PACIFIC JOURNAL OF MATHEMATICS
Vol, 59, No. 2, 1975

METRIZATION OF SPACES WITH COUNTABLE LARGE BASIS DIMENSION

GARY GRUENHAGE AND PHILLIP ZENOR

With the following results, we generalize known metrization theorems for spaces with large basis dimension 0 i.e., non-archimedean spaces) to the higher dimensions: *Theorem.* **If X is a normal Σ-space with countable large basis dimension, then X is metrizable.** *Theorem.* **If X is a normal $w\Delta$-space with countable large basis dimension, then X is metrizable.**

I. Introduction. A collection Γ of subsets of a set X is said to have *rank* 1 if whenever g_1 and g_2 are in Γ with $g_1 \cap g_2 \neq \emptyset$ then $g_1 \subset g_2$ or $g_2 \subset g_1$. According to P. J. Nyikos [13], a topological space X has *large basis dimension* $\leqq n$ (denoted $\operatorname{Bad} X \leqq n$) if X has a basis which is the union of $n+1$ rank 1 collections of open sets. X has *countable large basis dimension* ($\operatorname{Bad} X \leqq \aleph_0$) if X has a basis which is the union of a countable number of rank 1 collections such that each point of X has a basis belonging to one of the collections (a property which is automatically true in the finite case). $\operatorname{Bad} X$ coincides with $\operatorname{Ind} X$ and $\dim X$ for metric spaces.

Spaces having large basis dimension 0 are usually called *non-archimedean* spaces. Theorems of Nyikos [11] and A. V. Archangelskii [3] show that a non-archimedean space is metrizable if and only if it is a Σ-space or a $w\Delta$-space. In this paper we show that these results are valid, under mild assumptions, for the higher dimensions. Our results also improve a result of G. Gruenhage [6], who showed that compact spaces having finite large basis dimension are metrizable.

II. Main results. According to Nyikos [11], a *tree of open sets* is a collection Γ of open sets such that if $g \in \Gamma$, then the set $\{g' \in \Gamma \mid g' \supset g\}$ is well-ordered by reverse inclusion; that is, $g \leqq g'$ if and only if $g \supset g'$. Nyikos shows that the rank 1 collections for spaces with $\operatorname{Bad} X \leqq \aleph_0$ can be considered as rank 1 trees of open sets. The following fact will be used in our proofs:

LEMMA 1. *Let T be a rank 1 tree of open subsets of a regular space X which contains a basis at each point of a subset X' of X. Then if $\mathscr{U}$ is a cover of X' by open subsets of X, there exists a subset T' of T such that*

(i) *T' is a cover of X';*

(ii) *the elements of T' are pairwise disjoint;*

(iii) *$t \in T'$ implies that either t is degenerate or $\bar{t}$ is a proper subset of some member of $\mathscr{U}$.*

Proof. Put t in T' if and only if (a) either t is degenerate or there is a member U of $\mathscr{U}$ such that $\bar{t}$ is a proper subset of U and (b) there is no predecessor of t in T whose closure is a proper subset of some element of $\mathscr{U}$. Since T contains a basis at each point of X' and since the predecessors of a given $t \in T$ are well-ordered, it is easy to see that T' covers X'. Further, since T is a tree, the members of T' are mutually exclusive.

Nyikos calls a space *basically screenable* if it has a basis which is the union of countably many rank 1 trees of open sets. Every space X with Bad $X \leqq \aleph_0$ is basically screenable. Basically screenable spaces are, of course, screenable; that is, every open cover has a σ-pairwise disjoint open refinement. While the following result is known, for the sake of completeness, we include its easy proof:

LEMMA 2. *A screenable countably compact space X is compact* |2|.

Proof. Let $\mathscr{U}$ be an open cover of X and let $\mathscr{V} = \cup \{ \mathscr{V}_n | n = 1, 2, \cdots \}$ be an open refinement of U covering X such that, for each i, the members of $\mathscr{V}_i$ are mutually exclusive. The set $\{ V_n = \bigcup \mathscr{V}_n | n = 1, 2, \cdots \}$ is a countable open cover of X; hence, there exists a finite subcover $\{ V_{n_1}, V_{n_2}, \cdots, V_{n_k} \}$. Then $\mathscr{V}_{n_1} \cup \mathscr{V}_{n_2} \cup \cdots \cup \mathscr{V}_{n_k}$ is a point-finite refinement of $\mathscr{U}$. Thus, X is metacompact and it is well-known that a metacompact countably compact space is compact.

According to C. R. Borges [4], a space X is a *$w\Delta$-space* if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that whenever $x \in X$ and $x_n \in \mathrm{St}(x, \mathscr{G}_n)$ for each n, then $\{x_1, x_2, \cdots\}$ has a cluster point.

THEOREM 1. *If X is a regular $w\Delta$-space with countable large basis dimension, then X has a point countable basis.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a sequence of open covers of X satisfying the properties given in the definition of a $w\Delta$-space. Let $\mathscr{B}_1, \mathscr{B}_2, \cdots$ and $X_1, X_2, \cdots$ be sequences such that $X = \bigcup \{X_i | i = 1, 2, \cdots\}$ and, for each i, $\mathscr{B}_i$ is a rank 1 tree of open sets containing a basis at each point of X_i.

For each $i < \omega_0$ and $\alpha < \omega_1$, we construct a collection $\mathscr{B}(i, \alpha)$ as follows: let $\mathscr{B}(i, 1)$ be a collection of mutually exclusive members of $\mathscr{B}_i$ that refines $\mathscr{G}_1$ and covers X_i.

Suppose $\mathscr{B}(i, \beta)$ has been defined for $\beta < \alpha$. If α is not a limit ordinal, applying Lemma 1, let $\mathscr{B}(i, \alpha)$ be a collection of mutually exclusive members of $\mathscr{B}_i$ such that

(i) if $j < \omega_0$, then $\mathscr{B}(i, j)$ refines $\mathscr{G}_j$;

(ii) $\mathscr{B}(i, \alpha)$ covers $(\cup \mathscr{B}(i, \alpha - 1)) \cap X_i$;

and (iii) $g \in \mathscr{B}(i, \alpha)$ implies $\bar{g}$ is a proper subset of some member of $\mathscr{B}(i, \alpha - 1)$, or g is degenerate. If α is a limit ordinal, for each $x \in X_i$, let $B(\alpha, x) = \operatorname{Int}(\bigcap_{\beta<\alpha}\{g \in \mathscr{B}(i, \beta) | x \in g\})$. Note that if x and y are in X_i, then either $B(\alpha, x) = B(\alpha, y)$ or $B(\alpha, x) \cap B(\alpha, y) = \emptyset$. Let $\mathscr{B}(i, \alpha) = \{B(\alpha, x) | x \in X_i\}$.

Let $\mathscr{B}_i^* = \bigcup_{\alpha<\omega_1} \mathscr{B}(i, \alpha)$. We will show that $\mathscr{B}_i^*$ is a point countable collection forming a basis for X_i in X.

We will say that g is a *chain in* $\mathscr{B}_i^*$ if g is a function from an initial segment of ω_1 into $\mathscr{B}_i^*$ so that (1) $g(\alpha) \in \mathscr{B}(i, \alpha)$ and (2) if $\beta < \alpha$, then $g(\beta) \supset g(\alpha)$. Note that by our construction, if $\beta < \alpha$, then $g(\beta) \supset \overline{g(\alpha)}$. Furthermore, if $x \in X_i$, then there is exactly one maximal chain, say g, such that $g(\alpha)$ contains x for every α in the domain of g.

Claim 1. The domain of each maximal chain in $\mathscr{B}_i^*$ is countable (and so, $\mathscr{B}_i^*$ is point countable in X).

Proof of Claim 1. Suppose the contrary; i.e., there is a chain, say g, of length ω_1.

Note that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact. To prove this, we will only show that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is countably compact; that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact will then follow from Lemma 2. To this end, let N denote a countable subset of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$. There is an α so that $g(\alpha)$ does not meet N. In particular then, no point of $\overline{g(\alpha + 1)}$ is a limit point of N. Because of property (i), it must be the case that N has a limit point in $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$; and so, $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact. But, $\{\overline{g(\omega_0 + 1)} - \overline{g(\alpha)} | \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ with no finite subcover, which is a contradiction from which Claim 1 follows.

Claim 2: $\mathscr{B}_1^*$ is a basis for X_i in X; in particular, if $x \in X_i$ and g is the maximal chain in $\mathscr{B}_i^*$ centered at x, then $\{g(\alpha) | \alpha$ is in the domain of $g\}$ is a local basis for x in X.

Proof of Claim 2. Suppose otherwise. Then there is a point x of X_i so that the maximal chain, g, centered at x does not yield a basis at x in X; i.e., $\{g(\alpha) | \alpha \in \text{domain of } g\}$ is not a local basis for x in X. Since the domain of g is countable, there is a first $\alpha_0 < \omega_1$ not in the domain g. There is a member B of $\mathscr{B}_i$ so that if $\alpha < \alpha_0$, then $g(\alpha)$ is not a subset of B but this means that B is a subset of each $g(\alpha)$. Then x is in the interior of $\bigcap_{\alpha<\alpha_0} g(\alpha)$. Thus, by our construction of $\mathscr{B}(i, \alpha_0)$, there is a member of $\mathscr{B}(i, \alpha_0)$ that contains x. This contradicts the maximality of g and it follows that $\{g(\alpha) \mid \alpha$ is in the domain of $g\}$ is a local basis for x in X.

We now have that $\bigcup_{i<\omega_0} \mathscr{B}_i^*$ is a point countable basis for X.

If $\mathscr{H}$ is a cover of the space X and if $x \in X$, then $C(x, H)$ will denote the set $\cap \{H \in \mathscr{H} \mid x \in H\}$. According to K. Nagami [9], the space X is a Σ-space if there is a sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$ of locally finite closed covers of such that if $x_0, x_1, x_2, , \cdots$ is a sequence with $x_i \in C(x_0, \mathscr{F}_i)$ for each $0 < i < \omega_0$, then $\{x_i\}$ has a cluster point. The sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$ is called a spectral Σ-sequence for X.

We will, without loss of generality, assume that each $\mathscr{F}_i$ is closed under intersections and, for each i, $\mathscr{F}_{i+1}$ refines $\mathscr{F}_i$.

LEMMA 3. *If X is a space with countable large basis dimension such that each uncountable subset of X has a limit point, then X is Lindelof.*

Proof. Since X has countable large basis dimension, X is screenable. G. Aquaro [1] has proved that every meta-Lindelof (and thus every screenable) space in which every uncountable set has a limit point is Lindelof.

THEOREM 2. *If X is a regular Σ-space with countable large basis dimension then X has a point countable basis.*

Proof. Let $\mathscr{F}_1, \mathscr{F}_2, \cdots$ be a sequence of locally finite closed coverings of X given in the definition of a Σ-space. For each n, let $\mathscr{G}_n$ be an open cover of X such that each member of $\mathscr{G}_n$ intersects only finitely many members of $\mathscr{F}_n$. Let $\mathscr{B}_1, \mathscr{B}_2, \cdots$ and $X_1, X_2, \cdots$ be sequences such that $X = \bigcup_{i<\omega_0} X_i$ and $\mathscr{B}_i$ is a rank 1 tree of open sets which contains a basis for each point of X_i.

Define $\mathscr{B}(i, \alpha)$, $i < \omega_0$, $\alpha < \omega_1$, exactly as in the proof of Theorem 1. Let $\mathscr{B}_i^* = \bigcup_{\alpha<\omega_1} \mathscr{B}(i, \alpha)$ and define chain in $\mathscr{B}_i^*$ as in the proof to Theorem 1.

Claim 1. Every chain in $\mathscr{B}_i^*$ is countable.

Proof of Claim 1. Suppose otherwise; i.e., suppose that g is a chain in $\mathscr{B}_i^*$ with length ω_1. Let $K = \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$. Every uncountable of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$ for suppose otherwise; that is, suppose that H is an uncountable subset of $\overline{g(\omega_0)} - K$ with no limit point in $\overline{g(\omega_0)} - K$.

Suppose that there is a point, h, of H such that, for each n,

$C(h, \mathscr{F}_n)$ intersects infinitely many points of H. Then there is a countable subset N of H with a limit point. Since N is countable, there is an $\alpha < \omega_1$ so that $g(\alpha)$ does not intersect N. It follows that no point of K is a limit point of N. Hence, no point of K is a limit point of N; and so, H has a limit point in $\overline{g(\omega_0)} - K$. This is a contradiction from which it follows that, for each h in H, there is an integer $n(h)$ such that $C(h, n(h))$ intersects only finitely many members of H. Thus, there is an N and an uncountable subset H^* of H so that if $h \in H^*$, then $n(h) = N$ and $\{C(h, F_N) | h \in H^*\}$ is an infinite subcollection of $\mathscr{F}_N$, each member of which intersects $g(N)$. But, $g(N)$ is in $\mathscr{B}(i, N)$ which contradicts the fact that $\mathscr{B}(i, N)$ refines $\mathscr{G}_N$. It follows that each uncountable subset of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$; and so, by Lemma 3, $\overline{g(\omega_0)} - K$ is Lindelof. But $\{\overline{g(\omega_0)} - \overline{g(\alpha)} | \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0)} - K$ with no countable subcover which is a contradiction from which Claim 1 follows.

That $\mathscr{B}_i^*$ contains a basis at each point of X_i follows exactly as in the proof of Theorem 1. Thus Theorem 2 is proved.

THEOREM 3. *If X is a normal Σ-space with countable large basis dimension, then X is metrizable.*

Proof. R. E. Hodel has proved that every Σ-space is a β-space [8], and that every β-space is countably metacompact [7]. A screenable countably metacompact space is metacompact. Nagami [10] has shown that a normal screenable metacompact space is paracompact. But a paracompact Σ-space with a point-countable base is metrizable [9].

THEOREM 4. *If X is a normal $w\Delta$-space with countable large basis dimension, then X is metrizable.*

Proof. As above, X is normal, screenable, and metacompact (since every $w\Delta$-space is a β-space), hence paracompact. But a papacompact $w\Delta$-space is an M-space, hence a Σ-space. Thus X is metrizable.

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Received January 22, 1975 and in revised form June 3, 1975.

AUBURN UNIVERSITY