PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

METRICS OF NEGATIVE CURVATURE ON VECTOR BUNDLES

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ABSTRACT. It is shown that any vector bundle E over a compact base manifold M admits a complete metric of negative (respectively nonpositive) curvature provided M admits a metric of negative (nonpositive) curvature.

1. Introduction.

The purpose of this note is to prove the following

THEOREM. *Let B be a compact n-dimensional manifold of negative sectional curvature. Then any vector bundle $\Pi: E \to B$ admits a complete metric of negative sectional curvature K_E satisfying $-a \leq K_E \leq -1$ for some constant $a \geq 1$. (Here a depends on the geometry of B and the topology of the bundle $\Pi: E \to B$.)*

If B is a compact manifold of nonpositive sectional curvature, then any vector bundle $\Pi: E \to B$ admits a complete metric of nonpositive sectional curvature K_E satisfying $-b \leq K \leq 0$ for some positive constant b.

This result should be compared with a well-known open problem of Gromoll: If M is a compact manifold of positive sectional curvature, does every vector bundle over M admit a complete metric of nonnegative sectional curvature?

The theorem was motivated by, and partially answers, a question of M. Gromov: Does every vector bundle over a compact base B, with a possibly *singular* metric of negative curvature on B, admit a smooth complete metric of negative curvature (cf. [**3**] for a discussion of a singular metrics). For example, let T be a hyperbolic group, in the sense of [**2**], and let X be a metric space on which T acts freely with compact quotient. One may ask if there is an embedding of X in $\mathbf{R}^n$ such that a tubular neighborhood of $X \subset \mathbf{R}^n$ admits a complete metric of negative sectional curvature. This approach is relevant for the Novikov conjecture for such hyperbolic groups.

It is of interest to note that Gromov, Lawson and Thurston [**4**] have recently shown that most 2-plane bundles E over a compact Riemann surface M_g, of genus greater than one, admit complete metrics of constant curvature -1, provided $|\chi(E)| \leq |\chi(M_g)|$.

I am grateful to M. Gromov for suggesting this problem and for interesting discussions.

2. Preliminaries.

We begin with the standard topological description of vector bundles. Let $\Pi_0: P \to B$ be a right principal $O(m)$ bundle, $m \geq 1$, over a smooth n-dimensional manifold B. Let $G = O(m)$ act on $\mathbf{R}^m$ on the left in the usual

Received by the editors November 23, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20; Secondary 58G30.
The author was partially supported by an NSF Postdoctoral Fellowship.

way by orthogonal transformations. Define an action of G on $P \times \mathbf{R}^m$ by $g(p,f) = (pg, g^{-1}f)$. Then the quotient space $E = P \times \mathbf{R}^m/G$ is a vector bundle $\Pi_B : E \to B$ with fiber F diffeomorphic to $\mathbf{R}^m$ and structure group G. E is called the vector bundle associated to P. Conversely, given a vector bundle V over B, we may assume without loss of generality that its structure group is $O(m)$. Then there is a principal $O(m)$ bundle over B such that associated bundle constructed above is equivalent to V.

Let $\langle\ ,\ \rangle_G$ denote the negative of the killing form of the Lie algebra $L(G)$ of G; we will also let $\langle\ ,\ \rangle_G$ denote the corresponding bi-invariant metric on G. Let $\langle\ ,\ \rangle_B = ds_B^2$ denote a smooth Riemannian metric on B. If $\Theta\colon TP \to\colon L(G)$ is any connection 1-form on P, we define a metric on P by

$$ds_P^2 = \Pi_0^*(ds_B^2) + \Theta \cdot \Theta, \tag{2.1}$$

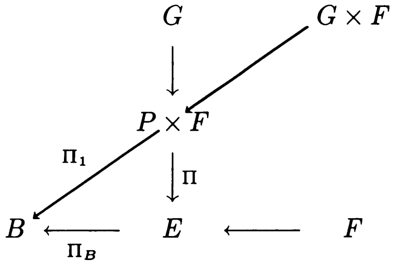
i.e. for vectors $x, y \in T_pP$, $\langle x, y\rangle_P = \langle \Pi_* x, \Pi_* y\rangle_B + \langle \Theta(x), \Theta(y)\rangle_G$.

It is well known (cf. [**5**]) that $\Pi_0\colon P \to B$ is a Riemannian submersion, with totally goedesic fibers, with respect to the metrics ds_p^2 and ds_B^2. Let H^1 denote the orthogonal complement of the tangent space to the orbits $G \subset P$. Then H^1 coincides with the horizontal spaces for the submersion Π_0, as well as the horizontal spaces for the connection 1-form.

Next we consider the product metric

$$ds_{P\times F}^2 = ds_P^2 + ds_F^2 \tag{22}$$

on $P \times F$, where ds_F^2 is the metric of constant curvature $-a^2$ on $F \approx \mathbf{R}^m$; of course $a = 0$ if $m = 1$. Note that $ds_{P\times F}^2$ is invariant under the action of G on $P \times F$, so that $ds_{P\times F}^2$ descends to give a metric ds_E^2 on E. We have the following commutative diagram:



With respect to the metrics defined above, each map Π, Π_1, Π_B is a Riemannian submersion with totally geodesic fibers (cf. [**5**] for a proof).

For later purposes, we recall a formula of O'Neill [**6**] relating the curvature of the base and total space of Riemannian submersion. Let $S \to M$ be a Riemannian submersion. Let X, Y be horizontal vector fields on S and let $X_* = \Pi_* X, Y_* = \Pi_* Y$. Then if K denotes sectional curvature, we have

$$K^S(X,Y) = K^M(X_*, Y_*) - \frac{3}{4}\frac{|[X,Y]^\vee|^2}{|X \wedge Y|^2}, \tag{2.3}$$

where $[X,Y]^\vee$ denotes the orthogonal projection of the Lie bracket $[X,Y]$ onto the vertical subspaces of $T(S)$.

3. Construction of metrics. The metric ds_E^2 constructed in §2 does not have negative sectional curvature. In fact, the O'Neill formulas [**6**] imply that the "mixed" curvature $K^E(X,V)$ for X horizontal and V vertical with respect to Π_B are nonnegative.

In order to construct metrics of negative curvature on E, we consider warped product metrics on $P \times F$. Let $g\colon F \to \mathbf{R}$ be an $O(m)$-invariant smooth function, with $g > 0$. Thus, $g = g(r)$, where r is the distance function to $0 \in F$ with respect to the metric ds_F^2. We will specify g more precisely later in this section. Extend g to a function $g\colon P \times F \to \mathbf{R}$ by first projecting on the second factor. We consider metrics of the form

$$d\tilde{s}^2_{P\times F} = g^2 \cdot ds_P^2 + ds_F^2. \tag{3.1}$$

Note that $d\tilde{s}^2$ is also a G-invariant metric and so gives a metric $d\tilde{s}^2_E$ on E. The projection $\Pi\colon P\times F \to E$ is a Riemannian submersion with respect to these metrics; the fibers are no longer totally geodesic however. Nevertheless, one may still use (2.3) to relate the curvatures.

We will need explicit descriptions of the horizontal and vertical spaces of Π in these metrics. Thus, let $X(M)$ denote the space of C^∞ vector fields on M. Define maps

$$L(G) \to X(P), \quad E \to \tilde{E}(p), \quad \text{and} \quad L(G) \to X(F), \quad E \to \tilde{E}(f),$$

where

$$\tilde{E}(p) = \frac{d}{dt}(p \cdot \exp tE)\bigg|_{t=0}, \qquad \tilde{E}(f) = \frac{d}{dt}(\exp tE \cdot f)\bigg|_{t=0}. \tag{3.2}$$

It is well known, and easy to verify, that these maps are Lie algebra homomorphisms. We note there is a constant $C > 0$ such that

$$\frac{1}{C} < \frac{|\tilde{E}(p)|}{|E|} < C \tag{3.3}$$

for all $p \in F$, for any given smooth metric on P. For any $f \in F$, we may choose $(m-1)$ unit vectors $e_i \in L(G)$, depending on f, such that $\{\tilde{e}_i(f)/\psi(r)\}^{m-1}$ is an orthonormal basis of $T_fS \subset T_fF$, where S is the geodesic r-sphere through f centered at 0. One calculates that

$$\psi(r) = \frac{1}{a}\sinh ar. \tag{3.4}$$

Thus $\{\tilde{e}_i(f)/\psi(r), \nabla r\}$ forms an orthonormal basis of T_pF. Note that $\tilde{E}(f) = 0$ for any $E \notin \operatorname{span}\{e_i\}_1^{m-1}$.

One easily sees that the vertical space $V_{p,f} \subset T_{(p,f)}P \times F$ for Π is given by

$$V_{p,f} = \operatorname*{span}_{E\in L(G)} [\tilde{E}(p) - \tilde{E}(f)].$$

By the remarks above,, we may choose a basis $\{e_i\} \in L(G)$, depending on f, such that

$$V_{p,f} = \operatorname*{span}_{i=1}^{m-1}[\tilde{e}_i(p) - \tilde{e}_i(f)] \oplus \operatorname*{span}_{i=m}^{N}[\tilde{e}_i(p)],$$

where $N = \dim G$. Note that $\dim V_{p,f} = N$. Let $H^1_{p,f} = (T_pG)^\perp \subset T_pP$ as in §2, $H^2 = \operatorname{span}_{i=1}^{m-1}[\alpha\tilde{e}_i(p) + \tilde{e}_i(f)]$, where

$$\alpha(p,f) = \frac{1}{g^2(p,f)}\frac{\langle\tilde{E}(f), \tilde{E}(f)\rangle}{\langle\tilde{E}(p), \tilde{E}(p)\rangle},$$

and let $H^3 = \operatorname{span}\nabla r$.

Then there is an orthogonal splitting, with respect to $d\tilde{s}^2_{P\times F}$, of the form

$$T(P \times F) = V \oplus H^1 \oplus H^2 \oplus H^3. \tag{3.5}$$

The subspace $H^1 \oplus H^2 \oplus H^3$ is the horizontal space for the submersion $\Pi\colon P\times F \to E$ with respect to the metrics $d\tilde{s}^2_{P\times F}$ and $d\tilde{s}^2_E$.

We now begin with the computation of the curvature of $d\tilde{s}^2_E$. First, by (2.3), the curvature of $(P \times F,\, d\tilde{s}^2_{P\times F})$ and $(E,\, d\tilde{s}^2_E)$ are related by

$$\tilde{K}^E(X_*, Y_*) = \tilde{K}^{P\times F}(X, Y) + \frac{3}{4}\frac{|[X,Y]^{\vee}|^2_{\sim}}{|X \wedge Y|_{\sim}} \tag{3.6}$$

for horizontal vectors $X, Y \in T(P \times F)$. To estimate the first term, we use the formula for the sectional curvature of a warped product given in [**1**]. Write $X = X_p + X_F$, where X_p (resp. X_F) is the orthogonal projection of X onto TP (resp. TF). If the pair $\{X, Y\}$ is orthonormal with respect to $d\tilde{s}^2$, then

$$\begin{aligned}\tilde{K}^{P\times F}(X,Y) = K^F(X_F, Y_F)\cdot |X_F \wedge Y_F|^2 - g[|Y_p|^2 D^2 g(X_F, X_F) - 2\langle X_P, Y_P\rangle \\ \cdot D^2 g(X_F, Y_F) + |X_P|^2 D^2 g(X_F, Y_F)] \\ + g^2[K^P(X_P, Y_P) - |\nabla g|^2]|X_P \wedge Y_P|^2\end{aligned} \tag{3.7}$$

Let B_ε denote the geodesic ball of radius ε about $0 \in F$. The function g will depend on a parameter ε, to be fixed below, and chosen to satisfy the following properties:

(3.8)
- (i) g is convex, i.e. $D^2 g \geq 0$ on F and $D^2 g < C_0 g$ outside B_ε.
- (ii) $|\nabla g|^2/g^2 > C_1$ outside B_ε.
- (iii) $|\nabla g|^2/g^2 < C_2$ outside some compact set of F.
- (iv) $|\nabla g|^2(x) > C_3 \cdot r(x)$ for $x \in B_\varepsilon$.
- (v) $g \leq 1$ in B_ε, $g > 1$ outside B_ε.

Here C_0, C_1, C_2, C_3 are constants, also to be specified below. For example, one may choose g of the form

$$g = \{a_1 + a_2 r^{3/2}\} e^{a_3 r}$$

and adjust $\{a_i\}$ to satisfy (3.8). Basically, a_1 is small and a_2, a_3 large.

Using (3.8) we may estimate (3.7). First, since g is convex, the second term in (3.7) within the brackets is nonnegative. Since F has curvature $-a^2$, we find

$$\tilde{K}^{P\times F}(X,Y) \leq -a^2|X_F \wedge Y_F|^2 + g^2[K^p(X_p, Y_p) - |\nabla g|^2]|X_p \wedge Y_p|^2.$$

We now consider several cases. Suppose $f \notin B_\varepsilon$. Choose $C_1 = a^2 + \sup K^p(X_p, Y_p)$. By (3.8)(ii) and (v) we obtain

$$\begin{aligned}\tilde{K}^{P\times F}(X,Y) &\leq -a^2|X_F \wedge Y_F|^2 - a^2 g^4 |X_P \wedge Y_P|^2 \\ &\leq a^2|X_F \wedge Y_F|^2_{\sim} - a^2|X_P \wedge Y_P|^2_{\sim} \leq -a^2/4.\end{aligned} \tag{3.9}$$

Next suppose $f \in B_\varepsilon$. If $|X_F \wedge Y_F|^2 \geq |X_p \wedge Y_p|^2_{\sim}$, then setting $b = \sup K^p(X_p, Y_p)$ we have

$$\begin{aligned}\tilde{K}^{P\times F}(X,Y) &\leq -a^2|X_F \wedge Y_F|^2 + \frac{b}{g^2}|X_p \wedge Y_p|^2_{\sim} \\ &\leq \left[-a^2 + \frac{b}{g^2}\right]|X_F \wedge Y_F|^2 \leq \frac{1}{4}\left[-a^2 + \frac{b}{g^2}\right]\end{aligned} \tag{3.10}$$

assuming $-a^2 + b/g^2 \leq 0$.

Finally, suppose $|X_F \wedge Y_F|^2 < |X_p \wedge Y_p|^2_\sim$ and $f \in B_\varepsilon$. We may write $X_p = X_B + X_2$, where $X_B \in H^1$ and $X_2 = \alpha \sum_1^{n-1} a_i e_i(p) \in (H^2)_p$. It is important to note that $|X_2|_\sim \to 0$ as $\varepsilon \to 0$. To see this, we have $|X_F| < 1$, so that $|\sum a_i e_i(f)| < 1$. Since, by definition, $\alpha = O(|e(f)|^2)$, the claim follows. Note also that $|X_B|_\sim$ is bounded away from zero as $f \to 0$, since by our assumption $|X_p|$ is bounded away from zero. These same remarks apply to Y_p and we obtain the estimate

$$K^P(X_p, Y_p) = K^p(X_B, Y_B) + O(\varepsilon).$$

Now (2.3) applied to the Riemannian submersion $\Pi_0 \colon P \to B$ gives

$$K^P(X_B, Y_B) = K^B(X_B, Y_B) - \frac{3}{4}\frac{|[X_B, Y_B]^\vee|^2}{|X_B \wedge Y_B|^2}.$$

Thus, for the last case, we obtain

$$\begin{aligned}(3.11)\quad \tilde{K}^{P\times F}(X, Y) &\le -a^2 |X_F \wedge Y_F|^2 \\ &+ \left[K^B(X_B, Y_B) - \frac{3}{4}\frac{|[X_B, Y_B]^\vee|^2}{|X_B \wedge Y_B|^2} - |\nabla g|^2 + O(\varepsilon)\right] g^2 |X_P \wedge Y_p|^2.\end{aligned}$$

In order to estimate the second term of (3.6), we use the following Lemma.

LEMMA. *Let X, Y be horizontal fields on $(P \times F, d\tilde{s}^2)$. Then there is a constant k, depending on ds_P^2 and* $\inf g$, *but not on a, such that*

$$(3.12)\qquad |[X, Y]^\vee|^2_\sim < k \cdot |X \wedge Y|^2_\sim.$$

PROOF. Since both sides of (3.12) are bilinear, it is sufficient to check (3.12) on a basis for the horizontal fields. Thus, let $X = \sum X_i$, $Y = \sum Y_i$, where $X_i, Y_i \in H^i$. One verifies that

$$[X_3, Y_i] = [X_i, Y_3] = 0, \qquad [X_1, Y_2] = [X_2, Y_1] = 0.$$

Thus $[X, Y] = [X_1, Y_1] + [X_2, Y_2]$ and

$$(3.13)\qquad |[X, Y]^\vee|^2_\sim = |[X_1, Y_1]^\vee|^2_\sim + |[X_2, Y_2]^\vee|^2_\sim.$$

Applying (2.3) to the submersion $\Pi_0 \colon P \to B$ gives

$$(3.14)\qquad K^p(X_1, Y_1) = K^B(X_1, Y_1) - \frac{3}{4}\frac{|[X_1, Y_1]^\vee|^2}{|X_1 \wedge Y_1|^2};$$

note that since X_1, Y_1, and $[X_1, Y_1] \in TP$, the vertical projections for Π_0 and Π agree. Since K^P and K^B are bounded, we have

$$|[X_1, Y_1]^\vee|^2 < k|X_1 \wedge Y_1|^2,$$

and thus

$$(3.15)\qquad |[X_1, Y_1]^\vee|^2_\sim < k|X_1 \wedge Y_1|^2_\sim.$$

We estimate the second term in (3.13) on a basis of the form $B_i = \alpha e_i(p) + e_i(f)$, where at a given $p_0 \in P$, we assume $\langle B_i, B_j\rangle(p_0, f) = 0$ if $i \neq j$. We have

$$\begin{aligned}[B_i, B_j] &= \alpha^2 [e_i(p), e_j(p)] + [e_i(f), e_j(f)] \\ &= \alpha^2 C_{ij}^k e_k(p) + C_{ij}^k e_k(f),\end{aligned}$$

where we have used the fact that the maps $E \to \tilde{E}(p)$ and $E \to \tilde{E}(f)$ are Lie algebra homomorphisms; here C_{ij}^k are the structure constants of $L(G)$. Thus

$$|[B_i, B_j]^\vee|_\sim^2 = \sum_{k,l,m} \frac{[\alpha^2 C_{ij}^k \langle e_k(p), e_k(m)\rangle_\sim - C_{ij}^l \langle e_l(f), e_m(f)\rangle_\sim]^2}{|e_m(p) - e_m(f)|_\sim^2}$$
$$\le C \cdot \frac{\alpha^4 |e(p)|_\sim^4 + |e(f)|_\sim^4}{|e(p) - e(f)|_\sim^2},$$

where C is a constant independent of the metrics. Since $|e(p) - e(f)|_\sim^2 \ge L$ for some constant L depending only on $\inf g$, we have

$$|[B_i, B_j]^\vee|^2 \le C^1[\alpha^4 |e(p)|_\sim^4 + |e(f)|_\sim^4].$$

On the other hand,

$$\begin{aligned} |B_i \wedge B_j|_\sim^2 &= |B_i|_\sim^2 |B_j|_\sim^2 - \langle B_i, B_j\rangle_\sim^2 \\ &= |\alpha e_i(p) + e_i(f)|_\sim^2 \cdot |\alpha e_j(p) + e_j(f)|_\sim^2 \\ &\le C[\alpha^4 |e(p)|_\sim^4 + |e(f)|_\sim^4]. \end{aligned}$$

Combining the last two estimates with (3.15) gives the result.

We now combine the above estimates to determine $\tilde{K}^E(X, Y)$. As before, we deal with several cases. We assume $m \ge 2$ and will discuss the case $m = 1$ at the end

(i) $f \notin B_\varepsilon$: Combining (3.9) and (3.12) and substituting into (3.6) gives

$$\tilde{K}^E(X, Y) \le -a^2/4 + k. \tag{3.16}$$

(ii) $f \in B_\varepsilon$ and $|X_F \wedge Y_F|^2 \ge |X_p \wedge Y_p|_\sim^2$: Using (3.10) and (3.12) as above gives

$$\tilde{K}^E(X, Y) \le \tfrac{1}{4}[-a^2 + b/g^2] + k. \tag{3.17}$$

Thus, making a choice of g satisfying (3.8), we see that we may choose a sufficiently large so that $\tilde{K}^E(X, Y) < 0$ in the above two cases. In particular, the curvature of E may be made negative outside a neighborhood of the 0-section of $\Pi_B \colon E \to B$, regardless of the curvature of B.

(iii) $f \in B_\varepsilon$ and $|X_F \wedge Y_F|^2 < |X_p \wedge Y_p|_\sim^2$: Using (3.11) and the fact that $|X \wedge Y|_\sim = 1$, we estimate (3.6) as

(3.18)
$$\begin{aligned} \tilde{K}^E(X, Y) \le -a^2 |X_F \wedge Y_F|^2 &+ \left[K^B(X_B, Y_B) - \frac{3}{4}\frac{|[X_B, X_B]^\vee|_\sim^2}{|X_B \wedge X_B|_\sim^2} - |\nabla g|^2 + O(\varepsilon)\right] \\ &\cdot \frac{1}{g^2}|X_p \wedge Y_p|_\sim^2 + \frac{3}{4}|[X_B, Y_B]^\vee|_\sim^2 + \frac{3}{4}|X_2, Y_2]^\vee|_\sim^2 \\ \le -a^2|X_F \wedge Y_F|^2 &+ [K^B(X_B, Y_B) + O(\varepsilon) - |\nabla g|^2]\frac{1}{g^2}|X_p \wedge Y_p|^2 + O(\varepsilon), \end{aligned}$$

where we have used (3.8)(v).

Now suppose first that $K^B(X_B, Y_B) < 0$, say $K^B(X_B, Y_B) \le -m^2 < 0$. Choosing ε sufficiently small in (3.18), we obtain

$$\tilde{K}^E(X, Y) \le -C \tag{3.19}$$

for some constant $C > 0$. We may combine (3.19) with (3.16) and (3.17) and rescale the metric if necessary to obtain $\tilde{K}^E(X,Y) \leq -1$ for all $X, Y \in T(E)$.

Next suppose only $K^B(X_B, Y_B) \leq 0$. Then by (3.18)

$$\tilde{K}^E(X,Y) \leq -a^2|X_F \wedge Y_F|^2 + [O(\varepsilon) - C_3\varepsilon]\frac{1}{g^2}|X_p \wedge Y_p|^2_\sim + O(\varepsilon).$$

We may choose ε sufficiently small and C_3 sufficiently large in (3.8) (iv) so that $[O(\varepsilon) - C_3\varepsilon]|X_p \wedge Y_p|^2_\sim/g^2$ is sufficiently negative for $\varepsilon \neq 0$, to dominate the last $O(\varepsilon)$ term. We than obtain $\tilde{K}^E(X,Y) \leq 0$. Combining this with (3.16) and (3.17) gives a complete metric on E of nonpositive sectional curvature.

Finally, it is straightforward to verify that the condition $D^2g < C_0 g$ outside B_ε for some constant C_0 implies in both cases $K^B < 0$ and $K^B \leq 0$ that

$$\tilde{K}^E(X,Y) \geq -M^2$$

for some constant M. This proves the theorem in the case $m \geq 2$.

Suppose finally that $m = 1$. In the notation above, any horizontal 2-plane for $\Pi\colon P \times \mathbf{R} \to E$ has a basis of the form $X = X_B + c \cdot \nabla r$, $Y = Y_B$. Since $[X,Y]^\vee = 0$, we obtain from (3.7)

$$\tilde{K}^E(X,Y) = \tilde{K}^{P\times\mathbf{R}}(X,Y) = -gc^2|Y_B|^2 g^{11} + g^2[K^B(X_B,Y_B) - |\nabla g|^2]|X_B \wedge Y_B|^2.$$

This can be made negative, respectively nonpositive, depending on the curvature of B, by choosing g to be any convex function. In particular, g satisfying (3.8) suffices to prove the theorem in this case also.

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