MICROBUNDLES AND S-DUALITY

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This paper contains a generalization of the S-duality theorem proved independently by Atiyah, Bott and A. Shapiro, [1]. The S-duality theorem may be stated as follows: *Let X be a compact differentiable manifold and let ξ_1, ξ_2 be vector bundles over X such that $\tau(X)\oplus\xi_1\oplus\xi_2$ is J-equivalent to a trivial vector bundle. Then the Thom spaces of ξ_1 and ξ_2 are S-duals in the sense of Spanier and Whitehead.* Our generalization goes in two directions. First we drop the differentiability condition on X and allow ξ_1, ξ_2 to be microbundles ($\tau(X)$ now meaning the tangent microbundle). Secondly we drop the compactness condition on X and compensate this by working with relativized Thom spaces over compact pairs. More precisely we show that if (A, B) is any compact pair in X, sufficiently nicely embedded, and (U, V) is a sufficiently nice open neighbourhood of (A, B), then

$$\tilde{H}^*(T_{\xi_1}(A, B)\wedge T_{\xi_2}(X-V, X-U))$$

has a spherical class with which slant product defines an (ungraded) isomorphism

$$\tilde{H}_*(T_{\xi_2}(X-V, X-U))\approx\tilde{H}^*(T_{\xi_1}(A, B)).$$

In fact this isomorphism is nothing but the Alexander–Spanier duality in the base manifold lifted by Thom isomorphisms, i.e. we have a commutative diagram

$$\begin{array}{ccc} \tilde{H}_*(T_{\xi_2}(X-V, X-U)) & \approx & \tilde{H}^*(T_{\xi_1}(A, B)) \\ \approx\downarrow & & \downarrow\approx \\ H_*(X-V, X-U) & \approx & H^*(A, B) \end{array}$$

(with the proper coefficients). This gives the generalized S-duality theorem.

An expression for the Thom class of a composite microbundle is important in the proof of the generalized duality theorem. In section 2 we deduce this expression from a cup product formula of Thom classes. Another application of the formula is given to the Gysin sequences of a Whitney sum of microbundles.

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1. Preliminaries

We keep the notation and terminology in [7] with one exception; all microbundles are assumed to have *closed* zero sections. This implies no serious loss of generality since any microbundle is isomorphic to an $\mathbf{R}^q$-bundle, and in an $\mathbf{R}^q$-bundle the zero section is automatically closed. (Notice that bundles and microbundles by definition always have trivializing partitions of unity, [7], sec. 1. By [7], Lemma 1.1 they always admit locally finite closed trivializing covers of the base. For a bundle this implies immediately that the zero section is closed.) Otherwise our general sources of reference in topology will be [2] and [12]. In accordance with [2], "normal", "compact", "locally compact", "paracompact" implies "Hausdorff".

Given a topological space X a subset A is called a *nondegenerate subset* of X if it is closed and the inclusion $A \subset X$ is a cofibration. It will be convenient to have listed a few basic properties of nondegenerate subsets. Proofs of the following lemmas can be found in [6] or [13].

1.1. LEMMA. *Let A be a subset of a space X. A is nondegenerate if and only if there is a function $\phi: X \to I$ such that $A = \phi^{-1}\{0\}$ and a deformation $D: X \times I \to X$ relative to A such that $D(\phi^{-1}[0, 1) \times 1) \subset A$.*

1.2. LEMMA. *Let A be a nondegenerate subset of a space X and let $p: E \to B$ be a fibration. Let $G: X \times I \to B$ be a homotopy and $h: X \times 0 \cup A \times I \to E$ a partial lifting of G. There is a lifting $H: X \times I \to E$ of G which is an extension of h, and any two such are homotopic rel $X \times 0 \cup A \times I$.*

1.3. LEMMA. *Let A be a nondegenerate subset of a space X. Any local system on A is extendable to a local system on some neighbourhood of A, and any two extensions are isomorphic on some common subneighbourhood by an isomorphism which is the identity over A.*

If $\mathcal{S}$ is a local system on some neighbourhood of A, then $\varprojlim H(U; \mathcal{S}|U) \approx H(A; \mathcal{S}|A)$ and $\varinjlim H^*(U; \mathcal{S}|U) \approx H^*(A; \mathcal{S}|A)$, the limits taken over all sufficiently small neighbourhoods U. Similarly, if $\mathcal{S}$ is defined on all of X, then $\varprojlim H(X, U; \mathcal{S}) \approx H(X, A; \mathcal{S})$ and $\varinjlim H^*(X, U; \mathcal{S}) \approx H^*(X, A; \mathcal{S})$.

In this paper (co-)homology means *singular* (co-)homology, with coefficients in a fixed principal ideal domain R if not otherwise indicated. A *local system* on a space X means a local system of R-modules on X, i.e. a contravariant functor from the fundamental groupoid of X to the category of R-modules ([12] p. 46). In the sequel we shall briefly write $H(A; \mathcal{S})$, $H^*(U; \mathcal{S})$, ... for $H(A; \mathcal{S}|A)$, $H^*(U; \mathcal{S}|U)$, ...

From Lemma 1.3 follows easily that collapsing a nondegenerate subset of a space X induces an isomorphism $H(X, A) \approx \tilde{H}(X/A)$, cf. [6]. In fact the proofs hold under slightly more general conditions:

1.4. Lemma. *Let A be a subset of a space X. Assume there is a function $\phi: X \to I$ such that $A = \phi^{-1}\{0\}$ and a deformation $D: (X, A) \times I \to (X, A)$ such that $D(\phi^{-1}[0, 1) \times 1) \subset A$. Then the collapsing k: $(X, A) \to (X/A, *)$ induces an isomorphism k^*: $H(X, A) \approx \tilde{H}(X/A)$.*

A space X is an *absolute neighbourhood retract* if any map from a closed subset of a metrizable space into X can be extended to a neighbourhood of the subset.

1.5. Lemma. *Let A be a closed subset of a metrizable space X. Assume that X and A are absolute neighbourhood retracts. Then A is nondegenerate.*

Following are some examples of nondegenerate subsets.

1.6. By a theorem of O. Hanner a (topological) manifold is an absolute neighbourhood retract, [5]. Therefore a closed submanifold of a manifold is nondegenerate.

1.7. If (X, A) is a relative CW complex, then A is nondegenerate in X, [12] p. 402.

1.8. If $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ is an $\mathbf{R}^q$-bundle and ξ_∞ is the associated S^q-bundle ([7] p. 29), then $s_\infty X$ is nondegenerate in E_∞. In fact if $W \subset X$ is trivializing for ξ_∞ it is easy to construct a map $E_\infty | W \to I$ which is 0 exactly on $s_\infty W$ and 1 exactly on sW. Gluing together such maps by a partition of unity pulled up from the base give a map $\phi': E_\infty \to I$ which is 0 exactly on $s_\infty X$ and 1 exactly on sX. Finally $s_\infty X$ is a strong fibre deformation retract of $E_\infty - sX$ by a deformation D' ([7] thm. 3.6). From ϕ' and D' one easily construct ϕ and D as in Lemma 1.1.

Let $\mu: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-microbundle. We shall write E^0 for $E - sX$. Consider a bundle neighbourhood V of sX in E. There is a contravariant functor from the fundamental groupoid of X to the homotopy category of pairs which to $x \in X$ assigns $(V|x, V^0|x)$ and to a path class $[\omega]$ from x' to x'' assigns the homotopy class

$$h[\omega] \in [V|x', V^0|x'; V|x'', V^0|x'']$$

(cf. [12] p. 101). Define a local system $\mathcal{O} = \mathcal{O}(\mu)$ on X as follows. For $x \in X$ let $\mathcal{O}_x = H^q(E|x, E^0|x) \approx H^q(V|x, V^0|x)$, and for a path class $[\omega]$ from x' to x'' let $\mathcal{O}_{[\omega]}: \mathcal{O}_{x'} \to \mathcal{O}_{x''}$ be the map induced from $h[\omega]^{-1}$ composed with excision isomorphisms. One easily verifies that 0 is constant over each trivializing subset in X but in general not constant over X. Similarly there is a local system $O^* = O^*(\mu)$ with $O^*_x = H_q(E|x, E^0|x)$ and $O^*_{(\omega)}$ the (homology) map induced from $h[\omega]$ composed with excision isomorphisms. We call O and O^* the *orientation systems* of μ. They are functorial, e.g. $f^*O(\mu) = O(f^*\mu)$ for any map f into X, and invariant under isomorphisms of microbundles. Finally each is naturally isomorphic to the dual of the other, and their tensor products $O \otimes O^*$ and $O^* \otimes O$ are naturally isomorphic to the standard constant local system R on X. It is also possible (but more laborous) to define the orientation systems directly without using the result that microbundles contain bundles.

If X is a manifold, the orientation systems of the tangent microbundle $\tau(X)$ will be written $O(X)$ and $O^*(X)$. If $X \xrightarrow{s} E \xrightarrow{p} X$ is an R^q-microbundle over an n-manifold then E is a $q+n$-manifold near the zero sections. More precisely there is a bundle neighbourhood V of sX in E which is a $q+n$-manifold. Therefore we may consider the orientation systems of V. It follows that there is a local system on X which assigns to x the R-module $H^{q+n}(E, E-x)$ and to a path class $[\omega]$ the map $O_{[\omega]}(V)$ composed with excision isomorphisms. This local system is obviously independent of the choice of V and will be denoted $s^*O(E)$ (by a slight abuse of notation). There is a similarly defined local system $s^*O^*(E)$, and one is naturally dual to the other, and their tensor product is naturally isomorphic to R.

1.9. Lemma. *Let* μ: $X \xrightarrow{s} E \xrightarrow{p} X$ *be a microbundle over a manifold* X. *There is a canonical isomorphism of local systems*

$$s^*O(E) \approx O(\mu) \otimes O(X).$$

Proof. We may as well assume μ to be a bundle. Let U be a neighbourhood of x in X trivializing for μ, and let Φ: $(E|U, E^0|U) \approx U \times (\mathbf{R}^q, \mathbf{R}^q - 0)$ be a trivialization. Then we have a composition of isomorphisms

$$\begin{aligned} H^{n+q}(E, E-s(x)) &\approx H^{n+q}(E|U, E|U-s(x)) \overset{\Phi^*}{\approx} H^{n+q}(U \times \mathbf{R}^q, U \times \mathbf{R}^q - (x, 0)) \\ &\approx H^n(U, U-x) \otimes H^q(\mathbf{R}^q, \mathbf{R}^q - 0) \approx H^n(X, X-x) \otimes H^q(\mathbf{R}^q, \mathbf{R}^q - 0) \\ &\overset{\mathrm{id} \otimes \Phi_x^{*-1}}{\approx} H^n(X, X-x) \otimes H^q(E|x, E^0|x). \end{aligned}$$

The composite map is independent of Φ and determines an isomorphism of local systems $s^*O(E) \approx O(X) \otimes O(\mu)$. The claim then follows.

Note that because of the natural isomorphism $O(X) \otimes O^*(X) \approx R$ the correspondence of Lemma 1.9 also takes the form

$$s^*O(E) \otimes O^*(X) \approx O(\mu).$$

There are similar formulas for the dual local systems.

Let $\mathcal{S}$ and $\mathcal{T}$ be local systems on spaces X and Y respectively. The local system $pr_1^*\mathcal{S}\otimes pr_2^*\mathcal{T}$ on $X\times Y$ will be denoted $\mathcal{S}\times\mathcal{T}$. Then $(\mathcal{S}\times\mathcal{T})^*$ is naturally isomorphic to $\mathcal{S}^*\times\mathcal{T}^*$, where * denotes the dual local system, e.g. $\mathcal{S}^*=\mathrm{Hom}\,(\mathcal{S}, R)$. If μ and ν are microbundles on X and Y respectively, then $\mathcal{O}(\mu\times\nu)$ is naturally isomorphic to $\mathcal{O}(\mu)\times\mathcal{O}(\nu)$, and if $X=Y$, then $\mathcal{O}(\mu\oplus\nu)$ is naturally isomorphic to $\mathcal{O}(\mu)\otimes\mathcal{O}(\nu)$.

An acyclic model argument shows that there are natural equivalences $\Delta(X; \mathcal{S})\otimes\Delta(Y; \mathcal{T})\to\Delta(X\times Y; \mathcal{S}\times\mathcal{T})$ and that any two such are naturally chain homotopic. In particular there are induced natural chain maps $\Delta(X, A; \mathcal{S})\otimes\Delta(Y, B; \mathcal{T})\to ((X, A)\times(Y, B); \mathcal{S}\times\mathcal{T})$, unique up to natural homotopy. Let τ be such a map.

Since $\Delta^*((X, A)\times(Y, B); \mathcal{S}^*\times\mathcal{T}^*)\approx\mathrm{Hom}\,(\Delta((X, A)\times(Y, B); \mathcal{S}\times\mathcal{T}), R)$, there is a slant product pairing

$$\Delta^*((X, A)\times(Y, B); \mathcal{S}^*\times\mathcal{T}^*)\otimes\Delta(Y, B; \mathcal{T})\to\Delta^*(X, A; \mathcal{S}^*)$$

which maps $c'\otimes c$ to $\tau^*(c')/c$ (cf. [12] p. 286). This operation is compatible with boundary- and coboundary operators, and defines a slant product

$$H^n((X, A)\times(Y, B); \mathcal{S}^*\times\mathcal{T}^*)\otimes H_i(Y, B; \mathcal{T})\to H^{n-i}(X, A; \mathcal{S}^*), \quad i, n\in\mathbf{Z}.$$

We write u/z for the image of $u\otimes z$ in $H^{n-i}(X, A; \mathcal{S}^*)$. The standard properties of slant products with constant coefficients carry automatically over to the operation defined above under reasonable excisiveness conditions, cf. [12] p. 287, properties 1–6. In sections 2 and 4 we will use properties of slant products and cap- and cup products without further reference.

1.10. LEMMA. *Let X be an n-manifold and $\mathcal{S}$ a local system on X which is free and finitely generated. For any pair (A, B) of compact nondegenerate subsets of X there is an isomorphism.*

$$H_i(X-B, X-A; \mathcal{S})\approx H^{n-i}(A, B; \mathcal{O}^*(X)\otimes\mathcal{S}), \quad i\in Z,$$

given by slant product with the Thom class of X. The isomorphism is functorial with respect to inclusions $(A', B')\subset(A, B)$.

Proof. The Thom class U_X of X lies in $H^n(X\times X, X\times X-\Delta X; \mathcal{O}^*(X)\times R)$. (Its definition is given in section 2.) Assume $\mathcal{S}=R$. Then the slant product map

$$z\mapsto U'_X/z, \quad U'_X=U_X\,|\,(A, B)\times(X-B, X-A)$$

is well defined in accordance with the outline above. The argument in [12] Ch. 6 sec. 2 then carries over almost verbally to show that for $B=\emptyset$ this map gives an isomorphism

$$H_i(X, X-A; R) \approx \bar{H}^{n-i}(A; O^*(X)),$$

where $\bar{H}^{n-i}A; O^*(X)) = \varinjlim H^{n-i}(U; O^*(X))$, U running through a cofinal collection of compact neighbourhoods of A. Since A is non-degenerate, however, $\bar{H}^*(A; O^*(X))$ is naturally isomorphic to $H^*(A; O^*(X))$ (Lemma 1.3). Naturality and the five lemma extends this result to pairs (A, B) where B is nonempty.

If S is not constant slant product with U'_X is not defined for elements in

$$H(X-B, X-A; S).$$

We show how to extend the definition in the case where S is free with one generator, which suffices for our applications. In this case $S \otimes S^*$ is naturally isomorphic to R which gives the isomorphism $O^*(X) \times R \approx O^*(X) \times (S \otimes S^*)$. On a sufficiently small neighbourhood of $\Delta X \subset X \times X$, $O^*(X) \times (S \otimes S^*)$ is canonically isomorphic to $(O^*(X) \otimes S) \times S^*$, and this isomorphism is natural with respect to inclusion maps. Therefore excision gives a canonical isomorphism

$$H^*(X \times X, X \times X - \Delta X; O^*(X) \otimes R) \approx H^*(X \times X, X \times X - \Delta X; (O^*(X) \otimes S) \times S^*)$$

under which U_X corresponds to a class $\bar{U}_X$. Define z/U'_X to equal $z/\bar{U}'_X$ for

$$z \in H(X-B, X-A; S).$$

The proof of the duality theorem now goes as before.

Let (Y_1, Y'_1) and (Y_2, Y'_2) be open pairs of a topological space and let S be a local system on $Y_1 \cup Y_2$. There is a Mayer–Vietoris exact sequence connecting the groups $H(Y_1 \cap Y_2, Y'_1 \cap Y_2; S)$, $H(Y_1, Y'_1; S)$, $H(Y_2, Y'_2; S)$ and $H(Y_1 \cup Y_2, Y'_1 \cup Y'_2; S)$ and a similar Mayer–Vietoris cohomology sequence.

1.11. Lemma. *Let $\mu: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-microbundle. For any local system S on X*

$$H_i(E, E^0; p^*S) = H^i(E, E^0; p^*S^*) = 0$$

for $i < q$.

Proof. If μ is the standard trivial $\mathbf{R}^q$-bundle, the result follows from the Künneth theorem. Thus it is true for $\mu | W$ where W is a trivializing open set in X. Applying the Mayer–Vietoris homology sequence to the pairs $(E | W_1, E^0 | W_1)$ and $(E | W_2, E^0 | W_2)$ where W_1, W_2 are open trivializing sets, extends the result to microbundles $\mu | W_1 \cup W_2$. By induction it must be true for $\mu | V$ where V is any finite union of trivializing open sets. Since $H(E, E^0; p^*S) \approx \lim H(E | V, E^0 | V; p^*S)$, the homology part follows. To get the cohomology part assume S to be free. Since $\Delta^*(E, E^0; p^*S^*) \approx \operatorname{Hom}(\Delta(E, E^0; p^*S), R)$ and since $\Delta(E, E^0; p^*S)$ is free, the usual universal coefficient theorem applies and shows that $H^i(E, E^0; p^*S^*) \approx \text{Hom}\,(H_i(E, E^0; p^*S), R)$ for $i<q$.

If S is arbitrary, less elementary methods seem necessary. By [8] or [10] there is a convergent spectral sequence whose first term is $E_2^{pq} = \varprojlim{}^p H^q(E|W, E^0|W; p^*S^*)$ which ends up in $H^*(E, E^0; p^*S^*)$, $\varprojlim{}^p$ being the pth derived functor of $\varprojlim$, $\varprojlim$ taken over the ordered collection of trivializing open sets W. Since $H^i(E|W, E^0|W; p^*S^*)=0$ for $i<q$, the result now follows. We shall not use the general cohomology case and content ourselves with this sketch.

Finally the following two results from point set topology will be useful.

1.12. Lemma. *Let $f\colon P\to X$ and $g\colon Q\to Y$ be quotient maps, and assume Q and X (or P and Y) locally compact. Then $f\times g\colon P\times Q\to X\times Y$ is a quotient map.*

1.13. Lemma. *Let $f\colon P\to X$ and $g\colon Q\to Y$ be quotient maps, and assume P and X compact. Then $f\times g\colon P\times Q\to X\times Y$ is a quotient map.*

For proofs see [4], theorems 1.5 and 1.6.

2. Thom Classes

Given an $\mathbf{R}^q$-microbundle $\mu\colon X \xrightarrow{s} E \xrightarrow{p} X$ a *Thom class* for μ is a cohomology class $U \in H^q(E, E^0; p^*O^*)$ whose restriction to any

$$H^q(E|x, E^0|x; p^*(O^*|x)) \approx \text{Hom}\,(H_q(E|x, E^0|x), H_q(E|x, E^0|x))$$

maps U to the generator corresponding to the identity automorphism on $H_q(E|x, E^0|x)$. We give an elementary proof of the existence and uniqueness of Thom classes.

2.1. Theorem. *Any microbundle admits a unique Thom class.*

Proof. Since the orientation system of the standard trivial microbundle $\varepsilon^q\colon X \xrightarrow{\times 0} X\times\mathbf{R}^q \xrightarrow{pr_1} X$ is naturally isomorphic to the constant local system $H^q(\mathbf{R}^q, \mathbf{R}^q-0)$, a Thom class of ε^q may be considered an element U of

$$H^q(X\times(\mathbf{R}^q, \mathbf{R}^q-0); H_q(\mathbf{R}^q, \mathbf{R}^q-0)) \approx H^0(X)\otimes \text{Hom}\,(H_q(\mathbf{R}^q, \mathbf{R}^q-0), H_q(\mathbf{R}^q, \mathbf{R}^q-0)).$$

Obviously the class U_{ε^q} corresponding under the Künneth isomorphism to $1_X \otimes \text{id}_{H_q(\mathbf{R}^q, \mathbf{R}^q-0)}$ is a Thom class for ε^q, and the only one. From this follows easily that if $\mu\colon X \xrightarrow{s} E \xrightarrow{p} X$ is a trivial $\mathbf{R}^q$-bundle, then μ has exactly one Thom class U_μ, which is the one corresponding to $U_{\varepsilon q}$ under any trivialization $\mu \approx \varepsilon^q$. By the excision property of singular theory this is still true if μ is a trivial microbundle. Assume next that μ is a general $\mathbf{R}^q$-microbundle and let W_1, W_2 be trivializing open sets in the base. Write $E_1 = E\,|\,W_1$, $E_2 = E\,|\,W_2$ and $E_1^0 = E^0\,|\,W_1$, $E_2^0 = E^0\,|\,W_2$. Then

$$(E_1 \cap E_2,\ E_1^0 \cap E_2^0) = (E,\ E^0)\,|\,(W_1 \cap W_2) \text{ and } (E_1 \cup E_2,\ E_1^0 \cup E_2^0) = (E,\ E^0)\,|\,W_1 \cup W_2).$$

Let U_1, U_2 be the (unique) Thom classes of $\mu\,|\,W_1$, $\mu\,|\,W_2$. By Lemma 1.11 the exact Mayer–Vietoris cohomology sequence starts in dimension q.

$$0 \to H^q(E_1 \cup E_2,\ E_1^0 \cup E_2^0;\ p^*(O^*\,|\,W_1 \cup W_2)) \xrightarrow{j^*} H^q(E_1,\ E_1^0;\ p^*(O^*\,|\,W_1)) \oplus H^q(E_2,\ E_2^0;\ p^*(O^*\,|\,W_2))$$
$$\xrightarrow{i^*} H^q(E_1 \cap E_2,\ E_1^0 \cap E_2^0;\ p^*(O^*\,|\,W_1 \cap W_2)) \xrightarrow{\delta^*} H^{q+1}(E_1 \cup E_2,\ E_1^0 \cup E_2^0;\ p^*(O^*\,|\,W_1 \cup W_2)) \to \dots.$$

From this we get that $i^*(U_1, U_2) = i_1^* U_1 - i_2^* U_2 = 0$ since clearly $i_1^* U_1 = i_2^* U_2 =$ the Thom class of $\mu\,|\,W_1 \cap W_2$. Therefore there is an element U_{12} in

$$H^q(E_1 \cup E_2,\ E_1^0 \cup E_2;\ p^*(O^*\,|\,W_1 \cup W_2))$$

such that $j^*(U_{12}) = (U_1, U_2)$, and since j^* is monic there is only one such element. Obviously this is a Thom class for $\mu\,|\,W_1 \cup W_2$, and obviously it is the only one. By induction it follows that for every $V \subset X$ which is a finite union of trivializing open sets there is a unique Thom class $U_V \in H^q(E\,|\,V,\ E^0\,|\,V;\ p^*(O^*\,|\,V))$ for $\mu\,|\,V$. By the universal coefficient theorem applied to Hom $(\Delta(E, E^0; p^*O), R) \approx \Delta^*(E, E^0; p^*(O^*)$

$$H^q(E,\ E^0;\ p^*O^*) \approx \operatorname{Hom}\,(H_q(E,\ E^0;\ p^*O),\ R) \approx \operatorname{Hom}\,(\varinjlim H_q(E\,|\,V,\ E^0\,|\,V;\ p^*O),\ R)$$
$$\approx \varprojlim \operatorname{Hom}\,(H_q(E\,|\,V,\ E^0\,|\,V;\ p^*O),\ R) = \varprojlim H^q(E\,|\,V,\ E^0\,|\,V;\ p^*O^*),$$

the limits taken over all V. Therefore there is a cohomology class $U_\mu \in H^q(E, E^0; p^*O^*)$ corresponding under these isomorphisms to $\varprojlim U_V$. It is obvious that U_μ has the properties of a Thom class for μ and that it is the only element of $H^q(E, E^0; p^*O^*)$ with these properties.

Remark 1. Clearly the Thom class of μ corresponds under excision maps to the Thom class of any $\mathbf{R}^q$-bundle contained in μ. Such exist according to the representation theorem (cf. [7]). Thus Thom classes of microbundles can be obtained from "ordinary" Thom classes of bundles. The proof of Theorem 2.1, however, also works if μ is a microbundle in the sense of Milnor, i.e. not necessarily with a trivializing partition of unity. More generally the Thom classes of microbundles over a given base space, correspond under isomorphisms (i.e. isogerms, cf. [7] p. 11). Again this follows immediately from the excision property of singular theory. Finally note that the correspondense $\mu \mapsto U_\mu$ is functorial if f is a map into the base X of μ, then $U_{f^*\mu} = f^* U_\mu$.

Remark 2. If $O^*(\mu)$ is isomorphic to the constant local system R on X, then p^*O^* is isomorphic to the constant local system R on E. Therefore any isomorphism $O^* \approx R$ defines an isomorphism $H^q(E, E^0; p^*O^*) \approx H^q(E, E^0; R)$. In this case μ is *orientable over* R, and a class $U \in H^q(E, E^0; R)$ corresponding to U_μ is called an *orientation class* of μ (over R). Notice, however, that the orientation class depends on the isomorphism $O^* \approx R$ and that an orientable microbundle may well have infinitely many orientation classes. Therefore, even in the case of an orientable microbundle it may be preferable to work with the canonical Thom class rather than with some arbitrarily chosen orientation class.

It will be convenient to have a short notation for the modules $H_q(\mathbf{R}^q, \mathbf{R}^q - 0)$ and $H^q(\mathbf{R}^q, \mathbf{R}^q - 0)$. We agree that single symbols H_q and H^q, respectively, shall mean these modules. Following are two examples.

2.2. Let X be a topological manifold (without boundary). The Thom class of its tangent microbundle $\tau(X): X \xrightarrow{\Delta} X \times X \xrightarrow{pr_1} X$ is written U_X and is called the Thom class of X. If $\dim X = q$, U_X lies in $H^q(X \times X, X \times X - \Delta X; O^*(X) \times R)$.

2.3. Let X be a closed submanifold of a manifold Y, $\dim X = q$, $\dim Y = r$. There is an integer $k \geq 0$, an open neighbourhood Y' of X in Y and a retraction $p: Y' \times \mathbf{R}^k \to X \times 0 \approx X$ which together with $s: X \approx X \times 0 \subset Y' \times \mathbf{R}^k$ defines an $\mathbf{R}^{r-q+k}$-microbundle $\nu: X \xrightarrow{s} Y' \times \mathbf{R}^k \xrightarrow{p} X$. The orientation system $O^*(\nu)$ may be identified with $s^*O^*(Y' \times \mathbf{R}^q) \otimes O(X)$ by Lemma 1.9. Then the Thom class U_ν lies in

$$H^{r-q+k}(Y' \times \mathbf{R}^k, Y' \times \mathbf{R}^k - X \times 0; p^*s^*O^*(Y' \times \mathbf{R}^q) \otimes p^*O(X))$$
$$\approx H^{r-q+k}((Y', Y' - X) \times (\mathbf{R}^k, \mathbf{R}^k - 0); (O^*(Y') \otimes p_0^*O(X)) \times H_k)$$
$$\approx H^{r-q}(Y' \ Y' - X; O^*(Y') \otimes p_0^*O(X)) \otimes \operatorname{Hom}(H_k, H_k), \quad p_0 = p \mid Y' \times 0.$$

Under these isomorphisms the Thom class U_ν corresponds to an element $U_{Y'X} \otimes \operatorname{id} H_k$ The groups $H^{r-q}(Y', Y' - X; O^*(Y') \otimes p_0^*O(X))$ form a direct system over the neighbourhoods Y'. Write the limit group $H^{r-q}(Y, Y - X; O^*(Y) \otimes p_0^*O(X))$. (This is clearly a harmless abuse of notation.) Since under inclusions $Y'' \subset Y'$, $U_{Y'X}$ is mapped onto $U_{Y''X}$, the cohomology classes $\{U_{Y'X}\}$ define an element

$$U_{YX} \in H^{r-q}(Y, Y - X; O^*(Y) \otimes p_0^*O(X))$$

called the *normal class* of X in Y. It depends on the way X is embedded in Y. It also depends on the germ of p, but not essentially so. In fact for any two retractions p_0, p_1 of neighbourhoods Y_0, Y_1 of X to X there is a canonical isomorphism

$$\Phi_{p_1 p_0}: p_0^* O(X) \mid Y_0 \cap Y_1 \approx p_1^* O(X) \mid Y_0 \cap Y_1$$

which is the identity over X such that always $\Phi_{p_2 p_1} \circ \Phi_{p_1 p_0} = \Phi_{p_2 p_0}$. The correponding canonical isomorphisms of limit groups map normal class to normal class.

2.4. Theorem. *Let $\mu: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-microbundle over a space X and let S be a local system on X. There are natural isomorphisms of graded modules of degrees $-q$ and q*

$$\Phi_\mu: H(E, E^0; p^* S) \approx H(X; O^*(\mu) \otimes S), \qquad \Phi_\mu^*: H^*(X; O(\mu) \otimes S) \approx H^*(E, E^0; p^* S),$$

defined by

$$\Phi_\mu(z) = p^*(U_\mu \cap z), \qquad \Phi_\mu^*(v) = p^*(v) \cup U_\mu.$$

Proof. Suppose μ is the standard trivial $\mathbf{R}^q$-bundle. Then it follows from standard properties of the cap product that

$$\Phi_\mu: H(X \times (\mathbf{R}^q, \mathbf{R}^q - 0); pr_1^* S) \to H(X; H_q(\mathbf{R}^q, \mathbf{R}^q - 0) \otimes S)$$

is an isomorphism. In the case of a general microbundle μ it then follows that $\Phi_{\mu | W}$ is an isomorphism for any trivializing open set $W \subset X$. By the Mayer–Victoris' gluing technique and the five lemma the conclusion extends to any $\Phi_{\mu | V}$ where V is the union of two trivializing open sets, hence (by induction) to the case where V is a finite union of trivializing open sets. The conclusion for Φ_μ now follows from the fact that $\Phi_\mu = \varinjlim \Phi_{\mu | V}$.

To get the cohomology part notice that Φ_μ is in fact induced from a chain map $\tau_\mu: \Delta(E, E^0; p^* S) \to \Delta(X; O^* \otimes S)$, namely $\tau_\mu(c) = \Delta(p)(u_\mu \cap c)$, where u_μ is some chosen cocycle representing U_μ. From the general identity Hom $(\Delta(Y, B; \mathcal{T}), R) \approx \Delta^*(Y, B; \mathcal{T}^*)$ it follows that the dual τ_μ^* is a map of cochain complexes $\tau_\mu^*: \Delta^*(X; O \otimes S^*) \to \Delta^*(E, E^0; p^* S^*)$ inducing Φ_μ^*. If S is free, the cochain complexes are free and fit into universal coefficient short exact sequences. The maps Φ_μ and Φ_μ^* being induced from a chain map define a map from the short exact sequence of $\Delta(X; O^* \otimes S)$

$$0 \to \operatorname{Ext}(H(X; O^* \otimes S), R) \to H^*(X; O \otimes S^*) \to \operatorname{Hom}(X; O^* \otimes S), R) \to 0$$

to that of $\Delta(E, E^0; p^* S^*)$. The five lemma then applies to show that Φ_μ^* is an isomorphism $H^*(X; O \otimes S^*) \approx H^*(E, E; p^* S^*)$. If in addition S is finitely generated, then S^* can be replaced with S.

If S is arbitrary, a spectral sequence argument is necessary. There is a spectral sequence for the covering $\{W\}$ of X, W trivializing for μ, ending in $H^*(X; O \otimes S)$ whose first term is $E_2^{pr} = \varprojlim^p H^r(W; O \otimes S)$ (cf. proof of Lemma 1.11). Similarly there is a spectral sequence for the covering $\{(E|W, E^0|W)\}$ of (E, E^0) ending in $H^{*+q}(E, E^0; p^*S)$ whose first term is $E_2^{pr} = \varprojlim^p H^{r+q}(E|W, E^0|W; p^*S)$. Then Φ_μ^* defines a map from the first spectral sequence to the second which is an isomorphism on first terms. It follows that Φ_μ^* is an isomorphism.

If μ is microbundle and ξ is a bundle contained in μ then the Thom classes (and Thom isomorphisms) of μ, ξ and ξ_∞ are turned into each other by excision isomorphisms. The sphere bundle ξ_∞ is used in defining the Thom space of μ. Let ξ be given by $\xi: X \xrightarrow{s} E \xrightarrow{p} X$. The ∞-section im $s_\infty \subset E_\infty$ is a strong deformation retract of E_∞^0 by a fibre preserving deformation ([7] p. 27). Therefore the inclusion $(E_\infty, \text{im } s_\infty) \subset (E_\infty, E_\infty^0)$ induces an isomorphism $H^*(E_\infty, E_\infty^0; p_\infty^* S) \approx H^*(E_\infty, \text{im } s_\infty; p_\infty^* S)$ for any local system S on X (since a local system $p_\infty^* S$ is necessarily constant along the fibres). In section 4 we shall need the following easy consequence of Theorem 2.4.

2.5. Corollary. *Let $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-bundle and A a closed subset of X. Let S be a local system on X. Then there are Thom isomorphisms*

$$\Phi_{\xi_\infty}(X, A): H(E_\infty, \text{im } s_\infty \cup E_\infty|A; p_\infty^* S) \approx H(X, A; O^* \otimes S)$$

$$\Phi_{\xi_\infty}^*(X, A): H^*(X, A; O \otimes S) \approx H^*(E_\infty, \text{im } s_\infty \cup E_\infty|A; p_\infty^* S).$$

Proof. When A is empty this results from theorem 2.4 by the remarks above. The general case follows from the special once we observe that the excision map $(E_\infty|A, s_\infty A) \subset (\text{im } s_\infty \cup E_\infty|A, \text{im } s_\infty)$ induce an isomorphism

$$H(E_\infty|A, s_\infty A; p_\infty^* S) \approx H(\text{im } s_\infty \cup E_\infty|A, \text{im } s_\infty; p_\infty^* S)$$

(and a similar isomorphism in cohomology). In fact then we get a map (of degree $-q$) from the exact homology sequence of the triple $(E_\infty, \text{im } s_\infty \cup E_\infty|A, \text{im } s_\infty)$ to the exact homology sequence of the pair (X, A) (coefficients $p_\infty^* S$ and S, respectively)

$$\begin{array}{ccccccc}
\dots \to & H_i(\text{im } s_\infty \cup E_\infty|A, \text{im } s_\infty) & \to & H_i(E_\infty, \text{im } s_\infty) & \to & H_i(E_\infty, \text{im } s_\infty \cup E_\infty|A) & \to \dots \\
& \approx \uparrow & & & & & \\
& H_i(E_\infty|A, s_\infty A) & & \approx \downarrow \Phi(X, \emptyset) & & \downarrow \Phi(X, A) & \\
& \approx \downarrow \Phi(A, \emptyset) & & & & & \\
\dots \to & H_{-q}(A) & \to & H_{i-q}(X) & \to & H_{i-q}(X, A) & \to \dots .
\end{array}$$

Thus, by the five lemma $\Phi(X, A)$ is an isomorphism.

To establish the excision isomorphism note that $s_\infty A$ is nondegenerate in $E_\infty|A$ (cf. example 1.8) and that im s_∞ is nondegenerate in im $s_\infty \cup E_\infty|A$. Also observe that there is a bijective correspondence between neighbourhoods V of im s_∞ in im $s_\infty \cup E_\infty|A$ and neighbourhoods $V_A = V \cap E_\infty|A$ of $s_\infty A$ in $E_\infty|A$. We get a commutative diagram (coefficients in $p_\infty^* S$)

$$\begin{array}{ccc} H(E_\infty|A, s_\infty A) & \rightarrow & H(E_\infty|A, V_A) \\ \downarrow & & \downarrow \approx \\ H(\text{im } s_\infty \cup E_\infty|A, \text{im } s_\infty) & \rightarrow & H(\text{im } s_\infty \cup E_\infty|A, V). \end{array}$$

The right vertical map is an isomorphism by the excision property. Passing to limits over V make the horizontal maps isomorphisms (Lemma 1.3), hence also the left vertical map.

For later applications it is important to compute the Thom class of a composite microbundle in terms of the Thom classes of the composing microbundles. By Corollary 3.7 in [7] this is equivalent to computing the Thom class of a Whitney sum. Thus, let

$$\mu_1: X \xrightarrow{s_1} E_1 \xrightarrow{p_1} X, \qquad \mu_2: X \xrightarrow{s_2} E_2 \xrightarrow{p_2} X$$

be two microbundles over X and let

$$\mu: X \xrightarrow{s} E \xrightarrow{p} X$$

be their Whitney sum. Let $\pi_1: E \to E_1$ and $\pi_2: E \to E_2$ be the canonical projections and $\sigma_1: E_1 \to E$, $\sigma_2: E_2 \to E$ their corresponding sections. The projections π_1, π_2 define maps of pairs

$$\bar{\pi}_1: (E, E - \sigma_2 E_2) \to (E_1, E_1^0), \qquad \bar{\pi}_2: (E, E - \sigma_1 E_1) \to (E_2, E_2^0)$$

which induce isomorphisms in homology. Given a pairing of local systems on X, $\psi: S_1 \otimes S_2 \to S$ there is a natural pairing

$$H^*(E_1, E_1^0; p_1^* S_1) \otimes H^*(E_2, E_2^0; p_2^* S_2) \to H^*(E, E^0; p^* S)$$

defined to the composite

$$\begin{array}{c} H^*(E_1, E_1^0; p_1^* S_1) \otimes H^*(E_2, E_2; p_2^* S_2) \\ \bar{\pi}_1^* \otimes \bar{\pi}_2^* \downarrow \\ H^*(E, E - \sigma_2 E_2; \bar{\pi}_1^* p_1^* S_1) \otimes H^*(E, E - \sigma_1 E_1; \bar{\pi}_2^* p_2^* S_2) \\ \smile \downarrow \\ H^*(E, E^0; \bar{\pi}_1^* p_1^* S_1 \otimes \bar{\pi}_2^* p_2^* S_2) \\ \tau^* \downarrow \\ H^*(E, E^0; p^* S). \end{array}$$

The map $\smile$ is the ordinary cup product, and τ_* is the homomorphism induced from the composite

$$\tau: \bar{\pi}_1^* p_1^* S_1 \otimes \bar{\pi}_2^* p_2^* S_2 = p^* S_1 \otimes p^* S_2 \approx p^*(S_1 \otimes S_2) \xrightarrow{p^* \circ \varphi} p^* S.$$

This pairing is invariant under isomorphism of microbundles, but in general dependent on the germs of p_i and s_i, $i=1, 2$. The image of an element $u \otimes v$ will (by a slight abuse of notation) be written $\bar{\pi}_1^* u \smile \bar{\pi}_2^* v$. It follows that there is always a pairing induced from the Künneth formula pairing $O_1^* \otimes O_2^* \to O^*$.

2.6. THEOREM. *If μ, μ_1, μ_2 are microbundles with Thom classes U, U_1, U_2 and $\mu = \mu_1 \oplus \mu_2$, then $U = \bar{\pi}_1^*(U_1) \smile \bar{\pi}_2^*(U_2)$.*

Proof. Since the cohomology class $\bar{\pi}_1^* U_1 \smile \bar{\pi}_2^* U_2$ belongs to $H^{q_1+q_2}(E, E^0; p^* O^*)$, with $\dim \mu_1 = q_1$, $\dim \mu_2 = q_2$, it suffices to show that it has the restriction properties characteristic for the Thom class U. Let (F_1, F_1^0), (F_2, F_2^0) and (F, F^0) be the fibre pairs of μ_1, μ_2 and μ over some point x in X and consider the following commutative diagram

$$\begin{array}{ccccccc} (E, E-\sigma_2 E_2) & \supset & (F, F^0) & \subset & (E, E-\sigma_1, E_1) \\ \bar{\pi}_1 \swarrow & & \bar{\bar{\pi}}_1 \swarrow \quad \searrow \bar{\bar{\pi}}_2 & & \searrow \bar{\pi}_2 \\ (E_1, E_1^0) \supset (F_1, F_1^0) & & & & (F_2, F_2^0) \subset (E_2, E_2^0). \end{array}$$

We have

$$(\bar{\pi}_1^* U_1 \smile \bar{\pi}_2^* U_2) \,|\, (F, F^0) = (\bar{\pi}_1^* U_1 \,|\, (F, F^0)) \smile (\bar{\pi}_2^* U_2 \,|\, (F, F^0))$$
$$= \bar{\bar{\pi}}_1^*(U_1 \,|\, (F_1, F_1^0)) \smile \bar{\bar{\pi}}_2^*(U_2 \,|\, (F_2, F_2^0)) = U_1 \,|\, (F_1, F_1^0) \times U_2 \,|\, (F_2, F_2^0)$$

since $(F, F^0) = (F_1, F_1^0) \times (F_2, F_2^0)$. But $U_i \,|\, (F_i, F_i)$ is the canonical generator of

$$H^{q_i}(F_i, F_i^0; H_{q_i}(F_i, F_i^0)),$$

and so $U_1 \,|\, (F_1, F_1^0) \times U_2 \,|\, (F_2, F_2^0)$ is the canonical generator of $H^{q_1+q_2}(F, F^0; H_{q_1+q_2}(F, F^0)) \approx H^{q_1}(F_1, F_1^0; H_{q_1}(F_1, F_1^0)) \otimes H^{q_2}(F_2, F_2^0; H_{q_2}(F_2, F_2^0))$.

For microbundles μ, ν whose composite $\varkappa = \mu \circ \nu$ is defined there is a natural cup product pairing $H^*(E(\mu), E(\mu)^0; p_\mu^* S) \otimes H^*(E(\nu), E(\nu)^0; p_\nu^* \mathcal{T}) \to H^*(E(\varkappa), E(\varkappa)^0; p_\varkappa^* \mathcal{U})$ associated to each pairing $p_\mu^* S \otimes \mathcal{T} \to p_\mu^* \mathcal{U}$ of local systems $S, \mathcal{U}$ on the base of μ, $\mathcal{T}$ on the base of ν. These pairings are equivalent to those introduced above. The one type is turned into the other by the isomorphism $\varkappa \approx \mu \oplus s_\mu^* \nu$, $\mu \approx \mu_1 \circ p_1^* \mu_2$. If $\mu: X \xrightarrow{s} E \xrightarrow{p} X$ and $\nu: E \xrightarrow{t} E' \xrightarrow{q} E$, let $\bar{q}: (E', E' - q^{-1} s X) \to (E, E^0)$ be the map defined by q. Then we have

2.7. COROLLARY. *If $\varkappa, \mu, \nu$ are microbundles with Thom classes $U_\varkappa, U_\mu, U_\nu$, and $\varkappa = \mu \circ \nu$, then $U_\varkappa = \bar{q}^* U_\mu \smile U_\nu$.*

If μ is an $\mathbf{R}^q$-microbundle over a space X, the restriction of the Thom class U_μ to the zero-section defines an element $\Omega_\mu \in H^q(X; \mathcal{O}^*)$ called the *characteristic class* of μ.

Chern and Lashof have given relations between the Gysin sequences of a Whitney sum of vector bundles and those of the components (orientable vector bundles over a compact manifold, actually). These relations drop easily out of Theorem 2.6.

2.7. THEOREM. *Let ξ_i, $i=1, 2$ be $\mathbf{R}^{q_i}$-bundles and ξ their Whitney sum. The canonical embeddings $\sigma_i : \xi_i \to \xi$ induce maps between the Gysin homology sequences of ξ_i and ξ*

$$\begin{array}{ccccccc} \dots \to & H_n(E_i^0; p_i^{0*}\mathcal{S}) & \to & H_n(X; \mathcal{S}) & \to & H_{n-q_i}(X; \mathcal{O}_i^* \otimes \mathcal{S}) & \to \dots \\ & \sigma_{i*}^0 \downarrow & & \mathrm{id} \downarrow & & \psi_i \downarrow & \\ \dots \to & H_n(E^0; p^{0*}\mathcal{S}) & \to & H_n(X; \mathcal{S}) & \to & H_{n-q}(X; \mathcal{O}^* \otimes \mathcal{S}) & \to \dots \end{array}$$

where the maps ψ_i are given by

$$\psi_1(z) = (-1)^{q_1 q_2} \Omega_2 \frown z, \qquad \psi_2(z) = \Omega_1 \frown z.$$

Similarly, in the diagrams of the Gysin cohomology sequences corresponding maps ψ_i^* are given by

$$\psi_1^*(v) = (-1)^{q_1 q_2} v \smile \Omega_2, \qquad \psi_2^*(v) = v \smile \Omega_1.$$

Proof. Let $\bar{\sigma}_i: (E_i, E_i^0) \to (E, E^0)$ and $\sigma_i^0: E_i^0 \to E^0$ be the maps defined by $\sigma_i: E_i \to E$, $i=1, 2$. Also keep the notation of theorem 2.6. Then we have maps

$$\Phi \bar{\sigma}_{i*} \Phi_i^{-1}: H_{n-q_i}(X; \mathcal{O}_i^* \otimes \mathcal{S}) \to H_{n-q}(X; \mathcal{O}^* \otimes \mathcal{S})$$

and $p_* \sigma_{i*} p_{i*}^{-1}: H_n(X; \mathcal{S}) \to H_n(X; \mathcal{S})$ between the Gysin sequences of ξ_i and ξ, commuting with σ_{i*}^0, $i=1, 2$. It is clear that $p_* \sigma_{i*} p_{i*}^{-1}$ is the identity map. To determine $\Phi \bar{\sigma}_{i*} \Phi_i^{-1}$ note that $p^* \Omega_i = p^* \sigma_i^* j_i^* U_i = \pi_i^* j_i^* U_i$, where $j_i: E_i \subset (E_i, E_i^0)$. Also writing $k_1: E \subset (E, E - \sigma_2 E_2)$, $k_2: E \subset (E, E - \sigma_1 E_1)$ we have $\bar{\pi}_i k_i = j_i \pi_i$. Thus for $\bar{z}_1 \in H_n(E_1, E_1^0; p_1^* \mathcal{S})$ we get

$$\begin{aligned} \Phi \sigma_{1*} \bar{z}_1 &= p_*(U \frown \bar{\sigma}_{1*} \bar{z}_1) = p_* \sigma_{1*}(\bar{\sigma}_1^*(\bar{\pi}_1^* U_1 \smile \bar{\pi}_2^* U_2) \frown \bar{z}_1) \\ &= p_{1*}((U \smile \sigma_1^* k_2^* \bar{\pi}_2^* U_2) \frown \bar{z}_1) = p_{1*}((U_1 \smile \sigma_1^* \pi_2^* j_2^* U_2) \frown \bar{z}_1) \\ &= p_{1*}((U_1 \smile \sigma_1^* p^* \Omega_2) \frown \bar{z}_1) = (-1)^{q_1 q_2} p_{1*}(p_1^* \Omega_2 \frown (U_1 \frown \bar{z}_1)) \\ &= (-1)^{q_1 q_2} \Omega_2 \frown p_{1*}(U_1 \frown \bar{z}_1). \end{aligned}$$

With $\bar{z}_1 = \Phi_1^{-1} z_1$ we have $z_1 = p_{1*}(U_1 \frown \bar{z}_1)$, and so $\Phi \bar{\sigma}_{1*} \Phi_1^{-1} z_1 = (-1)^{q_1 q_2} \Omega_2 \frown z_1$. The corresponding cohomology formula is gotten by duality or is deduced from an even simpler computation. The case $i=2$ is the same (except for a possible shift in sign).

We conclude this section with two illustrations of the use of Theorem 2.5.

2.8. Let X, Y, Z be manifolds of dimension q, r, s respectively, X a closet submanifold of Y and Y a closed submanifold of Z. Then X is a closed submanifold of Z and we have normal classes

$$U_{YX} \in H^{r-q}(Y, Y-X; O^*(Y) \otimes p_0^* O(X)),$$
$$U_{ZY} \in H^{s-r}(Z, Z-Y; O^*(Z) \otimes p_1^* O(Y)),$$
$$U_{ZX} \in H^{s-q}(Z, Z-X; O^*(Z) \otimes p^* O(X)),$$

where p_0, p_1 and p are neighbourhood retractions such that germ $p_0 \circ$ germ $p_1 =$ germ p (cf. ex. 2.3). Let V be a neighbourhood of Y in Z. Then we have a cup product map

$$\begin{array}{c} H^*(V, V-Y; O^*(V) \otimes p_1^* O(Y)) \otimes H^*(V, Y-X; p_1^* O^*(Y) \otimes p^* O(X)) \\ \smile \downarrow \\ H^*(V, V-X; O^*(V) \otimes p^* O(X)). \end{array}$$

By the excision property

$H^*(V, V-Y; O^*(V) \otimes p_1^* O(Y))$ is isomorphic to $H^*(Z, Z-Y; O^*(Z) \otimes p_1^* O(Y))$

and

$H^*(V, V-X; O^*(V) \otimes p^* O(X))$ is isomorphic to $H^*(Z, Z-X; O^*(Z) \otimes p^* O(X))$.

Passing to limits with respect to V therefore defines a cup product

$$\begin{array}{c} H^*(Z, Z-Y; O^*(Z) \otimes p_1^* O(Y)) \otimes H^*(Y, Y-X; O^*(Y) \otimes p_0^* O(X)) \\ \smile \downarrow \\ H^*(Z, Z-X; O^*(Z) \otimes p^* O(X)) \end{array}$$

since $\varinjlim H^*(V, Y-X; p_1^* O^*(Y) \otimes p^* O(X)) = H^*(Y, Y-X; O^*(Y) \otimes p_0^* O(X))$. (This follows from the five lemma and the fact that Y is taut in Z with respect to any local system.) By corollary 2.6 we have the relation

$$U_{ZX} = U_{ZY} \smile U_{YX}.$$

2.9. If the coefficient ring R equals $\mathbf{Z}_2$ then for any space X there are natural isomorphisms of $O(\mu)$ and $O^*(\mu)$ to the constant local system $\mathbf{Z}_2$ for microbundles μ over X. Consider an $\mathbf{R}^q$-microbundle μ and its Thom class $U_\mu \in H^q(E(\mu), E(\mu)^0; \mathbf{Z}_2)$. Define the Stiefel-Whitney classes $w_i(\mu) \in H^i(B(\mu); \mathbf{Z}_2)$, $i \geq 0$, by

$$\Phi_\mu^* w_i(\mu) = Sq^i U_\mu.$$

An immediate consequence of theorem 2.5 and the Cartan formula for Steenrod squares is the Whitney duality theorem

$$w_k(\mu \oplus \mu') = \sum_{i+j=k} w_i(\mu) \smile w_j(\mu'), \quad k \geq 0.$$

If X is a closed submanifold of Y as in 2.3, X has a stable normal bundle, and there is an associated Thom isomorphism in cohomology. By the Künneth theorem the stable part splits off and defines a modified relative Thom isomorphism

$$\Phi^*_{YX}: H^i(X, X-A; \mathbf{Z}_2) \approx H^{r-q+i}(Y, Y-A; \mathbf{Z}_2)$$

for any closed subset A in X. This is given by

$$\Phi^*_{YX}\bar{v} = \bar{v} \smile U_{YX},$$

where the cup product is taken in the sense of 2.8. Take $A = X$ and define the normal Stiefel–Whitney classes $\bar{w}_i \in H^i(X; \mathbf{Z}_2)$, $i \geq 0$, of X in Y by

$$\Phi_{YX}\bar{w}_i = Sq^i U_{YX}.$$

Then Theorem 2.5 and the Cartan formula give the Whitney duality theorem

$$w_k(Y) \mid X = \sum_{i+j=k} \bar{w}_i \smile w_j(X), \quad k \geq 0.$$

Our main application of Theorem 2.5 (or its corollary) will be to the S-duality theorem. We first make a digression on proper fibre homotopy equivalence.

3. Proper fibre homotopy type

In this section we are concerned with $\mathbf{R}^q$-bundles (of different dimensions) and their associated sphere bundles. As usual a bundle map $\xi \to \xi'$ is required to respect zero sections although this particular aspect is not important here, i.e. the definitions work without that restriction and give equivalent results. The following is easily verified.

3.1. Lemma. *Let $f: \xi \to \xi'$ be a bundle map between $\mathbf{R}^q$-bundles. Then f extends to a bundle map $f_\infty: \xi_\infty \to \xi'_\infty$ sending* im s_∞ *into* im s'_∞ *if and only if f is fibrewise proper.*

Consider a homotopy $H: E \times I \to E$, where E is the total space of the $\mathbf{R}^q$-bundle ξ. If H is fibre preserving and respects zero-sections, it defines a bundle homotopy $H: \xi \times I \to \xi$. Since $(\xi \times I)_\infty = \xi_\infty \times I$, by Lemma 3.1 H extends to a bundle homotopy $H_\infty: \xi_\infty \times I \to \xi_\infty$ sending im $s_\infty \times I$ to im s_∞ if and only if H is fibrewise proper. In other words H extends to a fibre preserving homotopy H_∞: $E_\infty \times I \to E_\infty$ which is the identity over im s_∞ if and only if H_t: $E \to E$ is fibrewise proper for every $t \in I$.

Now, define two $\mathbf{R}^q$-bundles ξ_1, ξ_2 over a common base X to be *properly fibre homotopy equivalent* if there exist fibrewise proper bundle maps $f_1: \xi_1 \to \xi_2$, $f_2: \xi_2 \to \xi_1$ (covering the identity) and fibrewise proper bundle homotopies $H_1\colon \xi_1 \times I \to \xi_1$, $H_2\colon \xi_2 \times I \to \xi_2$ (covering the projection) from $f_2 \circ f_1$ to id_{ξ_1} and from $f_1 \circ f_2$ to id_{ξ_2}, respectively. Note that if ξ_1, ξ_2 are properly fibre homotopy equivalent, then so are $\xi_1|A, \xi_2|A$ for any subset A in the base. (In particular the fibres of ξ_1 and ξ_2 over any point x in X must have the same proper fibre homotopy type. Hence it is impossible to extend the definition to bundles of different dimensions.)

Next we discuss the S-type of Thom spaces, following Atiyah [1]. Recall that the Thom space $T_\xi(A, B)$, where (A, B) is a pair of subsets in the base of $\xi\colon X \xrightarrow{s} E \xrightarrow{p} X$, ξ an $\mathbf{R}^q$-bundle, is defined to be the pointed space

$$T_\xi(A, B) = p_\infty^{-1}A / s_\infty A \cup p_\infty^{-1}B.$$

We write T_ξ for $T_\xi(X, \varnothing)$.

3.2. Lemma. *Let ξ, ξ' be $\mathbf{R}^q$-, $\mathbf{R}^{q'}$-bundles over X, X', respectively, and let (A, B), (A', B') be closed pairs in X, X'. There is a natural relative homeomorphism of pointed spaces*

$$T_{\xi\times\xi'}((A, B) \times (A', B')) \to T_\xi(A, B) \wedge T_{\xi'}(A', B').$$

If A and A' are locally compact or if A or A' is compact this correspondence is a homeomorphism.

Proof. Without loss of generality we may suppose $A = X$ and $A' = X'$. Clearly there is a natural continuous bijection

$$(E \times E')_\infty / \mathrm{im}\,(s \times s')_\infty \to E_\infty \times E'_\infty / (E_\infty \times \mathrm{im}\, s'_\infty) \cup (\mathrm{in}\; s_\infty \times E'_\infty)$$

induced from the inclusion $E \times E' \subset E_\infty \times E'_\infty$. Similarly the quotient maps $E_\infty \to E_\infty/\mathrm{im}\, s_\infty$ and $E'_\infty \to E'_\infty/\mathrm{im}\, s'_\infty$ define the natural map

$$E_\infty \times E'_\infty \to (E_\infty/\mathrm{im}\, s_\infty) \times (E'_\infty/\mathrm{im}\, s'_\infty) \to (E_\infty/\mathrm{im}\, s_\infty) \wedge (E'_\infty/\mathrm{im}\, s'_\infty)$$

sending $E_\infty \times \mathrm{im}\, s'_\infty \cup \mathrm{im}\, s_\infty \times E'_\infty$ to the base point, Therefore we also have a natural continuous bijection

$$E_\infty \times E'_\infty / (E_\infty \times \mathrm{im}\, s'_\infty) \cup (\mathrm{im}\, s_\infty \times E'_\infty) \to (E_\infty/\mathrm{im}\, s_\infty) \wedge (E'_\infty/\mathrm{im}\, s'_\infty).$$

Composing the two bijections give the desired map $T_{\xi\times\xi'} \to T_\xi \wedge T_{\xi'}$ in the absolute case. Obviously this is a homeomorphism outside the base points. That this map is not in general a homeomorphism stems from the fact that a product of quotient topologies need not be a quotient topology. It follows from Lemmas 1.12 and 1.13, however, that if X, X' are locally compact or at least one space is compact, then the natural bijections are in fact homeomorphisms. The relative case now follows from the general identity $T_\xi(X, B) = T_\xi / T_{\xi|B}$.

Under the conditions above on A and A' we agree to identify $T_{\xi\times\xi'}((A, B)\times(A', B'))$ with $T_\xi(A, B) \wedge T_{\xi'}(A', B')$.

3.3. Corollary. *Let ξ be an $\mathbf{R}^q$-bundle. Then $T_{\xi\oplus\varepsilon^n}(A, B) = S^n T_\xi(A, B)$ for any closed pair (A, B) in the base.*

Proof. There is a natural isomorphism $\xi\oplus\varepsilon^n(X) \approx \xi\times\varepsilon^n(pt)$. In particular, if $B\subset X$ then $\xi\oplus\varepsilon^n(X)\,|\,B = \xi\,|\,B\times\varepsilon^n(pt)$. Thus $T_{\xi\oplus\varepsilon^n}(X, B) = T_\xi(X, B)\wedge T_{\varepsilon^n(pt)}$ by 3.2. But $T_{\varepsilon^n(pt)} = S^n$ therefore $T_{\xi\oplus\varepsilon^n}(X, B) = S^n T_\xi(X, B)$.

3.4. Lemma. *Let ξ, ξ' be $\mathbf{R}^q$-bundles over a space X, and suppose ξ, ξ' are properly fibre homotopy equivalent. Then $T_\xi(A, B)$ and $T_{\xi'}(A, B)$ are homotopy equivalent for any closed pair (A, B) in X.*

This follows easily from the definition of proper fibre homotopy equivalence together with Lemma 3.1.

Finally define an $\mathbf{R}^q$- and an $\mathbf{R}^{q'}$-bundle with common base to be *stably fibre homotopy equivalent* or *J-equivalent* if for some integers $n, n'\geqslant 0$ $\xi\oplus\varepsilon^n$ and $\xi'\oplus\varepsilon^{n'}$ are properly fibre homotopy equivalent. Then, from 3.3 and 3.4 we have

3.5. Lemma. *Let ξ, ξ' be stably fibre homotopy equivalent. Then $T_\xi(A, B)$ and $T_{\xi'}(A, B)$ are of the same S-type for any closed pair (A, B) in the base X.*

It is perhaps not immediately clear that the above definition of J-equivalence restricts to the classical one over the category of ortogonal bundles. However, the equivalence follows from the Hirsch–Mazur theorem, cf. [7]:

3.6. Lemma. *ξ and ξ' are stably fibre homotopy equivalent if and only if for some integers n, n' the enveloping disk bundles of $\xi\oplus\varepsilon^{n+1}$, $\xi'\oplus\varepsilon^{n'+1}$ have fibre homotopy equivalent boundary bundles.*

Proof. If ξ, ξ' are stably fibre homotopy equivalent, then for some positive integers n, n', $(\xi\oplus\varepsilon^n)_\infty$ and $(\xi'\oplus\varepsilon^{n'})_\infty$ are fibre homotopy equivalent. But by the Hirsch–Mazur theorem $(\xi\oplus\varepsilon^n)_\infty$ and $(\xi'\oplus\varepsilon^{n'})_\infty$ are the boundaries of disk bundles whose interiors are $\xi\oplus\varepsilon^{n+1}$ and $\xi'\oplus\varepsilon^{n'+1}$, respectively. Thus ξ and ξ' are J-equivalent in the classical sense. Conversely, any fibre homotopy equivalence of the bounding sphere bundles of $\xi\oplus\varepsilon^{n+1}$ and $\xi'\oplus\varepsilon^{n'+1}$ extends radially to a fibre homotopy equivalence of $(\xi\oplus\varepsilon^n)_{\infty C}$ and $(\xi'\oplus\varepsilon^{n'})_{\infty C}$.

Collapsing fibre boundaries gives a fibre homotopy equivalence of $(\xi\oplus\varepsilon^{n+1})_\infty$ and $(\xi'\oplus\varepsilon^{n'+1})_\infty$ respecting ∞-sections, and therefore a proper fibre homotopy equivalence of $\xi\oplus\varepsilon^{n+1}$ and $\xi'\oplus\varepsilon^{n'+1}$. Thus ξ and ξ' are stably fibre homotopy equivalent.

We have defined the Thom spaces $T_\xi(A, B)$ for an $\mathbf{R}^q$-bundle ξ. If B is nicely imbedded in A, then $T_\xi(A, B)$ reflects the homology properties of the bundle pair $(\xi|A, \xi|B)$.

3.7. Lemma. *Let $\xi: X \xrightarrow{s} E \xrightarrow{p} X$ be an $\mathbf{R}^q$-bundle and A a nondegenerate subset of X. The collapsing $(E_\infty, \operatorname{im} s_\infty \cup E_\infty|A) \to (T_\xi(X, A), *)$ induces isomorphisms in homology (constant coefficients).*

Proof. If A is empty the claim follows from Lemma 1.4 since im s_∞ is nondegenerate in E_∞. Assume that A is nonempty and write im $s_\infty = X_\infty$, $s_\infty A = A_\infty$. Then A_∞ is nondegenerate in X_∞, and so there is a map $\phi: X_\infty \to I$ which is 0 exactly on A and a deformation $D: (X_\infty, A_\infty)\times I \to (X_\infty, A_\infty)$ relative to A_∞ such that $D(\phi^{-1}[0, 1)\times 1)\subset A_\infty$. Since $s_\infty \circ p: E_\infty \to X_\infty$ is a bundle, by the homotopy lifting property there is a deformation

$$D': (E_\infty, E_\infty|A)\times I \to (E_\infty, E_\infty|A)$$

covering D. Thus if $\phi': E_\infty \to I$ is the map $\phi\circ s_\infty \circ p_\infty$, then $D'(\phi'^{-1}[0, 1)\times 1)\subset E_\infty|A$. Moreover, since X_∞ is nondegenerate in E_∞, D' can be chosen so as to be an extension of D, i.e. so that $D'(X_\infty \times I)\subset X_\infty$([13], Theorem 4).

Also there is a *fibre* deformation

$$D'': (E_\infty, X_\infty)\times I \to (E_\infty, X_\infty)$$

relative to X_∞ and a map $\phi'': E_\infty \to I$ which is zero exactly on X_∞ such that

$$D''(\phi''^{-1}[0, 1)\times 1)\subset X_\infty.$$

Let $\psi = \phi'\cdot\phi''$. Then $\psi^{-1}\{0\} = X_\infty \cup E_\infty|A$ and $\psi^{-1}[0, 1) = \phi''^{-1}[0, 1)\cup \phi'^{-1}[0, 1)$. Let

$$F: (E_\infty, X_\infty \cup E_\infty|A)\times I \to (E_\infty, X_\infty \cup E_\infty|A)$$

be the deformation $F(e, t) = D'(D''(e, t), t)$. Then $F(\psi^{-1}[0, 1)\times 1)\subset X_\infty \cup E_\infty|A$. Thus F and ψ would make $X_\infty \cup E_\infty|A$ nondegenerate in E_∞ except that F is not constant on $X_\infty \cup E_\infty|A$. Nevertheless, by Lemma 1.4 the collapsing

$$(E_\infty, X_\infty \cup E_\infty|A) \to (E_\infty/X_\infty \cup E_\infty|A, *)$$

induce an isomorphism in homology.

Note that although we have not proved $X_\infty \cup E_\infty|A$ nondegenerate in E_∞ (which is probably true), the proof shows that after collapsing $X_\infty \cup E_\infty|A$ the base point $*$ is nondegenerate in $E_\infty/X_\infty \cup E_\infty|A$. In other words if A is nondegenerate in X, then $T_\xi(X, A)$ is a space with nondegenerate base point.

We close this section by defining dual pairs. Let X be a topological space and consider the collection of closed pairs (A, B) in X such that B is nondegenerate in A and A/B is compact. Call two such pairs (A, B) and (A', B') *duals* if the conditions

$A \cap B' = \emptyset$

The inclusions $A \subset X - B'$, $B' \subset X - A$ *are homotopy equivalences*

as well as the dual conditions (where the priming of A and B is interchanged) are satisfied. Even if the conditions make sense for broader classes of pairs we will not extend the definition, i.e. the term "duals" will imply that the pairs concerned are of the restricted type. In the end we restrict our interest to the even smaller class of pairs (A, B) which in addition have homotopy of relative CW-complexes.

Following are some examples

3.8. If X is a compact space, then X (i.e. $(X, \emptyset)$) is self dual.

3.9. Let (X, A, B) be a polyhedral triple with A (hence B) compact. Let

$$(|K|, |L|, |M|) \approx (X, A, B)$$

be a triangulation. Let L' be the subcomplex of sd^2K consisting of simplices with no faces in sd^2L and M' the subcomplex of simplices with no faces in sd^2M. Let A' be the image of $|M'|$ and B' the image of $|L'|$ under the given triangulation. Then (A, B) and (A', B') are duals.

3.10. Let X be a manifold with (possibly empty) boundary $\dot{X}$. Assume that $\dot{X}$ is compact. By the collaring theorem (cf. [3]) there is a relatively compact open collaring V of $\dot{X}$ in X. (If $\dot{X} = \emptyset$, take $V = \emptyset$.) Then $\dot{X}$ and $(X, X - V)$ are duals. If X is compact, then $(X, \dot{X})$ and $X - V$ are also duals.

3.11. Let X be a compact manifold with boundary $\dot{X}$, and let DX be its double. Then DX is a compact manifold without boundary containing X and a copy X_- of X, such that $DX = X \cup X_-$ and $\dot{X} = X \cap X_-$. Let V be an open collaring of $\dot{X}$ in DX. Then (DX, X_-) and $X - V$ are duals. Similarly $(X, \dot{X})$ and $(DX - V, X_- - V)$ are duals.

Let X_1, X_2 be compact pointed spaces with nondegenerate base points. Then $X_1 \wedge X_2$ is a space of the same type. Similarly the suspensions $SX_i = X_i \wedge S$, $i = 1, 2$, are spaces with nondegenerate base points. Note that there is a natural homeomorphism

$$S^k(X_1 \wedge X_2) \approx X_1 \wedge S^k X_2$$

for any $k \geqslant 0$ by which we identify the two spaces. An *n-duality map* (or briefly a duality map) for X_1, X_2, $n \geqslant 0$, is a map

$$\phi: X_1 \wedge S^k X_2 \rightarrow S^{k+n}$$

for some $k \geqslant 0$ such that for some (hence for any) generator s^* in $\tilde{H}^{k+n}(S^{k+n})$ slant product with the spherical class $\phi^* s^* \in \tilde{H}^{k+n}(X_1 \wedge S^k X_2) \approx H^n(X_1 \wedge X_2)$ defines an isomorphism

$$\tilde{H}_i(X_2) \approx \tilde{H}^{n-i}(X_1), \quad i \in \mathbf{Z}.$$

When X_1, X_2 have S-type of finite CW-complexes, the existence of an n-duality map for X_1, X_2 implies that X_1 and X_2 are n-duals in the sense of S-theory ([11] p. 360, [12] p. 462).

Before stating the S-duality theorem for microbundles observe that the definitions and properties of $\mathbf{R}^q$-bundles in this section carry automatically over to microbundles (with some care about the naturality properties) by the microbundle representation theorem.

4. The S-duality Theorem

The aim of this section is to prove the following

4.1. S-DUALITY THEOREM. *Let X be a topological manifold and let μ_1, μ_2 be microbundles over X such that $\tau(X) \oplus \mu_1 \oplus \mu_2$ is stably fibre homotopy equivalent to the trivial $\mathbf{R}^{Q'}$-bundle over X. Let (A, B) and (A', B') be dual pairs in X, one of which is compact. Then there is a duality map*

$$\phi: T_{\mu_1}(A, B) \wedge S^{Q'-Q} T_{\mu_2}(A', B') \rightarrow S^{Q'},$$

where $Q = \dim X + \dim \mu_1 + \dim \mu_2$.

4.2. COROLLARY. *Under the condition of Theorem 4.1 if (A, B) and (A', B') are relative CW-complexes, then $T_{\mu_1}(A, B)$ and $T_{\mu_2}(A', B')$ are Q-duals in the sense of S-theory.*

4.3. COROLLARY. *Let X be a compact manifold with boundary $\dot{X}$ and let $-\tau$ be a stable inverse to $\tau(X)$. Then there is a duality map*

$$\phi: T_{-\tau} \wedge X/\dot{X} \rightarrow S^Q,$$

where $Q = \dim \tau(X) + \dim -\tau$.

This follows easily from 4.1 and examples 3.10 or 3.11. If $\dot{X}$ is empty, $X/\dot{X}$ means X^+, i.e. X with an isolated base point adjoined. In general A/B means A^+ if B is empty.

In a q-dimensional sphere a subpolyhedron and its complement (deformed to a compact polyhedron) are q-duals in the sense of S-theory. This situation has the following generalization

4.4. Corollary. *Let X be a q-dimensional π-manifold, and let (A, B) and (A', B') be non-empty dual pairs in X, one of which is compact. Then there is a duality map*

$$\phi: A/B \wedge S^{Q'-q}(A'/B') \to S^{Q'}.$$

Here a π-manifold means a topological manifold with trivial stable tangent microbundle.

To simplify the presentation of the proof of Theorem 4.1 first recall that the Thom space of a microbundle is defined by a representing bundle. Therefore we may without loss of generality assume that μ_1, μ_2 are actually bundles; say μ_1 is an $\mathbf{R}^{q_1}$-bundle and μ_2 an $\mathbf{R}^{q_2}$-bundle. Also we may as well assume that $\tau(X) \oplus \mu_1 \oplus \mu_2$ is properly fibre homotopy trivial since additional trivial bundles can be absorbed in μ_2 without changing its Thom spaces except for suspensions (Corollary 3.3). Finally, to obtain maximum clarity we carry through the proof in detail in the absolute case $(A, B) = X, \varnothing)$ and then make the necessary additional remarks for the relative case. Of course, in the absolute case we must assume X compact.

Thus, let X be a compact q-dimensional manifold, and let μ_1, μ_2 be an $\mathbf{R}^{q_1}$-, respectively an $\mathbf{R}^{q_2}$-bundle. Assume that the (microbundle) Whitney sum $\tau(X) \oplus \mu_1 \oplus \mu_2$ is properly fibre homotopy trivial. Consider the composite microbundle $\mu = \tau(X) \circ (\mu_1 \times \mu_2)$,

$$\mu: X \xrightarrow{s} E_1 \times E_2 \xrightarrow{p} X$$

with $s = (s_1 \times s_2) \circ \Delta$, $p = pr_1 \circ (p_1 \times p_2)$. By Corollary 3.7 in [7], μ is isomorphic to

$$\tau(X) \oplus \Delta^*(\mu_1 \times \mu_2) \approx \tau(X) \oplus \mu_1 \oplus \mu_2,$$

hence μ is properly fibre homotopy trivial. Therefore there is a bundle neighbourhood W of sX in $E_1 \times E_2$ and a fibrewise proper bundle map

$$W \to X \times \mathbf{R}^Q, \quad Q = q + q_1 + q_2 \tag{1}$$

which is a fibre homotopy equivalence by proper fibre homotopies. By Lemma 3.3 this extends uniquely to a bundle map

$$(W_\infty, \text{im } s_\infty) \to X \times (S^Q, \infty) \tag{2}$$

which is fibre homotopy equivalence of pairs. Observe also that this map sends W^0 into $X \times (\mathbf{R}^Q, \mathbf{R}^Q - 0)$ and in fact defines a fibre homotopy equivalence of pairs

$$(W, W^0) \to X \times (\mathbf{R}^Q, \mathbf{R}^Q - 0). \tag{3}$$

Since $\mu_1 \times \mu_2$ is a bundle, $(E_1 \times E_2)_\infty$ and $(s_1 \times s_2)_\infty$ is well defined, and (1) defines a continuous map

$$(E_1 \times E_2)_\infty \to X \times S^Q \tag{4}$$

which equals (1) on W and maps $(E_1 \times E_1)_\infty - W$ onto $X \times \infty$. Since $\operatorname{im}(s_1 \times s_2)_\infty$ is certainly outside W, (4) can be considered a map of pairs

$$((E_1 \times E_2)_\infty, \operatorname{im}(s_1 \times s_2)_\infty) \to X \times (S^Q, \infty). \tag{5}$$

Composing with the projection map gives

$$\phi': ((E_1 \times E_2)_\infty, \operatorname{im}(s_1 \times s_2)_\infty) \to (S^Q, \infty), \tag{6}$$

and collapsing $\operatorname{im}(s_1 \times s_2)_\infty$ defines

$$\phi: (T_{\mu_1 \times \mu_2}, *) \to (S^Q, \infty). \tag{7}$$

In the same way one gets a map

$$\phi'': (E_{1\infty}, \operatorname{im} s_{1\infty}) \times (E_{2\infty}, \operatorname{im} s_{2\infty}) \to (S^Q, \infty) \tag{8}$$

which is an extension of (1) and such that if k is the canonical bundle map

$$k: (E_{1\infty}, \operatorname{im} s_{1\infty}) \times (E_{2\infty}, \operatorname{im} s_{2\infty}) \to ((E_1 \times E_2)_\infty, \operatorname{im}(s_1 \times s_2)_\infty)$$

then $\phi'' \circ k = \phi'$. (The continuity of k is a local question and so follows easily from Lemma 1.12.) Moreover, by Lemma 3.2 and 3.7 the so indicated maps in the diagram below are isomorphisms, and therefore k_* is an isomorphism

$$\begin{array}{ccc} H((E_{1\infty}, \operatorname{im} s_{1\infty}) \times (E_{2\infty}, \operatorname{im} s_{2\infty})) & \xrightarrow{k_*} & H((E_1 \times E_2)_\infty, \operatorname{im}(s_1 \times s_2)_\infty) \\ \approx \downarrow & & \approx \downarrow \\ \tilde{H}(T_{\mu_1} \wedge T_{\mu_2}) & \approx & \tilde{H}(T_{\mu_1 \times \mu_2}) \end{array}$$

(Note that this works equally well in the relative case.) We claim that the map

$$\phi: T_{\mu_1} \wedge T_{\mu_2} \to S^Q$$

obtained from (7) by identifying $T_{\mu_1 \times \mu_2}$ with $T_{\mu_1} \wedge T_{\mu_2}$ is a duality map.

To see this let $s^* \in \tilde{H}^Q(S^Q; \tilde{H}_Q(S^Q))$ be the canonical generator corresponding under the Künneth isomorphism to the identity operator on $\tilde{H}_Q(S^Q)$. In accordance with our previous conventions let us write $\tilde{H}_Q$ for $\tilde{H}_Q(S^Q)$. We will show that slanting with $\phi^* s^*$ defines isomorphisms

$$\tilde{H}_i(T_{\mu_2}) \approx \tilde{H}^{Q-i}(T_{\mu_1}; \tilde{H}_Q), \quad i \in \mathbf{Z}.$$

To do so it is convenient to replace the Thom spaces T_{μ_1}, T_{μ_2} and $T_{\mu_1\times\mu_2}$ with their associated pairs of total spaces $(E_{1\infty}, \operatorname{im} s_{1\infty})$, $(E_{2\infty}, \operatorname{im} s_{2\infty})$ and $((E_1\times E_2)_\infty, \operatorname{im}(s_1\times s_2)_\infty)$. According to Lemma 3.7 this does not change the homology groups. By our remarks above we may also replace the last pair with the product $(E_{1\infty}, \operatorname{im} s_{1\infty})\times(E_{2\infty}, \operatorname{im} s_{2\infty})$ without changing the homology. In this new context we have to show that slanting with ϕ''^*s^* (coefficients $\tilde{H}_Q$) defines isomorphisms

$$H_i(E_{2\infty}, \operatorname{im} s_{2\infty}) \approx H^{Q-i}(E_{1\infty}, \operatorname{im} s_{1\infty}; \tilde{H}_Q), \quad i\in\mathbf{Z}.$$

First notice that since μ is properly fibre homotopy trivial its orientation system $O^*(\mu)$ is isomorphic to the constant local system H_Q. In fact the map (3) defines an explicit isomorphism $O^*(\mu)\approx H_Q$. Identify $O^*(\mu)$ with H_Q by this isomorphism. Then U_μ lies in $H^Q(E_1\times E_2, E_1\times E_2 - sX; H_Q)$. Since $(E_1, E_1^0)\times(E_2, E_2^0)$ is contained in

$$(E_1\times E_2, E_1\times E - sX),$$

U_μ restricts to a class U'_μ in $H^Q((E_1, E_1^0)\times(E_2, E_2^0); H^Q)$. Let U''_μ be the image of U'_μ under the canonical isomorphism

$$H^*((E_1, E_1^0)\times(E_2, E_2^0); H_Q)\approx H^*((E_{1\infty}, \operatorname{im} s_{1\infty})\times(E_{2\infty}, \operatorname{im} s_{2\infty}); \tilde{H}_Q).$$

By Lemma 4.6 below this element is the spherical class ϕ''^*s^*. Similarly the Thom class U_X of $\tau(X)$ restricts to an element U'_X in $H^q(X\times X; O^*(X)\times R)$. The absolute case of Theorem 4.1 now follows from

4.5. Lemma. *The diagram*

$$\begin{array}{ccc} H_i(E_{2\infty}, \operatorname{im} s_{2\infty}) & \xrightarrow{U''_\mu/} & H^{Q-i}(E_{1\infty}, \operatorname{im} s_{1\infty}); \tilde{H}_Q) \\ \Phi_2\downarrow\approx & & \Phi_1^*\uparrow\approx \\ H_{i-q_2}(X; O_2^*) & \underset{\approx}{\xrightarrow{U'_X/}} & H^{q+q_1-i}(X\, O^*(X)\otimes O_2^*) \end{array}$$

is commutative up to sign for all $i\in\mathbf{Z}$.

Proof. Notice first that the Thom isomorphism $\Phi^*_{\mu_1}$ maps from coefficients

$$O^*(X)\otimes O^*(\mu_2)$$

to coefficients $p^*(O^*(\mu_1)\otimes O^*(X)\otimes O^*(\mu_2))$. This local system, however, is canonically isomorphic to $p^*O^*(\mu)$ which we have identified with H_Q. Similarly $\Phi_1^*=\Phi^*_{\mu_{1\infty}}$ maps to coefficients $\tilde{H}_Q$. Now by Corollary 2.7 $U''_\mu=(p_{1\infty}\times p_{2\infty})^*U'_X\smile(U_{\mu_{1\infty}}\times U_{\mu_{2\infty}})$, since clearly

$U_{\mu_{1\infty}\times\mu_{2\infty}} = U_{\mu_{1\infty}} \times U_{\mu_{2\infty}}$. Therefore, for $\bar{z}_2 \in H_i(E_{2\infty}, \text{ im } s_{2\infty})$ we have (suppressing all subscripts ∞ and writing U_i for $U_{\mu_{i\infty}}$)

$$\Phi_1^*(U'_X/\Phi_2\bar{z}) = U_1 \smile p_1^*(U'_X/\Phi_2\bar{z}) = U_1 \smile p_1^*(U'_X/p_{2*}(U_2 \frown \bar{z}_2)).$$

But $p_1^*(U'_X/p_{2*}(U_2\frown\bar{z})) = (p_1\times p_2)^* U'_X/U_2\frown\bar{z}_2 = ((p_1\times p_2)^* U'_X \smile (1\times U_2))/\bar{z}$. Thus

$$\begin{aligned}\Phi_1^*(U'_X/\Phi_2\bar{z}_2) &= U_1 \smile (((p_1\times p_2)^* U'_X \smile (1\times U_2))/\bar{z}_2) \\ &= ((U_1\times 1)\smile(p_1\times p_2)^* U'_X \smile (1\times U_2))\bar{z}_2 = \pm((p_1\times p_2)^* U'_X \smile (U_1\times U_2))/\bar{z}_2 = \pm U''_\mu/\bar{z}.\end{aligned}$$

In the proof of 4.1 we anticipated the following result

4.6. LEMMA. *Under the isomorphism.*

$$\tilde{H}^*(T_{\mu_1} \wedge T_{\mu_2}; \tilde{H}_Q) \approx H((E_{1\infty}, \text{ im } s_{1\infty}) \times (E_{2\infty}, \text{ im } s_{2\infty}); \tilde{H}_Q)$$

the spherical class $\phi''^* s^*$ corresponds to the restricted Thom class U''_μ.

Proof. The image of the spherical class in $H^Q((E_1, E_1^0) \times (E_2, E_2^0); H_Q)$ is by construction the image of the canonical generator by the composite map

$$\begin{aligned} H^Q(\mathbf{R}^Q, \mathbf{R}^Q - 0; H_Q) &\xrightarrow{\text{pr}_2^*} H^Q(X \times (\mathbf{R}^Q, \mathbf{R}^Q - 0); H_Q) \\ &\xrightarrow{(3)^*} H^Q(W, W^0; p^* O^*(\mu)) \overset{\text{exc.}}{\approx} H^Q(E_1\times E_2, E_1\times E_2 - sX; p^* O^*(\mu)) \\ &\xrightarrow{\text{restr.}} H^Q((E_1, E_1^0) \times (E_2, E_2^0); p^* O^*(\mu)). \end{aligned}$$

But this image is clearly U'_1.

The extension of the proofs above to the relative case requires little new. We show how to modify the map ϕ. Note that the assumptions are as before (except that X need not be compact) so that we still have the maps (1)–(8). In particular we get from (6) by restriction a map

$$((E_1\times E_2)_\infty | A \times A', (s_1\times s_2)_\infty A \times A') \to (S^Q, \infty). \tag{9}$$

Secondly we note that the zero section sX of μ and nothing else is mapped by (6) to 0 in S^Q (the antipodal point of ∞). In particular the subset $(E_1 \times E_2)_\infty | A \times B' \cup B \times A'$ is mapped to a closet subset of S^Q bounded away from 0. (Recall that one of the pairs (A, B), (A', B') is compact). Let D be a disk around ∞ in S^Q containing this closed set. Collapsing D gives a new sphere which we also denotes S^Q. Composing (9) with the collapsing map gives a map of $(E_1 \times E_2)_\infty$ to S^Q, which sends both $(s_1 \times s_2)_\infty A \times A'$ and

$$(E_1 \times E_2)_\infty | A \times B' \cup B \times A' \quad \text{to } \infty.$$

Therefore there is a quotient map

$$(T_{\mu_1 \times \mu_2}((A, B) \times (A', B')), *) \to (S^Q, \infty).$$

Identifying the Thom space with the smash product $T_{\mu_1}(A, B) \wedge T_{\mu_2}(A', B')$ finally defines the duality map

$$\phi \colon T_{\mu_1}(A, B) \wedge T_{\mu_2}(A', B') \to S^Q.$$

The rest of the proof now proceeds approximately as before. Lemma 4.5 is replaced by its relativized version, with relative Thom isomorphisms and the Alexander duality (Lemma 1.6)

$$H_{i-q_2}(A', B'; O_2^*) \overset{U_X''/}{\approx} H^{q+q_2-i}(A, B; O^*(X) \otimes O_2^*),$$

where $U_X'' = U_X | (A, B) \times (A', B')$ (since we can replace $H_{i-q_2}(X - B, X - A; O_2^*)$ with $H_{i-q_2}(A', B'; O_2^*)$). Similarly, the spherical class defined by ϕ may be identified with the restriction of U_μ to $(E_{1\infty} | A, s_{1\infty} A \cup E_{1\infty} | B) \times (E_{2\infty} | A', s_{2\infty} A' \cup E_{2\infty} | B')$. The computation in the proof of Lemma 4.5 remains the same. This completes the proof of the S-duality theorem.

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