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MICROBUNDLES ARE FIBRE BUNDLES[1]

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Introduction. In [1], Milnor develops a theory for structures, known as microbundles which generalize vector bundles. It is shown there that this is a proper generalization; that some microbundles cannot be derived from any vector bundle. It is then possible, for instance, to find a substitute (tangent microbundle) for the tangent bundle over a manifold M even though M admits no differential structure.

A well-known and more general class of structures than vector bundles (but less general than microbundles) is the class of fibre bundles with a Euclidean fibre and structural group the origin-preserving homeomorphisms of Euclidean space topologized by the compact-open topology (cf. [2]). In this note such structures will be

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denoted simply as bundles.

Briefly then, our main result is that every microbundle over a complex is isomorphic to a bundle, in fact it contains a bundle and that this bundle is unique. The same result for microbundles over manifolds and, more generally, ANR's in Euclidean space follows easily. In the case of the tangent microbundle over a topological manifold M this means that for each point x in M there is selected a neighborhood U_x which is homeomorphic to Euclidean space and U_x varies continuously with x. In the case of a differentiable manifold this selection is accomplished by means of a Riemannian metric. It was the absence of such a selection, however, that provided impetus to the consideration of microbundles (cf. Introduction in [1]).

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Statement of results. Let R^l denote l-dimensional Euclidean space and $\mathcal{G}$ the space of all imbeddings of R^l into R^l provided with the compact-open topology. Let $\mathcal{G}_0$ be the origin-preserving elements of $\mathcal{G}$, $\mathcal{H}$ those elements of $\mathcal{G}$ whose images are R^l, and $\mathcal{H}_0 = \mathcal{G}_0 \cap \mathcal{H}$.

A microbundle X, having fibre dimension l, $B \to^i E \to^j B$, *admits* a bundle providing there is an open set E_1 in E containing the 0-section $i(B)$ such that $j | E_1 : E_1 \to B$ is a fibre bundle with fibre R^l and structural group $\mathcal{H}_0$. The fibre bundle in this case will be called an *admissible bundle.*

Let T_n be the statement that every microbundle over a locally-finite n-dimensional complex admits a bundle. Let U_n be the statement that any two admissible bundles for the same microbundle over a locally-finite n-dimensional complex are isomorphic. An isomorphism in this case is a homeomorphism between the total spaces preserving fibres and which is the identity on the 0-section.

THEOREM. *T_n and U_n are true for all n.*

The proof will be sketched, with details appearing in a later paper. T_0 and U_0 follow immediately from the fact that microbundles over a 0-dimensional set are all trivial. We finish the proof by showing

(1) T_{n-1} and U_{n-1} imply T_n, and

(2) T_n implies U_n.

The following lemma is used repeatedly.

LEMMA. *There is a map $F: \mathcal{G}_0 \times I \to \mathcal{G}_0$ such that*

(a) $F(g, 0) = g$, *all* $g \in \mathcal{G}_0$,

(b) $F(g, 1) \in \mathcal{H}_0$, *all* $g \in \mathcal{G}_0$,

(c) $F(h, t) \in \mathcal{H}_0$, *all* $h \in \mathcal{H}_0$, $t \in I$.

The proof for the Lemma consists of analyzing the isotopy, which is relatively easy to produce, taking a single element in $\mathcal{G}_0$ into $\mathcal{H}_0$ and expressing the process in a canonical fashion. Then one has to verify that this process yields nearby isotopies if the elements in $\mathcal{G}_0$ are close in the compact-open topology. The full verification is long and somewhat tedious and is therefore suppressed here.

We return to the proof of (1). Let X be a microbundle over a locally-finite n-complex K with diagram: $K \to^i E \to^j K$. For each n-simplex σ in K we find an admissible trivial bundle ξ_σ for $X|\sigma$. Next let D be an open set in E containing $i(K)$ such that $j^{-1}(\sigma) \cap D \subset E(\xi_\sigma)$ the total space of ξ_σ. Let K^{n-1} denote the $(n-1)$-skeleton of K and Y the microbundle $K^{n-1} \to^{i'} j^{-1}(K^{n-1}) \cap D \to^{j'} K^{n-1}$, where i' and j' are the restrictions of i and j. By T_{n-1}, Y admits a bundle η, and by the choice of D, for each point z in $\partial\sigma$, the η-fibre over z is contained in the ξ_σ-fibre over z. $\eta|\partial\sigma$ and $\xi_\sigma|\partial\sigma$ are admissible bundles for $X|\partial\sigma$ and since the second is trivial, by U_{n-1} it follows that $\eta|\partial\sigma$ is trivial also. This permits us to coordinatize both the η-fibres and ξ_σ-fibres over $\partial\sigma$. The inclusion of the former in the latter determines a map of $\partial\sigma$ into $\mathcal{G}_0$ which by the Lemma can be deformed into a map of $\partial\sigma$ into $\mathcal{H}_0$. If σ_1 is a smaller concentric simplex in σ we may regard this deformation as assigning elements in $\mathcal{G}_0$ to points in σ-int σ_1 so that the elements assigned to points in $\partial\sigma_1$ are all in $\mathcal{H}_0$. This correspondence permits us to smooth the fibres in going from the η-fibres over $\partial\sigma$ to the ξ_σ-fibres over σ_1. By repeating this process on each n-simplex σ the bundle η over K^{n-1} can be extended to a bundle over K.

To prove (2) let $\sigma_1, \sigma_2, \cdots, \sigma_\alpha, \cdots$ be a well-ordering of those simplexes in the n-complex K which are not faces of some higher dimensional simplex in K. Let ξ_1 and ξ_2 be two admissible bundles for X, a microbundle over K. By T_n there is no loss in generality in assuming $E(\xi_1) \subset E(\xi_2)$. Let $f_0: E(\xi_1) \to E(\xi_2)$ be the inclusion. Let $N(\sigma_\alpha)$ be the closed star neighborhood of σ_α in the second barycentric subdivision. Let $K_\alpha = \bigcup_{\beta \leqq \alpha} \sigma_\beta$, a subcomplex.

Suppose for each $\beta < \alpha$ we have defined $f_\beta: E(\xi_1) \to E(\xi_2)$, an imbedding, taking fibres into fibres, and f_β is the identity on $i(K)$. Suppose further that $f_\beta|K_\beta$ is an isomorphism from $\xi_1|K_\beta$ onto $\xi_2|K_\beta$ and that for each point p in $E(\xi_1) - j^{-1}(N(\sigma_\beta))$ there is a $\gamma < \beta$ and a neighborhood N of p such that $f_\beta|N = f_{\beta'}|N$ for $\gamma \leqq \beta' \leqq \beta$. We construct f_α satisfying these properties.

Let $g_\alpha: E(\xi_1) \to E(\xi_2)$ be $f_{\alpha-1}$ if $\alpha - 1$ exists. Otherwise $g_\alpha = \text{limit}_{\beta \to \alpha} f_\beta$, which exists since each point in K lies in only finitely-many $N(\sigma_\beta)$. Then $g_\alpha(E(\xi_1))$ is the total space of a bundle η_α in a natural way. Since

$N(\sigma_\alpha)$ is contractible $\eta_\alpha | N(\sigma_\alpha)$ and $\xi_2 | N(\sigma_\alpha)$ are trivial. This allows us to coordinatize the two sets of fibres and the inclusion of the η_α-fibres in the ξ_2-fibres assigns an element g^z in $\mathcal{G}_0$ to every z in $N(\sigma_\alpha)$. Let $t: K \rightarrow I$ be a map such that $t(\sigma_\alpha) = 1$ and $t(K - N(\sigma_\alpha)) = 0$. Using the Lemma we associate with each z in $N(\sigma_\alpha)$ the imbedding $h^z = F(g^z, t(z))$. Then for each z in σ_α, h^z is in $\mathcal{H}_0$, and for z in $\mathrm{Cl}(K - N(\sigma_\alpha)) \cap N(\sigma_\alpha)$, $h^z = g^z$. If we define a homeomorphism h_α from $E(\eta_\alpha)$ to $E(\xi_2)$ by taking the η_α-fibre over z into the ξ_2-fibre over z according to h^z, for each z in $N(\sigma_\alpha)$ and using the inclusion elsewhere, then $f_\alpha = h_\alpha g_\alpha$ satisfies the desired properties of the induction.

The isomorphism from ξ_1 onto ξ_2 is the limit of the f_α. This proves (2) and finishes the proof of the Theorem.

COROLLARY 1. *If B is a neighborhood retract in some Euclidean space, for example a manifold, then any microbundle over B admits a unique bundle.*

PROOF. Let B be a subset of R^n and V an open set in R^n containing B and $\rho: V \rightarrow B$, a retraction. Then if X is a microbundle over B, $\rho^*(X)$ may be regarded as an extension of X to all of V. But V can be triangulated and the Theorem applied to give both the existence and uniqueness.

Let $\mathcal{H}_0^+(n)$ be the orientation-preserving, origin-preserving homeomorphisms of R^n onto R^n. As a consequence of the Theorem and the fact that $k_0 S^8 \rightarrow k_{\mathrm{top}} S^8$ is not an isomorphism [1], we have:

COROLLARY 2. *For large enough n, the homomorphism $\pi_7(\mathrm{SO}(n)) \rightarrow \pi_7(\mathcal{H}_0^+(n))$ induced by inclusion is not an isomorphism.*

Added in proof. The author has learned that B. Mazur has obtained independently a somewhat different proof of the Theorem.

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