therefore, by the continuity of the function Φ, we conclude that $\Phi(\rho_{i,n})\to\Phi(\rho_i)$ as $n\to\infty$. Hence

$$V_\Phi(f_n,T)\to V_\Phi(f,T)\quad\text{as}\quad n\to\infty.$$

Taking the lower limit in both parts of inequality (2.11), we obtain the inequality

$$V_\Phi(f,T)\le\liminf_{n\to\infty}\mathsf{V}_\Phi(f_n,E)\qquad\forall\, T\in\mathcal{T}(E),$$

which just implies (d).

(e) By virtue of item (a), it is clear that the left-hand side in (e) is not less than the right-hand side. Conversely, for any number $\alpha<\mathsf{V}_\Phi(f,E)$, by (2.3), there exists a decomposition $T=\{t_i\}_{i=0}^m\in\mathcal{T}(E)$ such that $V_\Phi(f,T)\ge\alpha$, but $T\in\mathcal{T}(E_{t_0}^{t_m})$; therefore, $\mathsf{V}_\Phi(f,E_{t_0}^{t_m})\ge V_\Phi(f,T)\ge\alpha$, which is what was required.

(f) Since $s\notin E$, s is a limit point of the set E. The function $E\ni t\mapsto\mathsf{V}_\Phi(f,E_t^-)\in[0,\infty]$ is nondecreasing by virtue of (a); therefore, the limit written in (f) exists in $[0,\infty]$; moreover, it is clear that this limit $\le\mathsf{V}_\Phi(f,E)$. On the other hand, by (e), for any $\alpha<\mathsf{V}_\Phi(f,E)$, there exist $a,b\in E$, $a<b<s$, such that $\mathsf{V}_\Phi(f,E_a^b)\ge\alpha$; this implies that for any $t\in E\cap[b,s)\ne\varnothing$, $\mathsf{V}_\Phi(f,E_t^-)\ge\mathsf{V}_\Phi(f,E_a^b)\ge\alpha$ by virtue of (a), and the relation in (f) follows.

(g) is proved similarly to (f).

(h) The first relation is proved similarly to (f). We prove the second relation in (h) in the following way:

$$\mathsf{V}_\Phi(f,E)=\lim_{E\ni b\to s}\mathsf{V}_\Phi(f,E_b^-)=\lim_{E\ni b\to s}\lim_{E\ni a\to i}\mathsf{V}_\Phi(f,(E_b^-)_a^+)=\lim_{E\ni b\to s}\lim_{E\ni a\to i}\mathsf{V}_\Phi(f,E_a^b).$$

The last inequality in (h) can be proved in a similar way. □

The mapping $\mathsf{V}_\Phi(\cdot,\cdot)$ is *minimal* in the following sense.

Proposition 2.3. *Let $\Phi\in\mathcal{M}$, and let a mapping $W:X^E\times 2^E\to[0,\infty]$ satisfy the following conditions for all $f:E\to X$ and $\varnothing\ne A\subset E$ (we assume that $W(f,\varnothing):=0$):*

(a) $\Phi\Bigl(\frac{d(f(t),f(s))}{t-s}\Bigr)\le\frac{1}{t-s}W(f,A)$ *for all $t,\ s\in A$, $s<t$;*
(b) $W(f,A)\le W(f,B)$ *for all $A\subset B\subset E$;*
(c) $W(f,A_t^-)+W(f,A_t^+)=W(f,A)$ *for all $t\in A$.*

Then $\mathsf{V}_\Phi(f,A)\le W(f,A)$ for all $f:E\to X$ and $A\subset E$.

Proof. The mapping $(f,A)\mapsto\mathsf{V}_\Phi(f,A)$ satisfies all the conditions listed above by Proposition 2.2(a,b,c). Now, if $f:E\to X$, $\varnothing\ne A\subset E$ and $T=\{t_i\}_{i=0}^m\in\mathcal{T}(A)$, then we have

$$V_\Phi(f,T)=\sum_{i=1}^m\Phi\biggl(\frac{d(f(t_i),f(t_{i-1}))}{t_i-t_{i-1}}\biggr)\cdot(t_i-t_{i-1})\overset{\text{(a)}}{\le}\sum_{i=1}^m W(f,A_{t_{i-1}}^{t_i})\overset{\text{(c)}}{=}W(f,A_{t_0}^{t_m})\overset{\text{(b)}}{\le}W(f,A),$$

from which the assertion follows if we take the supremum over all $T\in\mathcal{T}(A)$. □

2.3. Relations between function spaces.

Proposition 2.4. (a) *If $f \in C^{0,1}(E;X)$, then $f \in AC(E;X)$.*

(b) *If E is a bounded set and $f \in C^{0,1}(E;X)$, then $f \in BV_\Phi(E;X)$ for any function $\Phi \in \mathcal{M}$ and the following inequality holds*:

$$\mathsf{V}_\Phi(f,E) \le \Phi(\mathrm{Lip}(f)) \cdot (\sup E - \inf E).$$

(c) *If E is a compact set, $\Phi \in \mathcal{M}$, and $f \in BV_\Phi(E;X)$, then $f \in BV_1(E;X)$ and the following inequality holds*:

$$\mathsf{V}_1(f,E) \le \Phi^{-1}\left(\frac{1}{\max E - \min E}\,\mathsf{V}_\Phi(f,E)\right) \cdot (\max E - \min E). \tag{2.12}$$

(d) *If $\Phi \in \mathcal{N}$ and $f \in BV_\Phi(E;X)$, then $f \in AC(E;X)$.*

(e) *If E is a compact set and $f \in AC(E;X)$, then $f \in BV_1(E;X)$.*

Proof. (a) For $\varepsilon > 0$, we set $\delta(\varepsilon) = \varepsilon/\max\{1, \mathrm{Lip}(f)\} > 0$. Then, if points $\{a_i, b_i\}_{i=1}^n \subset E$ are such that $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and $\sum\limits_{i=1}^n (b_i - a_i) \le \delta(\varepsilon)$, we have

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \mathrm{Lip}(f) \cdot \sum_{i=1}^n (b_i - a_i) \le \mathrm{Lip}(f) \cdot \delta(\varepsilon) \le \varepsilon.$$

(b) It suffices to note that for any partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we have

$$\begin{aligned} V_\Phi(f,T) &= \sum_{i=1}^m \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &\le \Phi(\mathrm{Lip}(f)) \cdot \sum_{i=1}^m (t_i - t_{i-1}) = \Phi(\mathrm{Lip}(f)) \cdot (t_m - t_0). \end{aligned}$$

(d) We first prove (d) and then (c). Let $\{a_i, b_i\}_{i=1}^n \subset E$ be such that $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$. Applying the Jensen inequality (A.1) for sums with

$$\alpha_i = b_i - a_i \qquad \text{and} \qquad x_i = \frac{d(f(b_i), f(a_i))}{b_i - a_i}$$

for $i \in \{1, \ldots, n\}$, we obtain

$$\Phi\left(\frac{\sum\limits_{i=1}^n d(f(b_i), f(a_i))}{\sum\limits_{i=1}^n (b_i - a_i)}\right) \le \frac{1}{\sum\limits_{i=1}^n (b_i - a_i)} \cdot \sum_{i=1}^n \Phi\left(\frac{d(f(b_i), f(a_i))}{b_i - a_i}\right) \cdot (b_i - a_i) \le \frac{1}{\sum\limits_{i=1}^n (b_i - a_i)} \cdot \mathsf{V}_\Phi(f,E).$$

Since $\Phi \in \mathcal{M}$, it is strictly increasing; therefore, taking the inverse function Φ^{-1} of both parts of the last inequality, we obtain

$$\sum_{i=1}^n d(f(b_i), f(a_i)) \le \left[\sum_{i=1}^n (b_i - a_i)\right] \cdot \Phi^{-1}\left(\frac{1}{\sum\limits_{i=1}^n (b_i - a_i)} \cdot \mathsf{V}_\Phi(f,E)\right). \tag{2.13}$$

Setting $v := \mathsf{V}_\Phi(f,E)$ and taking into account that $\lim\limits_{\rho\to\infty} \Phi(\rho)/\rho = \infty$, we have

$$\lim_{t\to 0} t\,\Phi^{-1}(v/t) = v \lim_{\rho\to\infty} \rho/\Phi(\rho) = 0. \tag{2.14}$$

Therefore, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $0 < t \le \delta(\varepsilon)$, then $t\,\Phi^{-1}(v/t) \le \varepsilon$. This and (2.13) imply the property

$$\text{if} \quad \sum_{i=1}^{n}(b_i - a_i) \le \delta(\varepsilon), \quad \text{then} \quad \sum_{i=1}^{n} d(f(b_i), f(a_i)) \le \varepsilon.$$

Thus, $f \in AC(E; X)$.

(c) Let $T = \{t_i\}_{i=0}^{n} \in \mathcal{T}(E)$ be an arbitrary partition. By Proposition 2.1(d), we fix two points $t_0 = \min E \in E$ and $t_n = \max E \in E$. Now, inequality (2.12) and assertion (c) follows from (2.13) if we set $a_1 = t_0$ and $b_i = a_{i+1} = t_i$ for $i = 1, \dots, n-1$, $b_n = t_n$, and note that $\sum\limits_{i=1}^{n}(b_i - a_i) = t_n - t_0 = \max E - \min E$.

(e) Let the mapping f be $\delta(\cdot)$-absolutely continuous. For $t \in E$, we set $\varphi(t) := \mathsf{V}_1(f, E_t^-)$ (the function φ a priori can take infinite values) and show that the function φ is also $\delta(\cdot)$-absolutely continuous on E. Let $\varepsilon > 0$, and let the tuple of points $\{a_i, b_i\}_{i=1}^{n} \subset E$ be such that $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$ and $\sum\limits_{i=1}^{n}(b_i - a_i) \le \delta(\varepsilon)$. For any $i \in \{1, \dots, n\}$ and any $\alpha_i < \mathsf{V}_1(f, E_{a_i}^{b_i})$, by the definition of the 1-variation $\mathsf{V}_1(f, E_{a_i}^{b_i})$ (2.3), we find a partition $T_i = \{t_{i,j}\}_{j=0}^{m_i}$ of the set $E_{a_i}^{b_i}$ such that $a_i = t_{i,0} < t_{i,1} < \dots < t_{i,m_i-1} < t_{i,m_i} = b_i$ and $V_1(f, T_i) \ge \alpha_i$. Since

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i}(t_{i,j} - t_{i,j-1}) = \sum_{i=1}^{n}(b_i - a_i) \le \delta(\varepsilon),$$

the initial assumption on the mapping f implies

$$\sum_{i=1}^{n}\alpha_i \le \sum_{i=1}^{n} V_1(f, T_i) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} d(f(t_{i,j}), f(t_{i,j-1})) \le \varepsilon.$$

Letting α_i tend to $\mathsf{V}_1(f, E_{a_i}^{b_i})$ and applying Proposition 2.2(c), we obtain

$$\sum_{i=1}^{n}|\varphi(b_i) - \varphi(a_i)| = \sum_{i=1}^{n}\big(\mathsf{V}_1(f, E_{b_i}^-) - \mathsf{V}_1(f, E_{a_i}^-)\big) = \sum_{i=1}^{n}\mathsf{V}_1(f, E_{a_i}^{b_i}) \le \varepsilon;$$

this implies the $\delta(\cdot)$-absolute continuity of the function φ.

Since φ is absolutely continuous on the set E, this function is (uniformly) continuous on E, and since E is compact, φ is bounded on E, and, in particular, $\mathsf{V}_1(f, E) = \varphi(\max E) < \infty$. □

Corollary 2.5. *Let E be a compact set, and let $\Phi \in \mathcal{M}$. Then*

(a) $C^{0,1}(E; X) \subset BV_\Phi(E; X) \subset BV_1(E; X)$ *and* $C^{0,1}(E; X) \subset AC(E; X) \subset BV_1(E; X)$;

(b) *if* $\Phi \in \mathcal{N}$, *then* $C^{0,1}(E; X) \subset BV_\Phi(E; X) \subset AC(E; X) \subset BV_1(E; X)$.

Corollary 2.6. *Let E be a compact set, $\Phi \in \mathcal{M}$, and $f \in BV_\Phi(E; X)$. Then*

(a) *the image $f(E)$ of the mapping f is a completely bounded and separable subset of X, and if it is known in addition that X is a complete metric space, then $f(E)$ is precompact* (*i.e., the closure of $f(E)$ in X is compact*);

(b) *f is continuous on E possibly outside the subset of E that is no more than countable.*

Proof. It is sufficient to take into account the embedding $BV_\Phi(E; X) \subset BV_1(E; X)$ and use Theorem A.2 given in the Appendix (see also Theorem 5.3(b)). □

The following proposition is a generalization of Proposition 2.4(c).

Proposition 2.7. *Let Φ, $\Psi \in \mathcal{M}$.*

(a) *If E is a bounded set, X is a metric space, and the condition*

$$\exists\, \rho_0 \geq 0 \text{ and } C > 0 \quad \text{such that} \quad \Psi(\rho) \leq C\Phi(\rho) \quad \forall\, \rho \geq \rho_0 \tag{2.15}$$

holds, then $BV_\Phi(E;X) \subset BV_\Psi(E;X)$.

(b) *Conversely, if $E = [a,b]$, X is a Banach space with the norm $\|\cdot\|$, and $BV_\Phi([a,b];X) \subset BV_\Psi([a,b];X)$, then condition (2.15) holds.*

Proof. (a) Suppose that (2.15) holds. If $T = \{t_i\}_{i=0}^m$ is a partition of E, then, for any mapping $f \in BV_\Phi(E;X)$, we have

$$V_\Psi(f,T) \leq \Psi(\rho_0) \cdot (\sup E - \inf E) + C \mathsf{V}_\Phi(f,E),$$

i.e., $f \in BV_\Psi(E;X)$.

(b) Now, let $BV_\Phi([a,b];X) \subset BV_\Psi([a,b];X)$, where X is a Banach space. Suppose the contrary, i.e., let condition (2.15) be violated. Then there exists an increasing sequence of positive numbers $\{\rho_n\}_{n=1}^\infty$ for which $\lim\limits_{n\to\infty} \rho_n = \infty$ and $\Psi(\rho_n) > 2^n\Phi(\rho_n)$ for all $n \in \mathbb{N}$. Consider an increasing sequence $\{a_n\}_{n=0}^\infty \subset [a,b]$ such that $a_0 = a$ and

$$a_n - a_{n-1} = \frac{(b-a)\Phi(\rho_1)}{2^n\Phi(\rho_n)}, \qquad n \in \mathbb{N}.$$

For $t \in [a,b]$, we set

$$g(t) = \begin{cases} \rho_n & \text{if } a_{n-1} \leq t < a_n,\ n \in \mathbb{N}, \\ 0 & \text{if } \lim\limits_{n\to\infty} a_n \leq t \leq b, \end{cases}$$

and define the mapping $f : [a,b] \to X$ by

$$f(t) = x_0 \int_a^t g(\tau)\, d\tau, \quad t \in [a,b], \quad \text{where} \quad x_0 \in X, \quad \|x_0\| = 1.$$

We show that $f \in BV_\Phi([a,b];X)$, but $f \notin BV_\Psi([a,b];X)$, which contradicts the assumption. In fact, for any $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$, we have

$$\begin{aligned} V_\Phi(f,T) &\leq \sum_{n=1}^\infty \Phi\left(\frac{\|f(a_n) - f(a_{n-1})\|}{a_n - a_{n-1}}\right) \cdot (a_n - a_{n-1}) \\ &= \sum_{n=1}^\infty \Phi(\rho_n) \cdot (a_n - a_{n-1}) = (b-a)\Phi(\rho_1) < \infty. \end{aligned}$$

On the other hand, if $m \in \mathbb{N}$ and $T_m = \{a_n\}_{n=0}^m$, then

$$\begin{aligned} \mathop{\mathsf{V}}_a^b{}_\Psi(f) \geq V_\Psi(f,T_m) &= \sum_{n=1}^m \Psi\left(\frac{\|f(a_n) - f(a_{n-1})\|}{a_n - a_{n-1}}\right) \cdot (a_n - a_{n-1}) \\ &= \sum_{n=1}^m \Psi(\rho_n) \frac{(b-a)\Phi(\rho_1)}{2^n\Phi(\rho_n)} \geq \sum_{n=1}^m (b-a)\Phi(\rho_1) = m(b-a)\Phi(\rho_1), \end{aligned}$$

which, by virtue of the arbitrariness of $m \in \mathbb{N}$, yields $\mathop{\mathsf{V}}_a^b{}_\Psi(f) = \infty$. □

Remark 2.1. Since the function $\Phi \in \mathcal{M}$ is convex, it has the right derivative $\Phi'_+(\rho)$ for any $\rho \in \mathbb{R}^+$, $\Phi'_+(\rho) > 0$ for $\rho > 0$, which is nondecreasing and right continuous. Therefore, there exist $\rho_0 \geq 0$ and $c_0 > 0$ such that $c_0\rho \leq \Phi(\rho)$ for $\rho \geq \rho_0$ since

$$\Phi(\rho) = \int_0^\rho \Phi'_+(\tau)\, d\tau > \int_{\rho/2}^\rho \Phi'_+(\tau)\, d\tau > \frac{\rho}{2}\, \Phi'_+\left(\frac{\rho}{2}\right) \geq \frac{1}{2}\, \Phi'_+\left(\frac{\rho_0}{2}\right) \cdot \rho, \quad \rho \geq \rho_0 > 0.$$

Then this fact and Proposition 2.7(a) imply the embedding from Proposition 2.4(c).

In concluding of this section, we prove an auxiliary lemma, which will be needed below (in Theorems 6.4 and 6.6).

Lemma 2.8. *Let $\Phi \in \mathcal{M}$ and $f \in BV_\Phi([a,b];X)$. Then, for any $h \in (0, b-a)$, we have the inequality*

$$\int_a^{b-h} \Phi\left(\frac{d(f(t+h), f(t))}{h}\right) dt = \int_{a+h}^{b} \Phi\left(\frac{d(f(t), f(t-h))}{h}\right) dt \le \bigvee_a^b{}_\Phi(f).$$

In particular, if X is a linear normed space (over $\mathbb{R}$ or $\mathbb{C}$) with the norm $\|\cdot\|$, then

$$\int_a^{b-h} \Phi\left(\left\|\frac{f(t+h) - f(t)}{h}\right\|\right) dt \le \bigvee_a^b{}_\Phi(f) \qquad \forall\, 0 < h < b-a. \tag{2.16}$$

Proof. By Proposition 2.2(a), the function $t \mapsto \bigvee_a^t{}_\Phi(f)$ is nondecreasing on the closed interval $[a,b]$; therefore, it is Riemann integrable on $[a,b]$. Let $0 < h < b-a$. Since the mapping f is continuous everywhere on $[a,b]$, possibly except for a finite set of points (Corollary 2.6(b)), the mapping $[a, b-h] \ni t \mapsto d(f(t+h), f(t))$ also has this property on $[a, b-h]$. Applying Proposition 2.2(b,c), we obtain

$$\Phi\left(\frac{d(f(t+h), f(t))}{h}\right) \le \frac{1}{h} \bigvee_t^{t+h}{}_\Phi(f) \le \frac{1}{h}\left(\bigvee_a^{t+h}{}_\Phi(f) - \bigvee_a^t{}_\Phi(f)\right)$$

for all $t \in [a, b-h]$. Now, the inequality of the lemma follows if we integrate the last equation with respect to t on the closed interval $[a, b-h]$:

$$\int_a^{b-h} \Phi\left(\frac{d(f(t+h), f(t))}{h}\right) dt \le \frac{1}{h}\left(\int_a^{b-h} \bigvee_a^{t+h}{}_\Phi(f)\, dt - \int_a^{b-h} \bigvee_a^t{}_\Phi(f)\, dt\right)$$

$$= \frac{1}{h}\left(\int_{b-h}^{b} \bigvee_a^t{}_\Phi(f)\, dt - \int_a^{a+h} \bigvee_a^t{}_\Phi(f)\, dt\right) \le \frac{1}{h} \int_{b-h}^{b} \bigvee_a^t{}_\Phi(f)\, dt \le \bigvee_a^b{}_\Phi(f).$$

□

3. Structural Theorem

Let $E \subset \mathbb{R}$, and let (X, d) be a metric space. A mapping $g : E \to X$ is called *natural* if $\mathsf{V}_1(g, E_a^b) = b - a$ for all $a, b \in E$, $a \le b$. The natural mapping $g : E \to X$ is Lipschitzian with Lipschitz constant $\mathrm{Lip}(g) \le 1$, since, by Proposition 2.2(b), for $\Phi(\rho) = \rho$, we have

$$d(g(b), g(a)) \le \mathsf{V}_1(g, E_a^b) = b - a, \qquad a, b \in E, \quad a \le b.$$

The main result of this section is the following *structural theorem*.

Theorem 3.1. *Let $E \subset \mathbb{R}$ be a compact set, and let X be a metric space. Denote by $\mathcal{F}(E;X)$ one of the classes of sets $C^{0,1}(E;X)$, $BV_\Phi(E;X)$ for $\Phi \in \mathcal{M}$, or $AC(E;X)$. The mapping $f : E \to X$ belongs to the class $\mathcal{F}(E;X)$ if and only if there exist a nondecreasing bounded function $\varphi \in \mathcal{F}(E;\mathbb{R}^+)$ and a natural mapping $g : E_1 \to X$ with $E_1 = \varphi(E)$ such that $f = g \circ \varphi\ E$. In addition, one can explicitly set $\varphi(t) = \mathsf{V}_1(f, E_t^-)$ for $t \in E$; in this case, the function φ preserves the main characteristics of the mapping f (i.e., the Lipschitz constant, the total Φ-variation, or the function $\delta(\cdot)$ from the definition of absolute continuity).*

The proof of this theorem is carried out in two steps and is contained in Lemmas 3.2 and 3.3. All examples of mappings of bounded variation are described in the following lemma, which gives the sufficient condition of Theorem 3.1.

Lemma 3.2. *Let $\varphi : E \to \mathbb{R}$, $E_1 = \varphi(E)$, $g \in C^{0,1}(E_1; X)$, $\mathrm{Lip}(g) \le 1$, and $f = g \circ \varphi$.*

(a) *If $\varphi \in C^{0,1}(E;\mathbb{R})$, then $f \in C^{0,1}(E;X)$ and $\mathrm{Lip}(f) \le \mathrm{Lip}(\varphi)$.*

(b) *If $\Phi \in \mathcal{M}$ and $\varphi \in BV_\Phi(E;\mathbb{R})$, then $f \in BV_\Phi(E;X)$ and $\mathsf{V}_\Phi(f,E) \le \mathsf{V}_\Phi(\varphi,E)$.*

(c) *If $\varphi \in AC(E;\mathbb{R})$, then $f \in AC(E;X)$; moreover, for the mapping f, the function $\delta(\cdot)$ from the definition of absolute continuity can be taken the same as for the function φ (which is written in the form $\delta_f(\cdot) = \delta_\varphi(\cdot)$).*

Proof. (a) For t, $s \in E$, we have

$$d(f(t), f(s)) = d\big(g(\varphi(t)), g(\varphi(s))\big) \le \mathrm{Lip}(g) \cdot |\varphi(t) - \varphi(s)| \le \mathrm{Lip}(g) \cdot \mathrm{Lip}(\varphi) \cdot |t-s| \le \mathrm{Lip}(\varphi) \cdot |t-s|.$$

(b) If $T = \{t_i\}_{i=0}^m$ is an arbitrary partition of E, then

$$\begin{aligned} V_\Phi(f,T) &= \sum_{i=1}^m \Phi\left(\frac{d\big(g(\varphi(t_i)), g(\varphi(t_{i-1}))\big)}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &\le \sum_{i=1}^m \Phi\left(\mathrm{Lip}(g) \cdot \frac{|\varphi(t_i) - \varphi(t_{i-1})|}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &\le \mathsf{V}_\Phi(\mathrm{Lip}(g) \cdot \varphi, E) \le \mathsf{V}_\Phi(\varphi, E). \end{aligned}$$

(c) Let $\varepsilon > 0$, $\{a_i, b_i\}_{i=1}^n \subset E$, $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$, and $\sum\limits_{i=1}^n (b_i - a_i) \le \delta_\varphi(\varepsilon)$. Then

$$\begin{aligned} \sum_{i=1}^n d(f(b_i), f(a_i)) &= \sum_{i=1}^n d\big(g(\varphi(b_i)), g(\varphi(a_i))\big) \\ &\le \mathrm{Lip}(g) \cdot \sum_{i=1}^n |\varphi(b_i) - \varphi(a_i)| \le \mathrm{Lip}(g) \cdot \varepsilon \le \varepsilon. \end{aligned}$$

□

Remark 3.1. Generally speaking, the fact that $\varphi \in BV_\Phi(E;\mathbb{R})$ and $\alpha > 1$ does not imply that $\alpha\varphi \in BV_\Phi(E;\mathbb{R})$ (cf. the proof of Lemma 3.2(b)).

The second lemma, which gives the necessary condition of Theorem 3.1, presents the canonical decomposition of mappings of bounded variation.

Lemma 3.3. *Let $f \in BV_1(E;X)$; for $t \in E$, we set $\varphi(t) = \mathsf{V}_1(f, E_t^-)$ and let $E_1 = \varphi(E)$. Then $\varphi : E \to \mathbb{R}^+$ is a nondecreasing bounded function; moreover, there exists a natural mapping $g : E_1 \to X$ (and, therefore, $g \in C^{0,1}(E_1;X)$ and $\mathrm{Lip}(g) \le 1$) such that*

(i) *$f = g \circ \varphi$ on E;*

(ii) *$g(E_1) = f(E)$ in X;*

(iii) *$\mathsf{V}_1(g, E_1) = \mathsf{V}_1(f, E)$.*

Moreover, for the case of a compact set $E \subset \mathbb{R}$, we have

(a) *if $f \in C^{0,1}(E;X)$, then $\varphi \in C^{0,1}(E;\mathbb{R})$ and $\mathrm{Lip}(\varphi) = \mathrm{Lip}(f)$;*

(b) *if $\Phi \in \mathcal{M}$ and $f \in BV_\Phi(E;X)$, then $\varphi \in BV_\Phi(E;\mathbb{R})$ and $\mathsf{V}_\Phi(\varphi, E) = \mathsf{V}_\Phi(f, E)$;*

(c) *if $f \in AC(E;X)$, then $\varphi \in AC(E;\mathbb{R})$ and $\delta_\varphi(\cdot) = \delta_f(\cdot)$.*

Proof. 1. In the first part of this proof (up to item (iii)) we follow [8, Lemma 3.3]. The function $\varphi : E \to \mathbb{R}$ is well defined, bounded ($\varphi(t) \le \mathsf{V}_1(f, E)$ for $t \in E$), nonnegative, and nondecreasing on E due to Proposition 2.2(a). For $\tau \in E_1$, we denote by $\varphi^{-1}(\{\tau\}) = \{t \in E \mid \varphi(t) = \tau\}$ the inverse image of the singleton $\{\tau\}$ under the mapping φ. We define the desired mapping $g : E_1 \to X$ as follows: if $\tau \in E_1$, we set

$$g(\tau) = f(t) \quad \text{for any} \quad t \in \varphi^{-1}(\{\tau\}). \tag{3.1}$$

Such a definition is correct since $\bigcup_{\tau \in E_1} \varphi^{-1}(\{\tau\}) = E$ and the value of $f(t) \in X$ does not depend on $t \in \varphi^{-1}(\{\tau\})$ since, in view of Proposition 2.2(b,c) for $t,\ s \in E,\ s \le t$, we have

$$d(f(t), f(s)) \le \mathsf{V}_1(f, E_s^t) = \varphi(t) - \varphi(s),$$

or, in abbreviated form,

$$d(f(t), f(s)) \le |\varphi(t) - \varphi(s)| \qquad \forall\, t,\ s \in E; \tag{3.2}$$

therefore, if $t,\ s \in \varphi^{-1}(\{\tau\})$, then $\varphi(t) = \tau = \varphi(s)$, and hence $f(t) = f(s)$.

The representation of f in the form of the composition from (i) is implied by (3.1) since for $t \in E$ we have $t \in \varphi^{-1}(\{\varphi(t)\})$; therefore, $f(t) = g(\varphi(t)) = (g \circ \varphi)(t)$.

Assertion (ii) is readily implied by (i). Assertion (iii) is implied by the formula of the change of the variable in the variation (Proposition A.3) and from (i):

$$\mathsf{V}_1(g, E_1) = \mathsf{V}_1(g, \varphi(E)) = \mathsf{V}_1(g \circ \varphi, E) = \mathsf{V}_1(f, E).$$

We find from inequality (3.2) that g is Lipschitzian with $\mathrm{Lip}(g) \le 1$ since for $\tau_1,\ \tau_2 \in E_1$ we have $\tau_1 = \varphi(t_1)$ and $\tau_2 = \varphi(t_2)$ for certain $t_1,\ t_2 \in E$, whence

$$d(g(\tau_1), g(\tau_2)) = d\big(g(\varphi(t_1)), g(\varphi(t_2))\big) = d(f(t_1), f(t_2)) \le |\varphi(t_1) - \varphi(t_2)| = |\tau_1 - \tau_2|.$$

We show that g is actually a natural mapping on E_1. Observing that $(E_1)_\tau^- = \varphi(E_t^-)$ for any $\tau \in E_1$ and $t \in \varphi^{-1}(\{\tau\})$ and applying Proposition A.3, we have

$$\mathsf{V}_1(g, (E_1)_\tau^-) = \mathsf{V}_1(g, \varphi(E_t^-)) = \mathsf{V}_1(g \circ \varphi, E_t^-) = \mathsf{V}_1(f, E_t^-) = \varphi(t) = \tau,$$

for such τ and t. For any $\alpha,\ \beta \in E_1,\ \alpha \le \beta$, by virtue of Proposition 2.2(c), this implies

$$\mathsf{V}_1(g, (E_1)_\alpha^\beta) = \mathsf{V}_1(g, (E_1)_\beta^-) - \mathsf{V}_1(g, (E_1)_\alpha^-) = \beta - \alpha,$$

which is what was required.

2. In the second part of the proof, we will prove (a)–(c). Here we take into account the embeddings that are mentioned in Corollary 2.5.

(a) Let $t,\ s \in E,\ s \le t$. If $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E_s^t)$, then

$$V_1(f, T) = \sum_{i=1}^m d(f(t_i), f(t_{i-1})) \le \mathrm{Lip}(f) \cdot (t_m - t_0) \le \mathrm{Lip}(f) \cdot (t - s);$$

therefore, $\mathsf{V}_1(f, E_s^t) \le \mathrm{Lip}(f) \cdot (t - s)$, and from Proposition 2.2(c) we obtain

$$|\varphi(t) - \varphi(s)| = \mathsf{V}_1(f, E_t^-) - \mathsf{V}_1(f, E_s^-) = \mathsf{V}_1(f, E_s^t) \le \mathrm{Lip}(f) \cdot (t - s).$$

Therefore, $\varphi \in C^{0,1}(E; \mathbb{R})$ and $\mathrm{Lip}(\varphi) \le \mathrm{Lip}(f)$. Taking into account the decomposition $f = g \circ \varphi$, where $\mathrm{Lip}(g) \le 1$, and Lemma 3.2(a), we conclude that $\mathrm{Lip}(\varphi) = \mathrm{Lip}(f)$.

(b) We show that $\varphi \in BV_\Phi(E; \mathbb{R})$. If $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, then, using inequality (2.12), for $i \in \{1, \ldots, m\}$, we find that

$$\varphi(t_i) - \varphi(t_{i-1}) = \mathsf{V}_1(f, E_{t_i}^-) - \mathsf{V}_1(f, E_{t_{i-1}}^-) = \mathsf{V}_1(f, E_{t_{i-1}}^{t_i})$$

$$\leq \Phi^{-1}\biggl(\frac{1}{t_i - t_{i-1}}\,\mathsf{V}_\Phi(f, E_{t_{i-1}}^{t_i})\biggr)\cdot (t_i - t_{i-1}),$$

from which, by virtue of the monotonicity of Φ and the additivity of the Φ-variation, we obtain

$$V_\Phi(\varphi, T) = \sum_{i=1}^{m} \Phi\biggl(\frac{|\varphi(t_i) - \varphi(t_{i-1})|}{t_i - t_{i-1}}\biggr)\cdot (t_i - t_{i-1})$$

$$\leq \sum_{i=1}^{m} \mathsf{V}_\Phi(f, E_{t_{i-1}}^{t_i}) = \mathsf{V}_\Phi(f, E_{t_0}^{t_m}) \leq \mathsf{V}_\Phi(f, E).$$

Therefore, $\mathsf{V}_\Phi(\varphi, E) \leq \mathsf{V}_\Phi(f, E)$. The decomposition $f = g \circ \varphi$ and Lemma 3.2(b) imply the relation $\mathsf{V}_\Phi(\varphi, E) = \mathsf{V}_\Phi(f, E)$.

(c) This assertion was proved in the proof of Proposition 2.4(e); see also Lemma 3.2(c). □

Remark 3.2. Theorem 3.1 generalizes the results of the structure of mappings of bounded variation that were obtained by the author earlier in [6–11]. In addition, the version of Theorem 3.1 is valid for mappings of bounded variation in the sense of Wiener [13] and Yang [14]; however, mappings of this kind do not have the additivity property of the variation, which does not allow one to prove the fact that multivalued mappings of bounded variation have selections in this case. In the case under consideration, the existence of regular selections of multivalued mappings of bounded variation in the sense of Jordan–Riesz–Orlicz will be proved in Theorem 7.1 below. Note that the algebraic aspects of the construction of the natural mapping g in Lemma 3.3, which dates back to the concept of factorization of a mapping, is described in detail at the end of Sec. 3 of [8].

4. The Generalized Selection Principle

In this section, we prove the following two theorems: the *strong selection principle* (Theorem 4.1) and the *weak selection principle* (Theorem 4.2). These theorems generalize the classical Helly selection principle (Theorem A.4 in the Appendix) to metric- and Banach-space-valued mappings. Theorem 4.1 is a consequence of the well-known Arzela–Ascoli theorem on the compactness of continuous mappings in a space (Theorem A.5 given in the Appendix), whereas the method for proving Theorem 4.2 is essentially different from the classical one (given, e.g., in [28]); it is based primarily on the representation of a mapping in the form of a composition from Theorem 3.1, which allows us to apply the Arzela–Ascoli theorem and the classical Helly section principle. The theorems considered in this section generalize and strengthen the author's results obtained in [7–9, 11]. Note that some analogs of the weak selection principle are valid for mappings of bounded variation in the sense of Wiener [13] and Yang [14].

Theorem 4.1 (strong selection principle)**.** *Let E be a compact set in $\mathbb{R}$, let (X, d) be a complete metric space, and let $\mathfrak{F} \subset X^E$ be an infinite family of mappings from E into X such that for any $t \in E$, the set $\{f(t) \mid f \in \mathfrak{F}\} =: \mathfrak{F}(t)$ is precompact in X.*

(a) *If $\Phi \in \mathcal{N}$ and the family $\mathfrak{F} \subset BV_\Phi(E; X)$ has a uniformly bounded Φ-variation, i.e., if*

$$v := \sup_{f \in \mathfrak{F}} \mathsf{V}_\Phi(f, E) < \infty, \tag{4.1}$$

then the family $\mathfrak{F}$ contains a sequence of mappings that converges uniformly on E to some mapping $f \in BV_\Phi(E; X)$ such that $\mathsf{V}_\Phi(f, E) \leq v$.

(b) *If the family $\mathfrak{F} \subset AC(E; X)$ is absolutely equicontinuous, i.e., if*

$$\delta(\varepsilon) := \inf_{f \in \mathfrak{F}} \delta_f(\varepsilon) > 0 \qquad \forall\, \varepsilon > 0, \tag{4.2}$$

then in $\mathfrak{F}$, there is a sequence that converges uniformly on E to some $\delta(\cdot)$-absolutely continuous mapping from E into X.

Proof. (a) We show that the family $\mathfrak{F}$ is equicontinuous. By Proposition 2.2(a,b) and condition (4.1), for all $f \in \mathfrak{F}$ and $t,\, s \in E,\ s < t$, we have

$$d(f(t), f(s)) \le (t-s)\Phi^{-1}\Big(\frac{1}{t-s}\,\mathsf{V}_\Phi(f,E)\Big) \le (t-s)\Phi^{-1}\Big(\frac{v}{t-s}\Big). \tag{4.3}$$

Since $\Phi \in \mathcal{N}$, it follows from (2.14) that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) = \delta(\varepsilon, v)$ such that $\rho\Phi^{-1}(v/\rho) \le \varepsilon$ for all $0 < \rho \le \delta(\varepsilon)$. We obtain from (4.3) that if $t,\, s \in E$ and $0 < t-s \le \delta(\varepsilon)$, then $\sup\limits_{f\in\mathfrak{F}} d(f(t), f(s)) \le \varepsilon$.

Taking into account that X is complete and the sets $\mathfrak{F}(t)$ are precompact and applying the Arzela–Ascoli theorem (Theorem A.5), we conclude that the family $\mathfrak{F}$ is precompact in $C(E;X)$; therefore, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset \mathfrak{F}$ that converges uniformly on E to some mapping $f \in C(E;X)$. It remains to note that, by (4.1) and Proposition 2.2(d), we have

$$\mathsf{V}_\Phi(f,E) \le \liminf_{k\to\infty} \mathsf{V}_\Phi(f_k,E) \le v < \infty.$$

(b) Condition (4.2) implies that $\mathfrak{F}$ is equicontinuous: if $f \in \mathfrak{F}$, $\varepsilon > 0$, and $t, s \in E$ are such that $|t-s| \le \delta_f(\varepsilon)$, then, since f is absolutely continuous, we have $d(f(t), f(s)) \le \varepsilon$, so that for any $\varepsilon > 0$, the condition $|t-s| \le \delta(\varepsilon)$ implies $\sup\limits_{f\in\mathfrak{F}} d(f(t), f(s)) \le \varepsilon$. Taking into account the assumptions made above, we see that by the Arzela–Ascoli theorem, there exists a sequence $\{f_k\}_{k=1}^{\infty}$ from $\mathfrak{F}$ that converges uniformly on E to some mapping $f \in C(E;X)$. The limit mapping f is $\delta(\cdot)$-absolutely continuous: if $\varepsilon > 0$ and $\{a_i, b_i\}_{i=1}^{n} \subset E$ are such that $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and $\sum\limits_{i=1}^{n}(b_i - a_i) \le \delta(\varepsilon)$, then $\sum\limits_{i=1}^{n} d(f_k(b_i), f_k(a_i)) \le \varepsilon$ (since $\delta_{f_k}(\cdot) \ge \delta(\cdot)$) for any $k \in \mathbb{N}$, and it remains to pass to the limit in the last inequality as $k \to \infty$. □

Theorem 4.2 (weak selection principle)**.** *Let $E = [a,b]$ be a closed interval in $\mathbb{R}$, K be a subset of a metric space (X,d), $\Phi \in \mathcal{M}$, and $\mathfrak{F} \subset C([a,b];K)$ be an infinite family*

$$v := \sup_{f\in\mathfrak{F}} \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) < \infty \tag{4.4}$$

of mappings of uniformly bounded Φ-variation. Then the family $\mathfrak{F}$ contains a sequence of mappings that converges pointwise on $[a,b]$ to a certain mapping $f \in BV_\Phi([a,b];X)$ for which $\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) \le v$.

If, in addition, X is a Banach space (over $\mathbb{R}$ or $\mathbb{C}$), then the theorem is valid also without the assumption of continuity of mappings from the family $\mathfrak{F}$.

Proof. The proof of this theorem is divided into three steps. At the first (preparatory) step, we summarize the general information that is then used at the main steps 2 and 3.

1. By Lemma 3.3, any mapping $f \in \mathfrak{F}$ can be represented as the composition $f = g_f \circ \varphi_f$ on $[a,b]$, where the nondecreasing, nonnegative function $\varphi_f \in BV_\Phi([a,b];\mathbb{R})$ is defined by the relation $\varphi_f(t) = \overset{t}{\underset{a}{\mathsf{V}}}_1(f)$ for $t \in [a,b]$ (and, therefore, $\varphi_f(a) = 0$), $E_{1,f} := \varphi_f([a,b])$, and $g_f \in C^{0,1}(E_{1,f};K)$; moreover, $\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(\varphi_f) = \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f)$ (by Lemma 3.3(b)) and $\mathrm{Lip}(g_f) \le 1$. The family $\mathfrak{F}_1 := \{\varphi_f \mid f \in \mathfrak{F}\}$ of nondecreasing functions on $[a,b]$ is infinite and uniformly bounded since, by virtue of Proposition 2.2(b) for $\Phi(\rho) = \rho$, inequality (2.12), the monotonicity of Φ, and (4.4), we have the following for all $t \in [a,b]$:

$$0 \le \varphi_f(t) \le \overset{b}{\underset{a}{\mathsf{V}}}_1(\varphi_f) \le (b-a)\Phi^{-1}\Big(\frac{1}{b-a}\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(\varphi_f)\Big) \le (b-a)\Phi^{-1}\Big(\frac{v}{b-a}\Big) =: L. \tag{4.5}$$

Therefore, applying Theorem A.4(a), we find that the family $\mathfrak{F}_1$ contains a sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ corresponding to the decompositions $f_n = g_n \circ \varphi_n$ (i.e., we set $\varphi_n = \varphi_{f_n}$ and $g_n = g_{f_n}$) for all $n \in \mathbb{N}$

that converges pointwise on $[a,b]$ to a certain nondecreasing bounded function $\varphi : [a,b] \to \mathbb{R}$ such that $0 \le \varphi(t) \le L$ for $t \in [a,b]$. Then it follows from Proposition 2.2(d) that $\varphi \in BV_\Phi([a,b];\mathbb{R})$ since

$$\bigvee_a^b{}_\Phi(\varphi) \le \liminf_{n\to\infty} \bigvee_a^b{}_\Phi(\varphi_n) = \liminf_{n\to\infty} \bigvee_a^b{}_\Phi(f_n) \le v < \infty.$$

Therefore, $\ell := \bigvee_a^b{}_1(\varphi) = \varphi(b)$ is finite (here we take into account (2.12) and the fact that φ is nondecreasing) and $\ell_n := \bigvee_a^b{}_1(\varphi_n) = \varphi_n(b) \to \varphi(b) = \ell$ for $n \to \infty$ by virtue of the pointwise convergence of the sequence $\{\varphi_n\}_{n=1}^\infty$.

2. Suppose that the conditions of the first part of the theorem are fulfilled. Since the mapping $f_n \in \mathfrak{F}$ is continuous, the function φ_n is also continuous on $[a,b]$ (by Theorem 4.3(a) in [8] or by Theorem 5.3(a) below); therefore, the natural mapping g_n is defined on the closed interval $E_{1,f_n} = \varphi_n([a,b]) = [0,\ell_n]$. If $\ell_n \ge \ell$, then we consider g_n only on the closed interval $[0,\ell]$, and if $\ell_n < \ell$, then we extend g_n to the semiopen interval $(\ell_n,\ell]$ by a “constant”: $g_n(\tau) = g_n(\ell_n)$ for all $\tau \in (\ell_n,\ell]$. By the Arzela–Ascoli theorem, the sequence of mappings $\{g_n\}_{n=1}^\infty \subset C^{0,1}([0,\ell];K)$ with Lipschitz constants $\mathrm{Lip}(g_n) \le 1$, $n \in \mathbb{N}$, is precompact in $C([0,\ell];K)$; therefore, it contains a subsequence $\{g_{n_k}\}_{k=1}^\infty$ that converges uniformly on $[0,\ell]$; we denote by g the limit of this subsequence. It is clear that $g \in C^{0,1}([0,\ell];K)$ and $\mathrm{Lip}(g) \le 1$; therefore, using Lemma 3.2(b) and the fact that $\varphi([a,b]) \subset [0,\ell]$ and $\varphi \in BV_\Phi([a,b];\mathbb{R})$, we find that there exists (and is well defined) the composition $f := g \circ \varphi \in BV_\Phi([a,b];K)$ and $\bigvee_a^b{}_\Phi(f) \le \bigvee_a^b{}_\Phi(\varphi) \le v$. Now, observing that for all $t \in [a,b]$,

$$\begin{aligned} d(f_{n_k}(t), f(t)) &= d\big((g_{n_k} \circ \varphi_{n_k})(t), (g \circ \varphi)(t)\big) \\ &\le d\big(g_{n_k}(\varphi_{n_k}(t)), g_{n_k}(\varphi(t))\big) + d\big(g_{n_k}(\varphi(t)), g(\varphi(t))\big) \\ &\le |\varphi_{n_k}(t) - \varphi(t)| + \sup_{\tau\in[0,\ell]} d(g_{n_k}(\tau), g(\tau)), \end{aligned}$$

and taking into account the kinds of convergence of φ_{n_k} and g_{n_k} indicated earlier, we conclude that the sequence $\{f_{n_k}\}_{k=1}^\infty \subset \mathfrak{F}$ converges pointwise to f on $[a,b]$.

3. Now let X be a Banach space and let condition (4.4) be fulfilled for the family $\mathfrak{F}$ from $K^{[a,b]}$. Initially, we argue as in Step 1. Note that in our case, $E_{1,f_n} = \varphi_n([a,b]) \subset [0,L]$, where L is the constant from (4.5), so that $\ell \le L$. By Lemma A.6, for any $n \in \mathbb{N}$, there exists a Lipschitzian mapping $\widetilde{g_n}$ from $\mathbb{R}$ into X that is an extension of g_n to $\mathbb{R}$ such that $\mathrm{Lip}(\widetilde{g_n}) \le \mathrm{Lip}(g_n) \le 1$. Denote by $\overline{g_n}$ the restriction of $\widetilde{g_n}$ to the closed interval $[0,L]$. By the Arzela–Ascoli theorem, The sequence of Lipschitzian mappings $\{\overline{g_n}\}_{n=1}^\infty$, which act from $[0,L]$ into a fixed compact subset of X and are such that $\mathrm{Lip}(\overline{g_n}) \le 1$, has a uniformly convergent subsequence $\{\overline{g_{n_k}}\}_{k=1}^\infty$ with the uniform limit, which is denoted by $\overline{g}$. It is clear that $\overline{g} \in C^{0,1}([0,L];X)$ and $\mathrm{Lip}(\overline{g}) \le 1$. By Lemma 3.2(b), for $f := \overline{g} \circ \varphi$ (as in step 2), we have $f \in BV_\Phi([a,b];X)$ and $\bigvee_a^b{}_\Phi(f) \le v$. Finally, for $t \in [a,b]$ (again as in Step 2), we obtain that

$$d(f_{n_k}(t), f(t)) = d\big(\overline{g_{n_k}}(\varphi_{n_k}(t)), \overline{g}(\varphi(t))\big) \le |\varphi_{n_k}(t) - \varphi(t)| + d\big(\overline{g_{n_k}}(\varphi(t)), \overline{g}(\varphi(t))\big),$$

which completes the proof. □

Remark 4.1. The author does not know whether the condition $E = [a,b]$ and the condition $\mathfrak{F} \subset K^{[a,b]}$, where $K \subset X$ is a compact set, in Theorem 4.2 can be replaced by the following weaker conditions: $E \subset \mathbb{R}$ is a compact subset and the sets $\mathfrak{F}(t) = \{f(t) \mid f \in \mathfrak{F}\}$ are precompact in a complete metric space X for any $t \in E$.

Remark 4.2. In the framework of Theorem 4.2, for $\Phi \in \mathcal{M}$ (in particular, for $\Phi(\rho) = \rho$), even a continuous sequence of mappings $\mathfrak{F}$ can converge pointwise to a discontinuous mapping from $BV_\Phi([a,b];X)$.

Theorems 4.1 and 4.2 are extended to mappings with unbounded domains in a standard way. As an example, we present the corresponding version for Theorem 4.2.

Corollary 4.3. *Suppose that in the statement of Theorem* 4.2*, the set* $E = \mathbb{R}$ *and the closed interval* $[a, b]$ *is everywhere replaced by* $\mathbb{R}$*. Then the pointwise convergence in this theorem takes place on* $\mathbb{R}$*.*

Proof. The proof is based on the construction of a Cantor diagonal sequence. Consider the exhaustion of $\mathbb{R}$ by closed intervals $[-n, n]$, $n \in \mathbb{N}$, and note that

$$\sup_{f\in\mathfrak{F}} \overset{n}{\underset{-n}{\mathsf{V}}}_{\Phi}(f) \le \sup_{f\in\mathfrak{F}} \mathsf{V}_{\Phi}(f, \mathbb{R}) =: v < \infty \qquad \forall\, n \in \mathbb{N}.$$

By Theorem 4.2, the family $\mathfrak{F}$ contains a sequence $\{f_n^1\}_{n=1}^{\infty}$ that is pointwise convergent on the closed interval $[-1, 1]$. On the basis of the same considerations, one extracts from $\{f_n^1\}_{n=1}^{\infty}$ a subsequence $\{f_n^2\}_{n=1}^{\infty}$ that is pointwise convergent on $[-2, 2]$. From the latter, one isolates the subsequence $\{f_n^3\}_{n=1}^{\infty}$ for the closed interval $[-3, 3]$, and so on. Then the diagonal sequence $\{f_n^n\}_{n=1}^{\infty}$ has all the properties that are required (see Proposition 2.2(d)). □

Similar assertions can be obtained for (bounded or unbounded) intervals or semiopen intervals.

Another consequence of the (strong) selection principle (Theorem 4.1(a)) is the following assertion on existence of the geodesic path of bounded Φ-variation between two points of a compact metric space ([11, Corollary 2.3]):

Corollary 4.4. *Let* $\Phi \in \mathcal{N}$*,* X *be a compact metric space, and* x*,* $y \in X$*,* $x \neq y$*. If there exists* $f_0 \in BV_{\Phi}([a, b]; X)$ *such that* $f_0(a) = x$ *and* $f_0(b) = y$*, then there exists a mapping* $g \in BV_{\Phi}([a, b]; X)$ *such that* $g(a) = x$*,* $g(b) = y$*, and*

$$\overset{b}{\underset{a}{\mathsf{V}}}_{\Phi}(g) = \min\Big\{ \overset{b}{\underset{a}{\mathsf{V}}}_{\Phi}(f) \,\Big|\, f \in BV_{\Phi}([a, b]; X),\ f(a) = x,\ f(b) = y \Big\}.$$

Note that in the case where $\Phi(\rho) = \rho$, a geodesic path between two points is always Lipschitzian ([7, Theorem 6.1]):

Theorem 4.5. *If* X *is a compact metric space,* x*,* $y \in X$*, and there exists a continuous mapping* $f_0 \in BV_1([a, b]; X)$ *such that* $f_0(a) = x$ *and* $f_0(b) = y$*, then there exists the mapping* $g \in C^{0,1}([a, b]; X)$ *such that* $g(a) = x$*,* $g(b) = y$*, and*

$$\overset{b}{\underset{a}{\mathsf{V}}}_{1}(g) = \min\Big\{ \overset{b}{\underset{a}{\mathsf{V}}}_{1}(f) \,\Big|\, f \in C([a, b]; X),\ f(a) = x,\ f(b) = y \Big\}.$$

5. Formulas for Jumps of the Functions of Φ-Variation

Everywhere in this section, we assume that the following objects are fixed: E is a subset of $\mathbb{R}$, (X, d) is a metric space, $\Phi \in \mathcal{M}$, and $f \in BV_{\Phi}(E; X)$. We define the nondecreasing bounded function $\phi : E \to \mathbb{R}^+$ by the relation $\phi(t) = \mathsf{V}_{\Phi}(f, E_t^-)$, $t \in E$; this function is called the *function of* Φ*-variation* of the mapping f. In this section, the continuity properties of the mapping f are studied; it is shown that discontinuity points of f exactly coincide with discontinuity points of ϕ, the relation between the jumps of the mapping f and the jumps of its function of Φ-variation ϕ is found, and the formulas for the Φ-variation of f on a set with a removed limit point are obtained (of course, the case where $\Phi \notin \mathcal{N}$ is of primary interest; see Proposition 2.4(d)).

We denote the set of all limit points of the set E by E', and for $t \in E$, we set $E_t^{-\prime} := (E_t^-)'$ and $E_t^{+\prime} := (E_t^+)'$. If $t \in E_t^{-\prime}$, $\alpha \in E$, $\alpha < t$, and α tends to t, then we say that α *tends on the left along the set* E *to the point* t and write $\alpha \to t - 0$ (instead of the more cumbersome notation $E \ni \alpha \to t - 0$); the concept of the *left limit at a point* t, which is denoted by $\lim_{\alpha\to t-0}$, is defined in this case. A similar convention is accepted for the right limit, $\lim_{\alpha\to t+0}$, at a point $t \in E_t^{+\prime}$.

The following notation is useful for us (Φ, f and d are fixed):

$$U(t,s) \equiv U_{\Phi,f,d}(t,s) := \Phi\biggl(\frac{d(f(t),f(s))}{t-s}\biggr) \cdot (t-s), \quad t,\ s \in E, \quad s < t. \tag{5.1}$$

We rewrite the Jensen inequality (2.6), which was written above for the points t_{k-1}, t, $t_k \in E$ such that $t_{k-1} < t < t_k$, in the following more convenient form:

$$\Phi\biggl(\frac{d(f(t_k),f(t_{k-1}))}{t_k - t_{k-1}}\biggr) \cdot (t_k - t_{k-1}) \le \Phi\biggl(\frac{d(f(t),f(t_{k-1}))}{t - t_{k-1}}\biggr) \cdot (t - t_{k-1}) + \Phi\biggl(\frac{d(f(t_k),f(t))}{t_k - t}\biggr) \cdot (t_k - t).$$

Replacing t_{k-1} by α, t by β, and t_k by γ here and taking (5.1) into account, we obtain the following (so-called) "ordered triangle inequality":

$$U(\gamma,\alpha) \le U(\gamma,\beta) + U(\beta,\alpha), \qquad \alpha,\ \beta,\ \gamma \in E, \quad \alpha < \beta < \gamma. \tag{5.2}$$

Lemma 5.1. *Let $t \in E$. We have*:

(a) *if $t \in {E_t^-}'$, then there exists the limit $U(t,t-0) := \lim\limits_{\alpha \to t-0} U(t,\alpha) \in \mathbb{R}^+$;*

(b) *if $t \in {E_t^+}'$, then there exists the limit $U(t+0,t) := \lim\limits_{\beta \to t+0} U(\beta,t) \in \mathbb{R}^+$;*

(c) *if $t \in {E_t^-}' \cap {E_t^+}'$, then there exists the limit $U(t+0,t-0) := \lim\limits_{\substack{\alpha \to t-0 \\ \beta \to t+0}} U(\beta,\alpha) \in \mathbb{R}^+$.*

Proof. (a) Since the function ϕ is nondecreasing and bounded on E, it has a finite one-sided left limit,

$$\phi(t-0) := \lim_{\alpha \to t-0} \phi(\alpha) = \sup\{\phi(\alpha) \mid \alpha \in E_t^-,\ \alpha \ne t\},$$

at the point t. On the other hand, using inequality (5.2) and Proposition 2.2(b,c), for any α, $\beta \in E$, $\alpha < \beta < t$, we have

$$U(t,\alpha) - U(t,\beta) \le U(\beta,\alpha) = \Phi\biggl(\frac{d(f(\beta),f(\alpha))}{\beta - \alpha}\biggr) \cdot (\beta - \alpha) \le \mathsf{V}_\Phi(f, E_\alpha^\beta) = \phi(\beta) - \phi(\alpha);$$

this implies

$$U(t,\alpha) + \phi(\alpha) \le U(t,\beta) + \phi(\beta) \qquad \forall\, \alpha,\ \beta \in E, \quad \alpha < \beta < t.$$

This means that the function $\alpha \mapsto U(t,\alpha) + \phi(\alpha)$ is not decreasing on $E_t^- \setminus \{t\}$; moreover, this function is bounded from above, i.e.,

$$U(t,\alpha) + \phi(\alpha) \le \mathsf{V}_\Phi(f, E_\alpha^t) + \mathsf{V}_\Phi(f, E_\alpha^-) = \mathsf{V}_\Phi(f, E_t^-) = \phi(t), \quad \alpha \in E, \quad \alpha < t;$$

therefore, it has a finite limit as $\alpha \to t - 0$, which implies (a).

(b) is proved similarly to (a); it is only worth noting that in this case, for $\beta \in E_t^+$ and $\beta > t$, the function $\phi(\beta)$ is not decreasing and is bounded from below by the number $\phi(t)$, the function $\beta \mapsto U(\beta,t) - \phi(\beta)$ is not increasing and is bounded from above by the number $-\phi(t)$.

(c) By (5.2) and Proposition 2.2(b,c), for any points α_2, α_1, β_1, and β_2 from the set E such that $\alpha_2 < \alpha_1 < t < \beta_1 < \beta_2$, we find that

$$\begin{aligned} U(\beta_2,\alpha_2) &\le U(\beta_2,\beta_1) + U(\beta_1,\alpha_1) + U(\alpha_1,\alpha_2) \\ &\le \phi(\beta_2) - \phi(\beta_1) + U(\beta_1,\alpha_1) + \phi(\alpha_1) - \phi(\alpha_2), \end{aligned}$$

i.e., the function $(\beta,\alpha) \mapsto U(\beta,\alpha) - \phi(\beta) + \phi(\alpha)$ is "monotone" in the following sense:

$$U(\beta_2,\alpha_2) - \phi(\beta_2) + \phi(\alpha_2) \le U(\beta_1,\alpha_1) - \phi(\beta_1) + \phi(\alpha_1), \quad \alpha_2 < \alpha_1 < t < \beta_1 < \beta_2. \tag{5.3}$$

We set

$$s := \sup\{U(\beta,\alpha) - \phi(\beta) + \phi(\alpha) \mid \alpha,\ \beta \in E,\ \alpha < t < \beta\}. \tag{5.4}$$

This quantity exists and is finite because, for α and β as in (5.4), we have

$$U(\beta,\alpha) - \phi(\beta) + \phi(\alpha) \le \mathsf{V}_\Phi(f, E_\alpha^\beta) - \mathsf{V}_\Phi(f, E_\beta^-) + \mathsf{V}_\Phi(f, E_\alpha^-) = 0.$$

Let us show that

$$\lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} \big(U(\beta,\alpha) - \phi(\beta) + \phi(\alpha)\big) = s. \tag{5.5}$$

From the definition of supremum in (5.4), we obtain that for any $\varepsilon > 0$, there exist $\alpha_0, \beta_0 \in E$, $\alpha_0 < t < \beta_0$, such that $s \le U(\beta_0, \alpha_0) - \phi(\beta_0) + \phi(\alpha_0) + \varepsilon$; by virtue of (5.3) and (5.4), for any $\alpha, \beta \in E$ such that $\alpha_0 \le \alpha < t < \beta \le \beta_0$, this implies

$$U(\beta,\alpha) - \phi(\beta) + \phi(\alpha) \le s \le U(\beta,\alpha) - \phi(\beta) + \phi(\alpha) + \varepsilon,$$

i.e.,

$$\big|\big(U(\beta,\alpha) - \phi(\beta) + \phi(\alpha)\big) - s\big| \le \varepsilon \qquad \forall\, \alpha \in E_{\alpha_0}^t \setminus \{t\}, \quad \forall\, \beta \in E_t^{\beta_0} \setminus \{t\}.$$

Assertion (c) now follows from the relation

$$U(\beta,\alpha) = \big(U(\beta,\alpha) - \phi(\beta) + \phi(\alpha)\big) + \phi(\beta) - \phi(\alpha), \quad \alpha,\ \beta \in E, \quad \alpha < t < \beta.$$

□

Lemma 5.2. *For $t \in E$, the following relations hold:*

(a) *if $t \in E_t^{-\prime}$, then $\phi(t) - \phi(t-0) = U(t, t-0)$;*
(b) *if $t \in E_t^{+\prime}$, then $\phi(t+0) - \phi(t) = U(t+0, t)$;*
(c) *if $t \in E_t^{-\prime} \cap E_t^{+\prime}$, then $\lim\limits_{\substack{\alpha\to t-0\\ \beta\to t+0}} \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) = U(t+0, t-0)$.*

Proof. (a) Passing to the limit as $\alpha \to t-0$ in the inequality

$$U(t,\alpha) \le \phi(t) - \phi(\alpha), \qquad \alpha \in E, \quad \alpha < t,$$

we find that $U(t, t-0) \le \phi(t) - \phi(t-0)$. To prove the converse inequality, it is sufficient to show that for any $\varepsilon > 0$, there exists $t' = t'(\varepsilon) \in E$, $t' < t$, such that

$$\phi(t) - \phi(\alpha) \le U(t,\alpha) + \varepsilon \qquad \forall\, \alpha \in E_{t'}^t \setminus \{t\}; \tag{5.6}$$

indeed, passing to the limit as $\alpha \to t-0$ in inequality (5.6), we find that $\phi(t) - \phi(t-0) \le U(t, t-0) + \varepsilon$, and it only remains to take into account the arbitrariness of $\varepsilon > 0$.

We now prove (5.6). Since $\phi(t) = \mathsf{V}_\Phi(f, E_t^-) \le \mathsf{V}_\Phi(f, E) < \infty$, for any $\varepsilon > 0$ there exists a partition $T = \{t_i\}_{i=0}^m \cup \{t\} \in \mathcal{T}(E_t^-)$ depending on ε with $t_m < t$ such that

$$\phi(t) \le U(t, t_m) + V_\Phi(f, T) + \varepsilon.$$

Observing that $T \in \mathcal{T}(E_{t_m}^-)$ in reality and applying (5.2) and Proposition 2.2(b,c), for all $\alpha \in E$ ($t_m < \alpha < t$) we obtain

$$\begin{aligned}\phi(t) &\le U(t,\alpha) + U(\alpha, t_m) + \mathsf{V}_\Phi(f, E_{t_m}^-) + \varepsilon\\ &\le U(t,\alpha) + \mathsf{V}_\Phi(f, E_{t_m}^\alpha) + \mathsf{V}_\Phi(f, E_{t_m}^-) + \varepsilon\\ &= U(t,\alpha) + \phi(\alpha) + \varepsilon,\end{aligned}$$

which yields (5.6) if one sets $t' = t'(\varepsilon) = t_m$.

The proof of (b) is somewhat different from that of (a); therefore, we present its main points. The inequality $U(t+0, t) \le \phi(t+0) - \phi(t)$ is obtained in the limit as $\beta \to t+0$ from the inequality $U(\beta, t) \le \phi(\beta) - \phi(t)$, $\beta \in E$, $\beta > t$. For the converse inequality, it is sufficient to show that $\forall\, \varepsilon > 0$,

$\exists\, t_0 = t_0(\varepsilon) \in E$, $t_0 > t$, such that $\forall\, \beta \in E_t^{t_0} \setminus \{t\}$, $\phi(\beta) - \phi(t) \le U(\beta, t) + \varepsilon$. Since $\mathsf{V}_\Phi(f, E_t^+) < \infty$, for $\varepsilon > 0$, $\exists\, T = \{t_i\}_{i=0}^m \cup \{t\} \in \mathcal{T}(E_t^+)$, $t_0 > t$, such that

$$\mathsf{V}_\Phi(f, E_t^+) \le U(t_0, t) + V_\Phi(f, T) + \varepsilon,$$

and since $T \in \mathcal{T}(E_{t_0}^+)$, for $\beta \in E$, $t < \beta < t_0$, we find that

$$\begin{aligned} \mathsf{V}_\Phi(f, E_t^+) &\le U(t_0, \beta) + U(\beta, t) + \mathsf{V}_\Phi(f, E_{t_0}^+) + \varepsilon \\ &\le \mathsf{V}_\Phi(f, E_\beta^{t_0}) + U(\beta, t) + \mathsf{V}_\Phi(f, E_{t_0}^+) + \varepsilon \\ &= \mathsf{V}_\Phi(f, E_\beta^+) + U(\beta, t) + \varepsilon; \end{aligned}$$

by Proposition 2.2(c), this implies

$$\phi(\beta) - \phi(t) = \mathsf{V}_\Phi(f, E_\beta^-) - \mathsf{V}_\Phi(f, E_t^-) = \mathsf{V}_\Phi(f, E_t^+) - \mathsf{V}_\Phi(f, E_\beta^+) \le U(\beta, t) + \varepsilon.$$

The proof of (c) is carried out in two steps.

1. From (2.3), we readily obtain the inequality

$$U(\beta, \alpha) \le \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) \qquad \forall\, \alpha \in E_t^-, \quad \forall\, \beta \in E_t^+, \quad \alpha < t < \beta, \tag{5.7}$$

since the set $\{\alpha, \beta\}$ is a partition of the set $E_\alpha^\beta \setminus \{t\}$.

We now show that $\forall\, \varepsilon > 0$, $\exists\, \alpha_0 = \alpha_0(\varepsilon)$, $\beta_0 = \beta_0(\varepsilon) \in E$, $\alpha_0 < t < \beta_0$, such that

$$\mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) \le U(\beta, \alpha) + \varepsilon \qquad \forall\, \alpha \in E_{\alpha_0}^t \setminus \{t\}, \quad \forall\, \beta \in E_t^{\beta_0} \setminus \{t\}. \tag{5.8}$$

We fix $\varepsilon > 0$. Using the definition of the Φ-variation $\mathsf{V}_\Phi(f, E \setminus \{t\})$, which does not exceed $\mathsf{V}_\Phi(f, E) < \infty$, we find a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E \setminus \{t\})$ depending on ε such that

$$t_0 < t_1 < \cdots < t_{k-1} < t < t_k < \cdots < t_{m-1} < t_m \quad \text{for a certain} \quad 1 \le k \le m$$

and

$$\mathsf{V}_\Phi(f, E \setminus \{t\}) \le V_\Phi(f, T) + \varepsilon = \sum_{i=1}^m U(t_i, t_{i-1}) + \varepsilon.$$

We set $T_1 = \{t_i\}_{i=0}^{k-1}$, $T_2 = \{t_i\}_{i=k}^m$, $\alpha_0 = t_{k-1}$, and $\beta_0 = t_k$. Now, if $\alpha, \beta \in E$ are such that $\alpha_0 < \alpha < t < \beta < \beta_0$, then, taking into account that $T_1 \cup \{\alpha\} \in \mathcal{T}(E_\alpha^-)$ and $T_2 \cup \{\beta\} \in \mathcal{T}(E_\beta^+)$, we obtain the following by (5.2):

$$\begin{aligned} \mathsf{V}_\Phi(f, E \setminus \{t\}) &\le V_\Phi(f, T_1) + U(t_k, t_{k-1}) + V_\Phi(f, T_2) + \varepsilon \\ &\le V_\Phi(f, T_1) + U(\alpha, t_{k-1}) + U(\beta, \alpha) + U(t_k, \beta) + V_\Phi(f, T_2) + \varepsilon \\ &\le \mathsf{V}_\Phi(f, E_\alpha^-) + U(\beta, \alpha) + \mathsf{V}_\Phi(f, E_\beta^+) + \varepsilon. \end{aligned} \tag{5.9}$$

Since the Φ-variation is additive, we find that

$$\mathsf{V}_\Phi(f, E \setminus \{t\}) = \mathsf{V}_\Phi(f, E_\alpha^-) + \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) + \mathsf{V}_\Phi(f, E_\beta^+), \tag{5.10}$$

and this relation, taken together with (5.9), yields (5.8).

2. Now we note that the limit on the left-hand side of the relation in (c) exists, is finite, and equals

$$\lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) = \inf\{\mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) \mid \alpha \in E_t^-,\ \beta \in E_t^+,\ \alpha < t < \beta\} \in \mathbb{R}^+, \tag{5.11}$$

by virtue of considerations that are similar to those with the help of which (5.5) was obtained from (5.4) In (5.7) and (5.8), we let $\alpha \to t-0$ and $\beta \to t+0$, and, taking Lemma 5.1(c) into account, we arrive at the following inequalities:

$$\lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} U(\beta,\alpha) \le \lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) \le \lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} U(\beta,\alpha) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (c) is proved. □

Theorem 5.3. (a) *A mapping f of bounded Φ-variation is right continuous at a point $t \in E \setminus \{\sup E\}$ (respectively, left continuous at the point $t \in E \setminus \{\inf E\}$) if and only if its function of Φ-variation ϕ has the same property at the point t.*

(b) *A mapping f of bounded Φ-variation is continuous on E, except for possibly the set of points from E which is no more than countable.*

Proof. Item (b) follows from (a), since the nondecreasing bounded function ϕ has the set of discontinuity points (of the first kind) on E which is no more than countable, and, by virtue of (a), the set of discontinuity points of the mapping f and that of the function ϕ coincide.

We prove (a). If t is an isolated point (i.e., if it is not a limit point) of E, then the statement is obvious. If $t \in E'$, then $t \in E_t^{-\prime}$ or $t \in E_t^{+\prime}$. For definiteness, we carry out the proof in the case where $t \in E_t^{-\prime}$.

1. We show that the limit $\lim\limits_{\alpha\to t-0} d(f(t), f(\alpha)) =: \omega$ exists in $\mathbb{R}^+$. Let $\varphi : E \to \mathbb{R}^+$ be a nondecreasing bounded function from Theorem 3.1. For $\alpha, \beta \in E$, $\alpha < \beta < t$, we have

$$\begin{aligned}|d(f(t), f(\alpha)) - d(f(t), f(\beta))| &\le d(f(\alpha), f(\beta)) \le \mathsf{V}_1(f, E_\alpha^\beta) \\ &= \mathsf{V}_1(f, E_\beta^-) - \mathsf{V}_1(f, E_\alpha^-) = \varphi(\beta) - \varphi(\alpha).\end{aligned}$$

The existence of the limit $\varphi(t-0) = \lim\limits_{\alpha\to t-0} \varphi(\alpha)$ implies that the Cauchy criterion for its existence is fulfilled. Then it follows from the obtained inequality that the Cauchy criterion for the existence of the limit ω is fulfilled for the function $E_t^- \ni \alpha \mapsto d(f(t), f(\alpha)) \in \mathbb{R}^+$.

2. We prove that if ϕ is left continuous at the point t, i.e., if $U(t, t-0) = 0$ by Lemma 5.2(a), then f is also left continuous at this point, i.e., $\omega = 0$. We suppose the contrary, i.e., let $\omega > 0$. Using properties of the limit, we find $t_0 \in E$, $t_0 < t$, such that $d(f(t), f(\alpha)) \ge \omega/2$ for all $\alpha \in E$, $t_0 < \alpha < t$, and, therefore, by the monotonicity of Φ, we have

$$\Phi\left(\frac{d(f(t), f(\alpha))}{t-\alpha}\right) \cdot (t-\alpha) \ge \Phi\left(\frac{\omega/2}{t-\alpha}\right) \cdot (t-\alpha).$$

Letting $\alpha \to t-0$ in this inequality and taking into account Lemma 5.1(a) and notation (5.1), we find that

$$\begin{aligned}U(t, t-0) &= \lim_{\alpha\to t-0} \Phi\left(\frac{d(f(t), f(\alpha))}{t-\alpha}\right) \cdot (t-\alpha) \\ &\ge \lim_{\alpha\to t-0} \Phi\left(\frac{\omega/2}{t-\alpha}\right) \cdot (t-\alpha) = \frac{\omega}{2} \cdot \lim_{\rho\to\infty} \frac{\Phi(\rho)}{\rho}.\end{aligned} \tag{5.12}$$

It is known from the theory of convex functions that this last limit always exists in $\mathbb{R}^+ \cup \{\infty\}$. It cannot be equal to zero (otherwise Φ is constant and $\Phi \notin \mathcal{M}$) and it cannot be equal to ∞ (otherwise, $\Phi \in \mathcal{N}$, and then f is absolutely continuous and $\omega = 0$). It then follows from (5.12) that $U(t, t-0) > 0$, and this leads to a contradiction.

Remark. We showed simultaneously that if a mapping of bounded Φ-variation is discontinuous at least at one point, then $0 < \lim\limits_{\rho\to\infty} \Phi(\rho)/\rho < \infty$.

3. Conversely, we now prove that if $\omega = 0$, then $\vartheta := U(t, t-0) = 0$. If $\vartheta > 0$, then there exists $t_0 \in E$, $t_0 < t$, such that for all $\alpha \in E$, $t_0 < \alpha < t$, we have

$$\Phi\left(\frac{d(f(t), f(\alpha))}{t-\alpha}\right) \cdot (t-\alpha) \geq \frac{\vartheta}{2} \quad \text{or} \quad d(f(t), f(\alpha)) \geq \Phi^{-1}\left(\frac{\vartheta/2}{t-\alpha}\right) \cdot (t-\alpha).$$

In the limit, as $\alpha \to t-0$, we obtain from the last inequality that

$$\omega = \lim_{\alpha \to t-0} d(f(t), f(\alpha)) \geq \lim_{\alpha \to t-0} \Phi^{-1}\left(\frac{\vartheta/2}{t-\alpha}\right) \cdot (t-\alpha) = \frac{\vartheta}{2} \cdot \lim_{\rho \to \infty} \frac{\rho}{\Phi(\rho)}.$$

Therefore, $\omega > 0$, and we arrive at a contradiction. □

Lemma 5.4. *For the point $t \in E$, we have*

(a) *if $t \in E_t^{-\prime}$, then $\mathsf{V}_\Phi(f, E_t^-) - \mathsf{V}_\Phi(f, E_t^- \setminus \{t\}) = \phi(t) - \phi(t-0) = \lim_{\alpha \to t-0} \mathsf{V}_\Phi(f, E_\alpha^t)$;*

(b) *if $t \in E_t^{+\prime}$, then $\mathsf{V}_\Phi(f, E_t^+) - \mathsf{V}_\Phi(f, E_t^+ \setminus \{t\}) = \phi(t+0) - \phi(t) = \lim_{\beta \to t+0} \mathsf{V}_\Phi(f, E_t^\beta)$;*

(c) *f is left continuous at the point $t \in E_t^{-\prime}$ (respectively, right continuous at the point $t \in E_t^{+\prime}$) if and only if $\mathsf{V}_\Phi(f, E_t^-) = \mathsf{V}_\Phi(f, E_t^- \setminus \{t\})$ (respectively, if $\mathsf{V}_\Phi(f, E_t^+) = \mathsf{V}_\Phi(f, E_t^+ \setminus \{t\})$).*

Proof. (a) Applying Proposition 2.2(f), where E is replaced by the set $E_t^- \setminus \{t\}$, we find that

$$\mathsf{V}_\Phi(f, E_t^- \setminus \{t\}) = \lim_{\alpha \to t-0} \mathsf{V}_\Phi(f, (E_t^- \setminus \{t\})_\alpha^-) = \lim_{\alpha \to t-0} \mathsf{V}_\Phi(f, E_\alpha^-) = \phi(t-0);$$

therefore, by Proposition 2.2(c) we obtain

$$\phi(t) - \phi(t-0) = \lim_{\alpha \to t-0} \bigl(\mathsf{V}_\Phi(f, E_t^-) - \mathsf{V}_\Phi(f, E_\alpha^-)\bigr) = \lim_{\alpha \to t-0} \mathsf{V}_\Phi(f, E_\alpha^t).$$

(b) By Proposition 2.2(g), in which E is replaced by $E_t^+ \setminus \{t\}$, and Proposition 2.2(c), we find that

$$\begin{aligned} \mathsf{V}_\Phi(f, E_t^+ \setminus \{t\}) &= \lim_{\beta \to t+0} \mathsf{V}_\Phi(f, (E_t^+ \setminus \{t\})_\beta^+) = \lim_{\beta \to t+0} \mathsf{V}_\Phi(f, E_\beta^+) \\ &= \mathsf{V}_\Phi(f, E) - \lim_{\beta \to t+0} \mathsf{V}_\Phi(f, E_\beta^-) = \mathsf{V}_\Phi(f, E_t^+) + \phi(t) - \phi(t+0). \end{aligned}$$

Applying Proposition 2.2(c) once again, we obtain

$$\phi(t+0) - \phi(t) = \lim_{\beta \to t+0} \bigl(\mathsf{V}_\Phi(f, E_t^+) - \mathsf{V}_\Phi(f, E_\beta^+)\bigr) = \lim_{\beta \to t+0} \mathsf{V}_\Phi(f, E_t^\beta).$$

(c) follows from items (a) and (b) and from Theorem 5.3(a). □

Lemma 5.5. *Let a point $t \in E$ be such that $t \in E_t^{-\prime} \cap E_t^{+\prime}$. Then*

(a) $\mathsf{V}_\Phi(f, E) = \mathsf{V}_\Phi(f, E_t^- \setminus \{t\}) + \mathsf{V}_\Phi(f, E_t^+ \setminus \{t\}) + \phi(t+0) - \phi(t-0)$;

(b) *f is continuous at the point t if and only if*

$$\mathsf{V}_\Phi(f, E) = \mathsf{V}_\Phi(f, E_t^- \setminus \{t\}) + \mathsf{V}_\Phi(f, E_t^+ \setminus \{t\});$$

(c) $\mathsf{V}_\Phi(f, E) = \mathsf{V}_\Phi(f, E \setminus \{t\}) + \phi(t+0) - \phi(t-0) - \lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} \mathsf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\})$;

(d) *if f is continuous at the point t, then $\mathsf{V}_\Phi(f, E) = \mathsf{V}_\Phi(f, E \setminus \{t\})$; the converse statement is not true in general.*

Proof. Relation (a) follows if we add relations (a) and (b) from Lemma 5.4 and then apply Proposition 2.2(c).

Assertion (b) follows from Lemma 5.4(c), Lemma 5.5(a), and the inequalities $\phi(t-0) \le \phi(t) \le \phi(t+0)$.

(c) Taking into account Proposition 2.2(c) and relation (5.10), we have the following for the points $\alpha,\ \beta \in E$ such that $\alpha < t < \beta$:

$$\begin{aligned} \mathsf{V}_\Phi(f,E) - \mathsf{V}_\Phi(f,E\setminus\{t\}) &= \mathsf{V}_\Phi(f,E_\alpha^-) + \mathsf{V}_\Phi(f,E_\alpha^\beta) + \mathsf{V}_\Phi(f,E_\beta^+) \\ &\quad - \big(\mathsf{V}_\Phi(f,E_\alpha^-) + \mathsf{V}_\Phi(f,E_\alpha^\beta\setminus\{t\}) + \mathsf{V}_\Phi(f,E_\beta^+)\big) \\ &= \phi(\beta) - \phi(\alpha) - \mathsf{V}_\Phi(f,E_\alpha^\beta\setminus\{t\}). \end{aligned}$$

Letting $\alpha \to t-0$ and $\beta \to t+0$ and taking into account Lemma 5.2(c), we arrive at relation (c).

(d) Let f be continuous at the point t. Then, by Theorem 5.3(a), we have that $\phi(t-0) = \phi(t) = \phi(t+0)$. Passing to the limit as $\alpha \to t-0$ and $\beta \to t+0$ in the inequality

$$U(\beta,\alpha) \le U(\beta,t) + U(t,\alpha), \qquad \alpha,\ \beta \in E, \quad \alpha < t < \beta,$$

and taking Lemma 5.2(a,b) into account, we find that

$$0 \le U(t+0,t-0) \le U(t+0,t) + U(t,t-0) = \big(\phi(t+0)-\phi(t)\big) + \big(\phi(t)-\phi(t-0)\big) = 0.$$

Therefore, $U(t+0,t-0) = 0$. It remains to use Lemmas 5.5(c) and 5.2(c).

We present an example showing that the converse statement is not true. Suppose that $\Phi(\rho) = \rho$, $E = [-1,1]$, and $f : [-1,1] \to \mathbb{R}$ is such that $f(t) = -1$ for $-1 \le t < 0$, $f(0) = 0$, and $f(t) = 1$ for $0 < t \le 1$ (the signum function). Then, by relation (5.14) given below, we have

$$\mathop{\mathsf{V}}_{-1}^{1}{}_1(f) = \mathsf{V}_1(f,[-1,1]\setminus\{0\}) = 2.$$

□

Finally, Lemmas 5.1, 5.2, 5.4, and 5.5 and notation (5.1) readily imply the following theorem (which is of special interest in the case where the function $\Phi \in \mathcal{M}\setminus\mathcal{N}$, so that there exists the limit $\lim\limits_{\rho\to\infty} \Phi(\rho)/\rho \in (0,\infty)$).

Theorem 5.6. *Let $f : E \to X$ be a mapping of bounded Φ-variation, and let $t \in E$. We set $[\Phi] = \lim\limits_{\rho\to\infty} \Phi(\rho)/\rho$ and assume that $\infty \cdot 0 := 0$. Then*

(a) *if $t \in E_t^{-\prime}$, we have* $\mathsf{V}_\Phi(f,E_t^-) = \mathsf{V}_\Phi(f,E_t^-\setminus\{t\}) + [\Phi]\cdot \lim\limits_{\alpha\to t-0} d(f(t),f(\alpha))$;

(b) *if $t \in E_t^{+\prime}$, we have* $\mathsf{V}_\Phi(f,E_t^+) = \mathsf{V}_\Phi(f,E_t^+\setminus\{t\}) + [\Phi]\cdot \lim\limits_{\beta\to t+0} d(f(\beta),f(t))$;

(c) *if $t \in E_t^{-\prime}\cap E_t^{+\prime}$, then, in addition to* (a) *and* (b), *we have*

$$\mathsf{V}_\Phi(f,E) = \mathsf{V}_\Phi(f,E_t^-\setminus\{t\}) + \mathsf{V}_\Phi(f,E_t^+\setminus\{t\}) + [\Phi]\cdot\Big(\lim_{\alpha\to t-0} d(f(t),f(\alpha)) + \lim_{\beta\to t+0} d(f(\beta),f(t))\Big);$$

$$\mathsf{V}_\Phi(f,E) = \mathsf{V}_\Phi(f,E\setminus\{t\}) + [\Phi]\cdot\Big(\lim_{\alpha\to t-0} d(f(t),f(\alpha)) + \lim_{\beta\to t+0} d(f(\beta),f(t)) - \lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} d(f(\beta),f(\alpha))\Big);$$

$$\mathsf{V}_\Phi(f,E\setminus\{t\}) = \mathsf{V}_\Phi(f,E_t^-\setminus\{t\}) + \mathsf{V}_\Phi(f,E_t^+\setminus\{t\}) + [\Phi]\cdot \lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} d(f(\beta),f(\alpha)). \tag{5.13}$$

If, in Theorem 5.6, the metric space X is *complete*, then, as can be easily seen from step 1 of the proof of Theorem 5.3, there exists the one-sided left limit $f(t-0) = \lim\limits_{\alpha\to t-0} f(\alpha) \in X$ at points $t \in E_t^{-\prime}$

and at points $t \in E_t^{+\prime}$, there exists the right limit $f(t+0) = \lim_{\beta \to t+0} f(\beta) \in X$. In this case, the relations given above are modified in a natural way if the limit sign is "pulled" under the sign of the metric d; thus, for example, relation (5.13) becomes

$$\begin{aligned} \mathsf{V}_\Phi(f, E) = \mathsf{V}_\Phi(f, E \backslash \{t\}) + [\Phi] \cdot \Big(& d(f(t), f(t-0)) \\ & + d(f(t+0), f(t)) - d(f(t+0), f(t-0)) \Big). \end{aligned} \tag{5.14}$$

The relations from Theorem 5.6 for the closed interval $E = [a, b]$ are also new and are easily rewritten for this case.

Remark 5.1. The formulas from Theorem 5.6 were first found by the author in [6, 2.23] in the case where $E = [a, b]$, $\Phi(\rho) = \rho$, and X is a Banach space. For the case where E is arbitrary and $\Phi(\rho) = \rho$, these formulas were generalized in [7] (for a complete metric space X) and in [8] (for an arbitrary metric space X). The method of proof in the general case ($\Phi \in \mathcal{M}$) is much like (with corresponding modifications) the method used in [8, Sec. 4].

For *continuous* mappings $f \in BV_\Phi([a, b]; X)$, where $\Phi \in \mathcal{M}$ and X is a metric space, the concept of the Φ-variation $\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f)$ coincides with the concept of an *integral of an interval function* in the sense of [33, Chapter 1, Sec. 3]. Indeed, an interval function $U(\beta, \alpha)$, $a \le \alpha < \beta \le b$, from (5.1) is *semiadditive* (inequality (5.2)) and *continuous* at any point $a < t < b$ (at the points $t = a$ and $t = b$, the unilateral continuity is meant), i.e., for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|U(\beta, \alpha)| \le \varepsilon$ for all α, $\beta \in [a, b]$, $\alpha < \beta$, such that $\alpha < t < \beta$ and $\beta - \alpha \le \delta$. The last assertion is implied by the continuity of f since $U(t+0, t-0) = 0$, as is shown in the proof of Lemma 5.5(d). For a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$, we set $\lambda(T) := \max_{1 \le i \le m} (t_i - t_{i-1})$, define the *oscillation* of the mapping f on the closed interval $[t_{i-1}, t_i]$ by the formula

$$\omega(f, [t_{i-1}, t_i]) := \sup\{d(f(t), f(s)) \mid t,\ s \in [t_{i-1}, t_i]\}, \quad i = 1, \dots, m,$$

and set

$$\Omega_\Phi(f, T) := \sum_{i=1}^m \Phi\left(\frac{\omega(f, [t_{i-1}, t_i])}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}).$$

Then we have

Corollary 5.7. *Let X be a metric space, $\Phi \in \mathcal{M}$, and f be a continuous mapping from $BV_\Phi([a, b]; X)$. Then the following relations hold*:

(a) $\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) = \lim_{\lambda(T) \to 0} V_\Phi(f, T)$, *i.e., for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\big|V_\Phi(f, T) - \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f)\big| \le \varepsilon$ for all $T \in \mathcal{T}_a^b$ and $\lambda(T) \le \delta$;*

(b) $\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) = \lim_{\lambda(T) \to 0} \Omega_\Phi(f, T)$.

Proof. With the remarks presented above being taken into account, item (a) is directly implied by [33, Chapter 1, Sec. 3.14, Theorem], and item (b) is implied by the inequalities

$$V_\Phi(f, T) \le \Omega_\Phi(f, T) \le \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) \qquad \forall\, T \in \mathcal{T}_a^b.$$

□

6. Mappings with Values in a Normed Space

In this section, we assume that X is a linear normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the norm $\|\cdot\|$. The set X^E of all mappings from $E \subset \mathbb{R}$ into X in a natural way becomes a linear space over $\mathbb{K}$ with respect to pointwise operations of addition and multiplication by a scalar value: if $f,\, g \in X^E$ and $c \in \mathbb{K}$, then

$$(f+g)(t) := f(t) + g(t), \qquad (cf)(t) := cf(t), \qquad t \in E.$$

6.1. The norm in the space $BV_\Phi(E;X)$. Let Φ be an $\mathcal{M}$-function. Since E is a fixed set in this section, we omit it in the notation of the Φ-variation and write $\mathsf{V}_\Phi(f)$ instead of $\mathsf{V}_\Phi(f,E)$. First of all, we note that the convexity of Φ implies that the set $BV_\Phi(E;X)$ is convex and $f \mapsto \mathsf{V}_\Phi(f)$ is a *convex* functional on this set, i.e.,

$$\mathsf{V}_\Phi(\theta f + (1-\theta)g) \le \theta\,\mathsf{V}_\Phi(f) + (1-\theta)\,\mathsf{V}_\Phi(g), \qquad f,\ g \in BV_\Phi(E;X), \quad 0 \le \theta \le 1, \tag{6.1}$$

and, in particular,

$$\mathsf{V}_\Phi(\theta f) \le \theta\,\mathsf{V}_\Phi(f), \qquad f \in BV_\Phi(E;X), \quad 0 \le \theta \le 1. \tag{6.2}$$

An $\mathcal{M}$-function Φ is called *tempered* if

$$\exists\, C \in (0,\infty), \quad \rho_0 \ge 0 \quad \text{such that} \quad \Phi(2\rho) \le C\Phi(\rho) \quad \forall\, \rho \ge \rho_0. \tag{6.3}$$

(Condition (6.3) in [24, Chapter 1, Sec. 4] is called the Δ_2-*condition.*) Note that always $C \ge 2$ since, by (6.2), for $\rho \ge 0$, we have $\Phi(2\rho) \ge 2\Phi(\rho)$. Note that the condition that the function $\Phi \in \mathcal{M}$ is tempered is equivalent to the condition

$$\forall\, s > 1\ \exists\, C(s) > 0, \text{ and } \rho_0(s) \ge 0 \quad \text{such that} \quad \Phi(s\rho) \le C(s)\Phi(\rho)\ \forall\, \rho \ge \rho_0(s). \tag{6.4}$$

Proposition 6.1. (a) *If Φ is a tempered $\mathcal{M}$-function, then the set $BV_\Phi(E;X)$ is a linear space over $\mathbb{K}$.*
(b) *If $E = [a,b]$, X is a Banach space, Φ is an $\mathcal{M}$-function, and $BV_\Phi(E;X)$ is a linear space, then the function Φ is tempered.*

Proof. (a) Let $f,\, g \in BV_\Phi(E;X)$, and let $c \in \mathbb{K}$. We note that $\mathsf{V}_\Phi(cf) = \mathsf{V}_\Phi(|c|f)$. If $|c| \le 1$, then from (6.2), we find that $cf \in BV_\Phi(E;X)$. On the other hand, if $|c| > 1$, then, by condition (6.4), we find $C > 0$ and $\rho_0 \ge 0$ such that $\Phi(|c|\rho) \le C\Phi(\rho)$ for $\rho \ge \rho_0$. Setting $\Psi(\rho) = \Phi(|c|\rho)$, $\rho \ge 0$, from Proposition 2.7(a), we obtain that $BV_\Phi(E;X) \subset BV_\Psi(E;X)$, and, in particular, $\mathsf{V}_\Phi(|c|f) = \mathsf{V}_\Psi(f) < \infty$, whence $cf \in BV_\Phi(E;X)$. From this and (6.1), it follows that $f + g \in BV_\Phi(E;X)$:

$$\mathsf{V}_\Phi(f+g) = \mathsf{V}_\Phi(\tfrac{1}{2}2f + \tfrac{1}{2}2g) \le \tfrac{1}{2}\,\mathsf{V}_\Phi(2f) + \tfrac{1}{2}\,\mathsf{V}_\Phi(2g) < \infty.$$

(b) If $\mathcal{L} := BV_\Phi([a,b];X)$ is a linear space, then this means, in particular, that the fact that $f \in \mathcal{L}$ implies $2f \in \mathcal{L}$, i.e., $\mathcal{L} \subset BV_\Psi([a,b];X)$, where $\Psi(\rho) := \Phi(2\rho)$, $\rho \ge 0$. From Proposition 2.7(b), for certain $C > 0$ and $\rho_0 \ge 0$, we then obtain that $\Phi(2\rho) = \Psi(\rho) \le C\Phi(\rho)$ for $\rho \ge \rho_0$, i.e., Φ satisfies (6.3). □

On the space $BV_\Phi(E;X)$, we define the following nonnegative functional (of Luxemburg's type [24, Chapter 2, Sec. 9.7]):

$$p(f) \equiv p_\Phi(f) := \inf\bigl\{r > 0 \,\big|\, \mathsf{V}_\Phi(f/r) \le 1\bigr\}, \qquad f \in BV_\Phi(E;X). \tag{6.5}$$

The quantity $p(f)$ is always finite ($\in \mathbb{R}^+$) since $\mathsf{V}_\Phi(f/r) \le \mathsf{V}_\Phi(f)/r$ for $r \ge 1$ by (6.2); therefore, $\lim\limits_{r\to\infty} \mathsf{V}_\Phi(f/r) = 0$. The main properties of the functional $p(\cdot)$ are summarized in the following proposition (everywhere in it Φ is a certain $\mathcal{M}$-function).

Proposition 6.2. (a) *For any $f \in BV_\Phi(E;X)$, the following inequality holds*:

$$\|f(t)-f(s)\| \le p(f)|t-s|\Phi^{-1}(1/|t-s|) \qquad \forall\, t,\ s \in E, \quad t \ne s.$$

(b) *If the mapping $f \in BV_\Phi(E;X)$ is such that $p(f) > 0$, then $\mathsf{V}_\Phi(f/p(f)) \le 1$ (and thus the infimum in (6.5) is reached for such mappings f).*

(c) *If $f \in BV_\Phi(E;X)$, $r_0 > 0$, and $\mathsf{V}_\Phi(f/r_0) = 1$, then $p(f) = r_0$.*

(d) *If Φ is a tempered $\mathcal{M}$-function, then the functional $p = p_\Phi$ is a seminorm on the linear space $BV_\Phi(E;X)$.*

(e) *If the sequence of mappings $f_n \in BV_\Phi(E;X)$, $n \in \mathbb{N}$, converges pointwise on E to $f \in X^E$ as $n \to \infty$, then $p(f) \le \limsup_{n\to\infty} p(f_n)$.*

Proof. (a) For t, $s \in E$, $s < t$, by Proposition 2.2(b) and definition (6.5), we have

$$\Phi\biggl(\frac{\|f(t)-f(s)\|}{r(t-s)}\biggr)\cdot(t-s) \le \mathsf{V}_\Phi(f/r) \le 1 \qquad \forall\, r > p(f),$$

from which, applying the inverse function Φ^{-1} to both parts of this inequality, we obtain

$$\|f(t)-f(s)\| \le r(t-s)\Phi^{-1}(1/(t-s)), \qquad r > p(f).$$

It remains to pass to the limit in this expression as $r \to p(f)$ and take into account the "symmetry" of the occurence of variables t and s.

(b) It follows from the definition of $p(f) > 0$ that $\mathsf{V}_\Phi(f/r) \le 1$ for all $r > p(f)$. Considering the sequence $r_n > p(f)$, $n \in \mathbb{N}$, such that $r_n \to p(f)$ as $n \to \infty$, observing that $f/r_n \to f/p(f)$ pointwise on E as $n \to \infty$, and applying Proposition 2.2(d), we find that

$$\mathsf{V}_\Phi(f/p(f)) \le \liminf_{n\to\infty} \mathsf{V}_\Phi(f/r_n) \le 1,$$

whence $p(f) \in \Lambda := \{r > 0 \mid \mathsf{V}_\Phi(f/r) \le 1\}$ and $p(f) = \min\Lambda$.

(c) Definition (6.5) yields $p(f) \le r_0$. Note that $p(f) > 0$ (otherwise, if $p(f) = 0$, then, by virtue of (a), f is a constant mapping, so that $\mathsf{V}_\Phi(f/r_0)=0\ne 1$). If $p(f) < r_0$, then, by (6.2) and (b), we have

$$1 = \mathsf{V}_\Phi\Bigl(\frac{f}{r_0}\Bigr) = \mathsf{V}_\Phi\Bigl(\frac{p(f)}{r_0}\cdot\frac{f}{p(f)}\Bigr) \le \frac{p(f)}{r_0}\cdot\mathsf{V}_\Phi\Bigl(\frac{f}{p(f)}\Bigr) \le \frac{p(f)}{r_0} < 1,$$

which is impossible. Therefore, $p(f) = r_0$.

(d) If $f = 0$, then $\mathsf{V}_\Phi(f/r) = 0$ for all $r > 0$; therefore, $p(f) = 0$. Conversely, if $p(f) = 0$, then from the inequality in (a) we find that $f(t) = f(s)$ for all t, $s \in E$.

For $f \in BV_\Phi(E;X)$ and $c \in \mathbb{K}$, we have

$$p(cf) = \inf\{r > 0 \mid \mathsf{V}_\Phi(cf/r) \le 1\} = |c|\inf\{\rho > 0 \mid \mathsf{V}_\Phi(f/\rho) \le 1\} = |c|p(f).$$

We now prove the triangle inequality $p(f+g) \le p(f) + p(g)$ for the mappings f, $g \in BV_\Phi(E;X)$. If $p(f) = 0$ or $p(g) = 0$, then this inequality is obvious. Let $p(f) > 0$ and $p(g) > 0$. Then, from (6.1) and (b), we obtain

$$\mathsf{V}_\Phi\Bigl(\frac{f+g}{p(f)+p(g)}\Bigr) \le \frac{p(f)}{p(f)+p(g)}\,\mathsf{V}_\Phi\Bigl(\frac{f}{p(f)}\Bigr) + \frac{p(g)}{p(f)+p(g)}\,\mathsf{V}_\Phi\Bigl(\frac{g}{p(g)}\Bigr) \le 1,$$

and the triangle inequality now follows from definition (6.5).

(e) If the limit mentioned above is infinite, then the inequality in (e) is obvious. Now, let this limit be finite. Then there exists a number $n_0 \in \mathbb{N}$ such that the value of $\sup_{k\geq n} p(f_k)$ is finite for all $n \geq n_0$. We fix $\varepsilon > 0$. Since $\varepsilon + \sup_{k\geq n} p(f_k) > p(f_n)$, we have the following from the definition of $p(f_n)$:

$$\mathsf{V}_\Phi\left(\frac{f_n}{\varepsilon + \sup_{k\geq n} p(f_k)}\right) \leq 1, \qquad n \geq n_0.$$

Since $f_n \to f$ pointwise on E as $n \to \infty$, we also have

$$\frac{f_n}{\varepsilon + \sup_{k\geq n} p(f_k)} \longrightarrow \frac{f}{\varepsilon + \limsup_{\nu\to\infty} p(f_\nu)} \quad \text{pointwise on } E \text{ as } \ n \to \infty;$$

therefore, it follows from Proposition 2.2(d) that

$$\mathsf{V}_\Phi\left(\frac{f}{\varepsilon + \limsup_{\nu\to\infty} p(f_\nu)}\right) \leq \liminf_{n\to\infty} \mathsf{V}_\Phi\left(\frac{f_n}{\varepsilon + \sup_{k\geq n} p(f_k)}\right) \leq 1.$$

Then, from the definition of $p(f)$, we obtain $p(f) \leq \varepsilon + \limsup_{\nu\to\infty} p(f_\nu)$, and it remains to take into account the arbitrariness of $\varepsilon > 0$. □

For a fixed $a \in E$ and a tempered $\mathcal{M}$-function Φ, we define the *norm* on the linear space $BV_\Phi(E;X)$ as

$$\|f\|_\Phi := \|f(a)\| + p(f), \qquad f \in BV_\Phi(E;X).$$

We have the following theorem.

Theorem 6.3. *If Φ is a tempered $\mathcal{M}$-function and X is a Banach space, then the space $BV_\Phi(E;X)$ with the norm $\|\cdot\|_\Phi$ is also a Banach space.*

Proof. It suffices to prove that $BV_\Phi(E;X)$ is a *complete* space. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $BV_\Phi(E;X)$:

$$\|f_n - f_m\|_\Phi = \|f_n(a) - f_m(a)\| + p(f_n - f_m) \to 0 \quad \text{as} \quad n,\, m \to \infty.$$

This fact and Proposition 6.2(a) imply that the sequence $\{f_n(t)\}_{n=1}^\infty$ is a Cauchy sequence in X for any $t \in E$, and since X is complete, there exists a mapping $f \in X^E$ such that f_n pointwise converges to f on E as $n \to \infty$. Since $f_n - f_m \to f_n - f$ pointwise on E as $m \to \infty$, then, using Proposition 6.2(e), we obtain

$$\|f_n - f\|_\Phi \leq \limsup_{m\to\infty} \|f_n - f_m\|_\Phi = \lim_{m\to\infty} \|f_n - f_m\|_\Phi \in \mathbb{R}^+ \qquad \forall\, n \in \mathbb{N},$$

from which, taking into account that $\{f_n\}_{n=1}^\infty$ is the Cauchy sequence, we have

$$\limsup_{n\to\infty} \|f_n - f\|_\Phi \leq \lim_{n\to\infty} \lim_{m\to\infty} \|f_n - f_m\|_\Phi = 0.$$

Therefore, $\|f_n - f\|_\Phi \to 0$ as $n \to \infty$. Hence there exists a number $N \in \mathbb{N}$ such that $\|f_N - f\|_\Phi \leq 1$, whence

$$\|f\|_\Phi \leq \|f - f_N\|_\Phi + \|f_N\|_\Phi \leq 1 + \|f_N\|_\Phi < \infty.$$

Thus, $f \in BV_\Phi(E;X)$, and the theorem is proved. □

6.2. The Φ-variation of smooth mappings. Let $C^1([a,b];X)$ denote the linear space of all mappings $f \in C([a,b];X)$ whose strong derivative f' (i.e., the derivative in the norm of the space X) exists and belongs to $C([a,b];X)$. In this section, assuming that X is a *not necessarily complete* linear normed space, we will prove the explicit formula (6.6) for the Φ-variation of smooth mappings $f \in C^1([a,b];X)$. For some special cases, this formula was earlier proved in [7, Theorem 8.7(b)] (for $\Phi(\rho) = \rho$), in [9, Theorem 4.2] (for $\Phi(\rho) = \rho^q$, $q > 1$), and also in [11, Corollary 3.2(c)] (if $\Phi \in \mathcal{N}$).

Theorem 6.4. *If $f \in C^1([a,b];X)$, then $f \in BV_\Phi([a,b];X)$ for all $\Phi \in \mathcal{M}$ and*

$$\mathop{\mathsf{V}}_a^b{}_\Phi(f) = \int_a^b \Phi\big(\|f'(t)\|\big)\,dt. \tag{6.6}$$

Proof. By the Lagrange mean value theorem A.7 (inequality (A.3)), we have an embedding $C^1([a,b];X) \subset C^{0,1}([a,b];X)$; therefore, the first assertion of the theorem is implied by Proposition 2.4(b). Formula (6.6) is proved in two steps.

1. Initially, suppose that X is complete, i.e., a Banach space. Let $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$. Taking into account that Φ increases and applying the Jensen integral inequality (A.2) in the case $\alpha(t) \equiv 1$, we obtain

$$\begin{aligned} V_\Phi(f,T) &= \sum_{i=1}^m \Phi\left(\frac{\|f(t_i) - f(t_{i-1})\|}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^m \Phi\left(\frac{\big\|\int_{t_{i-1}}^{t_i} f'(t)\,dt\big\|}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &\le \sum_{i=1}^m \Phi\left(\frac{\int_{t_{i-1}}^{t_i} \|f'(t)\|\,dt}{t_i - t_{i-1}}\right) \cdot (t_i - t_{i-1}) \\ &\le \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \Phi\big(\|f'(t)\|\big)\,dt = \int_a^b \Phi\big(\|f'(t)\|\big)\,dt. \end{aligned}$$

Thus, we have

$$\mathop{\mathsf{V}}_a^b{}_\Phi(f) \le \int_a^b \Phi\big(\|f'(t)\|\big)\,dt. \tag{6.7}$$

The completeness of X is required for the existence of the X-valued Riemannian integral $\int_{t_{i-1}}^{t_i} f'(t)\,dt$. If X is not complete, then, embedding X in its completion and observing that on the elements of the space X, the norm in the completion coincides with the norm in X, we obtain inequality (6.7) without the assumption on the completeness of X.

2. For the inequality converse to (6.7), the completeness of X is not needed. This inequality is immediately implied by inequality (2.16) if we show that

$$\lim_{h\to+0} \int_a^{b-h} \Phi\left(\left\|\frac{f(t+h) - f(t)}{h}\right\|\right) dt = \int_a^b \Phi\big(\|f'(t)\|\big)\,dt. \tag{6.8}$$

Fixing $0 < h < b-a$ and applying Theorem A.7 (Lagrange mean-value theorem) to the mapping $[0,h] \ni s \mapsto f(t+s) \in X$, for arbitrary $t \in [a, b-h]$, we find that

$$\|f(t+h) - f(t)\| \le h \sup_{t\in[a,b]} \|f'(t)\| = h\,\mathrm{Lip}(f), \tag{6.9}$$

$$\|f(t+h) - f(t) - f'(t)h\| \le h \sup_{s\in(0,h)} \|f'(t+s) - f'(t)\|. \tag{6.10}$$

Denoting by $\Phi'_+(\rho)$ the right derivative of Φ at the point $\rho \in \mathbb{R}^+$ (see Remark 2.1), by the Lagrange mean-value theorem we obtain the inequality

$$|\Phi(\rho_1) - \Phi(\rho_2)| \le |\rho_1 - \rho_2| \cdot \big(\Phi'_+(\rho_1) + \Phi'_+(\rho_2)\big), \quad \rho_1 \ge 0, \quad \rho_2 \ge 0. \tag{6.11}$$

The difference between the left- and right-hand sides of relation (6.8) is estimated as

$$\left| \int_a^{b-h} \Phi\left(\left\|\frac{f(t+h)-f(t)}{h}\right\|\right) dt - \int_a^b \Phi\big(\|f'(t)\|\big)\, dt \right| \\ \le \int_a^{b-h} \left| \Phi\left(\left\|\frac{f(t+h)-f(t)}{h}\right\|\right) - \Phi\big(\|f'(t)\|\big) \right| dt + \int_{b-h}^b \Phi\big(\|f'(t)\|\big)\, dt.$$

As $h \to +0$, the second integral on the right-hand side tends to zero. By (6.11), (6.9), and (6.10), the first integral does not exceed the following expression:

$$\int_a^{b-h} \left| \left\|\frac{f(t+h)-f(t)}{h}\right\| - \|f'(t)\| \right| \cdot \left(\Phi'_+\left(\left\|\frac{f(t+h)-f(t)}{h}\right\|\right) + \Phi'_+\big(\|f'(t)\|\big) \right) dt \\ \le 2\Phi'_+(\mathrm{Lip}(f)) \int_a^{b-h} \left\| \frac{f(t+h)-f(t)}{h} - f'(t) \right\| dt \le 2\Phi'_+(\mathrm{Lip}(f)) \int_a^{b-h} \sup_{s\in(0,h)} \|f'(t+s) - f'(t)\|\, dt.$$

Here the integrand tends to zero as $h \to +0$ due to the uniform continuity of the derivative f' on $[a,b]$. This completes the proof of relation (6.8); therefore, (6.6) is also proved. □

Corollary 6.5. *For $f \in C^1([a,b];X)$ and $\Phi \in \mathcal{M}$, the following estimates hold:*

$$r_1 := \frac{\int_a^b \|f'(t)\|\, dt}{(b-a)\Phi^{-1}\left(\frac{1}{b-a}\right)} \le p(f) = p_\Phi(f) \le \frac{\max_{t\in[a,b]} \|f'(t)\|}{\Phi^{-1}\left(\frac{1}{b-a}\right)} =: r_2. \tag{6.12}$$

Proof. The inequality on the right-hand side is implied by the fact that if $r \ge r_2$, then, by (6.6) and the monotonicity of Φ, we have

$$\bigvee_a^b{}_\Phi(f/r) = \int_a^b \Phi\big(\|f'(t)\|/r\big)\, dt \le \int_a^b \Phi\big(\|f'(t)\|/r_2\big)\, dt \le 1.$$

The inequality on the left-hand side is implied by the fact that for $r \le r_1$, by the Jensen inequality (A.2) we have

$$\bigvee_a^b{}_\Phi(f/r) \ge \int_a^b \Phi\big(\|f'(t)\|/r_1\big)\, dt \ge \Phi\left(\frac{\int_a^b \|f'(t)\|\, dt}{r_1(b-a)}\right)(b-a) = 1.$$

We now show that estimates (6.12) are sharp. Let $\Phi(\rho) = \rho^q$ and let $q > 1$. Then

$$\bigvee_a^b{}_q(f/r_0) = \frac{1}{r_0^q} \int_a^b \|f'(t)\|^q dt = 1, \qquad r_0 = \left(\int_a^b \|f'(t)\|^q dt\right)^{1/q};$$

therefore, $p(f) = r_0$ by Proposition 6.2(c). Inequalities (6.12) in this case take the form

$$(b-a)^{\frac{q}{q-1}} \int_a^b \|f'(t)\|\, dt \le \left(\int_a^b \|f'(t)\|^q dt\right)^{1/q} \le (b-a)^{1/q} \max_{t\in[a,b]} \|f'(t)\|. \tag{6.13}$$

Now let $X = \mathbb{R}$, and let $[a,b] = [0,1]$. If $f(t) = t$, then both inequalities in (6.13) turns into equalities. If $f(t) = t^2$, then inequalities in (6.13) become strict: $1 < 2\left(\frac{1}{q+1}\right)^{1/q} < 2$ for $q > 1$. □

6.3. Differentiation of mappings of bounded Φ-variation. In this subsection, we additionally assume that X is a Banach space. Let X^* denote the space that is strongly dual to X (i.e., a Banach space of all linear continuous functionals on X). The value of a functional $x^* \in X^*$ on an element $x \in X$ is written as (x^*, x) or (x, x^*). The norm in the space X^* is denoted by $\|\cdot\|_*$, and if it does not lead to confusion, then we denote it by $\|\cdot\|$.

The *weak derivative* of a mapping $f : [a, b] \to X$ at a point $t_0 \in (a, b)$ is the element $f^\bullet(t_0) \in X$ satisfying the condition

$$\Big(\frac{f(t_0+h) - f(t_0)}{h} - f^\bullet(t_0), x^*\Big) \to 0 \quad \text{as} \quad \mathbb{R} \ni h \to 0 \quad \forall\, x^* \in X^*.$$

In this case, the mapping f is said to be *weakly differentiable* at the point t_0. Any mapping f that is strongly differentiable at the point t_0 is also weakly differentiable at this point, and $f^\bullet(t_0) = f'(t_0)$.

It is well known that any scalar-valued absolutely continuous function on a closed interval $[a, b]$ is differentiable almost everywhere on (a, b) and can be represented in the form of the Lebesgue integral of its derivative. However, in the absence of additional constraints imposed on the Banach space X, even a *Lipschitzian* mapping $f : [a, b] \to X$ can be neither strongly nor weakly differentiable on (a, b) (see [23], [2, Chapter 1, Sec. 2.1], or [3, Chapter 1, Sec. 3.2]). In the same works, it is shown that if X is a *reflexive* Banach space, then the mapping $f \in AC([a, b]; X)$ is strongly differentiable almost everywhere on (a, b) and can be represented in the form of the Bochner indefinite integral of its strong derivative.

For $\Phi \in \mathcal{M}$, let $L_\Phi([a, b]; X)$ denote the space of all (equivalence classes) of strongly measurable mappings $f : [a, b] \to X$ for which the Lebesgue integral $\int_a^b \Phi\big(\|f(t)\|\big)\, dt$ is finite (this space can be not a linear one). By $A^1_\Phi([a, b]; X)$ we denote the space of all mappings $f \in AC([a, b]; X)$ that have the strong derivative f' (defined almost everywhere with respect to the Lebesgue measure on $[a, b]$), which belongs to the space $L_\Phi([a, b]; X)$.

Theorem 6.6. *Let $\Phi \in \mathcal{M}$, let X be a reflexive Banach space, and let $f \in BV_\Phi([a, b]; X)$. Then we have*

(a) *the mapping f is weakly differentiable almost everywhere on (a, b), its weak derivative $f^\bullet$ is strongly measurable, $f^\bullet \in L_\Phi([a, b]; X)$, and*

$$\int_a^b \Phi\big(\|f^\bullet(t)\|\big)\, dt \le \bigvee_a^b{}_\Phi(f); \tag{6.14}$$

(b) *moreover, if $\Phi \in \mathcal{N}$, then $f \in A^1_\Phi([a, b]; X)$, f can be represented in the form*

$$f(t) = f(a) + \int_a^t f'(\tau)\, d\tau \quad \textit{for all} \quad a \le t \le b \tag{6.15}$$

(*where the integral on the right-hand side is understood in the sense of Bochner*), *and*

$$\bigvee_a^b{}_\Phi(f) = \int_a^b \Phi\big(\|f'(t)\|\big)\, dt. \tag{6.16}$$

Conversely, if $\Phi \in \mathcal{N}$ and $f \in A^1_\Phi([a, b]; X)$, then $f \in BV_\Phi([a, b]; X)$.

Proof. (a) By Corollaries 2.5 and 2.6(a), we have $f \in BV_1([a, b]; X)$, and the image of $f([a, b])$ is a precompact (and, in particular, separable) set in X. Therefore, the strong closure of the linear hull of $f([a, b])$ is a separable Banach space. Thus, we assume without loss of generality that X is a separable reflexive Banach space.

Using the inequality $\|f(t) - f(s)\| \le \bigvee_a^t{}_1(f) - \bigvee_a^s{}_1(f)$, $a \le s \le t \le b$, the fact that the function $t \mapsto \bigvee_a^t{}_1(f)$ is not decreasing on $[a, b]$, and the Lebesgue theorem on differentiation of monotone scalar-valued functions, we find that the Lebesgue measure of the set

$$A_0 := \Big\{t \in [a, b]\, Bigl|\ \limsup_{h\to 0}\Big\|\frac{f(t+h) - f(t)}{h}\Big\| = \infty\Big\}$$

is equal to zero. Since X is reflexive and separable, its dual X^* is also separable [20, Chapter 5, Sec. 2, Lemma]. Let $\{x_n^*\}_{n=1}^{\infty}$ be a strongly dense sequence in X^*. For any $n \in \mathbb{N}$, the scalar-valued function $t \mapsto (f(t), x_n^*)$ is of bounded 1-variation on $[a,b]$; therefore, it is almost everywhere differentiable on the interval (a,b): there exists a set $A_n \subset [a,b]$ of zero Lebesgue measure such that the derivative $\frac{d}{dt}(f(t), x_n^*)$ exists at every point $t \in [a,b] \setminus A_n$. The Lebesgue measure of the set $A := A_0 \cup \bigcup_{n=1}^{\infty} A_n$ is also equal to zero, and, for all $n \in \mathbb{N}$ and all $t \in [a,b] \setminus A$, there exists the limit

$$\lim_{h \to 0} \Big(\frac{f(t+h) - f(t)}{h}, x_n^* \Big) = \frac{d}{dt}(f(t), x_n^*).$$

The fact that any reflexive Banach space is weakly sequentially dense ([20, Chapter 5, Sec. 1, Theorem 7]) implies that for any $t \in [a,b] \setminus A$, there exists an element $f^{\bullet}(t) \in X$ such that

$$\lim_{h \to 0} \Big(\frac{f(t+h) - f(t)}{h}, x^* \Big) = (f^{\bullet}(t), x^*) \qquad \forall\, x^* \in X^*. \tag{6.17}$$

Therefore, the mapping f has the weak derivative almost everywhere on (a,b), which is weakly measurable on (a,b). Since f has values in a separable space, $f^{\bullet}$ is almost separable space-valued; therefore, by the Pettis theorem, $f^{\bullet}$ is strongly measurable. Choosing (by virtue of the Hahn–Banach theorem) the element $x^* \in X^*$ in relation (6.17) such that $(f^{\bullet}(t), x^*) = \|f^{\bullet}(t)\|$ and $\|x^*\| = 1$, we find that

$$\|f^{\bullet}(t)\| \le \liminf_{h \to 0} \Big\| \frac{f(t+h) - f(t)}{h} \Big\| \qquad \forall\, t \in (a,b) \setminus A. \tag{6.18}$$

Using the Fatou lemma and inequality (2.16), we obtain

$$\int_a^b \Phi\big(\|f^{\bullet}(t)\|\big)\, dt \le \liminf_{h \to 0} \int_a^{b-h} \Phi\bigg(\Big\| \frac{f(t+h) - f(t)}{h} \Big\| \bigg)\, dt \le \bigvee_a^b{}_{\Phi}(f). \tag{6.19}$$

Therefore, $f^{\bullet} \in L_{\Phi}([a,b]; X)$.

(b) Now, let $\Phi \in \mathcal{N}$. Then $f \in AC([a,b]; X)$ by Proposition 2.4(d); therefore, the scalar-valued function $t \mapsto (f(t), x^*)$ is absolutely continuous on $[a,b]$ for any $x^* \in X^*$. Using (a), for all $t \in [a,b]$, we have

$$(f(t) - f(a), x^*) = \int_a^t (f^{\bullet}(\tau), x^*)\, d\tau = \Big(\int_a^t f^{\bullet}(\tau)\, d\tau, x^* \Big) \qquad \forall\, x^* \in X^*,$$

and, therefore,

$$f(t) = f(a) + \int_a^t f^{\bullet}(\tau)\, d\tau \qquad \forall\, t \in [a,b]. \tag{6.20}$$

This implies that the strong limit $\lim\limits_{h \to 0} \frac{1}{h}(f(t+h) - f(t)) = f^{\bullet}(t)$ coincides with the strong derivative $f'(t)$ for almost all $t \in (a,b)$. Therefore, the strong derivative f' exists almost everywhere on (a,b), $f' \in L_{\Phi}([a,b]; X)$, relation (6.15) holds by virtue of (6.20), and relation (6.16) is implied by inequality (6.19), where $f^{\bullet}$ is replaced by f', and inequality (6.7), which holds with (6.15) taken into account by the same arguments as in the first step of the proof of Theorem 6.4.

Finally, the last assertion of the theorem holds (without the assumption that X is reflexive) due to the fact that, proceeding from relation (6.15), we can apply the arguments of step 1 of the proof of Theorem 6.4 and obtain inequality (6.7), which just means that $f \in BV_{\Phi}([a,b]; X)$. The theorem is proved. □

Remark 6.1. (a) If $\Phi \in \mathcal{M}$ and $\varphi \in BV_{\Phi}([a,b]; \mathbb{R})$, then, by Theorem 6.6(a), the derivative $\varphi' \in L_{\Phi}([a,b]; \mathbb{R})$ exists almost everywhere on $[a,b]$; moreover,

$$\int_a^b \Phi\big(|\varphi'(t)|\big)\, dt \le \bigvee_a^b{}_{\Phi}(\varphi). \tag{6.21}$$

(If $\Phi(\rho) = \rho$, then the example in which φ is a "Cantor ladder" [28, Chapter 8, Sec. 2, Example] shows that we cannot assert anything more than this inequality in (6.21).) Thus, if $f \in BV_\Phi([a,b];X)$, where X is an arbitrary *metric* space and $\varphi(t) = \overset{t}{\underset{a}{\mathsf{V}}}_1(f)$ for $a \le t \le b$, then, by Theorem 3.1, we have

$$\int_a^b \Phi\Bigl(\bigl|\tfrac{d}{dt}\overset{t}{\underset{a}{\mathsf{V}}}_1(f)\bigr|\Bigr)\,dt = \int_a^b \Phi\bigl(|\varphi'(t)|\bigr)\,dt \le \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(\varphi) = \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f). \tag{6.22}$$

(b) Let $\Phi \in \mathcal{N}$. If X is a reflexive Banach space, then it follows from Theorem 6.6(b) that $BV_\Phi([a,b];X) = A^1_\Phi([a,b];X)$, or, in other words,

$$f \in BV_\Phi([a,b];X) \iff f \in AC([a,b];X) \quad \text{and} \quad f' \in L_\Phi([a,b];X). \tag{6.23}$$

For $X = \mathbb{R}$, criterion (6.23) is nothing other than the well-known criterion of F. Riesz [32] (see also [33, Chapter 2, Sec. 3.36, Lemma]), if $\Phi(\rho) = \rho^q$, $q > 1$; it is Yu. T. Medvedev's criterion [25] if $\Phi \in \mathcal{N}$. From (6.16), we also obtain

$$\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(\varphi) = \int_a^b \Phi\bigl(|\varphi'(t)|\bigr)\,dt, \qquad \Phi \in \mathcal{N}, \qquad \varphi \in BV_\Phi([a,b];\mathbb{R}). \tag{6.24}$$

Now, if X is an arbitrary metric space, $f : [a,b] \to X$, and $\varphi(t) = \overset{t}{\underset{a}{\mathsf{V}}}_1(f)$ for $a \le t \le b$, then, by Lemmas 3.2(b,c) and 3.3(b,c), Theorem 6.6(b), and relation (6.24), we have the following generalization of criterion (6.23):

$$\begin{aligned} f \in BV_\Phi([a,b];X) &\iff \varphi \in BV_\Phi([a,b];\mathbb{R}) \\ &\iff \varphi \in AC([a,b];\mathbb{R}) \quad \text{and} \quad \varphi' \in L_\Phi([a,b];\mathbb{R}), \end{aligned} \tag{6.25}$$

and we also obtain the equality in (6.22):

$$\overset{b}{\underset{a}{\mathsf{V}}}_\Phi(f) = \overset{b}{\underset{a}{\mathsf{V}}}_\Phi(\varphi) = \int_a^b \Phi\bigl(|\varphi'(t)|\bigr)\,dt = \int_a^b \Bigl(\bigl|\tfrac{d}{dt}\overset{t}{\underset{a}{\mathsf{V}}}_1(f)\bigr|\Bigr)\,dt.$$

Theorem 6.7. *If X is a metric space, then*

$$AC([a,b];X) = \bigcup_{\Phi \in \mathcal{N}} BV_\Phi([a,b];X).$$

Proof. The inclusion $\supset$ is proved in Proposition 2.4(d). Conversely, we show that for any mapping $f \in AC([a,b];X)$, there exists a function $\Phi \in \mathcal{N}$, depending on f such that $f \in BV_\Phi([a,b];X)$. Let $\varphi(t) = \overset{t}{\underset{a}{\mathsf{V}}}_1(f)$, and let $t \in [a,b]$. Then we have $\varphi \in AC([a,b];\mathbb{R})$ by Lemma 3.3(c) and $\varphi' \in L^1([a,b];\mathbb{R})$. By virtue of (6.25), it is sufficient to prove that $\varphi' \in L_\Phi([a,b];\mathbb{R})$ (in the proof of this assertion, we follow [24, Chapter 2, Sec. 8.1]). The sets

$$J_n = \{t \in [a,b] \mid n-1 \le |\varphi'(t)| < n\}, \qquad n \in \mathbb{N},$$

are pairwise disjoint, $\bigcup_{n=1}^{\infty} J_n = [a,b]$, and

$$\sum_{n=1}^{\infty} n\mu(J_n) \le \int_a^b |\varphi'(t)|\,dt + (b-a) < \infty,$$

where $\mu(J_n)$ stands for the Lebesgue measure of the set J_n. Let an increasing sequence $\{\rho_n\}_{n=1}^{\infty}$, $\rho_1 \ge 1$, $\lim_{n\to\infty} \rho_n = \infty$, be such that

$$\sum_{n=1}^{\infty} \rho_n n\mu(J_n) < \infty. \tag{6.26}$$

Setting

$$\widetilde{\Phi}(\tau) = \begin{cases} \tau & \text{if } 0 \le \tau < 1, \\ \rho_n & \text{if } n \le \tau < n+1,\ n \in \mathbb{N}, \end{cases} \qquad \tau \in \mathbb{R}^+,$$

and

$$\Phi(\rho) = \int_0^\rho \widetilde{\Phi}(\tau)\, d\tau, \qquad \rho \in \mathbb{R}^+,$$

we find that $\Phi \in \mathcal{N}$ (and even that $\lim\limits_{\rho\to 0} \Phi(\rho)/\rho = 0$). Since

$$\Phi(n) = \int_0^n \widetilde{\Phi}(\tau)\, d\tau \le \rho_n n,$$

we have by (6.26) that

$$\int_a^b \Phi\big(|\varphi'(t)|\big)\, dt = \sum_{n=1}^\infty \int_{J_n} \Phi\big(|\varphi'(t)|\big)\, dt \le \sum_{n=1}^\infty \Phi(n)\mu(J_n) \le \sum_{n=1}^\infty \rho_n n \mu(J_n) < \infty.$$

Therefore, $\varphi' \in L_\Phi([a,b];\mathbb{R})$, and the theorem is proved. □

7. Existence of Selections of Multivalued Mappings

In order to state and prove the main result of this section (Theorem 7.1), we first give some definitions and facts from the theory of multivalued mappings (for more detailed information, see [1, Chapter 1, Secs. 1 and 5] and [5, Chapter 2, Sec. 1]).

Let A and B be two nonempty subsets of a metric space (X,d). The quantity

$$\mathrm{e}(A,B) := \sup_{x\in A} \operatorname{dist}(x,B) \in [0,\infty], \quad \text{where} \quad \operatorname{dist}(x,B) := \inf_{y\in B} d(x,y), \tag{7.1}$$

is called the *access of the set A to the set B*. The *Hausdorff distance $D(A,B)$ between the sets A and B* is defined by the relation

$$D(A,B) = \max\big\{\mathrm{e}(A,B),\, \mathrm{e}(B,A)\big\}. \tag{7.2}$$

Since $\mathrm{e}(A,B) = 0$ if and only if A is contained in the closure of B and $\mathrm{e}(A,B) \le \mathrm{e}(A,C) + \mathrm{e}(C,B)$ for the nonempty set $C \subset X$, we have that $D(\cdot,\cdot)$ is the *pseudometric* on the set of all nonempty subsets of X, i.e., it satisfies all the metric axioms, and, possibly, takes infinite values. The mapping D is a *metric* (called the *Hausdorff metric*) on the set $\mathrm{cb}(X)$ of all nonempty closed bounded subsets of the space X and, in particular, on the set $\mathrm{c}(X)$ of all nonempty compact subsets in X.

A *multivalued mapping* from a metric space (E,d_E) into a metric space (X,d) is a mapping $F : E \to 2^X$, where 2^X stands for the set of all subsets of X, so that the set $F(t) \subset X$ is associated with every point $t \in E$. The *graph* of a mapping F is the set $\mathrm{Gr}(F) := \{(t,x) \in E \times X \mid x \in F(t)\}$, and the *range* of the mapping F is the set $\mathrm{R}(F) := \bigcup\limits_{t\in E} F(t) \subset X$.

We set $\dot{2}^X = 2^X \setminus \{\varnothing\}$. A multivalued mapping $F : E \to \dot{2}^X$ is said to be

(a) *upper semicontinuous at a point* $t_0 \in E$ if, for any neighborhood $\mathcal{O}(F(t_0))$ of the set $F(t_0)$, there exists a neighborhood $\mathcal{O}(t_0)$ of the point t_0 such that $F(t) \subset \mathcal{O}(F(t_0))$ for all $t \in \mathcal{O}(t_0)$;

(b) *lower semicontinuous at a point* $t_0 \in E$ if, for any $x_0 \in F(t_0)$ and any neighborhood $\mathcal{O}(x_0)$ of the point x_0, there exists a neighborhood $\mathcal{O}(t_0)$ of the point t_0 such that $F(t) \cap \mathcal{O}(x_0) \ne \varnothing$ for all $t \in \mathcal{O}(t_0)$;

(c) *continuous at a point* $t_0 \in E$ if it is simultaneously upper semicontinuous and lower semicontinuous at this point;

(d) *Hausdorff continuous at a point* $t_0 \in E$ if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $D(F(t), F(t_0)) \le \varepsilon$ for all $t \in E$ such that $d_E(t,t_0) \le \delta(\varepsilon)$;

(e) (e) *upper semicontinuous on* E (*lower semicontinuous on* E, *continuous on* E, and *Hausdorff continuous on* E), if this property holds at any point $t_0 \in E$;

(f) *compact-valued* if $F(t)$ is a compact subset of X, i.e., if $F(t) \in \mathrm{c}(X)$, for any $t \in E$ (for instance, if the graph $\mathrm{Gr}(F)$ is compact in $E \times X$, then F is compact space-valued, but the converse is not true).

It is known (see [1, Chapter 1, Sec. 5, Corollary 1]) that a compact-valued multivalued mapping $F : E \to \dot{2}^X$ from the metric space E into the metric space X is continuous on E in the sense of item (c) if and only if it is Hausdorff continuous on E in the sense of item (d).

We say that a multivalued mapping $F : E \to \dot{2}^X$ has a (*regular*) *selection* if there exists a mapping $f : E \to X$ such that $f(t) \in F(t)$ for all $t \in E$.

By Michael's theorem [26] (see also [1, Chapter 1, Sec. 11]), any lower semicontinuous multivalued mapping from the metric space E into the set of closed convex subsets of a Banach space X has a continuous selection. As is shown in [19,29], in the absence of the convexity condition for values of a multivalued mapping, continuous selections may not exist even for a compact-valued Lipschitzian mapping (e.g., when $E \subset \mathbb{R}^n$, $n \geq 2$). We are interested in the question of existence of regular selections of multivalued mappings F with compact graphs without the convexity condition for values of these mappings, and a key role is played by the fact that the domain of F is a connected subset of the real line $\mathbb{R}$.

If $E \subset \mathbb{R}$, then we say that a multivalued mapping $F : E \to \mathrm{cb}(X)$ is

(g) *Lipschitz continuous* (or a *Lipschitzian mapping*) if

$$\mathrm{Lip}(F) := \sup\left\{ \frac{D(F(t), F(s))}{|t-s|} \,\middle|\, t,\ s \in E, \quad t \neq s \right\} < \infty;$$

(h) *absolutely continuous* (more precisely, $\delta(\cdot)$-*absolutely continuous*) if, for a certain function $\delta : (0,\infty) \to (0,\infty)$, any $\varepsilon > 0$, and any finite tuple of points $\{a_i, b_i\}_{i=1}^n \subset E$ with $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$, the condition $\sum_{i=1}^n (b_i - a_i) \leq \delta(\varepsilon)$ implies $\sum_{i=1}^n D(F(b_i), F(a_i)) \leq \varepsilon$;

(i) a *mapping of bounded* Φ-*variation* for $\Phi \in \mathcal{M}$ if

$$\mathsf{V}_\Phi(F, E) := \sup\bigl\{ V_{\Phi,D}(F, T) \mid T \in \mathcal{T}(E) \bigr\} < \infty,$$

where, similarly to (2.2), we set

$$V_{\Phi,D}(F,T) := \sum_{i=1}^m \Phi\left(\frac{D(F(t_i), F(t_{i-1}))}{t_i - t_{i-1}} \right) \cdot (t_i - t_{i-1}), \qquad T = \{t_i\}_{i=0}^m \in \mathcal{T}(E).$$

Let $C(E; \mathrm{c}(X))$, $C^{0,1}(E; \mathrm{c}(X))$, $AC(E; \mathrm{c}(X))$, and $BV_\Phi(E; \mathrm{c}(X))$ with $\Phi \in \mathcal{M}$ denote, respectively, the spaces of all (Hausdorff) continuous, Lipschitz continuous, absolutely continuous mappings, and mappings of bounded Φ-variation with respect to the Hausdorff metric D acting from E into the metric space $\mathrm{c}(X)$ of nonempty compact subsets of X.

Now we are able to state the main result of this section, Theorem 7.1, which generalizes the results on the existence of selections of nonconvex-valued multivalued mappings of bounded 1-variation (in the sense of Jordan) presented in [19,22,35] for the finite-dimensional space X and in [7–9,11] and [27, Theorem D1.8] for an infinite-dimensional Banach space X.

Theorem 7.1. *Let X be a Banach space* (*over $\mathbb{R}$ or $\mathbb{C}$*) *with the norm $\|\cdot\|$, $E = [a,b]$ be a closed interval in $\mathbb{R}$, and let a multivalued mapping $F : E \to \mathrm{c}(X)$ have the compact graph $\mathrm{Gr}(F)$. If $\Phi \in \mathcal{M}$ and $F \in BV_\Phi(E; \mathrm{c}(X))$, then, for any $t_0 \in E$ and $x_0 \in F(t_0)$, there exists a selection $f \in BV_\Phi(E; X)$ of the mapping F such that $f(t) \in F(t)$ at all points of continuity $t \in E$ of the mapping F and*

$$f(t_0) = x_0, \qquad \mathsf{V}_\Phi(f, E) \leq \mathsf{V}_\Phi(F, E), \qquad \mathsf{V}_1(f, E) \leq \mathsf{V}_1(F, E).$$

If, in addition, it is known that F is continuous or $\Phi \in \mathcal{N}$, then the selection f is also continuous and $f(t) \in F(t)$ for all $t \in E$.

Proof. The proof is carried out in six steps. In the first three steps, Theorem 7.1 is proved for the case where $\Phi \in \mathcal{N}$.

1. For every $n \in \mathbb{N}$, let $T_n = \{t_i^n\}_{i=0}^n \in \mathcal{T}_a^b$ be a partition of a closed interval $[a, b]$ with the following properties:

(1) $t_0 \in T_n$, i.e., $t_0 = t_{k(n)}^n$ for a certain $k(n) \in \{0, 1, \ldots, n\}$;
(2) if $\lambda(T_n) := \max\limits_{1 \le i \le n} (t_i^n - t_{i-1}^n)$, then $\lambda(T_n) \to 0$ as $n \to \infty$.

By induction, we define elements $x_i^n \in F(t_i^n)$ in the following way. Let $n \in \mathbb{N}$, and let initially $a < t_0 < b$.

(a) We set $x_{k(n)}^n = x_0$.
(b) If $i \in \{1, \ldots, k(n)\}$ and the element $x_i^n \in F(t_i^n)$ is already defined, we choose an element $x_{i-1}^n \in F(t_{i-1}^n)$ such that $\|x_i^n - x_{i-1}^n\| = \operatorname{dist}(x_i^n, F(t_{i-1}^n))$.
(c) If $i \in \{k(n) + 1, \ldots, n\}$ and the element $x_{i-1}^n \in F(t_{i-1}^n)$ is already defined, we choose an element $x_i^n \in F(t_i^n)$ such that $\|x_{i-1}^n - x_i^n\| = \operatorname{dist}(x_{i-1}^n, F(t_i^n))$.

Now, if $t_0 = a$, i.e., if $k(n) = 0$, then we find x_i^n according to (a) and (c), and if $t_0 = b$, so that $k(n) = n$, we find x_i^n following (a) and (b).

We now define the sequence of mappings $f_n : [a, b] \to X$, $n \in \mathbb{N}$, in the following way:

$$f_n(t) = x_{i-1}^n + \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}(x_i^n - x_{i-1}^n), \qquad t \in [t_{i-1}^n, t_i^n], \quad i = 1, \ldots, n. \tag{7.3}$$

We note at once that $f_n(t_i^n) = x_i^n$, $f_n(t_{i-1}^n) = x_{i-1}^n$, and, therefore, $f_n(t_0) = x_0$ for all $n \in \mathbb{N}$, and also

$$\|x_i^n - x_{i-1}^n\| \le D(F(t_i^n), F(t_{i-1}^n)), \qquad n \in \mathbb{N}, \quad i = 1, \ldots, n, \tag{7.4}$$

by virtue of definitions (b), (c), (7.1), and (7.2).

All the mappings $f_n : [a, b] \to X$ are continuous, and the restriction of f_n to every closed interval $[t_{i-1}^n, t_i^n]$ is continuously differentiable. Taking into account that

$$f_n'(t) = \frac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} \qquad \text{for} \qquad t_{i-1}^n \le t \le t_i^n,$$

and applying Proposition 2.2(c), formula (6.6), and inequality (7.4), we find that

$$\begin{aligned} \mathop{\mathsf{V}}_a^b{}_\Phi(f_n) &= \sum_{i=1}^n \mathop{\mathsf{V}}_{t_{i-1}^n}^{t_i^n}{}_\Phi(f_n) = \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \Phi\big(\|f_n'(t)\|\big)\, dt \\ &= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \Phi\left(\frac{\|x_i^n - x_{i-1}^n\|}{t_i^n - t_{i-1}^n}\right) dt \\ &= \sum_{i=1}^n \Phi\left(\frac{\|x_i^n - x_{i-1}^n\|}{t_i^n - t_{i-1}^n}\right) \cdot (t_i^n - t_{i-1}^n) \\ &\le \sum_{i=1}^n \Phi\left(\frac{D(F(t_i^n), F(t_{i-1}^n))}{t_i^n - t_{i-1}^n}\right) \cdot (t_i^n - t_{i-1}^n) \\ &= V_{\Phi,D}(F, T_n) \le \mathop{\mathsf{V}}_a^b{}_\Phi(F) < \infty, \qquad \forall\, n \in \mathbb{N}. \end{aligned} \tag{7.5}$$

By Proposition 2.4(d,e), F has the Jordan bounded variation; therefore, the calculations performed for $\Phi(\rho)=\rho$ in (7.5) also give the estimate

$$\mathop{\mathsf{V}}_a^b{}_1(f_n) \le \mathop{\mathsf{V}}_a^b{}_1(F) \quad \text{for all} \quad n\in\mathbb{N}. \tag{7.6}$$

2. We show that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in X for any $t\in[a,b]$. We fix $t\in[a,b]$. For any $n\in\mathbb{N}$, there exists a number $i(n)\in\{1,\dots,n\}$ also depending on t such that $t^n_{i(n)-1}\le t\le t^n_{i(n)}$; therefore, it follows from the condition $\lim\limits_{n\to\infty}\lambda(T_n)=0$ that, as $n\to\infty$, the sequences $t^n_{i(n)-1}$ and $t^n_{i(n)}$ tend to the point t. It follows from (7.3), (7.4), and the (absolute) continuity of the mapping F that

$$\begin{aligned}\|f_n(t)-x^n_{i(n)}\| &= \frac{t^n_{i(n)}-t}{t^n_{i(n)}-t^n_{i(n)-1}}\|x^n_{i(n)}-x^n_{i(n)-1}\| \\ &\le D(F(t^n_{i(n)}),F(t^n_{i(n)-1}))\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{7.7}$$

Since F has a compact graph and $x^n_{i(n)}\in F(t^n_{i(n)})$, i.e., $(t^n_{i(n)},x^n_{i(n)})\in \mathrm{Gr}(F)$, there exists a subsequence of $\{(t^n_{i(n)},x^n_{i(n)})\}_{n=1}^\infty$ (which is denoted in the same way as the sequence itself) that converges in $[a,b]\times X$ as $n\to\infty$ to a certain point $(\tau,x)\in\mathrm{Gr}(F)$. But $t^n_{i(n)}\to t$ as $n\to\infty$; therefore, $\tau=t$, so that $(t,x)\in\mathrm{Gr}(F)$ or $x\in F(t)$. At the same time, $x^n_{i(n)}\to x$ in X as $n\to\infty$. From (7.7), we then obtain that as $n\to\infty$, the subsequence $f_n(t)$ converges in X to the element $x\in F(t)$, and this means that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact.

3. Applying the strong selection principle (Theorem 4.1(a)) to the family $\mathfrak{F}=\{f_n\}_{n=1}^\infty$, by virtue of (7.5) and step 2, we find a subsequence of $\{f_n\}_{n=1}^\infty$ (denoted by the same symbol) that converges uniformly on $[a,b]$ to a certain mapping $f\in BV_\Phi([a,b];X)$. It is clear here that $f(t_0)=x_0$, and, by Proposition 2.2(d) and inequalities (7.5) and (7.6), we have

$$\mathop{\mathsf{V}}_a^b{}_\Phi(f)\le \liminf_{n\to\infty}\mathop{\mathsf{V}}_a^b{}_\Phi(f_n)\le \mathop{\mathsf{V}}_a^b{}_\Phi(F), \tag{7.8}$$

$$\mathop{\mathsf{V}}_a^b{}_1(f)\le \liminf_{n\to\infty}\mathop{\mathsf{V}}_a^b{}_1(f_n)\le \mathop{\mathsf{V}}_a^b{}_1(F). \tag{7.9}$$

It remains to show that $f(t)\in F(t)$ for any $t\in[a,b]$. We fix such t. By the arguments from step 2, we have that $f_n(t)$ converges in X to a certain element $x\in F(t)$ as $n\to\infty$, and from the definition of f, we find that $f_n(t)\to f(t)$ in X as $n\to\infty$; therefore, $f(t)=x\in F(t)$, so that in the case where $\Phi\in\mathcal{N}$, Theorem 7.1 is proved.

4. In this auxiliary step, we show that the proof presented above can easily be adopted in order to obtain the following statement: if, under the hypotheses of Theorem 7.1, we have $F\in C^{0,1}([a,b];\mathrm{c}(X))$, then there exists a selection $f\in C^{0,1}([a,b];X)$ of the mapping F such that $f(t)\in F(t)$ for all $t\in[a,b]$, $f(t_0)=x_0$, and $\mathrm{Lip}(f)\le\mathrm{Lip}(F)$.

Initially, we argue as in step 1 up to inequality (7.4) and observe that the sequence $\{f_n\}_{n=1}^\infty$ is uniformly Lipschitzian on the closed interval $[a,b]$, i.e., $\mathrm{Lip}(f_n)\le\mathrm{Lip}(F)$ for all $n\in\mathbb{N}$; in fact, if $t,\ s\in[t^n_{i-1},t^n_i]$, then, from (7.3) and (7.4), we obtain

$$\|f_n(t)-f_n(s)\|\le \frac{x^n_i-x^n_{i-1}}{t^n_i-t^n_{i-1}}\,|t-s|\le \mathrm{Lip}(F)\cdot|t-s|.$$

Since $\{f_n\}_{n=1}^\infty$ is equicontinuous, by virtue of step 2 we can apply the Arzela–Ascoli theorem and find a subsequence of $\{f_n\}_{n=1}^\infty$ that converges uniformly on $[a,b]$ to a certain mapping $f\in C([a,b];X)$. It is clear that actually $f\in C^{0,1}([a,b];X)$, $\mathrm{Lip}(f)\le\mathrm{Lip}(F)$, $f(t_0)=x_0$, and the inclusion $f(t)\in F(t)$ follows from step 2 in the same way as in step 3.

5. Suppose that $\Phi\in\mathcal{M}$ and $F\in BV_\Phi([a,b];\mathrm{c}(X))\cap C([a,b];\mathrm{c}(X))$. By Lemma 3.3(b), we have the composition $F=G\circ\varphi$, where the function $\varphi(t)=\mathop{\mathsf{V}}_a^t{}_1(F)$, $t\in[a,b]$, belongs to the space $BV_\Phi([a,b];\mathbb{R})$

(and to the space $BV_1([a,b];\mathbb{R})$) and the multivalued mapping $G:[0,\ell]=\varphi([a,b])\to \mathrm{c}(X)$ is Lipschitz continuous; moreover, $\ell=\bigvee_a^b{}_1(F)=\bigvee_a^b{}_1(\varphi)$, $\mathrm{Lip}(G)\le 1$, and $\bigvee_a^b{}_\Phi(\varphi)=\bigvee_a^b{}_\Phi(F)$. Since F is continuous, the function φ is also continuous on $[a,b]$.

We now show that G has a compact graph. Consider a sequence $\{(\tau_n,y_n)\}_{n=1}^{\infty}$ from the graph $\mathrm{Gr}(G)$. Then $\tau_n=\varphi(t_n)$ for a certain point $t_n\in[a,b]$, and $y_n\in G(\varphi(t_n))=F(t_n)$, so that $\{(t_n,y_n)\}_{n=1}^{\infty}\subset \mathrm{Gr}(F)$. The compactness of the graph of F implies that there exists a subsequence of $\{(t_n,y_n)\}_{n=1}^{\infty}$ (this subsequence is denoted by the same symbol) such that $t_n\to t$ and $y_n\to y$ as $n\to\infty$, where $y\in F(t)$. We set $\tau=\varphi(t)$. Since φ is continuous, (τ_n,y_n) converges to $(\tau,y)\in \mathrm{Gr}(G)$; this is the required result.

Observing that $x_0\in F(t_0)=G(\tau_0)$, where $\tau_0=\varphi(t_0)$, by virtue of step 4 we find a mapping $g\in C^{0,1}([0,\ell];X)$ such that $g(\tau_0)=x_0$, $g(\tau)\in G(\tau)$ for all $\tau\in[0,\ell]$ and $\mathrm{Lip}(g)\le \mathrm{Lip}(G)\le 1$. We set $f=g\circ\varphi$. By Lemma 3.2(b), the mapping $f\in BV_\Phi([a,b];X)$ is continuous and

$$\bigvee_a^b{}_\Phi(f)=\bigvee_a^b{}_\Phi(g\circ\varphi)\le \bigvee_a^b{}_\Phi(\varphi)=\bigvee_a^b{}_\Phi(F),$$
$$\bigvee_a^b{}_1(f)=\bigvee_a^b{}_1(g\circ\varphi)\le \bigvee_a^b{}_1(\varphi)=\bigvee_a^b{}_1(F).$$

Finally, $f(t_0)=g(\varphi(t_0))=g(\tau_0)=x_0$ and $f(t)=g(\varphi(t))\in G(\varphi(t))=F(t)$ for all points $t\in[a,b]$.

6. We now consider the general case where $\Phi\in\mathcal{M}$ and $F\in BV_\Phi([a,b];\mathrm{c}(X))$. We argue as in Step 1 up to inequality (7.6). By (7.3), the image of each mapping f_n is contained in the closed convex hull $\overline{\mathrm{co}}\,\mathrm{R}(F)$ of the range $\mathrm{R}(F)$ of the mapping F, but $\mathrm{Gr}(F)$ is a compact set in $[a,b]\times X$; therefore, $\mathrm{R}(F)$ is also compact (in X), and hence, $\overline{\mathrm{co}}\,\mathrm{R}(F)$ is compact (the last assertion follows from Lemma A.8).

Applying the weak selection principle (Theorem 4.2) to the family $\mathfrak{F}=\{f_n\}_{n=1}^{\infty}$ in the space $BV_\Phi([a,b];X)$, we find that there exists a subsequence (denoted by the same symbol) of $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to a certain mapping $f\in BV_\Phi([a,b];X)$ on $[a,b]$. Moreover, it is clear that $f(t_0)=x_0$, and that inequalities (7.8) and (7.9) hold.

Let $t\in[a,b]$ be a point of continuity of F (see Corollary 2.6(b)). We show that $f(t)\in F(t)$ in this case. Since (7.7) holds at the point t, we have $f_n(t)\to x\in F(t)$ as $n\to\infty$ by virtue of step 2, and $f_n(t)\to f(t)$ by construction, from which we conclude that $f(t)=x\in F(t)$. This completes the proof. □

Remark 7.1. In Theorem 7.1, one could fix finitely many points $t_i\in[a,b]$ and $x_i\in F(t_i)$, $i=1,\dots,m$, $m\in\mathbb{N}$, and then we would additionally have the following for a selection f: $f(t_i)=x_i$, $i=1,\dots,m$.

Corollary 7.2. *Theorem* 7.1 *is valid if one sets* $E=\mathbb{R}$ *in it.*

Proof. Let $\{r_k\}_{k\in\mathbb{Z}}\subset\mathbb{R}$ be a strictly increasing sequence of points of continuity of the mapping F such that $r_0<t_0<r_1$, $\lim\limits_{k\to\infty}r_k=\infty$, and $\lim\limits_{k\to\infty}r_{-k}=-\infty$. We set $I_k=[r_k,r_{k+1}]$, $k\in\mathbb{Z}$, so that $\mathbb{R}=\bigcup\limits_{k\in\mathbb{Z}}I_k$. Applying Theorem 7.1 on the closed interval $E=I_0$, we find a selection $f_0\in BV_\Phi(I_0;X)$ of the mapping F (more precisely, that of the restriction $F|_{I_0}$ of the mapping F to the closed interval I_0) such that $f_0(t_0)=x_0$, $\mathsf{V}_\Phi(f_0,I_0)\le \mathsf{V}_\Phi(F,I_0)$, and $\mathsf{V}_1(f_0,I_0)\le\mathsf{V}_1(F,I_0)$. "Moving along the closed intervals I_k to the right" from the point r_1, we sequentially apply Theorem 7.1: first on the closed interval I_1 with the initial condition $x_0=f_0(r_1)\in F(r_1)$, then on the closed interval I_2 with the initial condition $x_0=f_1(r_2)\in F(r_2)$, $\dots$, on the closed interval I_k with the initial condition $x_0=f_{k-1}(r_k)\in F(r_k)$, and so on for $k\in\mathbb{N}$. As a result, for any $k\in\mathbb{N}$, we find a selection $f_k\in BV_\Phi(I_k;X)$ of the mapping F on the closed interval I_k such that

$$f_k(r_k)=f_{k-1}(r_k),\quad \mathsf{V}_\Phi(f_k,I_k)\le\mathsf{V}_\Phi(F,I_k),\quad \mathsf{V}_1(f_k,I_k)\le\mathsf{V}_1(F,I_k). \tag{7.10}$$

We carry out a similar construction "moving along the closed intervals I_k to the left" from the point r_0. Then, for any $k\in\mathbb{Z}$, on the closed interval I_k, there exists a selection $f_k\in BV_\Phi(I_k;X)$ of the mapping F

such that relations (7.10) hold. If $t \in \mathbb{R}$, so that $t \in I_k$ for a certain $k \in \mathbb{Z}$, we set $f(t) := f_k(t)$. It is clear that the mapping $f : \mathbb{R} \to X$ thus defined is a selection of F on $\mathbb{R}$, $f(t_0) = f_0(t_0) = x_0$, and, by virtue of the limit property and additivity (Proposition 2.2(h,c)), we have

$$\begin{aligned} \mathsf{V}_\Phi(f, \mathbb{R}) &= \lim_{k\to\infty} \mathsf{V}_\Phi(f, [r_{-k}, r_k]) = \lim_{k\to\infty} \sum_{i=-k}^{k-1} \mathsf{V}_\Phi(f_i, I_i) \\ &\le \lim_{k\to\infty} \sum_{i=-k}^{k-1} \mathsf{V}_\Phi(F, I_i) = \lim_{k\to\infty} \mathsf{V}_\Phi(F, [r_{-k}, r_k]) = \mathsf{V}_\Phi(F, \mathbb{R}), \end{aligned}$$

and, similarly, $\mathsf{V}_1(f, \mathbb{R}) \le \mathsf{V}_1(F, \mathbb{R})$. □

An assertion that is similar to Corollary 7.2 holds also for (bounded or unbounded) intervals and semiopen intervals.

8. Certain Generalizations

In this section, we propose certain generalizations of the results presented above. We are especially interested in further strengthening Theorem 7.1. For simplicity, we restrict ourselves to the consideration of mappings on the closed interval $E = [a, b]$. Let, as usual, (X, d) be a metric space, let $\Phi \in \mathcal{M}$, and let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with the derivative $\sigma'(t) > 0$ for all $t \in E$. For a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$ of our closed interval and for a mapping $f : E \to X$, we set

$$V_{\Phi,\sigma}(f, T) := \sum_{i=1}^m \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{\sigma(t_i) - \sigma(t_{i-1})}\right) \cdot \big(\sigma(t_i) - \sigma(t_{i-1})\big), \tag{8.1}$$

$$\mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f) := \sup\big\{V_{\Phi,\sigma}(f, T) \,\big|\, T \in \mathcal{T}_a^b\big\}, \tag{8.2}$$

and also

$$BV_{\Phi,\sigma}(E; X) := \Big\{f : E \to X \,\Big|\, \mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f) < \infty\Big\}.$$

The functional $f \mapsto \mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f)$ will be called the (Φ, σ)-*variation.* Note that for $X = \mathbb{R}$ and $\Phi(\rho) = \rho^q$, $q > 1$, the concept of the (Φ, σ)-variation (8.1), (8.2) considered as the *integral of a* (*continuous*) *interval function* (see the end of Sec. 5 after Remark 5.1) dates back to [18] and [30], where it was applied for problems of the theory of quadratic forms with an infinite number of variables.

First of all, we note that Propositions 2.1–2.3 are valid (with corresponding modifications in Proposition 2.2(b)) for the (Φ, σ)-variation as well. In Proposition 2.4(b), the inequality becomes

$$\mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f) \le \Phi\big(\mathrm{Lip}(f) / \min_{t\in[a,b]} \sigma'(t)\big) \cdot \big(\sigma(b) - \sigma(a)\big),$$

and the inequality (2.12) in Proposition 2.4(c) is modified as

$$\mathop{\mathsf{V}}_a^b{}_1(f) \le \big(\sigma(b) - \sigma(a)\big)\Phi^{-1}\left(\frac{1}{\sigma(b) - \sigma(a)} \cdot \mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f)\right), \quad f \in BV_{\Phi,\sigma}(E; X). \tag{8.3}$$

Proposition 2.4(d) is preserved with obvious changes in its proof for the (Φ, σ)-variation as well since the function σ is Lipschitzian on $[a, b]$, and, therefore, it is absolutely continuous. Inequality (2.16) in Lemma 2.8 is replaced by

$$\int_a^{b-h} \Phi\left(\frac{\|f(t+h) - f(t)\|}{\sigma(t+h) - \sigma(t)}\right) \cdot \frac{\sigma(t+h) - \sigma(t)}{h}\, dt \le \mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f), \quad 0 < h < b - a. \tag{8.4}$$

Theorem 3.1 is also carried over to the case of the (Φ,σ)-variation, but in the proof of necessity (in Lemma 3.3), one should use inequality (8.3) instead of (2.12). The same can be said about the weak and strong selection principles presented in Sec. 4. The formulas for jumps of the (Φ,σ)-variation are accordingly modified, if, instead of (5.1), we set

$$U(t,s)=\Phi\biggl(\frac{d(f(t),f(s))}{\sigma(t)-\sigma(s)}\biggr)\cdot\bigl(\sigma(t)-\sigma(s)\bigr),\qquad t,\ s\in E,\quad s<t.$$

The norm in $BV_{\Phi,\sigma}(E;X)$ is introduced as in Sec. 6.1, and formula (6.6) for the variation of a smooth mapping $f\in C^1(E;X)$ becomes

$$\mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f)=\int_a^b\Phi\biggl(\frac{\|f'(t)\|}{\sigma'(t)}\biggr)\cdot\sigma'(t)\,dt. \tag{8.5}$$

As an example, we present the calculations that prove the inequality $\le$ in (8.5) (i.e., an analog of inequality (6.7)) using the complete version of the Jensen inequality (A.2): if $T=\{t_i\}_{i=0}^m\in\mathcal{T}_a^b$, then

$$\begin{aligned}V_{\Phi,\sigma}(f,T)&=\sum_{i=1}^m\Phi\biggl(\frac{\bigl\|\int_{t_{i-1}}^{t_i}f'(t)\,dt\bigr\|}{\sigma(t_i)-\sigma(t_{i-1})}\biggr)\cdot\bigl(\sigma(t_i)-\sigma(t_{i-1})\bigr)\\&\le\sum_{i=1}^m\Phi\biggl(\frac{\int_{t_{i-1}}^{t_i}\bigl(\|f'(t)\|/\sigma'(t)\bigr)\sigma'(t)\,dt}{\int_{t_{i-1}}^{t_i}\sigma'(t)\,dt}\biggr)\cdot\int_{t_{i-1}}^{t_i}\sigma'(t)\,dt\\&\overset{(A.2)}{\le}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\Phi\biggl(\frac{\|f'(t)\|}{\sigma'(t)}\biggr)\sigma'(t)\,dt=\int_a^b\Phi\biggl(\frac{\|f'(t)\|}{\sigma'(t)}\biggr)\sigma'(t)\,dt.\end{aligned}$$

In order to prove the converse inequality, we use inequality (8.4).

It is easy to see that the corresponding analog of Theorem 6.6 holds for the (Φ,σ)-variation.

We now consider the question on the existence of selections for multivalued mappings of bounded (Φ,σ)-variation. The definition of the subsequence $\{f_n\}_{n=1}^\infty$ in (7.3) is modified in the following way: for $i=1,\dots,n$, we set

$$f_n(t)=x_{i-1}^n+\frac{\sigma(t)-\sigma(t_{i-1}^n)}{\sigma(t_i^n)-\sigma(t_{i-1}^n)}(x_i^n-x_{i-1}^n),\qquad t_{i-1}^n\le t\le t_i^n;$$

therefore, under the same constraints, we have

$$f_n'(t)=\frac{\sigma'(t)}{\sigma(t_i^n)-\sigma(t_{i-1}^n)}(x_i^n-x_{i-1}^n).$$

Instead of (7.5), we have

$$\begin{aligned}\mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(f_n)&=\sum_{i=1}^n\mathop{\mathsf{V}}_{t_{i-1}^n}^{t_i^n}{}_{\Phi,\sigma}(f_n)=\sum_{i=1}^n\int_{t_{i-1}^n}^{t_i^n}\Phi\biggl(\frac{\|f_n'(t)\|}{\sigma'(t)}\biggr)\sigma'(t)\,dt\\&=\sum_{i=1}^n\int_{t_{i-1}^n}^{t_i^n}\Phi\biggl(\frac{\|x_i^n-x_{i-1}^n\|}{\sigma(t_i^n)-\sigma(t_{i-1}^n)}\biggr)\sigma'(t)\,dt\\&=\sum_{i=1}^n\Phi\biggl(\frac{\|x_i^n-x_{i-1}^n\|}{\sigma(t_i^n)-\sigma(t_{i-1}^n)}\biggr)\bigl(\sigma(t_i^n)-\sigma(t_{i-1}^n)\bigr)\\&\overset{(7.4)}{\le}\sum_{i=1}^n\Phi\biggl(\frac{D(F(t_i^n),F(t_{i-1}^n))}{\sigma(t_i^n)-\sigma(t_{i-1}^n)}\biggr)\bigl(\sigma(t_i^n)-\sigma(t_{i-1}^n)\bigr)\\&=V_{\Phi,\sigma,D}(F,T_n)\le\mathop{\mathsf{V}}_a^b{}_{\Phi,\sigma}(F)<\infty\qquad\forall\,n\in\mathbb{N},\end{aligned}$$

in which (8.5) is used. Thus, we have the following generalization of Theorem 7.1:

Theorem 8.1. *Let $E \subset \mathbb{R}$ be a closed interval, an interval, or a semiopen interval, X be a Banach space, $\Phi \in \mathcal{M}$, and $\sigma \in C^1(\mathbb{R};\mathbb{R})$, $\sigma'(t) > 0$ for all $t \in E$. If a multivalued mapping $F \in BV_{\Phi,\sigma}(E;\mathrm{c}(X))$ has a compact graph, then, for any $m \in \mathbb{N}$ and any $t_i \in E$ $x_i \in F(t_i)$, $i = 1, \dots, m$, there exists a selection $f \in BV_{\Phi,\sigma}(E;X)$ of the mapping F such that $f(t) \in F(t)$ at all points $t \in E$ at which F is continuous, $f(t_i) = x_i$, $i = 1, \dots, m$, $\mathsf{V}_{\Phi,\sigma}(f,E) \le \mathsf{V}_{\Phi,\sigma}(F,E)$, and $\mathsf{V}_1(f,E) \le \mathsf{V}_1(F,E)$.*

If, in addition, F is continuous or $\Phi \in \mathcal{N}$, then the selection f is also continuous and $f(t) \in F(t)$ for all $t \in E$.

We briefly touch upon yet another generalization using the previous notation. Let Y be a metric space with metric d_Y, and let $\sigma : \mathbb{R} \to Y$ be an injective mapping (i.e., $\sigma(t) \ne \sigma(s)$ for t, $s \in E$, $t \ne s$). Setting

$$V_{\Phi,\sigma}(f,T) = \sum_{i=1}^{m} \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{d_Y(\sigma(t_i), \sigma(t_{i-1}))}\right) \cdot d_Y(\sigma(t_i), \sigma(t_{i-1})),$$

we define the (Φ,σ)-variation of the mapping $f : E \to X$ as in (8.2) (if $E = [a,b]$). The definition of such a kind for a measure variation can be found in [31, Chapter 5]. Note, for example, that property (d) in Proposition 2.4 in the case where $\Phi \in \mathcal{N}$ means that the mapping f is *absolutely continuous with respect to the mapping* σ in a sense that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$ and $\sum\limits_{i=1}^{n} d_Y(\sigma(b_i), \sigma(a_i)) \le \delta$, then $\sum\limits_{i=1}^{n} d(f(b_i), f(a_i)) \le \varepsilon$. However, such a generalization of the (Φ,σ)-variation does not introduce any changes in Theorem 8.1. Indeed, if Y is a Banach space with norm $\|\cdot\|_Y$, $\sigma \in C^1(\mathbb{R};Y)$, and $\sigma'(t) \ne 0$ when $t \in E$, then we additionally assume that σ has the following property:

$$\|\sigma(t) - \sigma(s)\|_Y = \left\| \int_s^t \sigma'(\tau)\, d\tau \right\|_Y = \int_s^t \|\sigma'(\tau)\|_Y d\tau, \quad t,\ s \in \mathbb{R}, \quad s < t,$$

and then the substitution $\sigma_1(t) = \int_0^t \|\sigma'(\tau)\|_Y d\tau$ reduces the considerations to the case of Theorem 8.1.

One can read in more detail about the generalizations of this section in [15].

A. Appendix. Auxiliary Statements

In the present Appendix, we have collected certain auxiliary statements (in the order in which they are cited) that are used in the main text of the paper.

Theorem A.1 (Jensen's inequalities). *If a function $\Phi \in \mathcal{M}$ is convex and continuous, the following Jensen inequalities hold:*

(a) *Jensen's inequality for sums*: *if $\alpha_i \ge 0$, $x_i \ge 0$, $i = 1, \dots, n$, and $\sum\limits_{i=1}^{n} \alpha_i > 0$, then*

$$\Phi\left(\frac{\sum\limits_{i=1}^{n} \alpha_i x_i}{\sum\limits_{i=1}^{n} \alpha_i}\right) \le \frac{\sum\limits_{i=1}^{n} \alpha_i \Phi(x_i)}{\sum\limits_{i=1}^{n} \alpha_i}; \tag{A.1}$$

(b) *the integral Jensen inequality*: *if functions α, $x : [a,b] \to \mathbb{R}^+$ are Lebesgue integrable on $[a,b]$ and $\int_a^b \alpha(t)\, dt > 0$, then we have (provided that all the integrals written do exist)*

$$\Phi\left(\frac{\int_a^b \alpha(t) x(t)\, dt}{\int_a^b \alpha(t)\, dt}\right) \le \frac{\int_a^b \alpha(t) \Phi(x(t))\, dt}{\int_a^b \alpha(t)\, dt}. \tag{A.2}$$

For the proof, see [28], Chapter 10, Sec. 5, Corollary of Theorem 4, and Theorem 6.

Theorem A.2. *Let $E \subset \mathbb{R}$, X be a metric space, and $f \in BV_1(E; X)$. Then*

(a) *the set $f(E)$ is completely bounded and separable in X, and if, in addition, X is complete, then $f(E)$ is precompact in X;*

(b) *f is continuous on E, except for (possibly) a set of points from E that is no more than countable.*

The proof is contained in [7], Proposition 2.1 and Theorem 4.1.

Proposition A.3 (change of a variable in a Jordan variation)**.** *If E_1, $E \subset \mathbb{R}$, $g : E_1 \to X$ (X is a metric space), and $\varphi : E \to E_1$ is a (not necessarily strictly) monotone function, then $\mathsf{V}_1(g, \varphi(E)) = \mathsf{V}_1(g \circ \varphi, E)$.*

See [8], Proposition 2.1(V4) for the proof.

Theorem A.4 (classical selection principle of E. Helly)**.** (a) *Let $\mathcal{F}$ be an infinite family of nondecreasing functions from the closed interval $[a, b]$ in $\mathbb{R}$. If the family $\mathcal{F}$ is uniformly bounded (i.e., if there exists a constant $C \geq 0$ such that $|f(t)| \leq C$ for all $t \in [a, b]$ and $f \in \mathcal{F}$), then it contains a sequence of functions that converges pointwise on $[a, b]$ to a certain nondecreasing bounded function from $[a, b]$ in $\mathbb{R}$.*

(b) *Let $\mathcal{F}$ be an infinite uniformly bounded family of functions from $[a, b]$ in $\mathbb{R}$. If $\mathcal{F}$ is a family of bounded 1-variation (i.e., if there exists $C \geq 0$ such that $\overset{b}{\underset{a}{\mathsf{V}_1}}(f) \leq C$ for all $f \in \mathcal{F}$), then it contains a sequence of functions that converges pointwise on $[a, b]$ to a certain function from $[a, b]$ in $\mathbb{R}$ of bounded 1-variation.*

The proof can be found in [28], Chapter 8, Sec. 4, Lemma 2 and the theorem (E. Helly's).

Theorem A.5 (Arzela–Ascoli)**.** *Let (E, d_E) be a compact metric space, and let (X, d) be a complete metric space. (Recall that the set $C(E; X)$ of all continuous mappings from E into X is the complete metric space with respect to the uniform metric d_u on $C(E; X)$, which is defined in the usual way: $d_u(f, g) = \sup_{t \in E} d(f(t), g(t))$ for f, $g \in C(E; X)$.) The family $\mathcal{F} \subset C(E; X)$ is precompact in the uniform metric d_u if and only if the following two conditions hold:*

(a) *the family $\mathcal{F}$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\sup\limits_{f \in \mathcal{F}} d(f(t), f(s)) \leq \varepsilon$ for all $t, s \in E$, $d_E(t, s) \leq \delta(\varepsilon)$;*

(b) *the family $\mathcal{F}$ is pointwise precompact in X, i.e., for any $t \in E$, the set $\{f(t) \mid f \in \mathcal{F}\}$ is precompact in X.*

The proof of this theorem can be found, for example, in [16, Theorem 0.4.13] or in [17, Chapter 4, Sec. 6, Theorem (4.44)].

Lemma A.6. *Let $E \subset \mathbb{R}$, and let X be a Banach space. If $g \in C^{0,1}(E; X)$, then there exists the mapping $\widetilde{g} \in C^{0,1}(\mathbb{R}; X)$ such that the restriction of $\widetilde{g}$ to the set E coincides with the mapping g; moreover, $\mathrm{Lip}(\widetilde{g}) = \mathrm{Lip}(g)$.*

The proof is contained in [8], Step 3 of the proof of Theorem 5.1.

Theorem A.7 (Lagrange mean-value theorem)**.** *Let X be a linear normed space with norm $\|\cdot\|$, I be a closed interval, an interval, or a semiopen interval in $\mathbb{R}$, $f \in C(I; X)$, and the right derivative $f'_+(t) \in X$ exist for any $t \in I \setminus Q$, where $Q \subset I$ is a no more than countable set. Then, for any a, $b \in I$, $a < b$, and $t_0 \in I \setminus Q$, the following inequalities hold:*

$$\|f(b) - f(a)\| \leq (b - a) \sup\{\|f'_+(t)\| : t \in]a, b[\setminus Q\}, \tag{A.3}$$

$$\|f(b) - f(a) - (b - a) f'_+(t_0)\| \leq (b - a) \sup\{\|f'_+(t) - f'_+(t_0)\| : t \in]a, b[\setminus Q\}. \tag{A.4}$$

(*A similar statement is valid also in the case where the mapping f is differentiable from the left.*)

For the proof, see [4, Chapter 1, Sec. 2.3].

Lemma A.8. *If R is a completely bounded subset of a linear normed space X, then its convex hull* $\mathrm{co}(R)$ *is also completely bounded.*

For the proof, see [8], Lemma 6.2.

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