J. DIFFERENTIAL GEOMETRY
17 (1982) 635–641

METRICS OF NEGATIVE RICCI CURVATURE ON SL(n, R), $n \geqslant 3$

M. L. LEITE & I. DOTTI DE MIATELLO

1. Introduction

In his survey article J. Milnor [8] treats the problem of classifying those Lie groups which admit left invariant metrics whose sectional or Ricci curvatures have constant sign.

Sectional curvature (denoted by k) classification theorems ($k > 0$, $k = 0$, $k < 0$, $k \geqslant 0$, $k \leqslant 0$) are presented in [9], [8], [6], [3] and [2]. The analogous Ricci curvature (henceforth denoted by r) classification problem is not completely solved. The cases $r = 0$ and $r \geqslant 0$ are equivalent to $k = 0$ and $k \geqslant 0$ (see [3]). The case $r > 0$ is settled through Myers theorem [8]. To our knowledge, the classification of Lie groups which admit a left invariant metric so that all Ricci curvatures satisfy $r < 0$ or $r \leqslant 0$ is still an open problem.

It is known [2] that a connected Lie group with a nonflat left invariant metric of sectional curvature $k \leqslant 0$ is necessarily *solvable* and *non-unimodular*. Contrasting with this, J. Milnor constructs in [8] nonflat metrics of Ricci curvature $r \leqslant 0$ on the 3-dimensional *simple* group $SL(2, R)$. He then asks whether any *higher dimensional simple group* admits such a metric.

In this article, we construct left invariant metrics of strictly negative Ricci curvature on the *simple Lie group* $SL(n, \mathbf{R})$, $n \geqslant 3$. The construction of such metrics is rather computational. To the reader's benefit, we include all necessary details.

We are indebted to R. J. Miatello for bringing to our attention that, for $n \geqslant 3$, the manifolds $\Gamma \setminus SL(n, \mathbf{R})$, where Γ is a uniform, torsion free, discrete subgroup of $SL(n, \mathbf{R})$, provide a family of examples of compact riemannian manifolds of negative Ricci curvature where $SO(n)$ acts freely (compare [10], [11, p. 10]). The fact that these manifolds also admit metrics of positive scalar curvature answers another question in [10].

Received December 21, 1981. The first author was supported in part by CNPq-Brazil, and the second by CNPq and Finep-Brazil.

2. Algebraic properties of $sl(n,\mathbf{R})$

Let $sl(n,\mathbf{R})$ be the Lie algebra of $n \times n$ real matrices of trace zero. We consider in $sl(n,\mathbf{R})$ the vector space decomposition

$$sl(n,\mathbf{R}) = k \oplus h \oplus a,$$

where k, h and a denote the subspaces of skew-symmetric, symmetric matrices with zero diagonal entries and diagonal matrices, respectively. Standard basis for k and h are given by

$$x_{ij} = e_{ij} - e_{ji}, \quad 1 \leqslant i < j \leqslant n,$$
$$y_{ij} = e_{ij} + e_{ji}, \quad 1 \leqslant i < j \leqslant n,$$

respectively, where e_{ij} is the $n \times n$ real matrix having 1 in the (i, j) position and 0's elsewhere, $1 \leqslant i, j \leqslant n$.

Let us state the algebraic properties of $sl(n,\mathbf{R})$ which will be needed to compute Ricci curvatures in §3.

It follows from the Lie bracket of matrices that

(2.1) $\quad [a, a] = 0, [a, k] = h, [a, h] = k, [k, k] = k, [h, h] = k.$

(2.2) $\quad [k, h] = h \oplus a$; precisely, the elements $[x_{ij}, y_{ij}]$, $i < j$, span a, and $[x_{ij}, y_{kl}], (k, l) \neq (i, j)$, span h.

The linear transformation $\mathrm{ad}(v)$, $v \in sl(n,\mathbf{R})$, maps an element w of $sl(n,\mathbf{R})$ into $[v, w]$, hence

$$(\mathrm{ad}(v))^2 w = v^2 w - 2vwv + wv^2.$$

This formula yields

$$\begin{aligned}
&\sum_{i<j} (\mathrm{ad}(x_{ij}))^2(v) = -(2n-4)v \quad (v \in k),\\
(2.3)\qquad &\sum_{i<j} (\mathrm{ad}(x_{ij}))^2(v) = -2nv \quad (v \in h \oplus a),\\
&(\mathrm{ad}(x_{ij}))^2(y_{ij}) = -4y_{ij}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\sum_{i<j} (\mathrm{ad}(y_{ij}))^2(v) = 2nv \quad (v \in k \oplus a),\\
(2.4)\qquad &\sum_{i<j} (\mathrm{ad}(y_{ij}))^2(v) = (2n-4)v \quad (v \in h),\\
&(\mathrm{ad}(y_{ij}))^2(x_{ij}) = 4x_{ij}.
\end{aligned}$$

The Killing form of any Lie algebra is defined by

$$B(x, y) = \operatorname{tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)).$$

In the case of $sl(n, \mathbf{R})$, this symmetric bilinear form is given by (see [5])

$$B(v, w) = 2n \operatorname{tr}(vw) \quad (v, w \in sl(n, \mathbf{R})). \tag{2.5}$$

3. Construction of metrics with negative Ricci curvature

We denote by $\langle\,,\rangle$ the canonical inner product of $sl(n, \mathbf{R})$ defined by

$$\langle v, w\rangle = \tfrac{1}{2} \operatorname{tr}(vw^t),$$

where w^t is the transpose matrix of w. With respect to this inner product, the vector space decomposition

$$sl(n, \mathbf{R}) = k \oplus h \oplus a$$

is orthogonal, and $\{x_{ij}\}$, $\{y_{ij}\}$ are orthonormal basis of k and h, respectively. Moreover, the adjoint of the linear transformation $\operatorname{ad}(v)$ coincides with $\operatorname{ad}(v^t)$, by properties of the trace. Thus the transformation $\operatorname{ad}(v)$ is skew-adjoint if v belongs to k, and self adjoint if v belongs to $a \oplus h$.

The inner product $\langle\,,\rangle$ induces a left invariant metric on $SL(n, \mathbf{R})$. As a particular case of next lemma, one obtains that, with respect to this metric, the Ricci curvature of a unit vector in k is equal to n, and in $a \oplus h$ is equal to $-3n$.

Now let us consider in $sl(n, \mathbf{R})$ the *modified inner product* $\langle \sigma(v), w\rangle$, where σ is the positive definite transformation which multiplies vectors of k, h and a by positive numbers ρ, β and, α respectively.

Left translation of it induces a *new* left invariant metric on $SL(n, \mathbf{R})$. As we will see later, the Ricci curvature with respect to this new metric is strictly negative, if $n \geqslant 3$, for many choices of ρ, β and α.

Given a left invariant metric, the self adjoint Ricci transformation, denoted by $\hat{r}$, may be expressed as (see [1])

$$\hat{r} = -\tfrac{1}{2} \sum_m \operatorname{ad}(v_m)^* \circ \operatorname{ad}(v_m) + \tfrac{1}{4} \sum_m \operatorname{ad}(v_m) \circ \operatorname{ad}(v_m)^* - \tfrac{1}{2}\hat{B},$$

where $\{v_m\}$ is any orthonormal basis of $sl(n, \mathbf{R})$, $\operatorname{ad}(v_m)^*$ is the adjoint tranformation of $\operatorname{ad}(v_m)$, and $\hat{B}$ is the self adjoint transformation associated to the Killing form B.

Let $\mathfrak{B}$ be the orthonormal basis of $sl(n, \mathbf{R})$, with respect to $\langle\,,\rangle$, consisting of x_{ij}, y_{ij} and u_k, where u_k, $1 \leqslant k \leqslant n-1$, is an orthonormal basis of a, and x_{ij}, y_{ij} are defined in §2.

Lemma. *With respect to the new metric of $SL(n,\mathbf{R})$, the orthonormal basis $x_{ij}/\sqrt{\rho}$, $y_{ij}/\sqrt{\beta}$, $u_k/\sqrt{\alpha}$ diagonalizes $\hat{r}$, the principal Ricci curvatures being given by*

$$r\left(\frac{x_{kl}}{\sqrt{\rho}}\right) = 2\frac{\rho^2-(\beta-\alpha)^2}{\rho\alpha\beta} + \frac{(n-2)\rho}{2\beta^2} + \frac{(n-2)}{2\rho},$$

$$r\left(\frac{y_{kl}}{\sqrt{\beta}}\right) = 2\frac{\beta^2-(\rho+\alpha)^2}{\rho\alpha\beta} - \frac{(n-2)\rho}{\beta^2} - \frac{2(n-2)}{\beta},$$

$$r\left(\frac{u_k}{\sqrt{\alpha}}\right) = n\frac{\alpha^2-(\rho+\beta)^2}{\rho\alpha\beta}.$$

Proof. It follows from (2.5) that $\langle \sigma\hat{B}v, w\rangle = 4n\langle v^t, w\rangle$, hence

$$\hat{B}(v) = -\frac{4n}{\rho}v \quad (v \in k),$$

$$\hat{B}(v) = \frac{4n}{\beta}v \quad (v \in h),$$

$$\hat{B}(v) = \frac{4n}{\alpha}v \quad (v \in a).$$

We recall from linear algebra that $\mathrm{ad}(v)^* = \sigma^{-1}\mathrm{ad}(v^t)\sigma$. Applying (2.1), (2.2) we easily see that the transformation $\mathrm{ad}(v)^*\mathrm{ad}(v)$, as well as $\mathrm{ad}(v)\mathrm{ad}(v)^*$, $v \in \mathfrak{B}$, restricted to direct summands of $sl(n,\mathbf{R})$ is a constant multiple of $(\mathrm{ad}(v))^2$. We tabulate the constants below. In the table, $\mathbf{R}x_{ij}{}^{\perp}$ and $\mathbf{R}y_{ij}{}^{\perp}$ denote the orthogonal complements of $\mathbf{R}x_{ij}$ and $\mathbf{R}y_{ij}$ in k and h, respectively.

<table>
<tr><th rowspan="2"></th><th colspan="2">k</th><th colspan="2">h</th><th rowspan="2">a</th></tr>
<tr><th>Rx_{ij}</th><th>$Rx_{ij}{}^{\perp}$</th><th>Ry_{ij}</th><th>$Ry_{ij}{}^{\perp}$</th></tr>
<tr><td>$\mathrm{ad}(x_{ij})^*\mathrm{ad}(x_{ij})$</td><td colspan="2">$-1$</td><td>$-\alpha/\beta$</td><td>$-1$</td><td>$-\beta/\alpha$</td></tr>
<tr><td>$\mathrm{ad}(x_{ij})\mathrm{ad}(x_{ij})^*$</td><td colspan="2">$-1$</td><td>$-\beta/\alpha$</td><td>$-1$</td><td>$-\alpha/\beta$</td></tr>
<tr><td>$\mathrm{ad}(y_{ij})^*\mathrm{ad}(y_{ij})$</td><td>$\alpha/\rho$</td><td>$\beta/\rho$</td><td colspan="2">$\rho/\beta$</td><td>$\rho/\alpha$</td></tr>
<tr><td>$\mathrm{ad}(y_{ij})\mathrm{ad}(y_{ij})^*$</td><td>$\rho/\alpha$</td><td>$\rho/\beta$</td><td colspan="2">$\beta/\rho$</td><td>$\alpha/\rho$</td></tr>
<tr><td>$\mathrm{ad}(u_k)^*\mathrm{ad}(u_k)$</td><td colspan="2">$\beta/\rho$</td><td colspan="2">$\rho/\beta$</td><td>$0$</td></tr>
<tr><td>$\mathrm{ad}(u_k)\mathrm{ad}(u_k)^*$</td><td colspan="2">$\rho/\beta$</td><td colspan="2">$\beta/\rho$</td><td>$0$</td></tr>
</table>

Since $x_{ij}/\sqrt{\rho}$, $y_{ij}/\sqrt{\beta}$, $u_k/\sqrt{\alpha}$, $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n-1$, form an orthonormal basis of $sl(n, \mathbf{R})$ with respect to the new metric, we have that

$$\begin{aligned}\hat{r}(x_{kl}) = \Bigg\{&\left[\frac{1}{4}\frac{1}{\rho}\sum\left(\mathrm{ad}(x_{ij})\right)^2\right] + \left[\left(-\frac{\alpha}{2\rho}+\frac{\rho}{4\alpha}\right)\frac{1}{\beta}\left(\mathrm{ad}(y_{kl})\right)^2\right] \\ &+\left[\left(\frac{-\beta}{2\rho}+\frac{\rho}{4\beta}\right)\frac{1}{\beta}\sum_{(i,j)\neq(k,l)}\left(\mathrm{ad}(y_{ij})\right)^2\right] \\ &+\left[\left(\frac{-\beta}{2\rho}+\frac{\rho}{4\beta}\right)\frac{1}{\alpha}\sum\left(\mathrm{ad}(u_k)\right)^2\right] - \frac{1}{2}\hat{B}\Bigg\}(x_{kl}).\end{aligned}$$

The property

$$\sum_k \left(\mathrm{ad}(u_k)\right)^2(v) = 4v \quad (v \in k \oplus h) \tag{3.1}$$

is also verified. Indeed,

$$\begin{aligned}&\left\langle\left(\sum\left(\mathrm{ad}(u_k)\right)^2 - \sum\left(\mathrm{ad}(x_{ij})\right)^2 + \sum\left(\mathrm{ad}(y_{ij})\right)^2\right)\cdot v, w\right\rangle \\ &\qquad = \mathrm{tr}\left(\mathrm{ad}(v^t)\mathrm{ad}(w)\right) = 4n\langle v, w\rangle,\end{aligned}$$

for all v and w in $sl(n, \mathbf{R})$, so (2.3), (2.4) imply (3.1).

Using (2.3), (2.4) once more and summing terms in the expression of $\hat{r}(x_{kl})$, we obtain that all $x_{kl}/\sqrt{\rho}$ are eigenvectors of constant eigenvalue

$$r\left(\frac{x_{kl}}{\sqrt{\rho}}\right) = \left\langle \hat{r}\left(\frac{x_{kl}}{\sqrt{\rho}}\right), \sigma\left(\frac{x_{kl}}{\sqrt{\rho}}\right)\right\rangle = 2\frac{\rho^2-(\beta-\alpha)^2}{\rho\alpha\beta} + \frac{(n-2)\rho}{2\beta^2} + \frac{(n-2)}{2\rho}.$$

Analogous computation shows that all $y_{kl}/\sqrt{\beta}$ are eigenvectors of $\hat{r}$, the eigenvalue as in lemma statement.

Finally, for an arbitrary u in a,

$$\begin{aligned}\hat{r}(u) &= \Bigg\{\left[\left(\frac{\beta}{2\alpha}-\frac{\alpha}{4\beta}\right)\frac{1}{\rho}\sum\left(\mathrm{ad}(x_{ij})\right)^2\right] \\ &\quad+\left[\left(-\frac{\rho}{2\alpha}+\frac{\alpha}{4\rho}\right)\frac{1}{\beta}\sum\left(\mathrm{ad}(y_{ij})\right)^2\right] - \frac{1}{2}\hat{B}\Bigg\}(u) \\ &= n\frac{\alpha^2-(\rho+\beta)^2}{\rho\alpha\beta}u.\end{aligned}$$

Theorem. *If $n \geqslant 3$, then the Ricci curvature of $SL(n, \mathbf{R})$ is strictly negative for many choices of ρ, α and β.*

Proof. The signs of the principal Ricci curvatures do not change if we multiply σ by a positive number, so we may assume $\beta = 1$. Therefore the Ricci curvature is strictly negative if and only if

$$f_1(\rho, \alpha) = 4\left[\rho^2 - (1-\alpha)^2\right] + (n-2)\alpha(1+\rho^2) < 0,$$
$$f_2(\rho, \alpha) = 2\left[1 - (\rho+\alpha)^2\right] - (n-2)\rho\alpha(\rho+2) < 0,$$
$$f_3(\rho, \alpha) = \alpha^2 - (\rho+1)^2 < 0.$$

Let us take the linear approximations of f_1 and f_2 around $(1,0)$,

$$L_1(\rho - 1, \alpha) = 8(\rho - 1) + (4 + 2n)\alpha,$$
$$L_2(\rho - 1, \alpha) = -4(\rho - 1) + (2 - 3n)\alpha.$$

The set of points (ρ, α), $\rho > 0$, $\alpha > 0$, where both linear functions become negative is a nonempty open triangle Δ of $\mathbf{R}^2$. Indeed, $L_i(\rho - 1, \alpha) < 0$, $i = 1, 2$, is equivalent to

$$\frac{2+n}{4}\alpha < 1 - \rho < \frac{3n-2}{4}\alpha,$$

and $\frac{1}{4}(3n-2) - \frac{1}{4}(2+n) \geqslant 1/2$, for $n \geqslant 3$. Using standard analysis we find a nonempty open subset of Δ where $f_i(\rho, \alpha) < 0$, $i = 1, 2$. Clearly $f_3(\rho, \alpha)$ is also negative there since $0 < \alpha < 1 - \rho$ on Δ.

Remark. For $n = 2$, there are no solutions of $f_i(\rho, \alpha) < 0$, $i = 1, 2, 3$ (compare [8], p. 308).

Corollary. *There exist compact riemannian manifolds of strictly negative Ricci curvature which admit a free differentiable $SO(n)$ action, $n \geqslant 3$. In particular, any connected compact Lie group acts freely on most of these manifolds.*

Proof. Let Γ be a torsion free, discrete subgroup of $SL(n, \mathbf{R})$ such that $\Gamma \setminus SL(n, \mathbf{R})$ is compact (see [4] for the existence of such subgroups). The right action of $SO(n)$ on $\Gamma \setminus SL(n, \mathbf{R})$ is free, since Γ is torsion free.

The left invariant metrics of negative Ricci curvature on $SL(n, \mathbf{R})$, $n \geqslant 3$, as constructed in the theorem, induce riemannian metrics on $\Gamma \setminus SL(n, \mathbf{R})$ with the same curvature properties, since Γ is discrete.

To prove the second assertion we recall that a connected compact Lie group may be imbedded as a subgroup of $SO(n, \mathbf{R})$, for n large.

Remark. It is proved in [8] that $SL(n, \mathbf{R})$, $n \geqslant 3$, admits a left invariant metric of strictly positive scalar curvature. Hence the manifolds $\Gamma \setminus SL(n, \mathbf{R})$, $n \geqslant 3$, provide examples of compact manifolds admitting both a metric of positive scalar curvature and a metric of negative Ricci curvature (compare [10,

p. 242]). The existence of metrics of nonnegative scalar curvature on such manifolds also follows from a theorem of Lawson-Yau [7], since $SO(n)$, $n \geqslant 3$, acts freely on them.

Added in proof. The construction of left invariant metrics of negative Ricci curvature on $SL(n, \mathbf{R})$, $n \geqslant 3$, has been recently extended to a larger class of simple Lie groups. Another work with precise statements and proof is under preparation.

References

[1] D. V. Alekseevskii, *Homogeneous Riemannian spaces of negative curvature*, Math. USSR-Sb. **25** (1975) 87–109.

[2] R. Azencott & E. Wilson, *Homogeneous manifolds with negative curvature. Part* I, Trans Amer. Math. Soc. **215** (1976) 323–362.

[3] L. B. Bergery, *Sur la courbure des métriques riemannienes invariantes des groupes de Lie et des espaces homogènes*, Ann. Sci. École Norm. Sup. (4) **11** (1978) 543–576.

[4] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963) 111–122.

[5] M. Goto & F. D. Grosshans, *Semisimple Lie algebras*, Marcell Dekker, New York, 1978.

[6] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974) 23–24.

[7] H. B. Lawson & S. T. Yau, *Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974) 232–244.

[8] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976) 293–329.

[9] N. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. Math. **96** (1972) 277–295.

[10] S. T. Yau, *The role of partial differential equations in differential geometry*, Proc. Intern. Congress Math. Helsinki, 1978, 235–250.

[11] ______, *Problem section*, Institute for Advanced Study, Princeton, NJ, 1980.

UNIVERSIDADE FEDERAL DE PERNAMBUCO
RECIFE, BRAZIL