METRICS OF CONSTANT SCALAR CURVATURE CONFORMAL TO RIEMANNIAN PRODUCTS

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ABSTRACT. We consider the conformal class of the Riemannian product $g_0 + g$, where g_0 is the constant curvature metric on S^m and g is a metric of constant scalar curvature on some closed manifold. We show that the number of metrics of constant scalar curvature in the conformal class grows at least linearly with respect to the square root of the scalar curvature of g. This is obtained by studying radial solutions of the equation $\Delta u - \lambda u + \lambda u^p = 0$ on S^m and the number of solutions in terms of λ.

1. INTRODUCTION

Any closed manifold admits metrics of constant scalar curvature. Given any Riemannian metric g on M^n we consider its conformal class $[g]$ and define the Yamabe constant of $[g]$ as the minimum of the (normalized) total scalar curvature functional restricted to $[g]$:

$$Y(M, [g]) = \inf_{h \in [g]} \frac{\int_M s_h \ dvol_h}{Vol(M, h)^{\frac{n-2}{n}}},$$

where s_h and $dvol_h$ are the scalar curvature and volume element of h.

It is elementary that the functional restricted to $[g]$ is bounded below, and the fact that the infimum is actually achieved is a fundamental result, obtained in a series of steps by Hidehiko Yamabe [17], Thierry Aubin [2], Neil Trudinger [16] and Richard Schoen [14]. Since the critical points of the functional (restricted to $[g]$) are the metrics of constant scalar curvature in $[g]$, it follows that minimizers are metrics of constant scalar curvature. These are called *Yamabe metrics*. So in any conformal class of metrics in any closed manifold there is at least one unit volume metric of constant scalar curvature. If the Yamabe constant of $[g]$ is non-positive, there is actually only one, the Yamabe metric of the conformal class. But when the Yamabe constant is positive there might be more. For instance, Daniel Pollack proved in [13] that every conformal class with positive Yamabe constant can be C^0 approximated by a conformal class with an arbitrarily large number of (non-isometric) metrics of constant scalar curvature. Uniqueness still holds for the conformal class of positive Einstein metrics different from the round metric on S^n by a result of Morio Obata [12]. For the conformal class of the round metric, all constant scalar curvature

metrics in the conformal class are obtained by conformal diffeomorphisms of the sphere (a non-compact family) and are all isometric. Examples of multiplicity of metrics of constant scalar curvature in a conformal class (which is a Riemannian covering of a number of manifolds) are also obtained by Emmanuel Hebey and Michel Vaugon in [7]. Recently, Simon Brendle gave examples of smooth conformal classes of (not round) Riemannian metrics on high dimensional spheres for which the space of unit volume constant scalar curvature metrics in the conformal class is non-compact [3].

But to determine all the metrics of constant scalar curvature in a given conformal class of positive Yamabe constant is a very difficult problem. The case we are particularly interested in is the Riemannian product of constant curvature metrics on S^2. When the curvature of the 2 factors is the same, the product metric is Einstein and it is the only constant scalar curvature metric in the conformal class. But as we mention in the next paragraph (see also Theorem 1.2 below), when we let the quotient of the curvature of the factors move away from 1, the product metric stops being a Yamabe minimizer, and we know there must be other metrics of constant scalar curvature in the conformal class. The question motivating this article is, how many of them are there and how do they look?

For any closed Riemannian manifold (M^n, g) the Yamabe constant of its conformal class is bounded above by $Y(S^n, [g_0])$, where g_0 is the round metric on the sphere [2]; so if g is a Yamabe metric we must have $s_g Vol(M, g)^{2/n} \leq Y(S^n, [g_0])$. Therefore if (M_1, g_1), (M_2, g_2) are Riemannian manifolds of constant scalar curvature and s_{g_1} is positive, then for δ positive and small the conformal class of the Riemannian product $\delta g_1 + g_2$ has at least two constant scalar curvature metrics: the Riemannian product and a Yamabe metric. The simplest case to consider is the Riemannian product $g_0 + dt^2$ on $S^n \times S^1$. In this case (see the articles by R. Schoen [15] and Osamu Kobayashi [9, 10]) all conformal factors producing metrics of constant scalar curvature are functions of S^1, and there is a sequence of values $\delta_i \to 0$ such that for $\delta \in (\delta_i, \delta_{i+1})$ the number of constant scalar curvature metrics in the conformal class of $\delta g_0 + dt^2$ is i.

In this article we will draw a similar picture in the case $S^k \times S^m$, $k, m > 1$. As usual we will call $p = p_N = \frac{2N}{N-2}$ and $a = a_N = \frac{4(N-1)}{N-2}$. We will consider the Riemannian product $\delta g_0^k + g_0^m$ which has scalar curvature $\mathbf{s}_\delta = (1/\delta)(k(k-1)) + m(m-1)$. For a positive, radial function $f : S^m \to \mathbf{R}$, let $u : [0, \pi] \to \mathbf{R}$ be the corresponding function (so $f(x) = u(d(x, P))$, where P is a fixed point in S^m). The conformal metric $f^{\frac{4}{k+m-2}} (\delta g_0^m + g_0^k)$ has constant scalar curvature K if and only if f satisfies the Yamabe equation

$$-a_{m+k} \Delta_{g_0^m} f + \mathbf{s}_\delta f = K f^{p_{m+k}-1} .$$

Then the function u must satisfy the equation

$$u'' + (m-1)\frac{\cos(t)}{\sin(t)} u' + \frac{K}{a} u^{p-1} - \frac{\mathbf{s}_\delta}{a} u = 0.$$

We normalize by taking $K = \mathbf{s}_\delta$, and we set $\lambda = K/a$. Then we are looking for positive solutions of the equation

$$u'' + (m-1)\frac{\cos(t)}{\sin(t)} u' + \lambda(u^{p-1} - u) = 0$$

with initial conditions $u(0) = \alpha$, $u'(0) = 0$ and such that $u'(\pi) = 0$.

The corresponding equation in $\mathbf{R}^m$ has been well studied. First one has to note that in this case the equations for different values of λ are all equivalent. So one only has to consider the equation $\Delta f - f + f^q = 0$. Then from the classical work of Basilis Gidas, Wei-Ming Ni and Louis Nirenberg [5, 6] it follows that all solutions which are positive and vanish at ∞ (ground states) must be radially symmetric. Then one is looking for solutions of the ordinary differential equation

$$u'' + \frac{m-1}{t}u' + u^q - u = 0,$$

with initial conditions $u(0) = \alpha$, $u'(0) = 0$ which are positive and $u(\infty) = 0$. This equation has been completely analyzed by Man Kam Kwong in [11], proving in particular that there exists exactly one such solution.

In our case we will see that the number of solutions grows at least linearly in $\sqrt{\lambda}$. We will build solutions which verify that $u'(\pi/2) = 0$ so the corresponding metric is invariant under the antipodal map, producing a metric in the projective space. More precisely, we will prove:

Theorem 1.1. *Let (M^k, g) be a Riemannian manifold of constant scalar curvature* **s***. The number of unit volume non-isometric metrics of constant scalar curvature in the conformal class $[g_0 + g]$ on $S^m \times M$ grows at least linearly with $\sqrt{\mathbf{s}}$. The same is true if we replace S^m with the projective space $\mathbf{P}^m$, with the metric of constant curvature. More explicitly, we will show that if $n \geq 1$ and*

$$(\mathbf{s} + m(m-1))\left(\frac{p-2}{a}\right) \in (2n(2n+m-1), (2n+2)(2n+2+m-1)],$$

then $[g_0 + g]$ contains at least $2n + 2$ unit volume non-isometric metrics of constant scalar curvature.

In order to construct the solutions in the previous theorem we will need to prove that there exists one radial solution which is strictly decreasing in $[0, \pi]$. To do so we will need the following elementary result, which we will prove in Section 4:

Theorem 1.2. *Let (M^k, g) be a closed Riemannian manifold of constant scalar curvature* **s***. If*

$$\mathbf{s} + m(m-1) > \frac{a\,m}{p_{k+m} - 2},$$

then the Riemannian product $g + g_0^m$ is not a Yamabe metric. Actually, the product metric is not a local minimum of the total scalar curvature functional restricted to the space $\{f(g + g_0^m) : f : S^m \to \mathbf{R}_{>0}\}$.

Let us consider the case $m = k = 2$. Then $p = p_4 = 4$ and $a = a_4 = 6$. We study the equation $u'' + (\cos(t)/\sin(t))u' + \lambda(u^3 - u) = 0$ and λ relates to the (constant) scalar curvature of (M, g) as $\lambda = (1/6)(\mathbf{s} + 2)$. Let $A_n = (1/2)n(n+1)$. We will show that for $\lambda \in (A_1, A_2]$ there are at least two solutions; one of them is the constant solution which is not a Yamabe minimizer and the other one is a strictly decreasing function. For $\lambda \in (A_2, A_3]$ there are at least 4 solutions, and in general for $\lambda \in (A_{2n}, A_{2n+2}]$ there are at least $2n + 2$ solutions. Except for one of them, all of these solutions verify that $u'(\pi/2) = 0$ and so produce constant scalar curvature metrics on $\mathbf{P}^2 \times M$.

The most interesting particular case for us is the conformal classes of the Riemannian product of metrics of constant curvature on $S^2 \times S^2$, which we will write

$g_0 + \delta g_0$. Moving δ in $(0, \infty)$ we obtain the values of λ in the range $(1/3, \infty)$. The previous comments translate into the following:

Theorem 1.3. *The metric $g_0 + \delta g_0$ is not a Yamabe metric for $\delta < 1/2$. Let $\delta_n = 2 \ (3n(n+1) - 2)^{-1}$. For $\delta \in [\delta_{2(n+1)}, \delta_{2n})$ the number of constant scalar curvature metrics in the conformal class of $g_0 + \delta g_0$ is at least $2n + 2$.*

Remark. For $\delta = 1/2$ the conformal class of $g_0 + (1/2)g_0$ attains the same value of the Yamabe functional, $12\sqrt{2}\pi$, as the conformal class of the Fubini-Study metric on $\mathbf{CP}^2$.

Remark. The results of Gidas, Ni and Nirenberg on the symmetry of solutions in $\mathbf{R}^n$ do not seem to apply to the case of S^n, and there is no adaptation of them in the literature which works in this case, at least to the author's knowledge. But there are techniques which apply to solutions on S^n with singularities. In particular the work of Caffarelli, Gidas and Spruck [4] implies that solutions to the Yamabe equation on $S^n \times S^1$ depend only on the S^1 variable that we mentioned before. It seems reasonable to expect that there must be some variation of their arguments proving that solutions on S^n are all radially symmetric. All but one of the solutions we are going to prove that exist have a local maximum at both 0 and π or a local minimum at both 0 and π. It is clear that in between any two of these solutions there exists one solution which has a minimum at 0 and a maximum at π (or vice versa). These should all be the radially symmetric solutions, but to prove this one should adapt many subtle ideas appearing in the work of M. K. Kwong [11]. This would describe all solutions which depend on only one of the factors. All in all it seems reasonable to conjecture that for $\delta \geq 1/2$ the metric $g_0 + \delta g_0$ is a Yamabe metric and the only unit volume metric of constant scalar curvature in its conformal class (Obata's Theorem [12] says that this is true for $\delta = 1$). For $\delta < 1/2$ the previous comments describe all the constant scalar curvature metrics in the conformal class for which the conformal factor depends on only one of the spheres. It is tempting to guess that these are actually all the solutions, but there is no real evidence to support that.

2. Sturm comparison

To study the differential equation we will need to apply some Sturm comparison techniques. For the convenience of the reader we state an appropriate version of the Sturm Theorem which appears in Ince's book [8]. It can also be found in [11, Lemma 1].

Theorem 2.1. *Let U and V be solutions of the equations*

$$U''(t) + f(t)U'(t) + g(t)U(t) = 0, \quad t \in (a, b),$$
$$V''(t) + f(t)V'(t) + G(t)V(t) = 0, \quad t \in (a, b).$$

Let (α, β) be a subinterval where $V(t) \neq 0$ and $U(t) \neq 0$ and such that $G(t) \geq g(t)$ for all $t \in (\alpha, \beta)$.
If

$$\frac{V'(\alpha)}{V(\alpha)} \leq \frac{U'(\alpha)}{U(\alpha)},$$

then

$$\frac{V'(t)}{V(t)} \leq \frac{U'(t)}{U(t)} \quad \forall t \in (\alpha, \beta).$$

If equality holds at any $x \in (\alpha, \beta)$, then $U \equiv V$ in $[\alpha, x]$.

3. Solutions near $u = 1$; the linear ODE

We want to study the differential equation

$$u'' + (m-1)\frac{\cos(t)}{\sin(t)}u' + \lambda(u^{p-1} - u) = 0,$$

where λ is positive, $p > 2$ and $m - 1$ is a positive integer. We set the initial conditions to be $u(0) = \alpha$ and $u'(0) = 0$. The interval of definition is $[0, \pi]$, and we are interested in positive solutions such that $u'(\pi) = 0$ (which give solutions in S^m). We consider $u = u(t, \alpha, \lambda)$.

There is a canonical solution, $u(t, 1, \lambda) = 1$. Our goal in this section is to understand the behavior of solutions near this canonical one, solutions $u(t, \alpha, \lambda)$ with α close to 1.

Consider the function

$$w(t) = \frac{\partial u}{\partial \alpha}(t, 1, \lambda).$$

Then w is the solution of the linear equation

$$w'' + (m-1)\frac{\cos(t)}{\sin(t)}w' + \lambda(p-2)w = 0,$$

with the initial conditions $w(0) = 1$, $w'(0) = 0$.

We let $A = (p-2)\lambda$ and call $w = w_A$ the corresponding solution. The solutions for $A = n(n + m - 1), n \geq 0$, can be given explicitly. For instance

$$w_0 = 1, w_m(t) = \cos(t), w_{2(m+1)}(t) = \frac{m+1}{m}\left(\cos^2(t) - \frac{1}{m+1}\right).$$

The formulas for $w_{n(n+m-1)}$, $n \geq 3$, can then be found recursively. If we call

$$H_A(f) = f'' + (m-1)\frac{\cos(t)}{\sin(t)}f' + Af,$$

then we have

$$H_A(\cos^n(t)) = (A - n(n + m - 1))\cos^n(t) + n(n-1)\cos^{n-2}(t).$$

It easily follows that

Lemma 3.1. *$w_{n(n+m-1)}$ is a linear combination (with rational coefficients) of powers of $\cos^{n-2k}(t)$, where $0 \leq 2k \leq n$.*

Therefore it follows that if n is odd, then $w_{n(n+m-1)}(\pi) = -1$, and if n is even, then $w_{n(n+m-1)}(\pi) = 1$. Moreover, if n is even, $w_{n(n+m-1)}$ is symmetric with respect to $t = \pi/2$ (and therefore $w_{n(n+m-1)}(\pi/2) \neq 0$, since in that case by the uniqueness of solutions it would have to vanish everywhere); and if n is odd, $w_{n(n+m-1)}$ is antisymmetric with respect to $t = \pi/2$ (and $w_{n(n+m-1)}(\pi/2) = 0$).

Lemma 3.2. *The solution $w_{n(n+m-1)}$ has exactly n zeros in the interval $(0, \pi)$. The number of zeros in the interval $(0, \pi/2)$ is equal to the number of zeros in the interval $(\pi/2, \pi)$.*

Proof. We use induction on n. We know it is true for the first values of n by explicit computation. By Sturm comparison (Theorem 2.1) we know that if $A < B$, then the solution w_B has at least one 0 in between any two zeros of w_A. Therefore w_B has at least the same number of zeros as w_A, and if it has exactly the same, then both must have the same sign after the last 0. Since when moving from n to $n+1$ the corresponding solutions change sign at the final value π it follows that $w_{(n+1)(m+n)}$ must have at least one more 0 than $w_{n(n+m-1)}$. By induction this means that $w_{(n+1)(m+n)}$ must have at least $n+1$ zeros. But $w_{(n+1)(m+n)}$ is a polynomial of degree $n+1$ in $\cos(t)$ and $\cos(t)$ is injective in $(0,\pi)$. Therefore $w_{(n+1)(m+n)}$ could have at most $n+1$ zeros.

The last statement follows directly from the previous comments.   □

The information we will use to prove the existence of constant scalar curvature metrics is about the number of local extrema of w_A in the interval $(0,\pi/2)$. We can give a complete analysis of this. For n even, $w'_{n(n+m-1)}(\pi/2) = 0$, and the number of local extrema in $(0,\pi/2)$ is $n/2-1$ (from the previous lemma). By Sturm comparison (Theorem 2.1) the number of local extrema of w_A is a non-decreasing function of A. This function jumps by one every time we cross a value $A = 2n(2n+m-1)$. Therefore if we call $C_n = (2n(2n+m-1), (2n+2)(2n+2+m-1)]$, we have proved

Theorem 3.3. *For $A \in C_n$ the solution w_A has exactly n local extrema in $(0,\pi/2)$.*

4. Proof of Theorem 1.2
and the existence of a strictly decreasing solution

First recall that if we have two conformal metrics, H and G, on an N-dimensional manifold and we express the conformal relation as $H = f^{\frac{4}{N-2}} G$, then the expression for the total scalar curvature functional of H, $S(H)$, in terms of G and f is

$$S(H) = Y_G(f) = \frac{4a \int |\nabla f|^2 dvol_G + \int s_G f^2 dvol_G}{\left(\int f^p dvol_G\right)^{2/p}} = \frac{E_G(f)}{\|f\|_p^2}.$$

Recall also that

$$(d/dt)|_{t=0}(Y_G(f+tu)) = \frac{2}{\|f\|_p^2} \int [-a\Delta f + sf - \|f\|_p^{-p} E_G(f) f^{p-1}] u \, dvol_G.$$

Given a Riemannian product of constant scalar curvature metrics $g_1 + g_2$ on $M_1 \times M_2$, one can consider conformal factors depending on only one of the variables and define, for instance, $[g_1 + g_2]_1 = \{f.(g_1 + g_2) : f : M_1 \to \mathbf{R}_{>0}\}$. Then we define [1]

$$Y_1(M_1 \times M_2, g_1 + g_2) = \inf_{h \in [g_1+g_2]_1} \frac{\int_{M_1 \times M_2} s_h \, dvol_h}{Vol(M_1 \times M_2, h)^{\frac{N-2}{N}}},$$

where $N = dim(M_1 \times M_2)$. It is easy to see that the infimum is realized [1, Proposition 2.2]. In the case $(M_1, g_1) = (S^m, g_0)$ given any positive function f on S^m, one can consider the spherical symmetrization f_*, which is the radial non-increasing function on S^m which verifies $Vol\{f_* > t\} = Vol\{f > t\}$ for all $t > 0$. Then it is well-known that the total scalar curvature functional is non-increasing by this symmetrization, namely $S(f_*(g_1 + g_2)) \le S(f(g_1 + g_2))$. This proves:

Lemma 4.1. *If (M^k, g) has constant scalar curvature there exists a radially symmetric non-increasing funcion on S^m which gives a minimizer for $Y_1(S^m \times M, g + g_0)$.*

Proof of Theorem 1.2. As we mentioned in the previous section, the function $u(x) = \cos(d(x, N))$ is an eigenfunction of the (negative) Laplacian operator of (S^m, g_0) (and hence of $(M \times S^m, g_0 + g)$) with eigenvalue $-m$. Moreover,

$$\int_{S^m} u \, dvol_{g_0} = 0.$$

Let $Y(t) = Y_{g_0+g}(1 + tu)$. Then $Y'(0) = 0$ and for some positive constant K,

$$Y''(0) = K \left(\int_{S^m} -a\Delta u \, u + (s_g + m(m-1))u^2 - (p-1)(s_g + m(m-1))u^2 \, dvol_{g_0} \right)$$

$$= K(am + (2-p)(s_g + m(m-1))) \int_{S^m} u^2 \, dvol_{g_0}.$$

The hypothesis says precisely that the previous expression is negative, and this proves the theorem. $\qquad\square$

Theorem 1.2 and Lemma 4.1 imply:

Corollary 4.2. *If $\lambda > \frac{m}{p_{m+k-2}}$, then there is $\alpha > 1$ such that the solution of the equation*

$$u'' + (m-1)\frac{\cos(t)}{\sin(t)}u' + \lambda(u^{p-1} - u) = 0$$

with initial condition $u(0) = \alpha$, $u'(0) = 0$, is positive, strictly decreasing in $[0, \pi]$, and $u'(\pi) = 0$.

5. THE NUMBER OF SOLUTIONS: PROOF OF THEOREM 1.1

Now we fix $\lambda > 0$ and study the dependence of the solution $u(t, \alpha, \lambda)$ of the equation

$$u'' + (m-1)\frac{\cos(t)}{\sin(t)}u' + \lambda(u^{p-1} - u) = 0$$

on α. It follows from Section 1 that if α is close to 1, then u intersects the canonical solution about $\sqrt{\lambda}$ times.

Let P be the subset $\{\alpha \in (0, \infty) : u_\alpha > 0 \text{ on } [0, \pi/2]\}$. Clearly P is an open subset of $(0, \infty)$ and $1 \in P$. Suppose $[1, A)$ is a maximal (to the right) interval included in P. Then u_A must be non-negative in $[0, \pi/2]$. Then u_A must be strictly positive in $[0, \pi/2)$ and $u_A(\pi/2) = 0$ (otherwise the interval would not be maximal, of course).

Now consider the energy function associated with u_A,

$$E_A(t) = \frac{(u'_A(t))^2}{2} + \lambda \left(\frac{u_A^p(t)}{p} - \frac{u_A^2(t)}{2} \right).$$

We have

$$E'_A(t) = -(m-1)\frac{\cos(t)}{\sin(t)}(u'_A(t))^2.$$

And so E_A is decreasing in the interval $[0, \pi/2]$. Since $E_A(\pi/2) > 0$ we must have positive energy on $[0, \pi/2]$. Consider the following simple lemma:

Lemma 5.1. *If u_α has a local minimum at t_0, then $u_\alpha(t_0) < 1$ and so $E_\alpha(t_0) < 0$.*

Then it follows:

Lemma 5.2. *If for some A, $[1, A)$ is a maximal (to the right) interval contained in P, then u_A has no local extrema in $(0, \pi/2)$.*

The following lemma will allow us to construct solutions without having to analyze the equation in the whole interval $[0, \pi]$.

Lemma 5.3. *Suppose that for some positive α the solution u_α verifies $u'_\alpha(\pi/2) = 0$. Then $u'_\alpha(\pi) = 0$ (and actually u_α is symmetric with respect to $t = \pi/2$).*

Proof. The function $v(t) = u_\alpha(\pi - t)$ for $t \in [\pi/2, \pi)$ is also a solution of the equation. Moreover $v(\pi/2) = u_\alpha(\pi/2)$ and $v'(\pi/2) = 0 = u'_\alpha(\pi/2)$. Therefore $v = u_\alpha$ and the lemma follows. □

Lemma 5.4. *Suppose that for some positive α_0 ($\neq 1$) the solution u_{α_0} has exactly k extrema in the open interval $(0, \pi/2)$ and $u'_{\alpha_0}(\pi/2) = 0$. Then there exists $\varepsilon > 0$ such that for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$ the number of local extrema of u_α in $(0, \pi/2]$ is k or $k + 1$.*

Proof. If $u_0 = u_{\alpha_0}(\pi/2) = 1$ we would have $\alpha_0 = 1$, and we have assumed this is not the case. Therefore $u_0 > 1$ or $u_0 > 1$. If $u_0 < 1 (> 1)$ there exists $\delta > 0$ such that for $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$ and $t \in (\pi/2 - \delta, \pi/2 + \delta)$, we have $u_\alpha(t) < 1 (> 1)$. This implies that such u_α cannot have any local maxima (minima) in $(\pi/2 - \delta, \pi/2 + \delta)$. Therefore it has at most 1 local extrema in that interval.

We can also assume the δ small enough so that u_{α_0} does not have any other extrema besides $\pi/2$ in $[\pi/2 - \delta, \pi/2 + \delta]$. Therefore u_{α_0} has k local extrema in that interval, and for $\varepsilon > 0$ small enough, $\varepsilon < \delta$, and $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, u_α also has k local extrema in $(0, \pi/2 - \delta)$ (and hence k or $k + 1$ in $(0, \pi/2]$). □

Lemma 5.5. *If α is close enough to zero, the solution u_α has no local extrema in $(0, \pi/2)$. If $\lambda(p - 2) > m$ there exists $\alpha > 1$ such that the solution u_α has no local extrema in $(0, \pi)$.*

Proof. For α close to 0 the solution u_α stays close to 0 and so stays less than 1 until $\pi/2$. Consequently, it does not have any local maxima and is increasing. This proves the first statement. The second statement is just Corollary 4.2. □

We are finally ready to prove Theorem 1.1.

Proof. If $\lambda(p - 2) \in (2n(2n + m - 1), (2n + 2)(2n + 2 + m - 1)]$, for α close to 1, it follows from Theorem 3.3 that the solution u_α has at least n local extrema in $(0, \pi/2)$. Increasing α from 1 to ∞ we bump into solutions for which $u'_\alpha(\pi/2) = 0$. Each one of these gives a constant scalar curvature metric. If for some value of α the solution u_α has at least one local minimum in $(0, \pi)$, then u'_α is positive somewhere in the interval. Let S be the set of values $\alpha \in (1, \infty)$ for which the solution u_α is positive in $[0, \pi/2]$ and has at least one local minimum in $(0, \pi/2)$. It follows from Lemma 5.5 that S is not the whole interval $(1, \infty)$. Let $(1, A)$ be a maximal subinterval contained in S. From the previous comments it follows that u'_α must be non-negative somewhere in $(0, \pi/2)$. But we can deduce from Lemma 5.2 that it cannot be the case that $u_A(\pi/2) = 0$. It follows that u_A is positive in $[0, \pi/2]$ and $u'_A(\pi/2) = 0$. Now we restrict to the interval $(1, A)$ where all the solutions remain positive in $(0, \pi/2)$. Let $U \subset (1, A)$ be the open subset of values α such that $u'_\alpha(\pi/2) \neq 0$. On a fixed subinterval of U all solutions have the same number

of local extrema in $(0, \pi/2)$. As we cross a value of α for which $u'_\alpha(\pi/2) = 0$, the number of local extrema before $\pi/2$ decreases at most by 1, from Lemma 5.3. We can make the same argument when α decreases from 1 to 0. Therefore the number of initial values α for which $u'_\alpha(\pi) = 0$ is at least $2n$. Note that for these solutions 0 and π are both local minima or maxima. Besides these, we have the constant solution and one strictly decreasing (or increasing) solution. $\qquad\square$

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References

[1] K. Akutagawa, L. Florit, J. Petean, *On Yamabe constants of Riemannian products*, Comm. Anal. Geom. **15** (2007), 947-969. MR2403191 (2009i:53030)

[2] T. Aubin, *Equations differentielles non-lineaires et probleme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269-296. MR0431287 (55:4288)

[3] S. Brendle, *Blow-up phenomena for the Yamabe equation*, Journal Amer. Math. Soc. **21** (2008), 951-979. MR2425176 (2009m:53084)

[4] A. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure. Appl. Math. **42** (1989), 271-297. MR982351 (90c:35075)

[5] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243. MR544879 (80h:35043)

[6] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$*, Advances in Math. Studies **7 A** (1981), 369-402. MR634248 (84a:35083)

[7] E. Hebey, M. Vaugon, *Meilleures constantes dans le theoreme dinclusion de Sobolev et multiplicite pour les problemes de Nirenberg et Yamabe*, Indiana Univ. Math. J. **41** (1992), 377-407. MR1183349 (93m:58121)

[8] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1956. MR0010757 (6:65f)

[9] O. Kobayashi, *On the large scalar curvature*, Research Report 11, Dept. Math., Keio Univ. (1985).

[10] O. Kobayashi, *Scalar curvature of a metric with unit volume*, Math. Ann. **279** (1987), 253-265. MR919505 (89a:53048)

[11] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$*, Arch. Rational Mech. Analysis **105** (1989), 243-266. MR969899 (90d:35015)

[12] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Diff. Geom. **6** (1971), 247-258. MR0303464 (46:2601)

[13] D. Pollack, *Nonuniqueness and high energy solutions for a conformally invariant scalar equation*, Comm. Anal. Geom. **1** (1993), 347-414. MR1266473 (94m:58051)

[14] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry **20** (1984), 479-495. MR788292 (86i:58137)

[15] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Lectures Notes in Math., 1365, Springer-Verlag, Berlin, 1989, 120-154. MR994021 (90g:58023)

[16] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 265-274. MR0240748 (39:2093)

[17] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21-37. MR0125546 (23:A2847)

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