MICROLOCAL ANALYSIS OF ASYMPTOTICALLY HYPERBOLIC AND KERR-DE SITTER SPACES

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with an appendix by SEMYON DYATLOV

Abstract. In this paper we develop a general, systematic, microlocal framework for the Fredholm analysis of non-elliptic problems, including high energy (or semiclassical) estimates, which is stable under perturbations. This framework, described in Section 2, resides on a compact manifold without boundary, hence in the standard setting of microlocal analysis.

Many natural applications arise in the setting of non-Riemannian b-metrics in the context of Melrose's b-structures. These include asymptotically de Sitter-type metrics on a blow-up of the natural compactification, Kerr-de Sitter-type metrics, as well as asymptotically Minkowski metrics.

The simplest application is a new approach to analysis on Riemannian or Lorentzian (or indeed, possibly of other signature) conformally compact spaces (such as asymptotically hyperbolic or de Sitter spaces), including a new construction of the meromorphic extension of the resolvent of the Laplacian in the Riemannian case, as well as high energy estimates for the spectral parameter in strips of the complex plane. These results are also available in a follow-up paper which is more expository in nature, [52].

The appendix written by Dyatlov relates his analysis of resonances on exact Kerr-de Sitter space (which then was used to analyze the wave equation in that setting) to the more general method described here.

1. Introduction

In this paper we develop a general microlocal framework which in particular allows us to analyze the asymptotic behavior of solutions of the wave equation on asymptotically Kerr-de Sitter and Minkowski space-times, as well as the behavior of the analytic continuation of the resolvent of the Laplacian on so-called conformally compact spaces. This framework is non-perturbative, and works, in particular, for black holes, for relatively large angular momenta (the restrictions come *purely* from dynamics, and not from methods of analysis of PDE), and also for perturbations of Kerr-de Sitter space, where 'perturbation' is only relevant to the extent that it guarantees that the relevant structures are preserved. In the context of analysis on conformally compact spaces, our framework establishes a Riemannian-Lorentzian duality; in this duality the spaces of different signature are smooth continuations of each other across a boundary at which the differential operator we study has some radial points in the sense of microlocal analysis.

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Since it is particularly easy to state, and only involves Riemannian geometry, we start by giving a result on manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of a compact manifold with boundary X_0 such that near the boundary Y, with a product decomposition nearby and a defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2},$$

where h is a family of metrics on ∂X_0 depending on x in an even manner, i.e. only even powers of x show up in the Taylor series. (There is a much more natural way to phrase the evenness condition, see [28, Definition 1.2].) We also write $X_{0,\mathrm{even}}$ for the manifold X_0 when the smooth structure has been changed so that x^2 is a boundary defining function; thus, a smooth function on X_0 is even if and only if it is smooth when regarded as a function on $X_{0,\mathrm{even}}$. The analytic continuation of the resolvent in this category (but without the evenness condition) was obtained by Mazzeo and Melrose [36], with possibly some essential singularities at pure imaginary half-integers as noticed by Borthwick and Perry [6]. Using methods of Graham and Zworski [26], Guillarmou [28] showed that for even metrics the latter do not exist, but generically they do exist for non-even metrics. Further, if the manifold is actually asymptotic to hyperbolic space (note that hyperbolic space is of this form in view of the Poincaré model), Melrose, Sá Barreto and Vasy [40] showed high energy resolvent estimates in strips around the real axis via a parametrix construction; these are exactly the estimates that allow expansions for solutions of the wave equation in terms of resonances. Estimates just on the real axis were obtained by Cardoso and Vodev for more general conformal infinities [7, 55]. One implication of our methods is a generalization of these results.

Below $\dot{\mathcal{C}}^\infty(X_0)$ denotes 'Schwartz functions' on X_0, i.e. $\mathcal{C}^\infty$ functions vanishing with all derivatives at ∂X_0, and $\mathcal{C}^{-\infty}(X_0)$ is the dual space of 'tempered distributions' (these spaces are naturally identified for X_0 and $X_{0,\mathrm{even}}$), while $H^s(X_{0,\mathrm{even}})$ is the standard Sobolev space on $X_{0,\mathrm{even}}$ (corresponding to extension across the boundary, see e.g. [31, Appendix B], where these are denoted by $\bar{H}^s(X^\circ_{0,\mathrm{even}})$) and $H^s_h(X_{0,\mathrm{even}})$ is the standard semiclassical Sobolev space, so for $h > 0$ fixed this is the same as $H^s(X_{0,\mathrm{even}})$; see [17, 21].

Theorem. *(See Theorem 4.3 for the full statement.) Suppose that X_0 is an $(n-1)$-dimensional manifold with boundary Y with an even Riemannian conformally compact metric g_0. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-2}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition (X_0, g_0) is non-trapping, then non-trapping estimates hold in every strip $|\operatorname{Im}\sigma| < C$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,

$$\|x^{-(n-2)/2+\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+2)/2+\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}. \tag{1.1}$$

If f has compact support in X°_0, the $s-1$ norm on f can be replaced by the $s-2$ norm.

Further, as stated in Theorem 4.3, the resolvent is *semiclassically outgoing* with a loss of h^{-1}, in the sense of recent results of Datchev and Vasy [15] and [16]. This means that for mild trapping (where, in a strip near the spectrum, one has polynomially bounded resolvent for a compactly localized version of the trapped model) one obtains resolvent bounds of the same kind as for the above-mentioned trapped models, and lossless estimates microlocally away from the trapping. In particular, one obtains logarithmic losses compared to non-trapping on the spectrum for hyperbolic trapping in the sense of [58, Section 1.2], and polynomial losses in strips, since for the compactly localized model this was recently shown by Wunsch and Zworski [58].

For conformally compact spaces, without using wave propagation as motivation, our method is to change the smooth structure, replacing x by $\mu = x^2$, conjugate the operator by an appropriate weight as well as remove a vanishing factor of μ, and show that the new operator continues smoothly and non-degenerately (in an appropriate sense) across $\mu = 0$, i.e. Y, to a (non-elliptic) problem which we can analyze utilizing by now almost standard tools of microlocal analysis. These steps are reflected in the form of the estimate (1.1); μ shows up in the evenness, conjugation due to the presence of $x^{-n/2+\imath\sigma}$, and the two halves of the vanishing factor of μ being removed in $x^{\pm 1}$ on the left and right hand sides. This approach is explained in full detail in the more expository and self-contained follow-up article, [52].

However, it is useful to think of a wave equation motivation — then $(n-1)$-dimensional hyperbolic space shows up (essentially) as a model at infinity inside a backward light cone from a fixed point q_+ at future infinity on n-dimensional de Sitter space $\hat{M}$, see [51, Section 7], where this was used to construct the Poisson operator. More precisely, the light cone is singular at q_+, so to desingularize it, consider $[\hat{M}; \{q_+\}]$. After a Mellin transform in the defining function of the front face; the model continues smoothly across the light cone Y inside the front face of $[\hat{M}; \{q_+\}]$. The inside of the light cone corresponds to $(n-1)$-dimensional hyperbolic space (after conjugation, etc.) while the exterior is (essentially) $(n-1)$-dimensional de Sitter space; Y is the 'boundary' separating them. Here Y should be thought of as the event horizon in black hole terms (there is nothing more to event horizons in terms of local geometry!).

The resulting operator P_σ has radial points at the conormal bundle $N^*Y \setminus o$ of Y in the sense of microlocal analysis, i.e. the Hamilton vector field is radial at these points, i.e. is a multiple of the generator of dilations of the fibers of the cotangent bundle there. However, tools exist to deal with these, going back to Melrose's geometric treatment of scattering theory on asymptotically Euclidean spaces [38]. Note that $N^*Y \setminus o$ consists of two components, Λ_+, resp. Λ_-, and in $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ the images, L_+, resp. L_-, of these are sinks, resp. sources, for the Hamilton flow. At $L_\pm$ one has choices regarding the direction one wants to propagate estimates (into or out of the radial points), which directly correspond to working with strong or weak Sobolev spaces. For the present problem, the relevant choice is propagating estimates *away from* the radial points, thus working with the 'good' Sobolev spaces (which can be taken to have as positive order as one wishes; there is a minimum amount of regularity imposed by our choice of propagation direction, cf. the requirement $s > \frac{1}{2} + C$ above (1.1)). All other points are either elliptic, or microhyperbolic. It remains to either deal with the non-compactness of the 'far end' of the $(n-1)$-dimensional de Sitter space — or instead, as is indeed more convenient when one wants to deal with more singular geometries, adding complex absorbing potentials, in the spirit of works of Nonnenmacher and Zworski [43] and Wunsch and Zworski [58]. In fact, the complex absorption could be replaced by adding a space-like boundary, see Remark 2.5, but for many microlocal purposes complex absorption is more desirable, hence we follow the latter method. However, crucially, these complex absorbing techniques (or the addition of a space-like boundary) already enter in the non-semiclassical problem in our case, as we are in a non-elliptic setting.

One can reverse the direction of the argument and analyze the wave equation on an $(n-1)$-dimensional even asymptotically de Sitter space X_0' by extending it across the boundary, much like the the Riemannian conformally compact space X_0 is extended in this approach. Then, performing microlocal propagation in the opposite direction, which amounts to working with the adjoint operators that we already need in order to prove existence of solutions for the Riemannian spaces[1], we obtain existence, uniqueness and structure results for asymptotically de Sitter spaces, recovering a large part[2] of the results of [51]. Here we only briefly indicate this method of analysis in Remark 4.6.

In other words, we establish a Riemannian-Lorentzian duality, that will have counterparts both in the pseudo-Riemannian setting of higher signature and in higher rank symmetric spaces, though in the latter the analysis might become more complicated. Note that asymptotically hyperbolic and de Sitter spaces are not connected by a 'complex rotation' (in the sense of an actual deformation); they are smooth continuations of each other in the sense we just discussed.

To emphasize the simplicity of our method, we list all of the microlocal techniques (which are relevant both in the classical and in the semiclassical setting) that we use on a *compact manifold without boundary*; in all cases *only microlocal Sobolev estimates* matter (not parametrices, etc.):

(i) Microlocal elliptic regularity.
(ii) Microhyperbolic propagation of singularities.
(iii) *Rough* analysis at a Lagrangian invariant under the Hamilton flow which roughly behaves like a collection of radial points, though the internal structure does not matter, in the spirit of [38, Section 9].
(iv) Complex absorbing 'potentials' in the spirit of [43] and [58].

These are almost 'off the shelf' in terms of modern microlocal analysis, and thus our approach, from a microlocal perspective, is quite simple. We use these to show that on the continuation across the boundary of the conformally compact space we have a Fredholm problem, on a perhaps slightly exotic function space, which however is (perhaps apart from the complex absorption) the simplest possible coisotropic function space based on a Sobolev space, with order dictated by the radial points. Also, we propagate the estimates along bicharacteristics in different directions depending on the component $\Sigma_\pm$ of the characteristic set under consideration; correspondingly the sign of the complex absorbing 'potential' will vary with

[1]This adjoint analysis also shows up for Minkowski space-time as the 'original' problem.

[2]Though not the parametrix construction for the Poisson operator, or for the forward fundamental solution of Baskin [1]; for these we would need a parametrix construction in the present compact boundaryless, but analytically non-trivial (for this purpose), setting.

$\Sigma_\pm$, which is perhaps slightly unusual. However, this is completely parallel to solving the standard Cauchy, or forward, problem for the wave equation, where one propagates estimates in *opposite* directions relative to the Hamilton vector field in the two components.

The complex absorption we use modifies the operator P_σ outside $X_{0,\mathrm{even}}$. However, while $(P_\sigma - \imath Q_\sigma)^{-1}$ depends on Q_σ, its behavior on $X_{0,\mathrm{even}}$, and even near $X_{0,\mathrm{even}}$, is independent of this choice; see the proof of Proposition 4.2 for a detailed explanation. In particular, although $(P_\sigma - \imath Q_\sigma)^{-1}$ may have resonances other than those of $\mathcal{R}(\sigma)$, the resonant states of these additional resonances are supported outside $X_{0,\mathrm{even}}$, hence do not affect the singular behavior of the resolvent in $X_{0,\mathrm{even}}$.

While the results are stated for the scalar equation, analogous results hold for operators on natural vector bundles, such as the Laplacian on differential forms. This is so because the results work if the principal symbol of the extended problem is scalar with the demanded properties, and the imaginary part of the subprincipal symbol is either scalar at the 'radial sets', or instead satisfies appropriate estimates (as an endomorphism of the pull-back of the vector bundle to the cotangent bundle) at this location; see Remark 2.1. The only change in terms of results on asymptotically hyperbolic spaces is that the threshold $(n-2)^2/4$ is shifted; in terms of the explicit conjugation of Subsection 4.9 this is so because of the change in the first order term in (4.29).

While here we mostly consider conformally compact Riemannian or Lorentzian spaces (such as hyperbolic space and de Sitter space) as appropriate boundary values (Mellin transform) of a blow-up of de Sitter space of one higher dimension, they also show up as a boundary value of Minkowski space. This is related to Wang's work on b-regularity [57], though Wang worked on a blown up version of Minkowski space-time; she also obtained her results for the (non-linear) Einstein equation there. It is also related to the work of Fefferman and Graham [22] on conformal invariants by extending an asymptotically hyperbolic manifold to Minkowski-type spaces of one higher dimension. We discuss asymptotically Minkowski spaces briefly in Section 5.

Apart from trapping — which is well away from the event horizons for black holes that do not rotate too fast — the microlocal structure on de Sitter space is *exactly* the same as on Kerr-de Sitter space, or indeed Kerr space near the event horizon. (Kerr space has a Minkowski-type end as well; although Minkowski space also fits into our framework, it does so a different way than Kerr at the event horizon, so the result there is not immediate; see the comments below.) This is to be understood as follows: from the perspective we present here (as opposed to the perspective of [51]), the tools that go into the analysis of de Sitter space-time suffice also for Kerr-de Sitter space, and indeed a much wider class, apart from the need to deal with trapping. The trapping itself was analyzed by Wunsch and Zworski [58]; their work fits immediately with our microlocal methods. Phenomena such as the ergosphere are mere shadows of dynamics in the phase space which is barely changed, but whose projection to the base space (physical space) undergoes serious changes. It is thus of great value to work microlocally, although it is certainly possible that for some non-linear purposes it is convenient to rely on physical space to the maximum possible extent, as was done in the recent (linear) works of Dafermos and Rodnianski [13, 14].

Below we state theorems for Kerr-de Sitter space time. However, it is important to note that all of these theorems have analogues in the general microlocal framework discussed in Section 2. In particular, analogous theorems hold on conjugated, re-weighted, and even versions of Laplacians on conformally compact spaces (of which one example was stated above as a theorem), and similar results apply on 'asymptotically Minkowski' spaces, with the slight twist that it is adjoints of operators considered here that play the direct role there.

We now turn to Kerr-de Sitter space-time and give some history. In exact Kerr-de Sitter space and for small angular momentum, Dyatlov [20, 19] has shown exponential decay to constants, even across the event horizon. This followed earlier work of Melrose, Sá Barreto and Vasy [39], where this was shown up to the event horizon in de Sitter-Schwarzschild space-times or spaces strongly asymptotic to these (in particular, no rotation of the black hole is allowed), and of Dafermos and Rodnianski in [11] who had shown polynomial decay in this setting. These in turn followed up pioneering work of Sá Barreto and Zworski [46] and Bony and Häfner [5] who studied resonances and decay away from the event horizon in these settings. (One can solve the wave equation explicitly on de Sitter space using special functions, see [44] and [59]; on asymptotically de Sitter spaces the forward fundamental solution was constructed as an appropriate Lagrangian distribution by Baskin [1].)

Also, polynomial decay on Kerr space was shown recently by Tataru and Tohaneanu [49, 48] and Dafermos and Rodnianski [13, 14], after pioneering work of Kay and Wald in [32] and [56] in the Schwarzschild setting. (There was also recent work by Marzuola, Metcalf, Tataru and Tohaneanu [35] on Strichartz estimates, and by Donninger, Schlag and Soffer [18] on L^∞ estimates on Schwarzschild black holes, following L^∞ estimates of Dafermos and Rodnianski [12, 10], of Blue and Soffer [4] on non-rotating charged black holes giving L^6 estimates, and Finster, Kamran, Smoller and Yau [23, 24] on Dirac waves on Kerr.) While some of these papers employ microlocal methods at the trapped set, they are mostly based on physical space where the phenomena are less clear than in phase space (unstable tools, such as separation of variables, are often used in phase space though). We remark that Kerr space is less amenable to immediate microlocal analysis to attack the decay of solutions of the wave equation due to the singular/degenerate behavior at zero frequency, which will be explained below briefly. This is closely related to the behavior of solutions of the wave equation on Minkowski space-times. Although our methods also deal with Minkowski space-times, this holds in a slightly different way than for de Sitter (or Kerr-de Sitter) type spaces at infinity, and combining the two ingredients requires some additional work. On perturbations of Minkowski space itself, the full non-linear analysis was done in the path-breaking work of Christodoulou and Klainerman [9], and Lindblad and Rodnianski simplified the analysis [33, 34], Bieri [2, 3] succeeded in relaxing the decay conditions, while Wang [57] obtained additional, b-type, regularity as already mentioned. Here we only give a linear result, but hopefully its simplicity will also shed new light on the non-linear problem.

As already mentioned, a microlocal study of the trapping in Kerr or Kerr-de Sitter was performed by Wunsch and Zworski in [58]. This is particularly important to us, as this is the only part of the phase space which does not fit directly into a relatively simple microlocal framework. Our general method is to use microlocal analysis to understand the rest of the phase space (with localization away from trapping realized via a complex absorbing potential), then use the gluing result of Datchev and Vasy [15] to obtain the full result.

Slightly more concretely, in the appropriate (partial) compactification of space-time, near the boundary of which space-time has the form $X_\delta \times [0, \tau_0)_\tau$, where X_δ denotes an extension of the space-time across the event horizon. Thus, there is a manifold with boundary X_0, whose boundary Y is the event horizon, such that X_0 is embedded into X_δ, a (non-compact) manifold without boundary. We write $X_+ = X_0^\circ$ for 'our side' of the event horizon and $X_- = X_\delta \setminus X_0$ for the 'far side'. Then the Kerr or Kerr-de Sitter d'Alembertians are b-operators in the sense of Melrose [42] that extend smoothly across the event horizon Y. Recall that in the Riemannian setting, b-operators are usually called 'cylindrical ends', see [42] for a general description; here the form at the boundary (i.e. 'infinity') is similar, modulo ellipticity (which is lost). Our results hold for small smooth perturbations of Kerr-de Sitter space in this b-sense. Here the role of 'perturbations' is simply to ensure that the microlocal picture, in particular the dynamics, has not changed drastically. Although b-analysis is the right conceptual framework, we mostly work with the Mellin transform, hence on manifolds without boundary, so the reader need not be concerned about the lack of familiarity with b-methods. However, we briefly discuss the basics in Section 3.

We *immediately* Mellin transform in the defining function of the boundary (which is temporal infinity, though is not space-like everywhere) — in Kerr and Kerr-de Sitter spaces this is operation is 'exact', corresponding to $\tau\partial_\tau$ being a Killing vector field, i.e. is not merely at the level of normal operators, but this makes little difference (i.e. the general case is similarly treatable). After this transform we get a family of operators that e.g. in de Sitter space is elliptic on X_+, but in Kerr space ellipticity is lost there. We consider the event horizon as a completely artificial boundary even in the de Sitter setting, i.e. work on a manifold that includes a neighborhood of $X_0 = \overline{X_+}$, hence a neighborhood of the event horizon Y.

As already mentioned, one feature of these space-times is some relatively mild trapping in X_+; this only plays a role in high energy (in the Mellin parameter, σ), or equivalently semiclassical (in $h = |\sigma|^{-1}$) estimates. We ignore a (semiclassical) microlocal neighborhood of the trapping for a moment; we place an absorbing 'potential' there. Another important feature of the space-times is that they are not naturally compact on the 'far side' of the event horizon (inside the black hole), i.e. X_-, and bicharacteristics from the event horizon (classical or semiclassical) propagate into this region. However, we place an absorbing 'potential' (a second order operator) there to annihilate such phenomena which do not affect what happens on 'our side' of the event horizon, X_+, in view of the characteristic nature of the latter. This absorbing 'potential' could *easily* be replaced by a space-like boundary, in the spirit of introducing a boundary $t = t_1$, where $t_1 > t_0$, when one solves the Cauchy problem from t_0 for the standard wave equation; note that such a boundary does not affect the solution of the equation in $[t_0, t_1]_t$. Alternatively, if X_- has a well-behaved infinity, such as in de Sitter space, the analysis could be carried out more globally. However, as we wish to emphasize the microlocal simplicity of the problem, we do not touch on these issues.

All of our results are in a general setting of microlocal analysis explained in Section 2, with the Mellin transform and Lorentzian connection explained in Section 3.

However, for the convenience of the reader here we state the results for perturbations of Kerr-de Sitter spaces. We refer to Section 6 for details. First, the general assumption is that

> P_σ, $\sigma \in \mathbb{C}$, is either the Mellin transform of the d'Alembertian $\Box_g$ for a Kerr-de Sitter spacetime, or more generally the Mellin transform of the normal operator of the d'Alembertian $\Box_g$ for a small perturbation, in the sense of b-metrics, of such a Kerr-de Sitter space-time;

see Section 3 for an explanation of these concepts. Note that for such perturbations the usual 'time' Killing vector field (denoted by $\partial_{\tilde{t}}$ in Section 6; this is indeed time-like in $X_+ \times [0,\epsilon)_{\tilde{t}}$ sufficiently far from ∂X_+) is no longer Killing. Our results on these space-times are proved by showing that the hypotheses of Section 2 are satisfied. We show this in general (under the conditions (6.2), which corresponds to $0 < \frac{9}{4}\Lambda r_s^2 < 1$ in de Sitter-Schwarzschild spaces, and (6.12), which corresponds to the lack of classical trapping in X_+; see Section 6), except where semiclassical dynamics matters. As in the analysis of Riemannian conformally compact spaces, we use a complex absorbing operator Q_σ; this means that its principal symbol in the relevant (classical, or semiclassical) sense has the correct sign on the characteristic set; see Section 2.

When semiclassical dynamics does matter, the *non-trapping assumption* with an absorbing operator Q_σ, $\sigma = h^{-1}z$, is

> in both the forward and backward directions, the bicharacteristics from any point in the semiclassical characteristic set of P_σ either enter the semiclassical elliptic set of Q_σ at some finite time, or tend to $L_\pm$;

see Definition 2.11. Here, as in the discussion above, $L_\pm$ are two components of the image of $N^*Y \setminus o$ in S^*X. (As L_+ is a sink while L_- is a source, even semiclassically, outside $L_\pm$ the 'tending' can only happen in the forward, resp. backward, directions.) Note that the semiclassical non-trapping assumption (in the precise sense used below) implies a classical non-trapping assumption, i.e. the analogous statement for classical bicharacteristics, i.e. those in S^*X. It is important to keep in mind that the classical non-trapping assumption can always be satisfied with Q_σ supported in X_-, far from Y.

In our first result in the Kerr-de Sitter type setting, to keep things simple, we ignore semiclassical trapping via the use of Q_σ; this means that Q_σ will have support in X_+. However, in X_+, Q_σ only matters in the semiclassical, or high energy, regime, and only for (almost) real σ. If the black hole is rotating relatively slowly, e.g. α satisfies the bound (6.22), the (semiclassical) trapping is always far from the event horizon, and one can make Q_σ supported away from there. Also, the Klein-Gordon parameter λ below is 'free' in the sense that it does not affect any of the relevant information in the analysis (principal and subprincipal symbol; see below). *Thus, we drop it in the following theorems for simplicity.*

Theorem 1.1. *Let Q_σ be an absorbing formally self-adjoint operator such that the semiclassical non-trapping assumption holds. Let $\sigma_0 \in \mathbb{C}$, and*

$$\mathcal{X}^s = \{u \in H^s :\ (P_{\sigma_0} - \imath Q_{\sigma_0})u \in H^{s-1}\},\ \mathcal{Y}^s = H^{s-1},$$
$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|(P_{\sigma_0} - \imath Q_{\sigma_0})u\|^2_{H^{s-1}}.$$

Let $\beta_\pm > 0$ be given by the geometry at conormal bundle of the black hole (−), resp. de Sitter (+) event horizons, see Subsection 6.1, and in particular (6.9)*. For $s \in \mathbb{R}$, let*[3] *$\beta = \max(\beta_+, \beta_-)$ if $s \geq 1/2$, $\beta = \min(\beta_+, \beta_-)$ if $s < 1/2$. Then, for $\lambda \in \mathbb{C}$,*

$$P_\sigma - \imath Q_\sigma - \lambda : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on

$$\mathbb{C}_s = \{\sigma \in \mathbb{C} : \operatorname{Im}\sigma > \beta^{-1}(1-2s)\} \tag{1.2}$$

and has a meromorphic inverse,

$$R(\sigma) = (P_\sigma - \imath Q_\sigma - \lambda)^{-1},$$

which is holomorphic in an upper half plane, $\operatorname{Im}\sigma > C$. Moreover, given any $C' > 0$, there are only finitely many poles in $\operatorname{Im}\sigma > -C'$, and the resolvent satisfies non-trapping estimates there, which e.g. with $s = 1$ (which might need a reduction in $C' > 0$) take the form

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)\|^2_{L^2} \leq C''|\sigma|^{-2}\|f\|^2_{L^2}.$$

The analogous result also holds on Kerr space-time if we suppress the Euclidean end by a complex absorption.

Dropping the semiclassical absorption in X_+, i.e. if we make Q_σ supported only in X_-, we have[4]

Theorem 1.2. *Let P_σ, β, $\mathbb{C}_s$ be as in Theorem 1.1, and let Q_σ be an absorbing formally self-adjoint operator supported in X_- which is classically non-trapping. Let $\sigma_0 \in \mathbb{C}$, and*

$$\mathcal{X}^s = \{u \in H^s : (P_{\sigma_0} - \imath Q_{\sigma_0})u \in H^{s-1}\},\ \mathcal{Y}^s = H^{s-1},$$

with

$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|\tilde{P}u\|^2_{H^{s-1}}.$$

Then,

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on $\mathbb{C}_s$, and has a meromorphic inverse,

$$R(\sigma) = (P_\sigma - \imath Q_\sigma)^{-1},$$

which for any $\epsilon > 0$ is holomorphic in a translated sector in the upper half plane, $\operatorname{Im}\sigma > C + \epsilon|\operatorname{Re}\sigma|$. The poles of the resolvent are called resonances*. In addition, taking $s = 1$ for instance, $R(\sigma)$ satisfies non-trapping estimates, e.g. with $s = 1$,*

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)\|^2_{L^2} \leq C'|\sigma|^{-2}\|f\|^2_{L^2}$$

in such a translated sector.

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[3]This means that we require the stronger of $\operatorname{Im}\sigma > \beta_\pm^{-1}(1-2s)$ to hold in (1.2). If we perturb Kerr-de Sitter space time, we need to increase the requirement on $\operatorname{Im}\sigma$ slightly, i.e. the size of the half space has to be slightly reduced.

[4]Since we are not making a statement for almost real σ, semiclassical trapping, discussed in the previous paragraph, does not matter.

It is in this setting that Q_σ could be replaced by working on a manifold with boundary, with the boundary being space-like, essentially as a time level set mentioned above, since it is supported in X_-.

Now we make the assumption that *the only semiclassical trapping is due to hyperbolic trapping with trapped set* Γ_z, $\sigma = h^{-1}z$, with hyperbolicity understood as in the 'Dynamical Hypotheses' part of [58, Section 1.2], i.e.

> in both the forward and backward directions, the bicharacteristics from any point in the semiclassical characteristic set of P_σ either enter the semiclassical elliptic set of Q_σ at some finite time, or tend to $L_\pm \cup \Gamma_z$.

We remark that just hyperbolicity of the trapped set suffices for the results of [58], see Section 1.2 of that paper; however, if one wants stability of the results under perturbations, one needs to assume that Γ_z is *normally hyperbolic.* We refer to [58, Section 1.2] for a discussion of these concepts. We show in Section 6 that for black holes satisfying (6.22) (so the angular momentum can be comparable to the mass) the operators Q_σ can be chosen so that they are supported in X_- (even quite far from Y) and the hyperbolicity requirement is satisfied. Further, we also show that for slowly rotating black holes the trapping is normally hyperbolic. Moreover, the (normally) hyperbolic trapping statement is purely in Hamiltonian dynamics, not regarding PDEs. It might be known for an even larger range of rotation speeds, but the author is not aware of this.

Under this assumption, one can combine Theorem 1.1 with the results of Wunsch and Zworski [58] about hyperbolic trapping and the gluing results of Datchev and Vasy [15] to obtain a better result for the merely spatially localized problem, Theorem 1.2:

Theorem 1.3. *Let P_σ, Q_σ, β, $\mathbb{C}_s$, $\mathcal{X}^s$ and $\mathcal{Y}^s$ be as in Theorem 1.2, and assume that the only semiclassical trapping is due to hyperbolic trapping. Then,*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on $\mathbb{C}_s$, and has a meromorphic inverse,

$$R(\sigma) = (P_\sigma - \imath Q_\sigma)^{-1},$$

which is holomorphic in an upper half plane, $\operatorname{Im}\sigma > C$. Moreover, there exists $C' > 0$ such that there are only finitely many poles in $\operatorname{Im}\sigma > -C'$, and the resolvent satisfies polynomial estimates there as $|\sigma| \to \infty$, $|\sigma|^{\varkappa}$, for some $\varkappa > 0$, compared to the non-trapping case, with merely a logarithmic loss compared to non-trapping for real σ, e.g. with $s = 1$:

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)\|^2_{L^2} \leq C''|\sigma|^{-2}(\log|\sigma|)^2\|f\|^2_{L^2}.$$

Farther, there are approximate lattices of poles generated by the trapping, as studied by Sá Barreto and Zworski in [46], and further by Bony and Häfner in [5], in the exact De Sitter-Schwarzschild and Schwarzschild settings, and in ongoing work by Dyatlov in the exact Kerr-de Sitter setting.

Theorem 1.3 immediately and directly gives the asymptotic behavior of solutions of the wave equation across the event horizon. Namely, the asymptotics of the wave equation depends on the finite number of resonances; their precise behavior depends on specifics of the space-time, i.e. on these resonances. This is true even in arbitrarily regular b-Sobolev spaces – in fact, the more decay we want to show, the higher Sobolev spaces we need to work in. Thus, a forteriori, this gives L^∞ estimates. We state this formally as a theorem in the simplest case of slow rotation; in the general case one needs to analyze the (finite!) set of resonances along the reals to obtain such a conclusion, and for the perturbation part also to show normal hyperbolicity (which we only show for slow rotation):

Theorem 1.4. *Let M_δ be the partial compactification of Kerr-de Sitter space as in Section 6, with τ the boundary defining function. Suppose that g is either a slowly rotating Kerr-de Sitter metric, or a small perturbation as a symmetric bilinear form on ${}^{\mathrm{b}}TM_\delta$. Then there exist $C' > 0$, $\varkappa > 0$ such that for $0 < \epsilon < C'$ and $s > (1+\beta\epsilon)/2$ solutions of $\Box_g u = f$ with $f \in \tau^\epsilon H_{\mathrm{b}}^{s-1+\varkappa}(M_\delta)$ vanishing in $\tau > \tau_0$, and with u vanishing in $\tau > \tau_0$, satisfy that for some constant c_0,*

$$u - c_0 \in \tau^\epsilon H^s_{\mathrm{b,loc}}(M_\delta).$$

In special geometries (without the ability to add perturbations) such decay has been described by delicate separation of variables techniques, again see Bony-Häfner [5] in the De Sitter-Schwarzschild and Schwarzschild settings, but only away from the event horizons, and by Dyatlov [20, 19] in the Kerr-de Sitter setting. Thus, in these settings, we recover in a direct manner Dyatlov's result across the event horizon [19], modulo a knowledge of resonances near the origin contained in [20]. In fact, for small angular momenta one can use the results from de Sitter-Schwarzschild space directly to describe these finitely many resonances, as exposed in the works of Sá Barreto and Zworski [46], Bony and Häfner [5] and Melrose, Sá Barreto and Vasy [39], since 0 is an isolated resonance with multiplicity 1 and eigenfunction 1; this persists under small deformations, i.e. for small angular momenta. Thus, exponential decay to constants, Theorem 1.4, follows immediately.

One can also work with Kerr space-time, apart from issues of analytic continuation. By using weighted spaces and Melrose's results from [38] as well as those of Vasy and Zworski in the semiclassical setting [54], one easily gets an analogue of Theorem 1.2 in $\operatorname{Im}\sigma > 0$, with smoothness and the almost non-trapping estimates corresponding to those of Wunsch and Zworski [58] down to $\operatorname{Im}\sigma = 0$ for $|\operatorname{Re}\sigma|$ large. Since a proper treatment of this would exceed the bounds of this paper, we refrain from this here. Unfortunately, even if this analysis were carried out, low energy problems would still remain, so the result is not strong enough to deduce the wave expansion. As already alluded to, Kerr space-time has features of both Minkowski and de Sitter space-times; though both of these fit into our framework, they do so in different ways, so a better way of dealing with the Kerr space-time, namely adapting our methods to it, requires additional work.

While de Sitter-Schwarzschild space (the special case of Kerr-de Sitter space with vanishing rotation), via the same methods as those on de Sitter space which give rise to the hyperbolic Laplacian and its continuation across infinity, gives rise essentially to the Laplacian of a conformally compact metric, with similar structure but different curvature at the two ends (this was used by Melrose, Sá Barreto and Vasy [39] to do analysis up to the event horizon there), the analogous problem for Kerr-de Sitter is of edge-type in the sense of Mazzeo's edge calculus [37] apart from a degeneracy at the poles corresponding to the axis of rotation, though it is not Riemannian. Note that edge operators have global properties in the fibers; in this case these fibers are the orbits of rotation. A reasonable interpretation of the appearance of this class of operators is that the global properties in the fibers capture non-constant (or non-radial) bicharacteristics (in the classical sense) in the conormal bundle of the event horizon, and also possibly the (classical) bicharacteristics entering X_+. This suggests that the methods of Melrose, Sá Barreto and Vasy [39] would be much harder to apply in the presence of rotation.

It is important to point out that the results of this paper are stable under small $\mathcal{C}^\infty$ perturbations[5] of the Lorentzian metric on the b-cotangent bundle at the cost of changing the function spaces slightly; this follows from the estimates being stable in these circumstances. Note that the function spaces depend on the principal symbol of the operator under consideration, and the range of σ depends on the subprincipal symbol at the conormal bundle of the event horizon; under general small smooth perturbations, defining the spaces exactly as before, the results remain valid if the range of σ is slightly restricted.

In addition, the method is stable under gluing: already Kerr-de Sitter space behaves as two separate black holes (the Kerr and the de Sitter end), connected by semiclassical dynamics; since only one component (say $\Sigma_{\hbar,+}$) of the semiclassical characteristic set moves far into X_+, one can easily add as many Kerr black holes as one wishes by gluing beyond the reach of the other component, $\Sigma_{\hbar,-}$. Theorems 1.1 and 1.2 automatically remain valid (for the semiclassical characteristic set is then irrelevant), while Theorem 1.3 remains valid provided that the resulting dynamics only exhibits mild trapping (so that compactly localized models have at most polynomial resolvent growth), such as normal hyperbolicity, found in Kerr-de Sitter space.

Since the specifics of Kerr-de Sitter space-time are, as already mentioned, irrelevant in the microlocal approach we take, we start with the abstract microlocal discussion in Section 2, which is translated into the setting of the wave equation on manifolds with a Lorentzian b-metric in Section 3, followed by the description of de Sitter, Minkowski and Kerr-de Sitter space-times in Sections 4, 5 and 6. Theorems 1.1-1.4 are proved in Section 6 by showing that they fit into the abstract framework of Section 2; the approach is completely analogous to de Sitter and Minkowski spaces, where the fact that they fit into the abstract framework is shown in Sections 4 and 5. As another option, we encourage the reader to read the discussion of de Sitter space first, which also includes the discussion of conformally compact spaces, presented in Section 4, as well as Minkowski space-time presented in the section afterwards, to gain some geometric insight, then the general microlocal machinery, and finally the Kerr discussion to see how that space-time fits into our setting. Finally, if the reader is interested how conformally compact metrics fit into the framework and wants to jump to the relevant calculation, a reasonable place to start is Subsection 4.9. We emphasize that for the conformally compact results, only Section 2 and Section 4.4-4.9, starting with the paragraph of (4.8), are strictly needed.

2. Microlocal framework

We now develop a setting which includes the geometry of the 'spatial' model of de Sitter space near its 'event horizon', as well as the model of Kerr and Kerr-de Sitter settings near the event horizon, and the model at infinity for Minkowski space-time near the light cone (corresponding to the adjoint of the problem described below in

[5]Certain kinds of perturbations conormal to the boundary, in particular polyhomogeneous ones, would only change the analysis and the conclusions slightly.

the last case). As a general reference for microlocal analysis, we refer to [31], while for semiclassical analysis, we refer to [17, 21]; see also [47] for the high-energy (or large parameter) point of view.

2.1. **Notation.** We recall the basic conversion between these frameworks. First, $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ is the set of $\mathcal{C}^\infty$ functions on $\mathbb{R}^p_z \times \mathbb{R}^\ell_\zeta$ satisfying uniform bounds

$$|D_z^\alpha D_\zeta^\beta a| \le C_{\alpha\beta}\langle\zeta\rangle^{k-|\beta|},\ \alpha\in\mathbb{N}^p,\ \beta\in\mathbb{N}^\ell.$$

If $O\subset\mathbb{R}^p$ and $\Gamma\subset\mathbb{R}^\ell_\zeta$ are open, we define $S^k(O;\Gamma)$ by requiring[6] these estimates to hold only for $z\in O$ and $\zeta\in\Gamma$. The class of classical (or one-step polyhomogeneous) symbols is the subset $S^k_{\mathrm{cl}}(\mathbb{R}^p;\mathbb{R}^\ell)$ of $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ consisting of symbols possessing an asymptotic expansion

$$a(z,r\omega)\sim\sum a_j(z,\omega)r^{k-j},$$

where $a_j\in\mathcal{C}^\infty(\mathbb{R}^p\times\mathbb{S}^{\ell-1})$. Then on $\mathbb{R}^n_z$, pseudodifferential operators $A\in\Psi^k(\mathbb{R}^n)$ are of the form

$$A=\mathrm{Op}(a);\ \mathrm{Op}(a)u(z)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta}a(z,\zeta)\,u(z')\,d\zeta\,dz',$$
$$u\in\mathcal{S}(\mathbb{R}^n),\ a\in S^k(\mathbb{R}^n;\mathbb{R}^n);$$

understood as an oscillatory integral. Classical pseudodifferential operators, $A\in\Psi^k_{\mathrm{cl}}(\mathbb{R}^n)$, form the subset where a is a classical symbol. The principal symbol $\sigma_k(A)$ of $A\in\Psi^k(\mathbb{R}^n)$ is the equivalence class $[a]$ of a in $S^k(\mathbb{R}^n;\mathbb{R}^n)/S^{k-1}(\mathbb{R}^n;\mathbb{R}^n)$. For classical a, one can instead consider $a_0(z,\omega)r^k$ as the principal symbol; it is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, which is homogeneous of degree k with respect to the $\mathbb{R}^+$-action given by dilations in the second factor, $\mathbb{R}^n\setminus\{0\}$.

Differential operators on $\mathbb{R}^n$ form the subset of $\Psi(\mathbb{R}^n)$ in which a is polynomial in the second factor, $\mathbb{R}^n_\zeta$, so locally

$$A=\sum_{|\alpha|\le k}a_\alpha(z)D_z^\alpha,\qquad \sigma_k(A)=\sum_{|\alpha|=k}a_\alpha(z)\zeta^\alpha.$$

If X is a manifold, one can transfer these definitions to X by localization and requiring that the Schwartz kernels are $\mathcal{C}^\infty$ densities away from the diagonal in $X^2=X\times X$; then $\sigma_k(A)$ is in $S^k(T^*X)/S^{k-1}(T^*X)$, resp. $S^k_{\mathrm{hom}}(T^*X\setminus o)$ when $A\in\Psi^k(X)$, resp. $A\in\Psi^k_{\mathrm{cl}}(X)$; here o is the zero section, and hom stands for symbols homogeneous with respect to the $\mathbb{R}^+$ action. If A is a differential operator, then the classical (i.e. homogeneous) version of the principal symbol is a homogeneous polynomial in the fibers of the cotangent bundle of degree k. We can also work with operators depending on a parameter $\lambda\in O$ by replacing $a\in S^k(\mathbb{R}^n;\mathbb{R}^n)$ by $a\in S^k(\mathbb{R}^n\times O;\mathbb{R}^n)$, with $\mathrm{Op}(a_\lambda)\in\Psi^k(\mathbb{R}^n)$ smoothly dependent on $\lambda\in O$. In the case of differential operators, a_α would simply depend smoothly on the parameter λ.

The large parameter, or high energy, version of this, with the large parameter denoted by σ, is that

$$A^{(\sigma)}=\mathrm{Op}^{(\sigma)}(a),\ \mathrm{Op}^{(\sigma)}(a)u(z)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta}a(z,\zeta,\sigma)\,u(z')\,d\zeta\,dz',$$
$$u\in\mathcal{S}(\mathbb{R}^n),\ a\in S^k(\mathbb{R}^n;\mathbb{R}^n_\zeta\times\Omega_\sigma),$$

[6]Another possibility would be to require uniform estimates on compact subsets; this makes no difference here.

where $\Omega\subset\mathbb{C}$, with $\mathbb{C}$ identified with $\mathbb{R}^2$; thus there are joint symbol estimates in ζ and σ. The high energy principal symbol now should be thought of as an equivalence class of functions on $\mathbb{R}^n_z\times\mathbb{R}^n_\zeta\times\Omega_\sigma$, or invariantly on $T^*X\times\Omega$. Differential operators with polynomial dependence on σ now take the form

$$A^{(\sigma)}=\sum_{|\alpha|+j\leq k}a_{\alpha,j}(z)\sigma^j D_z^\alpha,\qquad \sigma_k^{(\sigma)}(A)=\sum_{|\alpha|+j=k}a_{\alpha,j}(z)\sigma^j\zeta^\alpha. \tag{2.1}$$

Note that the principal symbol includes terms that would be subprincipal with $A^{(\sigma)}$ considered as a differential operator for a fixed value of σ.

The semiclassical operator algebra[7], $\Psi_\hbar(\mathbb{R}^n)$, is given by

$$A_h=\mathrm{Op}_\hbar(a);\ \mathrm{Op}_\hbar(a)u(z)=(2\pi h)^{-n}\int_{\mathbb{R}^n}e^{i(z-z')\cdot\zeta/h}a(z,\zeta,h)\,u(z')\,d\zeta\,dz',$$
$$u\in\mathcal{S}(\mathbb{R}^n),\ a\in\mathcal{C}^\infty([0,1)_h;S^k(\mathbb{R}^n;\mathbb{R}^n_\zeta));$$

its classical subalgebra, $\Psi_{\hbar,\mathrm{cl}}(\mathbb{R}^n)$ corresponds to $a\in\mathcal{C}^\infty([0,1)_h;S^k_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{R}^n_\zeta))$. The semiclassical principal symbol is now $\sigma_{\hbar,k}(A)=a|_{h=0}\in S^k(T^*X)$. We can again add an extra parameter $\lambda\in O$, so $a\in\mathcal{C}^\infty([0,1)_h;S^k(\mathbb{R}^n\times O;\mathbb{R}^n_\zeta))$; then in the invariant setting the principal symbol is $a|_{h=0}\in S^k(T^*X\times O)$. Note that if $A^{(\sigma)}=\mathrm{Op}^{(\sigma)}(a)$ is a classical operator with a large parameter, then for $\lambda\in O\subset\mathbb{C}$, $\overline{O}$ compact, $0\notin\overline{O}$,

$$h^k\,\mathrm{Op}^{(h^{-1}\lambda)}(a)=\mathrm{Op}_\hbar(\tilde a),\ \tilde a(z,\zeta,h)=h^k a(z,h^{-1}\zeta,h^{-1}\lambda),$$

and $\tilde a\in\mathcal{C}^\infty([0,1)_h;S^k_{\mathrm{cl}}(\mathbb{R}^n\times O_\lambda;\mathbb{R}^n_\zeta))$. The converse is not quite true: roughly speaking, the semiclassical algebra is a blow-up of the large parameter algebra; to obtain an equivalence, we would need to demand in the definition of the large parameter algebra merely that $a\in S^k(\mathbb{R}^n;[\overline{\mathbb{R}^n_\zeta\times\Omega_\sigma};\partial\overline{\mathbb{R}^n_\zeta\times\{0\}}])$, so in particular for bounded σ, a is merely a family of symbols depending smoothly on σ (not jointly symbolic); we do not discuss this here further. Note, however, that it is the (smaller, i.e. stronger) large parameter algebra that arises naturally when one Mellin transforms in the b-setting, see Subsection 3.1.

Differential operators now take the form

$$A_{h,\lambda}=\sum_{|\alpha|\leq k}a_\alpha(z,\lambda;h)(hD_z)^\alpha. \tag{2.2}$$

Such a family has two principal symbols, the standard one (but taking into account the semiclassical degeneration, i.e. based on $(hD_z)^\alpha$ rather than D_z^α), which depends on h and is homogeneous, and the semiclassical one, which is at $h=0$, and is not homogeneous:

$$\sigma_k(A_{h,\lambda})=\sum_{|\alpha|=k}a_\alpha(z,\lambda;h)\zeta^\alpha,$$
$$\sigma_\hbar(A_{h,\lambda})=\sum_{|\alpha|\leq k}a_\alpha(z,\lambda;0)\zeta^\alpha.$$

However, the restriction of $\sigma_k(A_{h,\lambda})$ to $h=0$ is the principal part of $\sigma_\hbar(A_{h,\lambda})$. In the special case in which $\sigma_k(A_{h,\lambda})$ is independent of h (which is true in the

[7]We adopt the convention that $\hbar$ denotes semiclassical objects, while h is the actual semiclassical parameter.

setting considered below), one can simply regard the usual principal symbol as the principal part of the semiclassical symbol. Note that for $A^{(\sigma)}$ as in (2.1),

$$h^k A^{(h^{-1}\lambda)} = \sum_{|\alpha|+j\leq k} h^{k-j-|\alpha|} a_{\alpha,j}(z)\lambda^j (hD_z)^\alpha,$$

which is indeed of the form (2.2), with polynomial dependence on both h and λ. Note that in this case the standard principal symbol is independent of h and λ.

2.2. **General assumptions.** Let X be a compact manifold and ν a smooth non-vanishing density on it; thus $L^2(X)$ is well-defined as a Hilbert space (and not only up to equivalence). We consider operators $P_\sigma \in \Psi^k_{\mathrm{cl}}(X)$ on X depending on a complex parameter σ, with the dependence being analytic (i.e. the coefficients depend analytically on σ). We also consider a complex absorbing 'potential', $Q_\sigma \in \Psi^k_{\mathrm{cl}}(X)$ which is formally self-adjoint. The operators we study are $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma$; P_σ^* depends on the choice of the density ν.

Typically we shall be interested in P_σ on an open subset U of X, and have Q_σ supported in the complement of U, such that over some subset K of $X \setminus U$, Q_σ is elliptic on the characteristic set of P_σ. In the Kerr-de Sitter setting, we would have $\overline{X_+} \subset U$. However, *this is not part of the general set-up.*

It is often convenient to work with the fiber-radial compactification $\overline{T}^* X$ of T^*X, in particular when discussing semiclassical analysis; see for instance [38, Sections 1 and 5]. Thus, S^*X should be considered as the boundary of $\overline{T}^*X$. When one is working with homogeneous objects, as is the case in classical microlocal analysis, one can think of S^*X as $(\overline{T}^*X \setminus o)/\mathbb{R}^+$, but this is not a useful point of view in semiclassical analysis[8]. Thus, if $\tilde\rho$ is a non-vanishing homogeneous degree -1 function on $T^*X \setminus o$, it is a defining function of S^*X in $\overline{T}^*X \setminus o$; if the homogeneity requirement is dropped it can be modified near the zero section to make it a defining function of S^*X in $\overline{T}^*X$. The principal symbols p, q of P_σ, Q_σ are homogeneous degree k functions on $T^*X \setminus o$, so $\tilde\rho^k p, \tilde\rho^k q$ are homogeneous degree 0 there, thus are functions[9] on $\overline{T}^*X$ near its boundary, S^*X, and in particular on S^*X. Moreover, H_p is homogeneous degree $k-1$ on $T^*X \setminus o$, thus $\tilde\rho^{k-1}\mathsf{H}_p$ a smooth vector field tangent to the boundary on $\overline{T}^*X$ (defined near the boundary), and in particular induces a smooth vector field on S^*X.

We assume that the principal symbol p, resp. q, of P_σ, resp. Q_σ, are real, are independent of σ, $p = 0$ implies $dp \neq 0$. We assume that the characteristic set of P_σ is of the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,\ \Sigma_+ \cap \Sigma_- = \emptyset,$$

[8]In fact, even in classical microlocal analysis it is better to keep at least a 'shadow' of the interior of S^*X by working with $T^*X \setminus o$ considered as a half-line bundle over S^*X with homogeneous objects on it; this keeps the action of the Hamilton vector field on the fiber-radial variable, i.e. the defining function of S^*X in $\overline{T}^*X$, non-trivial, which is important at radial points.

[9]This depends on choices unless $k = 0$; they are naturally sections of a line bundle that encodes the differential of the boundary defining function at S^*X. However, the only relevant notion here is ellipticity, and later the Hamilton vector field up to multiplication by a positive function, which is independent of choices. In fact, we emphasize that all the requirements listed for p, q and later $p_{\hbar,z}$ and $q_{\hbar,z}$, except possibly (2.5)-(2.6), are also fulfilled if $P_\sigma - \imath Q_\sigma$ is replaced by *any* smooth positive multiple, so one may factor out positive factors at will. This is useful in the Kerr-de Sitter space discussion. For (2.5)-(2.6), see Footnote 12.

$\Sigma_\pm$ are relatively open[10] in Σ, and

$$\mp q \geq 0 \text{ near } \Sigma_\pm.$$

We assume that there are conic submanifolds $\Lambda_\pm \subset \Sigma_\pm$ of $T^*X\setminus o$, outside which the Hamilton vector field H_p is not radial, and to which the Hamilton vector field H_p is tangent. Here $\Lambda_\pm$ are typically Lagrangian, but this is not needed[11]. The properties we want at $\Lambda_\pm$ are (probably) not stable under general smooth perturbations; the perturbations need to have certain properties at $\Lambda_\pm$. However, the estimates we then derive *are stable* under such perturbations. First, we want that for a homogeneous degree -1 defining function $\tilde\rho$ of S^*X near $L_\pm$, the image of $\Lambda_\pm$ in S^*X,

$$\tilde\rho^{k-2}\mathsf{H}_p\tilde\rho|_{L_\pm} = \mp\beta_0,\ \beta_0 \in \mathcal{C}^\infty(L_\pm),\ \beta_0 > 0. \tag{2.3}$$

Next, we require the existence of a non-negative homogeneous degree zero quadratic defining function ρ_0, of $\Lambda_\pm$ (i.e. it vanishes quadratically at $\Lambda_\pm$, and is non-degenerate) and $\beta_1 > 0$ such that

$$\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0 - \beta_1\rho_0 \tag{2.4}$$

is ≥ 0 modulo cubic vanishing terms at $\Lambda_\pm$. (The precise behavior of $\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0$, or of linear defining functions, is irrelevant, because we only need a relatively weak estimate. It would be relevant if one wanted to prove Lagrangian regularity.) Under these assumptions, L_- is a source and L_+ is a sink for the H_p-dynamics in the sense that nearby bicharacteristics tend to $L_\pm$ as the parameter along the bicharacteristic goes to $\pm\infty$. Finally, we assume that the imaginary part of the subprincipal symbol at $\Lambda_\pm$, which is the symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*) \in \Psi_{\mathrm{cl}}^{k-1}(X)$ as p is real, is[12]

$$\pm\tilde\beta\beta_0(\operatorname{Im}\sigma)\tilde\rho^{-k+1},\ \tilde\beta \in \mathcal{C}^\infty(L_\pm), \tag{2.5}$$

$\tilde\beta$ is positive along $L_\pm$, and write

$$\beta_{\mathrm{sup}} = \sup\tilde\beta,\ \beta_{\mathrm{inf}} = \inf\tilde\beta > 0. \tag{2.6}$$

If $\tilde\beta$ is a constant, we may write

$$\beta = \beta_{\mathrm{inf}} = \beta_{\mathrm{sup}}. \tag{2.7}$$

The results take a little nicer form in this case since depending on various signs, sometimes β_{inf} and sometimes β_{sup} is the relevant quantity.

We make the following *non-trapping* assumption. For $\alpha \in S^*X$, let $\gamma_+(\alpha)$, resp. $\gamma_-(\alpha)$ denote the image of the forward, resp. backward, half-bicharacteristic from α. We write $\gamma_\pm(\alpha) \to L_\pm$ (and say $\gamma_\pm(\alpha)$ tends to $L_\pm$) if given any neighborhood O of $L_\pm$, $\gamma_\pm(\alpha) \cap O \neq \emptyset$; by the source/sink property this implies that the points

[10] Thus, they are connected components in the extended sense that they may be empty.

[11] An extreme example would be $\Lambda_\pm = \Sigma_\pm$. Another extreme is if one or both are empty.

[12] If H_p is radial at $L_\pm$, this is independent of the choice of the density ν. Indeed, with respect to $f\nu$, the adjoint of P_σ is $f^{-1}P_\sigma^* f$, with P_σ^* denoting the adjoint with respect to ν. This is $P_\sigma^* + f^{-1}[P_\sigma^*, f]$, and the principal symbol of $f^{-1}[P_\sigma^*, f] \in \Psi_{\mathrm{cl}}^{k-1}(X)$ vanishes at $L_\pm$ as $\mathsf{H}_p f = 0$. In general, we can only change the density by factors f with $\mathsf{H}_p f|_{L_\pm} = 0$, which in Kerr-de Sitter space-times would mean factors independent of ϕ at the event horizon. A similar argument shows the independence of the condition from the choice of f when one replaces P_σ by fP_σ, under the same conditions: either radiality, or just $\mathsf{H}_p f|_{L_\pm} = 0$.

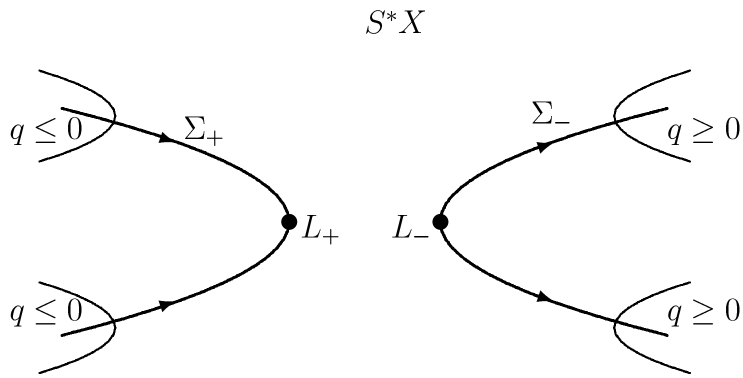

FIGURE 1. The components $\Sigma_\pm$ of the characteristic set in the cosphere bundle S^*X. The submanifolds $L_\pm$ are points here, with L_- a source, L_+ a sink. The thin lined parabolic regions near the edges show the absorbing region, i.e. the support of q. For $P_\sigma - \imath Q_\sigma$,the estimates are always propagated away from $L_\pm$ towards the support of q, so in the direction of the Hamilton flow in Σ_-, and in the direction opposite of the Hamilton flow in Σ_+; for $P_\sigma^* + \imath Q_\sigma$, the directions are reversed.

on the curve are in O for sufficiently large (in absolute value) parameter values. We assume that, with $\mathrm{ell}(Q_\sigma)$ denoting the elliptic set of Q_σ,

(2.8)
$$\alpha \in \Sigma_- \setminus L_- \Rightarrow \big(\gamma_-(\alpha) \to L_- \text{ or } \gamma_-(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset\big) \text{ and } \gamma_+(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset,$$
$$\alpha \in \Sigma_+ \setminus L_+ \Rightarrow \big(\gamma_+(\alpha) \to L_+ \text{ or } \gamma_+(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset\big) \text{ and } \gamma_-(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset.$$

That is, all forward and backward half-(null)bicharacteristics of P_σ either enter the elliptic set of Q_σ, or go to $\Lambda_\pm$, i.e. $L_\pm$ in S^*X. The point of the assumptions regarding Q_σ and the flow is that we are able to propagate estimates forward near where $q \ge 0$, backward near where $q \le 0$, so by our hypotheses we can always propagate estimates for $P_\sigma - \imath Q_\sigma$ from $\Lambda_\pm$ towards the elliptic set of Q_σ, and also if both ends of a bicharacteristic go to the elliptic set of Q_σ then we can propagate the estimates from one of the directions. On the other hand, for $P_\sigma^* + \imath Q_\sigma$, we can propagate estimates from the elliptic set of Q_σ towards $\Lambda_\pm$, and again if both ends of a bicharacteristic go to the elliptic set of Q_σ then we can propagate the estimates from one of the directions. This behavior of $P_\sigma - \imath Q_\sigma$ vs. $P_\sigma^* + \imath Q_\sigma$ is important for duality reasons.

Remark 2.1. For simplicity of notation we have not considered vector bundles on X. However, if E is a vector bundle on X with a positive definite inner product on the fibers and $P_\sigma, Q_\sigma \in \Psi^k_{\mathrm{cl}}(X;E)$ with scalar principal symbol p, and in case of P_σ the imaginary part of the subprincipal symbol is of the form (2.5) with $\tilde\beta$ a bundle-endomorphism satisfying an inequality in (2.6) as a bundle endomorphism, the arguments we present go through.

2.3. **Elliptic and microhyperbolic points.** We now turn to analysis. First, by the usual elliptic theory, on the elliptic set of $P_\sigma - \imath Q_\sigma$, so both on the elliptic set of P_σ and on the elliptic set of Q_σ, one has elliptic estimates[13]: for all s and N, and for all $B, G \in \Psi^0(X)$ with G elliptic on $\mathrm{WF}'(B)$,

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k}} + \|u\|_{H^{-N}}), \tag{2.9}$$

with the estimate also holding for $P_\sigma^* + \imath Q_\sigma$. By propagation of singularities, in $\Sigma \setminus (\mathrm{WF}'(Q_\sigma) \cup L_+ \cup L_-)$, one can propagate regularity estimates either forward or backward along bicharacteristics, i.e. for all s and N, and for all $A, B, G \in \Psi^0(X)$ such that $\mathrm{WF}'(G) \cap \mathrm{WF}'(Q_\sigma) = \emptyset$, and forward (or backward) bicharacteristics from $\mathrm{WF}'(B)$ reach the elliptic set of A, while remaining in the elliptic set of G, one has estimates

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}). \tag{2.10}$$

Here P_σ can be replaced by $P_\sigma - \imath Q_\sigma$ or $P_\sigma^* + \imath Q_\sigma$ by the condition on $\mathrm{WF}'(G)$; namely $GQ_\sigma \in \Psi^{-\infty}(X)$, and can thus be absorbed into the $\|u\|_{H^{-N}}$ term. As usual, there is a loss of one derivative compared to the elliptic estimate, i.e. the assumption on $P_\sigma u$ is H^{s-k+1}, not H^{s-k}, and one needs to make H^s assumptions on Au, i.e. regularity propagates.

2.4. **Analysis near $\Lambda_\pm$.** At $\Lambda_\pm$, for $s \geq m > (k - 1 - \beta \operatorname{Im} \sigma)/2$, β given by the subprincipal symbol at $\Lambda_\pm$, we can propagate estimates *away* from $\Lambda_\pm$:

Proposition 2.2. *For $\operatorname{Im}\sigma \geq 0$, let[14] $\beta = \beta_{\inf}$, for $\operatorname{Im}\sigma < 0$, let $\beta = \beta_{\sup}$. For all N, for $s \geq m > (k-1-\beta \operatorname{Im}\sigma)/2$, and for all $A, B, G \in \Psi^0(X)$ such that $\mathrm{WF}'(G) \cap \mathrm{WF}'(Q_\sigma) = \emptyset$, A elliptic at $\Lambda_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}'(B)$ tend to $\Lambda_\pm$, with closure in the elliptic set of G, one has estimates*

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|Au\|_{H^m} + \|u\|_{H^{-N}}), \tag{2.11}$$

in the sense that if $u \in H^{-N}$, $Au \in H^m$ and $GP_\sigma u \in H^{s-k+1}$, then $Bu \in H^s$, and (2.11) *holds. In fact, Au can be dropped from the right hand side (but one must assume $Au \in H^m$):*

$$Au \in H^m \Rightarrow \|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}), \tag{2.12}$$

where $u \in H^{-N}$ and $GP_\sigma u \in H^{s-k+1}$ is considered implied by the right hand side. Note that Au does not appear on the right hand side, hence the display before the estimate.

This is completely analogous to Melrose's estimates in asymptotically Euclidean scattering theory at the radial sets [38, Section 9]. Note that the H^s regularity of Bu is 'free' in the sense that we do not need to impose H^s assumptions on u anywhere; merely H^m at $\Lambda_\pm$ does the job; of course, on $P_\sigma u$ one must make the H^{s-k+1} assumption, i.e. the loss of one derivative compared to the elliptic setting. At the cost of changing regularity, one can propagate estimate *towards* $\Lambda_\pm$. Keeping in mind that for P_σ^* the subprincipal symbol becomes $\beta\overline{\sigma}$, we have the following:

[13]Our convention in estimates such as (2.9) and (2.10) is that if one assumes that all the quantities on the right hand side are in the function spaces indicated by the norms then so is the quantity on the left hand side, and the estimate holds. As we see below, at $\Lambda_\pm$ not all relevant function space statements appear in the estimate, so we need to be more explicit there.

[14]Note that this is consistent with (2.7).

Proposition 2.3. *For* $\operatorname{Im}\sigma > 0$, *let*[15] $\beta = \beta_{\mathrm{sup}}$, *for* $\operatorname{Im}\sigma \leq 0$, *let* $\beta = \beta_{\mathrm{inf}}$. *For* $s < (k-1-\beta\operatorname{Im}\sigma)/2$, *for all* N, *and for all* $A, B, G \in \Psi^0(X)$ *such that* $\mathrm{WF}'(G) \cap \mathrm{WF}'(Q_\sigma) = \emptyset$, B, G *elliptic at* $\Lambda_\pm$, *and forward (or backward) bicharacteristics from* $\mathrm{WF}'(B) \setminus \Lambda_\pm$ *reach* $\mathrm{WF}'(A)$, *while remaining in the elliptic set of* G, *one has estimates*

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma^* u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}). \tag{2.13}$$

Proof of Propositions 2.2-2.3. It suffices to prove that there exist O_j open with $L_\pm \subset O_{j+1} \subset O_j$, $\cap_{j=1}^\infty O_j = L_\pm$, and A_j, B_j, G_j with WF' in O_j, B_j elliptic on $L_\pm$, such that the statements of the propositions hold. Indeed, in case of Proposition 2.2 the general case follows by taking j such that A, G are elliptic on O_j, use the estimate for A_j, B_j, G_j, where the right hand side then can be estimated by A and G, and then use microlocal ellipticity, propagation of singularities and a covering argument to prove the proposition. In case of Proposition 2.3, the general case follows by taking j such that G, B are elliptic on O_j, so all forward (or backward) bicharacteristics from $O_j \setminus \Lambda_\pm$ reach $\mathrm{WF}'(A)$, thus microlocal ellipticity, propagation of singularities and a covering argument proves $\|A_j u\|_{H^s} \leq C(\|GP_\sigma^* u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}})$, and then the special case of the proposition for this O_j gives an estimate for $\|B_j u\|_{H^s}$ in terms of the same quantities. The full estimate for $\|Bu\|_{H^s}$ is then again a straightforward consequence of microlocal ellipticity, propagation of singularities and a covering argument.

We now consider commutants $C_\epsilon^* C_\epsilon$ with $C_\epsilon \in \Psi^{s-(k-1)/2-\delta}(X)$ for $\epsilon > 0$, uniformly bounded in $\Psi^{s-(k-1)/2}(X)$ as $\epsilon \to 0$; with the ϵ-dependence used to regularize the argument. More precisely, let

$$c = \phi(\rho_0)\tilde{\rho}^{-s+(k-1)/2}, \qquad c_\epsilon = c(1+\epsilon\tilde{\rho}^{-1})^{-\delta},$$

where $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi' \leq 0$ and ϕ is supported sufficiently close to 0 so that

$$\rho_0 \in \operatorname{supp} d\phi \Rightarrow \mp\tilde{\rho}^{k-1}\mathsf{H}_p\rho_0 > 0; \tag{2.14}$$

such ϕ exists by (2.4). Note that the sign of $\mathsf{H}_p\tilde{\rho}^{-s+(k-1)/2}$ depends on the sign of $-s+(k-1)/2$ which explains the difference between $s > (k-1)/2$ and $s < (k-1)/2$ in Propositions 2.2-2.3 when there are no other contributions to the threshold value of s. The contribution of the subprincipal symbol, however, shifts the critical value $(k-1)/2$.

Now let $C \in \Psi^{s-(k-1)/2}(X)$ have principal symbol c, and have $\mathrm{WF}'(C) \subset \operatorname{supp}\phi \circ \rho_0$, and let $C_\epsilon = CS_\epsilon$, $S_\epsilon \in \Psi^{-\delta}(X)$ uniformly bounded in $\Psi^0(X)$ for $\epsilon > 0$, converging to Id in $\Psi^{\delta'}(X)$ for $\delta' > 0$ as $\epsilon \to 0$, with principal symbol $(1+\epsilon\tilde{\rho}^{-1})^{-\delta}$. Thus, the principal symbol of C_ϵ is c_ϵ.

First, consider (2.11). Then

$$\begin{aligned}\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)) &= \sigma_{k-1}(\imath(P_\sigma^* - P_\sigma))c_\epsilon^2 + 2c_\epsilon \mathsf{H}_p c_\epsilon \\ &= \mp 2\left(-\tilde{\beta}\operatorname{Im}\sigma\beta_0\phi + \beta_0\left(-s + \frac{k-1}{2}\right)\phi \mp (\tilde{\rho}^{k-1}\mathsf{H}_p\rho_0)\phi' + \delta\beta_0\frac{\epsilon}{\tilde{\rho}+\epsilon}\phi\right) \\ &\qquad\qquad \phi\tilde{\rho}^{-2s}(1+\epsilon\tilde{\rho}^{-1})^{-\delta},\end{aligned}$$

[15]Note the switch compared to Proposition 2.2! Also, β does not matter when $\operatorname{Im}\sigma = 0$; we define it here so that the two Propositions are consistent via dualization, which reverses the sign of the imaginary part.

so

$$\begin{aligned}
&\mp\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma))\\
&\qquad \leq -2\beta_0\left(s - \frac{k-1}{2} + \tilde\beta \operatorname{Im}\sigma - \delta\right)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-\delta}\phi^2\\
&\qquad\qquad + 2(\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-\delta}\phi'\phi.
\end{aligned} \tag{2.15}$$

Here the first term on the right hand side is negative if $s-(k-1)/2+\beta\operatorname{Im}\sigma-\delta>0$ (since $\tilde\beta\operatorname{Im}\sigma\geq\beta\operatorname{Im}\sigma$ by our definition of β), and this is the same sign as that of ϕ' term; the presence of δ (needed for the regularization) is the reason for the appearance of m in the estimate. To avoid using the sharp Gårding inequality, we choose ϕ so that $\sqrt{-\phi\phi'}$ is $\mathcal{C}^\infty$, and then

$$\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma) = -S_\epsilon^*(B^*B + B_1^*B_1 + B_{2,\epsilon}^* B_{2,\epsilon})S_\epsilon + F_\epsilon,$$

with $B, B_1, B_{2,\epsilon} \in \Psi^s(X)$, $B_{2,\epsilon}$ uniformly bounded in $\Psi^s(X)$ as $\epsilon\to 0$, F_ϵ uniformly bounded in $\Psi^{2s-1}(X)$, and $\sigma_s(B)$ an elliptic multiple of $\phi(\rho_0)\tilde\rho^{-s}$. Computing the pairing, using an extra regularization (insert a regularizer $\Lambda_r\in\Psi^{-1}(X)$, uniformly bounded in $\Psi^0(X)$, converging to Id in $\Psi^\delta(X)$ to justify integration by parts, and use that $[\Lambda_r, P_\sigma^*]$ is uniformly bounded in $\Psi^1(X)$, converging to 0 strongly, cf. [50, Lemma 17.1] and its use in [50, Lemma 17.2]) yields

$$\langle \imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)u, u\rangle = \langle \imath C_\epsilon^* C_\epsilon u, P_\sigma u\rangle - \langle \imath P_\sigma, C_\epsilon^* C_\epsilon u\rangle.$$

Using Cauchy-Schwartz on the right hand side, a standard functional analytic argument (see, for instance, Melrose [38, Proof of Proposition 7 and Section 9]) gives an estimate for Bu, showing u is in H^s on the elliptic set of B, provided u is microlocally in $H^{s-\delta}$. A standard inductive argument, starting with $s-\delta=m$ and improving regularity by $\leq 1/2$ in each step proves (2.11).

For (2.13), the argument is similar, but we want to change the sign of the first term on the right hand side of (2.15), i.e. we want it to be positive. This is satisfied if $s-(k-1)/2+\beta\operatorname{Im}\sigma-\delta<0$ (since $\tilde\beta\operatorname{Im}\sigma\leq\beta\operatorname{Im}\sigma$ by our definition of β in Proposition 2.3), hence (as $\delta>0$) if $s-(k-1)/2+\beta\operatorname{Im}\sigma<0$, so regularization is not an issue. On the other hand, ϕ' now has the wrong sign, so one needs to make an assumption on $\operatorname{supp} d\phi$, which is the Au term in (2.13). Since the details are standard, see [38, Section 9], we leave these to the reader. □

Remark 2.4. Fixing a ϕ, it follows from the proof that the same ϕ works for (small) smooth perturbations of P_σ, even if those perturbations do not preserve the event horizon, namely even if (2.4) does not hold any more: only its implication, (2.14), on $\operatorname{supp} d\phi$ matters, which *is* stable under perturbations. Moreover, as the rescaled Hamilton vector field $\tilde\rho^{k-1}\mathsf{H}_p$ is a smooth vector field tangent to the boundary of the fiber-compactified cotangent bundle, i.e. a b-vector field, and as such depends smoothly on the principal symbol, and it is *non-degenerate* radially by (2.3), the weight, which provides the positivity at the radial points in the proof above, still gives a positive Hamilton derivative for small perturbations. Since this proposition thus holds for $\mathcal{C}^\infty$ perturbations of P_σ (indeed, even pseudodifferential ones), and this proposition is the only delicate estimate we use, and it is only marginally so, we deduce that all the other results below also hold in this generality.

2.5. **Complex absorption.** Finally, one has propagation estimates for complex absorbing operators, requiring a sign condition. (See for instance [43] and [15] in the semiclassical setting; the changes are minor in the 'classical' setting.) First, one can propagate regularity to $\mathrm{WF}'(Q_\sigma)$ (of course, in the elliptic set of Q_σ one has a priori regularity). Namely, for all s and N, and for all $A, B, G \in \Psi^0(X)$ such that $q \leq 0$, resp. $q \geq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of A, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}).$$

Thus, for $q \geq 0$ one can propagate regularity in the forward direction along the Hamilton flow, while for $q \leq 0$ one can do so in the backward direction.

On the other hand, one can propagate regularity away from the elliptic set of Q_σ. Namely, for all s and N, and for all $B, G \in \Psi^0(X)$ such that $q \leq 0$, resp. $q \geq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of Q_σ, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}).$$

Again, for $q \geq 0$ one can propagate regularity in the forward direction along the Hamilton flow, while for $q \leq 0$ one can do so in the backward direction. At the cost of reversing the signs of q, this also gives that for all s and N, and for all $B, G \in \Psi^0(X)$ such that $q \geq 0$, resp. $q \leq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of Q_σ, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma^* + \imath Q_\sigma)u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}).$$

Remark 2.5. As mentioned in the introduction, these complex absorption methods could be replaced in specific cases, including all the specific examples we discuss here, by adding a boundary $\tilde{Y}$ instead, provided that the Hamilton flow is well-behaved relative to the base space, namely inside the characteristic set H_p is not tangent to $T^*_{\tilde{Y}}X$ with orbits crossing $T^*_{\tilde{Y}}X$ in the opposite directions in $\Sigma_\pm$ in the following way. If $\tilde{Y}$ is defined by $\tilde{y}$ which is positive on 'our side' U with U as discussed at the beginning of Subsection 2.2, we need $\pm\mathsf{H}_p\tilde{y}|_{\tilde{Y}} > 0$ on $\Sigma_\pm$. Then the functional analysis described in [31, Proof of Theorem 23.2.2], see also [53, Proof of Lemma 4.14], can be used to prove analogues of the results we give below on $X_+ = \{\tilde{y} \geq 0\}$. For instance, if one has a Lorentzian metric on X near $\tilde{Y}$, and $\tilde{Y}$ is space-like, then (up to the sign) this statement holds with $\Sigma_\pm$ being the two components of the characteristic set. However, in the author's opinion, this detracts from the clarity of the microlocal analysis by introducing projection to physical space in an essential way.

2.6. **Global estimates.** Recall now that $q \geq 0$ near Σ_-, and $q \leq 0$ on Σ_+, and recall our non-trapping assumptions, i.e. (2.8). Thus, we can piece together the estimates described earlier (elliptic, microhyperbolic, radial points, complex absorption) to propagate estimates forward in Σ_- and backward in Σ_+, thus away from $\Lambda_\pm$ (as well as from one end of a bicharacteristic which intersects the elliptic set of q in both directions). This yields that for any N, and for any $s \geq m > (k-1-\beta\,\mathrm{Im}\,\sigma)/2$, and for any $A \in \Psi^0(X)$ elliptic at $\Lambda_+ \cup \Lambda_-$,

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|Au\|_{H^m} + \|u\|_{H^{-N}}).$$

This implies that for any $s > m > (k - 1 - \beta \operatorname{Im} \sigma)/2$,

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|u\|_{H^m}). \tag{2.16}$$

On the other hand, recalling that the adjoint switches the sign of the imaginary part of the principal symbol and also that of the subprincipal symbol at the radial sets, propagating the estimates in the other direction, i.e. backward in Σ_- and forward in Σ_+, thus towards $\Lambda_\pm$, from the elliptic set of q, we deduce that for any N (which we take to satisfy $s' > -N$) and for any $s' < (k - 1 + \beta \operatorname{Im} \sigma)/2$,

$$\|u\|_{H^{s'}} \leq C(\|(P_\sigma^* + \imath Q_\sigma)u\|_{H^{s'-k+1}} + \|u\|_{H^{-N}}). \tag{2.17}$$

Note that the dual of H^s, $s > (k-1-\beta \operatorname{Im} \sigma)/2$, is $H^{-s} = H^{s'-k+1}$, $s' = k-1-s$, so $s' < (k - 1 + \beta \operatorname{Im} \sigma)/2$, while the dual of H^{s-k+1}, $s > (k - 1 - \beta \operatorname{Im} \sigma)/2$, is $H^{k-1-s} = H^{s'}$, with $s' = k - 1 - s < (k - 1 + \beta \operatorname{Im} \sigma)/2$ again. Thus, the spaces (apart from the residual spaces, into which the inclusion is compact) in the left, resp. right, side of (2.17), are exactly the duals of those on the right, resp. left, side of (2.16). Thus, by a standard functional analytic argument, see e.g. [31, Proof of Theorem 26.1.7], namely dualization and using the compactness of the inclusion $H^{s'} \to H^{-N}$ for $s' > -N$, this gives the solvability of

$$(P_\sigma - \imath Q_\sigma)u = f,\ s > (k - 1 - \beta \operatorname{Im} \sigma)/2,$$

for f in the annihilator in H^{s-k+1} (via the duality between H^{s-k+1} and H^{-s+k-1} induced by the L^2-pairing) of the finite dimensional subspace $\operatorname{Ker}(P_\sigma + \imath Q_\sigma^*)$ of $H^{-s+k-1} = H^{s'}$, and indeed elements of this finite dimensional subspace have wave front set in $\Lambda_+ \cup \Lambda_-$ and lie in $\cap_{s'<(k-1+\beta \operatorname{Im} \sigma)/2} H^{s'}$. Thus, there is the usual real principal type loss of one derivative relative to the elliptic problem, and in addition, there are restrictions on the orders for which is valid.

In addition, one also has almost uniqueness by a standard compactness argument (using the compactness of the inclusion of H^s into H^m for $s > m$), by (2.16), namely not only is the space of f in the space as above is finite codimensional, but the nullspace of $P_\sigma - \imath Q_\sigma$ on H^s, $s > (k - 1 - \beta \operatorname{Im} \sigma)/2$, is also finite dimensional, and its elements are in $\mathcal{C}^\infty(X)$.

In order to analyze the σ-dependence of solvability of the PDE, we reformulate our problem as a more conventional Fredholm problem. Thus, let $\tilde{P}$ be any operator with principal symbol $p - \imath q$; e.g. $\tilde{P}$ is $P_{\sigma_0} - \imath Q_{\sigma_0}$ for some σ_0. Then consider

$$\mathcal{X}^s = \{u \in H^s :\ \tilde{P}u \in H^{s-k+1}\},\ \mathcal{Y}^s = H^{s-k+1}, \tag{2.18}$$

with

$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|\tilde{P}u\|^2_{H^{s-k+1}}.$$

Note that $\mathcal{X}^s$ only depends on the principal symbol of $\tilde{P}$. Moreover, $\mathcal{C}^\infty(X)$ is dense in $\mathcal{X}^s$; this follows by considering $R_\epsilon \in \Psi^{-\infty}(X)$, $\epsilon > 0$, such that $R_\epsilon \to \operatorname{Id}$ in $\Psi^\delta(X)$ for $\delta > 0$, R_ϵ uniformly bounded in $\Psi^0(X)$; thus $R_\epsilon \to \operatorname{Id}$ strongly (but not in the operator norm topology) on H^s and H^{s-k+1}. Then for $u \in \mathcal{X}^s$, $R_\epsilon u \in \mathcal{C}^\infty(X)$ for $\epsilon > 0$, $R_\epsilon u \to u$ in H^s and $\tilde{P}R_\epsilon u = R_\epsilon \tilde{P}u + [\tilde{P}, R_\epsilon]u$, so the first term on the right converges to $\tilde{P}u$ in H^{s-k+1}, while $[\tilde{P}, R_\epsilon]$ is uniformly bounded in $\Psi^{k-1}(X)$, converging to 0 in $\Psi^{k-1+\delta}(X)$ for $\delta > 0$, so converging to 0 strongly as a map $H^s \to H^{s-k+1}$. Thus, $[\tilde{P}, R_\epsilon]u \to 0$ in H^{s-k+1}, and we conclude that $R_\epsilon u \to u$ in

$\mathcal{X}^s$. (In fact, $\mathcal{X}^s$ is a first-order coisotropic space, more general function spaces of this nature are discussed by Melrose, Vasy and Wunsch in [41, Appendix A].)

With these preliminaries,

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is Fredholm for each σ with $s \geq m > (k - 1 - \beta \operatorname{Im} \sigma)/2$, and is an analytic family of bounded operators in this half-plane of σ's.

Theorem 2.6. *Let P_σ, Q_σ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.18)*. If $k-1-2s > 0$, let $\beta = \beta_{\inf}$, if $k - 1 - 2s < 0$, let $\beta = \beta_{\sup}$. Then*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on

$$\mathbb{C}_s = \{\sigma \in \mathbb{C} : \operatorname{Im} \sigma > \beta^{-1}(k - 1 - 2s)\}. \tag{2.19}$$

Thus, analytic Fredholm theory applies, giving meromorphy of the inverse provided the inverse exists for a particular value of σ.

Remark 2.7. Note that the Fredholm property means that $P_\sigma^* + \imath Q_\sigma$ is also Fredholm on the dual spaces; this can also be seen directly from the estimates. The analogue of this remark also applies to the semiclassical discussion below.

2.7. **Semiclassical estimates.** For this reason, and also for wave propagation, we also want to know the $|\sigma| \to \infty$ asymptotics of $P_\sigma - \imath\sigma$ and $P_\sigma^* + \imath Q_\sigma$; here P_σ, Q_σ are operators with a large parameter. As discussed earlier, this can be translated into a semiclassical problem, i.e. one obtains families of operators $P_{h,z}$, with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. As usual, we multiply through by h^k for convenient notation when we define $P_{h,z}$:

$$P_{h,z} = h^k P_{h^{-1}z} \in \Psi^k_{\hbar,\mathrm{cl}}(X).$$

From now on, we merely require $P_{h,z}, Q_{h,z} \in \Psi^k_{\hbar,\mathrm{cl}}(X)$. Then the semiclassical principal symbol $p_{\hbar,z}$, $z \in O \subset \mathbb{C}$, $0 \notin \overline{O}$ compact, which is a function on T^*X, has limit p at infinity in the fibers of the cotangent bundle, so is in particular real in the limit. More precisely, as in the classical setting, but now $\tilde{\rho}$ made smooth at the zero section as well (so is not homogeneous there), we consider

$$\tilde{\rho}^k p_{\hbar,z} \in \mathcal{C}^\infty(\overline{T}^* X \times O);$$

then $\tilde{\rho}^k p_{\hbar,z}|_{S^*X \times O} = \tilde{\rho}^k p$, where $S^*X = \partial \overline{T}^* X$. We assume that $p_{\hbar,z}$ is real when z is real. We shall be interested in $\operatorname{Im} z \geq -Ch$, which corresponds to $\operatorname{Im} \sigma \geq -C$ (recall that $\operatorname{Im} \sigma \gg 0$ is where we expect holomorphy). Note that when $\operatorname{Im} z = \mathcal{O}(h)$, $\operatorname{Im} p_{\hbar,z}$ still vanishes, as the contribution of $\operatorname{Im} z$ is semiclassically subprincipal in view of the order h vanishing.

We write the semiclassical characteristic set of $p_{\hbar,z}$ as $\Sigma_\hbar$; assume that

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-},\ \Sigma_{\hbar,+} \cap \Sigma_{\hbar,-} = \emptyset,$$

$\Sigma_{\hbar,\pm}$ are relatively open in $\Sigma_\hbar$, and

$$\pm \operatorname{Im} p_{\hbar,z} \geq 0 \text{ and } \mp q_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm}.$$

Microlocal results analogous to the classical results also exist in the semiclassical setting. In the interior of $\overline{T}^* X$, i.e. in T^*X, only the microlocal elliptic, microhyperbolic and complex absorption estimates are relevant. At $L_\pm \subset S^*X$ we in addition

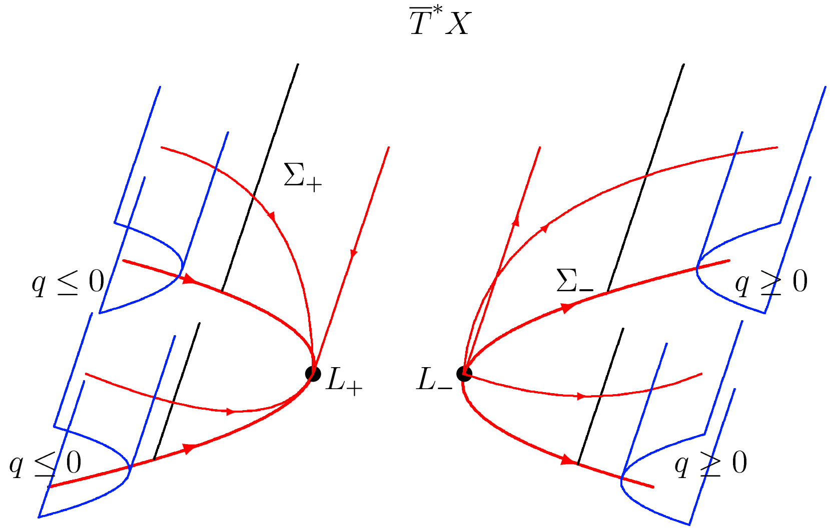

FIGURE 2. The components $\Sigma_{\hbar,\pm}$ of the semiclassical characteristic set in $\overline{T}^*X$, which are now two-dimensional in the figure. The cosphere bundle is the horizontal plane at the bottom of the picture; the intersection of this figure with the cosphere bundle is what is shown on Figure 1. The submanifolds $L_\pm$ are still points, with L_- a source, L_+ a sink. The red lines are bicharacteristics, with the thick ones inside $S^*X = \partial\overline{T}^*X$. The blue regions near the edges show the absorbing region, i.e. the support of q. For $P_{h,z} - \imath Q_{h,z}$, the estimates are always propagated away from $L_\pm$ towards the support of q, so in the direction of the Hamilton flow in $\Sigma_{\hbar,-}$, and in the direction opposite of the Hamilton flow in $\Sigma_{\hbar,+}$; for $P^*_{h,z} + \imath Q_{h,z}$, the directions are reversed.

need the analogue of Propositions 2.2-2.3. As these are the only non-standard estimates, though they are very similar to estimates of [54], where, however, only global estimates were stated, we explicitly state these here and indicate the very minor changes needed in the proof compared to Propositions 2.2-2.3.

Proposition 2.8. *For all N, for $s \ge m > (k-1-\beta\operatorname{Im}\sigma)/2$, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi^0_\hbar(X)$ such that $\mathrm{WF}'_h(G) \cap \mathrm{WF}'_h(Q_\sigma) = \emptyset$, A elliptic at $L_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}'_h(B)$ tend to $L_\pm$, with closure in the elliptic set of G, one has estimates*

$$Au \in H^m_h \Rightarrow \|Bu\|_{H^s_h} \le C(h^{-1}\|GP_{h,z}u\|_{H^{s-k+1}_h} + h\|u\|_{H^{-N}_h}), \tag{2.20}$$

where, as usual, $GP_{h,z}u \in H^{s-k+1}_h$ and $u \in H^{-N}_h$ are assumptions implied by the right hand side.

Proposition 2.9. *For $s < (k-1-\beta\operatorname{Im}\sigma)/2$, for all N, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi^0_\hbar(X)$ such that $\mathrm{WF}'_h(G) \cap \mathrm{WF}'_h(Q_\sigma) = \emptyset$, B, G elliptic at $L_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}'_h(B) \setminus L_\pm$ reach $\mathrm{WF}'_h(A)$, while* *remaining in the elliptic set of G, one has estimates*

$$\|Bu\|_{H_h^s} \leq C(h^{-1}\|GP_{h,z}^* u\|_{H_h^{s-k+1}} + \|Au\|_{H_h^s} + h\|u\|_{H_h^{-N}}). \tag{2.21}$$

Proof. We just need to localize in $\tilde\rho$ in addition to ρ_0; such a localization in the classical setting is implied by working on S^*X or with homogeneous symbols. We achieve this by modifying the localizer ϕ in the commutant constructed in the proof of Propositions 2.2-2.3. As already remarked, the proof is much like at radial points in semiclassical scattering on asymptotically Euclidean spaces, studied by Vasy and Zworski [54], but we need to be more careful about localization in ρ_0 and $\tilde\rho$ as we are assuming less about the structure.

First, note that $L_\pm$ is defined by $\tilde\rho = 0$, $\rho_0 = 0$, so $\tilde\rho^2 + \rho_0$ is a quadratic defining function of $L_\pm$. Thus, let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near 0, $\phi' \leq 0$ and ϕ supported sufficiently close to 0 so that

$$\tilde\rho^2 + \rho_0 \in \operatorname{supp} d\phi \Rightarrow \mp\tilde\rho^{k-1}(\mathsf{H}_p\rho_0 + 2\tilde\rho\mathsf{H}_p\tilde\rho) > 0$$

and

$$\tilde\rho^2 + \rho_0 \in \operatorname{supp}\phi \Rightarrow \mp\tilde\rho^{k-2}\mathsf{H}_p\tilde\rho > 0.$$

Such ϕ exists by (2.3) and (2.4) as

$$\mp\tilde\rho(\mathsf{H}_p\rho_0 + 2\tilde\rho\mathsf{H}_p\tilde\rho) \geq \beta_1\rho_0 + 2\beta_0\tilde\rho^2 - \mathcal{O}((\tilde\rho^2 + \rho_0)^{3/2}).$$

Then let c be given by

$$c = \phi(\rho_0 + \tilde\rho^2)\tilde\rho^{-s+(k-1)/2}, \qquad c_\epsilon = c(1 + \epsilon\tilde\rho^{-1})^{-\delta}.$$

The rest of the proof proceeds exactly as for Propositions 2.2-2.3. □

We first show that under extra assumptions, giving semiclassical ellipticity for $\operatorname{Im} z$ bounded away from 0, we have non-trapping estimates. So assume that for $|\operatorname{Im} z| > \epsilon > 0$, $p_{\hbar,z}$ is semiclassically elliptic on T^*X (but not necessarily at $S^*X = \partial\overline{T}^*X$, where the standard principal symbol p already describes the behavior). Also assume that $\pm\operatorname{Im} p_{\hbar,z} \geq 0$ near the classical characteristic set $\Sigma_{\hbar,\pm} \subset S^*X$. Then the semiclassical version of the classical results (with ellipticity in T^*X making these trivial except at S^*X) apply. Let H_h^s denote the usual semiclassical function spaces. Then, on the one hand, for any $s \geq m > (k - 1 - \beta \operatorname{Im} z/h)/2$, $h < h_0$,

$$\|u\|_{H_h^s} \leq Ch^{-1}(\|(P_{h,z} - \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^m}), \tag{2.22}$$

and on the other hand, for any N and for any $s < (k - 1 + \beta \operatorname{Im} z/h)/2$, $h < h_0$,

$$\|u\|_{H_h^s} \leq Ch^{-1}(\|(P_{h,z}^* + \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^{-N}}). \tag{2.23}$$

The h^2 term can be absorbed in the left hand side for sufficiently small h, so we automatically obtain invertibility of $P_{h,z} - \imath Q_{h,z}$.

In particular, $P_{h,z} - \imath Q_{h,z}$ is invertible for $z = \imath$ and h small, i.e. $P_\sigma - \imath Q_\sigma$ is such for σ pure imaginary with large positive imaginary part, proving the meromorphy of $P_\sigma - \imath Q_\sigma$ under these extra assumptions. Note also that for instance

$$\|u\|^2_{H^1_{|\sigma|^{-1}}} = \|u\|^2_{L^2} + |\sigma|^{-2}\|du\|^2_{L^2}, \qquad \|u\|_{H^0_{|\sigma|^{-1}}} = \|u\|_{L^2},$$

(with the norms with respect to any positive definite inner product).

Theorem 2.10. *Let P_σ, Q_σ, β, $\mathbb{C}_s$ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.18)*. Then, for $\sigma \in \mathbb{C}_s$,*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

has a meromorphic inverse

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s.$$

Moreover, for all $\epsilon > 0$ there is $C > 0$ such that it is invertible in $\operatorname{Im}\sigma > C+\epsilon|\operatorname{Re}\sigma|$, *and* non-trapping estimates *hold:*

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-k+1}\|f\|_{H^{s-1}_{|\sigma|^{-1}}}.$$

To deal with estimates for z (almost) real, we need additional assumptions. We make the non-trapping assumption into a definition:

Definition 2.11. We say that $p_{\hbar,z} - \imath q_{\hbar,z}$ is *semiclassically non-trapping* if the bicharacteristics from any point in $\Sigma_\hbar \setminus (L_+ \cup L_-)$ flow to $\operatorname{ell}(q_{\hbar,z}) \cup L_+$ (i.e. either enter $\operatorname{ell}(q_{\hbar,z})$ at some finite time, or tend to L_+) in the forward direction, and to $\operatorname{ell}(q_{\hbar,z}) \cup L_-$ in the backward direction.

Remark 2.12. The part of the semiclassically non-trapping property on S^*X is just the classical non-trapping property; thus, the point is its extension into to the interior T^*X of $\overline{T}^*X$. Since the classical principal symbol is assumed real, there did not need to be any additional restrictions on $\operatorname{Im} p_{\hbar,z}$ there.

The semiclassical version of all of the above estimates are then applicable for $\operatorname{Im} z \geq -Ch$, and one obtains on the one hand that for any $s \geq m > (k-1-\beta\operatorname{Im} z/h)/2$, $h < h_0$,

$$\|u\|_{H^s_h} \leq Ch^{-1}(\|(P_{h,z} - \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^m}), \tag{2.24}$$

On the other hand, for any N and for any $s < (k-1+\beta\operatorname{Im} z/h)/2$, $h < h_0$,

$$\|u\|_{H^s_h} \leq Ch^{-1}(\|(P^*_{h,z} + \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^{-N}}), \tag{2.25}$$

The h^2 term can again be absorbed in the left hand side for sufficiently small h, so we automatically obtain invertibility of $P_{h,z} - \imath Q_{h,z}$.

Translated into the classical setting this gives

Theorem 2.13. *Let P_σ, Q_σ, $\mathbb{C}_s$, β be as above, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.18)*. Let $C > 0$. Then there exists σ_0 such that*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma : \operatorname{Im}\sigma > -C,\ |\operatorname{Re}\sigma| > \sigma_0\}$, assumed to be a subset of $\mathbb{C}_s$, and non-trapping estimates

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-k+1}\|f\|_{H^{s-k+1}_{|\sigma|^{-1}}}$$

hold. For $s = 1$, $k = 2$ this states that for $|\operatorname{Re}\sigma| > \sigma_0$, $\operatorname{Im}\sigma > -C$,

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)\|^2_{L^2} \leq C''|\sigma|^{-2}\|f\|^2_{L^2}.$$

Analogous results work for other Sobolev spaces; H^1_h was chosen above for simplicity.

Remark 2.14. We emphasize that if semiclassical non-trapping assumptions are made, but not ellipticity for z non-real, meromorphy still follows by taking h small and $z > 0$, say, to get a point of invertibility. This is useful because one can eliminate the need to conjugate by a factor to induce such ellipticity when the resulting estimate is irrelevant. (Mostly estimates in strips for $h^{-1}z$, i.e. $\mathcal{O}(h)$ estimates for z, matter.) However, there is a cost: while only finitely many poles can lie in any strip $|\operatorname{Im}\sigma| < C$, there is no need for this statement to hold if we allow $\operatorname{Im}\sigma > -C$. Since, for the application to the wave equation, $\operatorname{Im}\sigma$ depends on the a priori growth rate of the solution u which we are Mellin transforming, this would mean that depending on the a priori growth rate one could get more (faster growing) terms in the expansion of u if one relaxes the growth condition on u.

While we stated just the global results here, of course one has microlocal estimates for the solution. In particular we have the following, stated in the semiclassical language, as immediate from the estimates used to derive from the Fredholm property:

Theorem 2.15. *Let P_σ, Q_σ, β be as above, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.18).

For $\operatorname{Re} z > 0$ *and $s' > s$, the resolvent $R_{h,z}$ is* semiclassically outgoing with a loss of h^{-1} *in the sense that if $\alpha \in \overline{T}^* X \cap \Sigma_{\hbar,\pm}$, and if for the forward $(+)$, resp. backward $(-)$, bicharacteristic $\gamma_\pm$, from α, $\mathrm{WF}_h^{s'-k+1}(f) \cap \overline{\gamma_\pm} = \emptyset$ then $\alpha \notin \mathrm{WF}_h^{s'}(hR_{h,z}f)$.*

In fact, for any $s' \in \mathbb{R}$, the resolvent $R_{h,z}$ extends to $f \in H_h^{s'}(X)$, with non-trapping bounds, provided that $\mathrm{WF}_h^s(f) \cap (L_+ \cup L_-) = \emptyset$. The semiclassically outgoing with a loss of h^{-1} result holds for such f and s' as well.

Proof. The only part that is not immediate by what has been discussed is the last claim. This follows immediately, however, by microlocal solvability in arbitrary ordered Sobolev spaces away from the radial points (i.e. solvability modulo $\mathcal{C}^\infty$, with semiclassical estimates), combined with our preceding results to deal with this smooth remainder plus the contribution near $L_+ \cup L_-$, which are assumed to be in $H_h^s(X)$. □

This result is needed for gluing constructions as in [15], namely polynomially bounded trapping with appropriate microlocal geometry can be glued to our resolvent. Furthermore, it gives non-trapping estimates microlocally away from the trapped set provided the overall (trapped) resolvent is polynomially bounded as shown by Datchev and Vasy [16].

Definition 2.16. Suppose $K_\pm \subset T^*X$ is compact, and $O_\pm$ is a neighborhood of K with compact closure and $O_\pm \cap \Sigma_\hbar \subset \Sigma_{\hbar,\pm}$. We say that $p_{\hbar,z}$ is *semiclassically locally mildly trapping of order $\varkappa$ in a C_0-strip* if

(i) there is a function[16] $F \in \mathcal{C}^\infty(T^*X)$, $F \geq 2$ on $K_\pm$, $F \leq 1$ on $T^*X \setminus O_\pm$, and for $\alpha \in (O_\pm \setminus K_\pm) \cap \Sigma_{\hbar,\pm}$, $(\mathsf{H}_{p_{\hbar,z}}F)(\alpha) = 0$ implies $(\mathsf{H}^2_{p_{\hbar,z}}F)(\alpha) < 0$; and

[16] For $\epsilon > 0$, such a function F provides an escape function, $\tilde{F} = e^{-CF}\mathsf{H}_{p_{\hbar,z}}F$ on the set where $1 + \epsilon \leq F \leq 2 - \epsilon$. Namely, by taking $C > 0$ sufficiently large, $\mathsf{H}_{p_{\hbar,z}}\tilde{F} < 0$ there; thus, every bicharacteristic must leave the compact set $F^{-1}([1+\epsilon, 2-\epsilon])$ in finite time. However, the existence of such an F is a stronger statement than that of an escape function: a bicharacteristic segment cannot leave $F^{-1}([1+\epsilon, 2-\epsilon])$ via the boundary $F = 2 - \epsilon$ in both directions since F cannot have a local minimum. This is exactly the way this condition is used in [15].

(ii) there exists $\tilde{Q}_{h,z} \in \Psi_\hbar(X)$ with $\mathrm{WF}'_h(\tilde{Q}_{h,z}) \cap K_\pm = \emptyset$, $\mp \tilde{q}_{\hbar,z} \geq 0$ near $\Sigma_{\hbar,\pm}$, $\tilde{q}_{\hbar,z}$ elliptic on $\Sigma_\hbar \setminus (O_+ \cup O_-)$ and $h_0 > 0$ such that if $\operatorname{Im} z > -C_0 h$ and $h < h_0$ then

$$\|(P_{h,z} - \imath \tilde{Q}_{h,z})^{-1} f\|_{H^s_h} \leq C h^{-\varkappa-1} \|f\|_{H_h^{s-k+1}}, \ f \in H_h^{s-k+1}. \tag{2.26}$$

We say that $p_{\hbar,z} - \imath q_{\hbar,z}$ is *semiclassically mildly trapping of order $\varkappa$ in a C_0-strip* if it is semiclassically locally mildly trapping of order $\varkappa$ in a C_0-strip and if the bicharacteristics from any point in $\Sigma_{\hbar,+} \setminus (L_+ \cup K_+)$ flow to $\{q_{\hbar,z} < 0\} \cup O_+$ in the backward direction and to $\{q_{\hbar,z} < 0\} \cup O_+ \cup L_+$ in the forward direction, while the bicharacteristics from any point in $\Sigma_{\hbar,-} \setminus (L_- \cup K_-)$ flow to $\{q_{\hbar,z} > 0\} \cup O_- \cup L_-$ in the backward direction and to $\{q_{\hbar,z} > 0\} \cup O_-$ in the forward direction.

An example[17] of locally mild trapping is hyperbolic trapping, studied by Wunsch and Zworski [58], which is of order $\varkappa$ for some $\varkappa > 0$. Note that (i) states that the sets $K_c = \{F \geq c\}$, $1 < c < 2$, are bicharacteristically convex in $O_\pm$, for by (i) any critical points of F along a bicharacteristic are strict local maxima.

As a corollary, we have:

Theorem 2.17. *Let P_σ, Q_σ, $\mathbb{C}_s$, β be as above, satisfying mild trapping assumptions with order $\varkappa$ estimates in a C_0-strip, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.18)*. Then there exists σ_0 such that*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma : \operatorname{Im}\sigma > -C_0,\ |\operatorname{Re}\sigma| > \sigma_0\}$, assumed to be a subset of $\mathbb{C}_s$, and

$$\|R(\sigma) f\|_{H^s_{|\sigma|^{-1}}} \leq C' |\sigma|^{\varkappa-k+1} \|f\|_{H^{s-k+1}_{|\sigma|^{-1}}}. \tag{2.27}$$

Further, if one has logarithmic loss in (2.26)*, i.e. if $h^{-\varkappa}$ can be replaced by $\log(h^{-1})$, for $\sigma \in \mathbb{R}$,* (2.27) *also holds with a logarithmic loss, i.e. $|\sigma|^\varkappa$ can be replaced by $\log|\sigma|$ for σ real.*

Proof. This is an almost immediate consequence of [15]. To get into that setting, we replace $Q_{h,z}$ by $Q'_{h,z}$ with $\mathrm{WF}'_h(Q_{h,z} - Q'_{h,z}) \subset O_+ \cup O_-$ and $Q'_{h,z}$ elliptic on $K_+ \cup K_-$, with $\mp q'_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. Then $P_{h,z} - \imath Q'_{h,z}$ is semiclassically non-trapping in the sense discussed earlier, so all of our estimates apply. With the polynomial resolvent bound assumption on $P_{h,z} - \imath \tilde{Q}_{h,z}$, and the function F in place of x used in [15], the results of [15] apply, taking into account Theorem 2.15 and [15, Lemma 5.1]. Note that the results of [15] are stated in a slightly different context for convenience, namely the function x is defined on the manifold X and not on T^*X, but this is a minor issue: the results and proofs apply verbatim in our setting. □

[17]Condition (i) follows by letting $\tilde{F} = \varphi_+^{2\kappa} + \varphi_-^{2\kappa}$ with the notation of [58, Lemma 4.1]; so

$$\mathsf{H}_p^2 \tilde{F} = 4\kappa^2\big((c_+^4 - \kappa^{-1} c_+ \mathsf{H}_p c_+)\varphi_+^{2\kappa} + 4\kappa^2\big((c_-^4 + \kappa^{-1} c_- \mathsf{H}_p c_-)\varphi_-^{2\kappa}$$

near the trapped set, $\varphi_+ = 0 = \varphi_-$. Thus, for sufficiently large κ, $\mathsf{H}_p \tilde{F} > 0$ outside $\tilde{F} = 0$. Since $\tilde{F} = 0$ defines the trapped set, in order to satisfy Definition 2.16, writing K and O instead of $K_\pm$ and $O_\pm$, one lets $K = \{\tilde{F} \leq \alpha\}$, $O = \{\tilde{F} < \beta\}$ for suitable (small) α and β, $\alpha < \beta$, and takes $F = G \circ \tilde{F}$ with G strictly decreasing, $G|_{[0,\alpha]} > 2$, $G|_{[\beta,\infty)} < 1$.

3. Mellin transform and Lorentzian b-metrics

3.1. **The Mellin transform.** In this section we discuss the basics of Melrose's b-analysis on an n-dimensional manifold with boundary $\bar{M}$, where the boundary is denoted by X. We refer to [42] as a general reference. In the main cases of interest here, the b-geometry is trivial, and $\bar{M} = X \times [0,\infty)_\tau$ with respect to some (almost) canonical (to the problem) product decomposition. Thus, the reader should feel comfortable in trivializing all the statements below with respect to this decomposition. In this trivial case, the main result on the Mellin transform, Lemma 3.1 is fairly standard, with possibly different notation of the function spaces; we include it here for completeness.

First, recall that the Lie algebra of b-vector fields, $\mathcal{V}_{\mathrm{b}}(\bar{M})$ consists of $\mathcal{C}^\infty$ vector fields on $\bar{M}$ tangent to the boundary. In local coordinates (τ, y), such that τ is a boundary defining function, they are of the form $a_n\tau\partial_\tau + \sum_{j=1}^{n-1} a_j\partial_{y_j}$, with a_j arbitrary $\mathcal{C}^\infty$ functions. Correspondingly, they are the set of all smooth sections of a $\mathcal{C}^\infty$ vector bundle, ${}^{\mathrm{b}}T\bar{M}$, with local basis $\tau\partial_\tau, \partial_{y_1}, \dots, \partial_{y_{n-1}}$. The dual bundle, ${}^{\mathrm{b}}T^*\bar{M}$, thus has a local basis given by $\frac{d\tau}{\tau}, dy_1, \dots, dy_{n-1}$. All tensorial constructions, such as form and density bundles, go through as usual.

The natural bundles related to the boundary are reversed in the b-setting. Thus, the b-normal bundle of the boundary X is well-defined as the span of $\tau\partial_\tau$ defined using any coordinates, or better yet, as the kernel of the natural map $\iota : {}^{\mathrm{b}}T_m\bar{M} \to T_m\bar{M}$, $m \in X$, induced via the inclusion $\mathcal{V}_{\mathrm{b}}(\bar{M}) \to \mathcal{V}(\bar{M})$, so

$$a_n\tau\partial_\tau + \sum_{j=1}^{n-1} a_j\partial_{y_j} \mapsto \sum_{j=1}^{n-1} a_j\partial_{y_j}, \qquad a_j \in \mathbb{R}.$$

Its annihilator in ${}^{\mathrm{b}}T^*_m\bar{M}$ is called the b-cotangent bundle of the boundary; in local coordinates (τ, y) it is spanned by $dy_1, \dots, dy_{n-1}$. Invariantly, it is the image of $T^*_m\bar{M}$ in ${}^{\mathrm{b}}T^*_m\bar{M}$ under the adjoint of the tangent bundle map ι; as this has kernel N^*_mX, ${}^{\mathrm{b}}T^*_m\bar{M}$ is naturally identified with $T^*_mX = T^*_m\bar{M}/N^*_mX$.

The algebra of differential operators generated by $\mathcal{V}_{\mathrm{b}}(\bar{M})$ over $\mathcal{C}^\infty(\bar{M})$ is denoted $\mathrm{Diff}_{\mathrm{b}}(\bar{M})$; in local coordinates as above, elements of $\mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$ are of the form

$$\mathcal{P} = \sum_{j+|\alpha|\leq k} a_{j\alpha}(\tau D_\tau)^j D_y^\alpha$$

in the usual multiindex notation, $\alpha \in \mathbb{N}^{n-1}$, with $a_{j\alpha} \in \mathcal{C}^\infty(\bar{M})$. Writing b-covectors as

$$\sigma\,\frac{d\tau}{\tau} + \sum_{j=1}^{n-1} \eta_j\,dy_j,$$

we obtain canonically dual coordinates to (τ, y), namely (τ, y, σ, η) are local coordinates on ${}^{\mathrm{b}}T^*\bar{M}$. The principal symbol of $\mathcal{P}$ is

$$\tilde{p} = \sigma_{\mathrm{b},k}(\mathcal{P}) = \sum_{j+|\alpha|=k} a_{j\alpha}\sigma^j\eta^\alpha; \tag{3.1}$$

it is a $\mathcal{C}^\infty$ function, which is a homogeneous polynomial of degree k in the fibers, on ${}^{\mathrm{b}}T^*\bar{M}$. Its Hamilton vector field, $\mathsf{H}_{\tilde{p}}$, is a $\mathcal{C}^\infty$ vector field, which is just the extension of the standard Hamilton vector field from $\bar{M}^\circ$, is homogeneous of degree

$k-1$, on ${}^{\mathrm{b}}T^*\bar{M}$, and it is tangent to ${}^{\mathrm{b}}T^*_X\bar{M}$. Explicitly, as a change of variables shows, in local coordinates,

$$\mathsf{H}_{\tilde{p}} = (\partial_\sigma\tilde{p})(\tau\partial_\tau) + \sum_j(\partial_{\eta_j}\tilde{p})\partial_{y_j} - (\tau\partial_\tau\tilde{p})\partial_\sigma - \sum_j(\partial_{y_j}\tilde{p})\partial_{\eta_j}, \tag{3.2}$$

so the restriction of $\mathsf{H}_{\tilde{p}}$ to $\tau=0$ is

$$\mathsf{H}_{\tilde{p}}|_{{}^{\mathrm{b}}T^*_X\bar{M}} = \sum_j(\partial_{\eta_j}\tilde{p})\partial_{y_j} - \sum_j(\partial_{y_j}\tilde{p})\partial_{\eta_j}, \tag{3.3}$$

and is thus tangent to the fibers (identified with T^*X) of ${}^{\mathrm{b}}T^*_XM$ over ${}^{\mathrm{b}}T^*_XM/T^*X$ (identified with $\mathbb{R}_\sigma$).

We next want to define normal operator[18] of $\mathcal{P}\in\mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$, obtained by freezing coefficients at $X=\partial\bar{M}$. To do this naturally, we want to extend the 'frozen operator' to one invariant under dilations in the fibers of the inward pointing normal bundle ${}_+N(X)$ of X; see [42, Equation (4.91)]. The latter can always be trivialized by the choice of an inward-pointing vector field V, which in turn fixes the differential of a boundary defining function τ at X by $V\tau|_X=1$; given such a choice we can identify ${}_+N(X)$ with a product

$$\bar{M}_\infty = X\times[0,\infty)_\tau,$$

with the normal operator being invariant under dilations in τ. Then for $m=(x,\tau)$, ${}^{\mathrm{b}}T_m\bar{M}_\infty$ is identified with ${}^{\mathrm{b}}T_{(x,0)}\bar{M}$.

On $\bar{M}_\infty$ operators of the form

$$\sum_{j+|\alpha|\le k} a_{j\alpha}(y)(\tau D_\tau)^jD_y^\alpha,$$

i.e. $a_{j\alpha}\in\mathcal{C}^\infty(X)$, are invariant under the $\mathbb{R}^+$-action on $[0,\infty)_\tau$; its elements are denoted by $\mathrm{Diff}_{\mathrm{b},I}(\bar{M}_\infty)$. The *normal operator* of $\mathcal{P}\in\mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$ is given by freezing the coefficients at X:

$$N(\mathcal{P}) = \sum_{j+|\alpha|\le k} a_{j\alpha}(0,y)(\tau D_\tau)^jD_y^\alpha \in \mathrm{Diff}^k_{\mathrm{b},I}(\bar{M}_\infty).$$

The *normal operator family* is then defined as

$$\hat{N}(\mathcal{P})(\sigma) = P_\sigma = \sum_{j+|\alpha|\le k} a_{j\alpha}(0,y)\sigma^jD_y^\alpha \in \mathrm{Diff}^k_{\mathrm{b},I}(\bar{M}_\infty).$$

Note also that we can identify a neighborhood of X in $\bar{M}$ with a neighborhood of $X\times\{0\}$ in $\bar{M}_\infty$ (this depends on choices), and then transfer $\mathcal{P}$ to an operator (still denoted by $\mathcal{P}$) on $\bar{M}_\infty$, extended in an arbitrary smooth manner; then $\mathcal{P}-N(\mathcal{P})\in\tau\mathrm{Diff}^k_{\mathrm{b}}(\bar{M}_\infty)$.

The principal symbol p of the normal operator family, including in the high energy (or, after rescaling, semiclassical) sense, is given by $\sigma_{\mathrm{b},k}(\mathcal{P})|_{{}^{\mathrm{b}}T^*_X\bar{M}}$. Correspondingly, the Hamilton vector field, including in the high-energy sense, of p is given by $\mathsf{H}_{\sigma_{\mathrm{b},k}(\mathcal{P})}|_{{}^{\mathrm{b}}T^*_X\bar{M}}$; see (3.3). It is useful to note that via this restriction we drop information about $\mathsf{H}_{\sigma_{\mathrm{b},k}(\mathcal{P})}$ as a b-vector field, namely the $\tau\partial_\tau$ component is neglected. Correspondingly, the dynamics (including at high energies) for the normal operator family is the same at radial points of the Hamilton flow regardless of the behavior of the $\tau\partial_\tau$ component, thus whether on ${}^{\mathrm{b}}S^*\bar{M}=({}^{\mathrm{b}}T^*\bar{M}\setminus o)/\mathbb{R}^+$,

[18] In fact, $\mathcal{P}\in\Psi^k_{\mathrm{b}}(\bar{M})$ works similarly.

with the τ variable included, we have a source/sink, or a saddle point, with the other (stable/unstable) direction being transversal to the boundary. This is reflected by the same normal operator family showing up in both de Sitter space and in Minkowski space, even though in de Sitter space (and also in Kerr-de Sitter space) in the full b-sense the radial points are saddle points, while in Minkowski space they are sources/sinks (with a neutral direction along the conormal bundle of the event horizon/light cone inside the boundary in both cases).

We now translate our results to solutions of $(\mathcal{P} - \imath\mathcal{Q})u = f$ when $P_\sigma - \imath Q_\sigma$ is the normal operator family of the b-operator $\mathcal{P} - \imath\mathcal{Q}$. A typical application is when $\mathcal{P} = \Box_g$ is the d'Alembertian of a Lorentzian b-metric on $\bar{M}$, discussed in Subsection 3.2.

Thus, consider the Mellin transform in τ, i.e. consider the map

$$\mathcal{M} : u \mapsto \hat{u}(\sigma, .) = \int_0^\infty \tau^{-\imath\sigma} u(\tau, .)\, \frac{d\tau}{\tau}, \tag{3.4}$$

with inverse transform

$$\mathcal{M}^{-1} : v \mapsto \check{v}(\tau, .) = \frac{1}{2\pi} \int_{\mathbb{R}+\imath\alpha} \tau^{\imath\sigma} v(\sigma, .)\, d\sigma, \tag{3.5}$$

with α chosen in the region of holomorphy. Note that for polynomially bounded (in τ) u (with values in a space, such as $\mathcal{C}^\infty(X)$, $L^2(X)$, $\mathcal{C}^{-\infty}(X)$), for u supported near $\tau = 0$, $\mathcal{M}u$ is holomorphic in $\operatorname{Im}\sigma > C$, $C > 0$ sufficiently large, with values in the same space (such as $\mathcal{C}^\infty(X)$, etc). We discuss more precise statements below. The Mellin transform is described in detail in [42, Section 5], but it is also merely a renormalized Fourier transform, so the results below are simply those for the Fourier transform (often of Paley-Wiener type) after suitable renormalization.

First, Plancherel's theorem is that if ν is a smooth non-degenerate density on X and r_c denotes restriction to the line $\operatorname{Im}\sigma = c$, then

$$r_{-\alpha} \circ \mathcal{M} : \tau^\alpha L^2(X \times [0,\infty); \frac{|d\tau|}{\tau}\nu) \to L^2(\mathbb{R}; L^2(X;\nu)) \tag{3.6}$$

is an isomorphism. We are interested in functions u supported near $\tau = 0$, in which case, with $r_{(c_1,c_2)}$ denoting restriction to the strip $c_1 < \operatorname{Im}\sigma < c_2$, for $N > 0$,

$$\begin{aligned} &r_{-\alpha,-\alpha+N} \circ \mathcal{M} : \tau^\alpha (1+\tau)^{-N} L^2(X \times [0,\infty); \frac{|d\tau|}{\tau}\nu) \\ &\quad \to \Big\{ v : \mathbb{R} \times \imath(-\alpha, -\alpha+N) \ni \sigma \to v(\sigma) \in L^2(X;\nu); \\ &\qquad v \text{ is holomorphic in } \sigma \text{ and } \sup_{-\alpha < r < -\alpha+N} \|v(. + \imath r, .)\|_{L^2(\mathbb{R}; L^2(X;\nu))} < \infty \Big\}, \end{aligned} \tag{3.7}$$

see [42, Lemma 5.18]. Note that in accordance with (3.6), v in (3.7) extends continuously to the boundary values, $r = -\alpha$ and $r = -\alpha - N$, with values in the same space as for holomorphy. Moreover, for functions supported in, say, $\tau < 1$, one can take N arbitrary.

Analogous results also hold for the b-Sobolev spaces $H^s_{\mathrm{b}}(X \times [0,\infty))$. For $s \geq 0$, these can be defined as in [42, Equation (5.41)]:

$$\begin{aligned} &r_{-\alpha} \circ \mathcal{M} : \tau^\alpha H^s_{\mathrm{b}}(X \times [0,\infty); \frac{|d\tau|}{\tau}\nu) \\ &\to \Big\{ v \in L^2(\mathbb{R}; H^s(X;\nu)) : \ v \in (1+|\sigma|^2)^{s/2} v \in L^2(\mathbb{R}; L^2(X;\nu)) \Big\}, \end{aligned}$$

with the analogue of (3.7) also holding; for $s < 0$ one needs to use the appropriate dual statements. See also [42, Equations (5.41)-(5.42)] for differential versions for integer order spaces.

If $\mathcal{P} - \imath\mathcal{Q}$ is invariant under dilations in τ on $\bar{M}_\infty = X \times [0,\infty)$ then $N(\mathcal{P} - \imath\mathcal{Q})$ can be identified with $\mathcal{P} - \imath\mathcal{Q}$ and we have the following simple lemma:

Lemma 3.1. *Suppose $\mathcal{P} - \imath\mathcal{Q}$ is invariant under dilations in τ for functions supported near $\tau = 0$, and the normal operator family $\hat{N}(\mathcal{P}-\imath\mathcal{Q})$ is of the form $P_\sigma - \imath Q_\sigma$ satisfying the conditions of Section 2, including semiclassical non-trapping. Let σ_j be the poles of the meromorphic family $(P_\sigma - \imath Q_\sigma)^{-1}$. Then for $\ell < \beta^{-1}(2s-k+1)$, $\ell \neq -\operatorname{Im}\sigma_j$ for any j, $(\mathcal{P} - \imath\mathcal{Q})u = f$, u tempered, supported near $\tau = 0$, $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, u has an asymptotic expansion*

$$u = \sum_j \sum_{\kappa \le m_j} \tau^{\imath\sigma_j} (\log|\tau|)^\kappa a_{j\kappa} + u' \tag{3.8}$$

with $a_{jk} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$.

If instead $N(\mathcal{P} - \imath\mathcal{Q})$ is semiclassically mildly trapping of order $\varkappa$ in a C_0-strip then for $\ell < C_0$ and $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$ one has

$$u = \sum_j \sum_{\kappa \le m_j} \tau^{\imath\sigma_j} (\log|\tau|)^\kappa a_{j\kappa} + u' \tag{3.9}$$

with $a_{jk} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$.

Conversely, given f in the indicated spaces, with f supported near $\tau = 0$, a solution u of $(\mathcal{P} - \imath\mathcal{Q})u = f$ of the form (3.8)*, resp.* (3.9)*, supported near $\tau = 0$ exists.*

In either case, the coefficients $a_{j\kappa}$ are given by the Laurent coefficients of $(\mathcal{P} - \imath\mathcal{Q})^{-1}$ at the poles σ_j applied to f, with simple poles corresponding to $m_j = 0$.

If $f = \sum_j \sum_{\kappa \le m'_j} \tau^{\alpha_j} (\log|\tau|)^\kappa b_{j\kappa} + f'$, with f' in the spaces indicated above for f, and $b_{jk} \in H^{s-k+1}(X)$, analogous results hold when the expansion of f is added to the form of (3.8) *and* (3.9)*, in the sense of the extended union of index sets, see* [42, Section 5.18].

For $\mathcal{P}^* + \imath\mathcal{Q}$ in place of $\mathcal{P} - \imath\mathcal{Q}$, analogous results apply, but we need $\ell < -\beta^{-1}(2s - k + 1)$.

Remark 3.2. Thus, for $\mathcal{P} - \imath\mathcal{Q}$, the more terms we wish to obtain in an expansion, the better Sobolev space we need to work in. For $\mathcal{P}^* + \imath\mathcal{Q}$, dually, we need to be in a weaker Sobolev space under the same circumstances. However, these spaces only need to be worse at the radial points, so under better regularity assumptions on f we still get the expansion in better Sobolev spaces away from the radial points — in particular in elliptic regions. This is relevant in our description of Minkowski space.

Proof. First consider the expansion. Suppose $\alpha, r \in \mathbb{R}$ are such that $u \in \tau^\alpha H_{\mathrm{b}}^r(\bar{M}_\infty)$ and $P_\sigma - \imath Q_\sigma$ has no poles in $\operatorname{Im}\sigma \ge -\alpha$; note that the vanishing of u for $\tau > 1$ means that this can be arranged, and then also $u \in \tau^\alpha(1+\tau)^{-N} H_{\mathrm{b}}^r(\bar{M}_\infty)$ for all N. The Mellin transform of the PDE, a priori in $\operatorname{Im}\sigma \ge -\alpha$, is $(P_\sigma - \imath Q_\sigma)\mathcal{M}u = \mathcal{M}f$. Thus,

$$\mathcal{M}u = (P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f \tag{3.10}$$

there. If $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, then shifting the contour of integration to $\operatorname{Im}\sigma = -\ell$, we obtain contributions from the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$, giving the expansion in (3.8) and (3.9) by Cauchy's theorem. The error term u' is what one obtains by integrating along the new contour in view of the high energy bounds on $(P_\sigma - \imath Q_\sigma)^{-1}$ (which differ as one changes one's assumption from non-trapping to mild trapping), and the assumptions on f.

Conversely, to obtain existence, let $\alpha < \min(\ell, -\sup \operatorname{Im}\sigma_j)$ and define $u \in \tau^\alpha H_{\mathrm{b}}^s(\bar{M}_\infty)$ by (3.10) using the inverse Mellin transform with $\operatorname{Im}\sigma = -\alpha$. Then u solves the PDE, hence the expansion follows by the first part of the argument. The support property of u follows from Paley-Wiener, taking into account holomorphy in $\operatorname{Im}\sigma > -\alpha$, and the estimates on $\mathcal{M}f$ and $(P_\sigma - \imath Q_\sigma)^{-1}$ there. □

One can iterate this to obtain a full expansion even when $\mathcal{P} - \imath\mathcal{Q}$ is not dilation invariant. Note that in most cases considered below, Lemma 3.1 suffices; the exception is if we allow general, non-stationary, b-perturbations of Kerr-de Sitter or Minkowski metrics.

Proposition 3.3. *Suppose $(\mathcal{P} - \imath\mathcal{Q})u = f$, and the normal operator family $\hat{N}(\mathcal{P} - \imath\mathcal{Q})$ is of the form $P_\sigma - \imath Q_\sigma$ satisfying the conditions of Section 2, including semiclassical non-trapping. Then for $\ell < \beta^{-1}(2(s - |\ell - \alpha|) - k + 1)$, $\ell \notin -\operatorname{Im}\sigma_j + \mathbb{N}$ for any j, $u \in \tau^\alpha H_{\mathrm{b}}^r(\bar{M}_\infty)$ supported near 0, $(\mathcal{P} - \imath\mathcal{Q})u = f$, $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, u has an asymptotic expansion*

$$u = \sum_j \sum_l \sum_{\kappa \le m_{jl}} \tau^{\imath\sigma_j + l}(\log|\tau|)^\kappa a_{j\kappa l} + u' \tag{3.11}$$

with $a_{jk} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H_{\mathrm{b}}^{s-[\ell-\alpha]}(\bar{M}_\infty)$, $[\ell - \alpha]$ being the integer part of $\ell - \alpha$.

If instead $N(\mathcal{P} - \imath\mathcal{Q})$ is semiclassically mildly trapping of order $\varkappa$ in a C_0-strip then for $\ell < C_0$ and $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$ one has

$$u = \sum_j \sum_l \sum_{\kappa \le m_{jl}} \tau^{\imath\sigma_j + l}(\log|\tau|)^\kappa a_{j\kappa l} + u' \tag{3.12}$$

with $a_{jkl} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H_{\mathrm{b}}^{s-[\ell-\alpha]}(\bar{M}_\infty)$.

If $f = \sum_j \sum_{\kappa \le m_j'} \tau^{\alpha_j}(\log|\tau|)^\kappa b_{j\kappa} + f'$, with f' in the spaces indicated above for f, and $b_{jk} \in H^{s-k+1}(X)$, analogous results hold when the expansion of f is added to the form of (3.11) *and* (3.12) *in the sense of the extended union of index sets, see* [42, Section 5.18].

If $\sigma_{\mathrm{b},k}(\mathcal{P} - \imath\mathcal{Q})$ vanishes on the characteristic set of $N(\mathcal{P} - \imath\mathcal{Q})$ to infinite order in Taylor series at $\tau = 0$, then there are no losses in the order of u', i.e. one can replace $u' \in \tau^\ell H_{\mathrm{b}}^{s-[\ell-\alpha]}(\bar{M}_\infty)$ by $u' \in \tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$, and $\ell < \beta^{-1}(2(s - |\ell - \alpha|) - k + 1)$ by $\ell < \beta^{-1}(2s - k + 1)$, giving the same form as in Lemma 3.1.

Conversely, under the characteristic assumption in the previous paragraph, given f in the indicated spaces, with f supported near $\tau = 0$, a solution u of $(\mathcal{P} - \imath\mathcal{Q})u = f + f^\sharp$ of the form (3.8), *resp.* (3.9), *$f^\sharp \in \tau^\infty H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, resp. $H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$, supported near $\tau = 0$, exists.*

Remark 3.4. The losses in the regularity of u' without further assumptions are natural due to the lack of ellipticity. Specifically, if, for instance, u is conormal to a hypersurface S transversal to X, as is the case in many interesting examples, the orbits of the $\mathbb{R}^+$-action on $\bar{M}_\infty$ must be tangent to S to avoid losses of regularity in the Taylor series expansion.

In particular, there are no losses if $(\mathcal{P} - \imath\mathcal{Q}) - N(\mathcal{P} - \imath\mathcal{Q}) \in \tau\mathrm{Diff}_{\mathrm{b}}^{k-1}(\bar{M}_\infty)$, rather than merely in $\tau\mathrm{Diff}_{\mathrm{b}}^{k}(\bar{M}_\infty)$.

We only stated the converse result under the extra characteristic assumption to avoid complications with the Sobolev orders. Global solvability depends on more than the normal operator, which is why we do not state such a result here.

Proof. One proceeds as in Lemma 3.1, Mellin transforming the problem, but replacing $\mathcal{P} - \imath\mathcal{Q}$ by $N(\mathcal{P} - \imath\mathcal{Q})$. Note that $(\mathcal{P} - \imath\mathcal{Q}) - N(\mathcal{P} - \imath\mathcal{Q}) \in \tau\mathrm{Diff}_{\mathrm{b}}^{k}(\bar{M}_\infty)$. We treat

$$\tilde{f} = ((\mathcal{P} - \imath\mathcal{Q}) - N(\mathcal{P} - \imath\mathcal{Q}))u$$

as part of the right hand side, subtracting it from f, so

$$N(\mathcal{P} - \imath\mathcal{Q})u = f - \tilde{f}.$$

If $u \in \tau^\alpha H_{\mathrm{b}}^r(\bar{M}_\infty)$ is supported near 0, then $\tilde{f} \in \tau^{\alpha+1} H_{\mathrm{b}}^r(\bar{M}_\infty)$, so Lemma 3.1 is applicable with ℓ replaced by $\min(\ell, \alpha+1)$. If $\ell \leq \alpha + 1$, we are done, otherwise we repeat the argument. Indeed, we now know that u is given by an expansion giving rise to poles of $\mathcal{M}u$ in $\mathrm{Im}\,\sigma > \alpha + 1$ plus an element of $\tau^{\alpha+1} H_{\mathrm{b}}^s(\bar{M}_\infty)$, so we also have better information on $\tilde{f}$, namely it is also given by a partial expansion, plus an element of $\tau^{\alpha+2} H_{\mathrm{b}}^{s-k}(\bar{M}_\infty)$, or indeed $\tau^{\alpha+2} H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$ under the characteristic assumption on $\mathcal{P} - \imath\mathcal{Q}$. Using the f with a partial expansion part of Lemma 3.1 to absorb the $\tilde{u}$ terms, we can work with ℓ replaced by $\min(\ell, \alpha + 2)$. It is this step that starts generating the sum over l in (3.11) and (3.12). The iteration stops in a finite number of steps, completing the proof.

For the existence, define a zeroth approximation u_0 to u using $N(\mathcal{P} - \imath\mathcal{Q})$ in Lemma 3.1, and iterate away the error $\tilde{f} = ((\mathcal{P} - \imath\mathcal{Q})u - N(\mathcal{P} - \imath\mathcal{Q}))u_0 - f$ in Taylor series. □

3.2. **Lorentzian metrics.** We now review common properties of Lorentzian b-metrics g on $\bar{M}$. Lorentzian b-metrics are symmetric non-degenerate bilinear forms on ${}^{\mathrm{b}}T_m\bar{M}$, $m \in \bar{M}$, of signature $(1, n-1)$, i.e. the maximal dimension of a subspace on which g is positive definite is *exactly* 1, which depend smoothly on m. In other words, they are symmetric sections of ${}^{\mathrm{b}}T^*\bar{M} \otimes {}^{\mathrm{b}}T^*\bar{M}$ which are in addition non-degenerate of Lorentzian signature. Usually it is more convenient to work with the dual metric G, which is then a symmetric section of ${}^{\mathrm{b}}T\bar{M} \otimes {}^{\mathrm{b}}T\bar{M}$ which is in addition non-degenerate of Lorentzian signature.

By non-degeneracy there is a nowhere vanishing b-density associated to the metric, $|dg|$, which in local coordinates (τ, y) is given by $\sqrt{|\det g|}\frac{|d\tau|}{\tau}\,|dy|$, and which gives rise to a Hermitian (positive definite!) inner product on functions. There is also a non-degenerate, but not positive definite, inner product on the fibers of the b-form bundle, ${}^{\mathrm{b}}\Lambda\bar{M}$, and thus, when combined with the aforementioned Hermitian inner product on functions, an inner product on differential forms which is not positive definite only due to the lack of definiteness of the fiber inner product. Thus, A^* is defined, as a formal adjoint, for any differential operator $A \in \mathrm{Diff}_{\mathrm{b}}^k(\bar{M}; {}^{\mathrm{b}}\Lambda\bar{M})$ acting on sections of the b-form bundle, such as the exterior derivative, d. Thus, g gives rise to the d'Alembertian,

$$\Box_g = d^*d + dd^* \in \mathrm{Diff}_{\mathrm{b}}^2(\bar{M}; \Lambda\bar{M}),$$

which preserves form degrees. The d'Alembertian on functions is also denoted by $\Box_g$. The principal symbol of $\Box_g$ is

$$\sigma_{\mathrm{b},2}(\Box_g) = G.$$

As discussed above, the normal operator of $\Box_g$ on $\bar{M}$ is $N(\Box_g) \in \mathrm{Diff}_{\mathrm{b},I}(\bar{M}_\infty)$, $\bar{M}_\infty = X \times [0,\infty)_\tau$. If $\bar{M} = \bar{M}_\infty$ (i.e. it is a product space to start with) and if $\Box_g$ already has this invariance property under a product decomposition, then the normal operator can be identified with $\Box_g$ itself. Taking the Mellin transform in τ, we obtain a family of operators, P_σ, on X, depending analytically on σ, the b-dual variable of τ. The semiclassical principal symbol of $P_{h,z} = h^2 P_{h^{-1}z}$ is just the dual metric G on the complexified cotangent bundle ${}^{\mathrm{b},\mathbb{C}}T^*_m\bar{M}$, $m = (x,\tau)$, evaluated on covectors $\varpi + z\,\frac{d\tau}{\tau}$, where ϖ is in the (real) span Π of the 'spatial variables' $T^*_x X$; thus Π and $\frac{d\tau}{\tau}$ are linearly independent. In general,

$$\begin{aligned}
&\langle \varpi + z\,\frac{d\tau}{\tau}, \varpi + z\,\frac{d\tau}{\tau}\rangle_G \\
&= \langle \varpi + \mathrm{Re}\, z\,\frac{d\tau}{\tau}, \varpi + \mathrm{Re}\, z\,\frac{d\tau}{\tau}\rangle_G - (\mathrm{Im}\, z)^2 \langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G \\
&\qquad + 2\imath\, \mathrm{Im}\, z \langle \varpi + \mathrm{Re}\, z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G.
\end{aligned} \tag{3.13}$$

For $\mathrm{Im}\, z \neq 0$, the vanishing of the imaginary part states that $\langle \varpi + \mathrm{Re}\, z\,\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = 0$; the real part is the first two terms on the right hand side of (3.13).

In the setting of Subsection 2.7 we want that when $\mathrm{Im}\, z \neq 0$ and $\mathrm{Im}\, p_{\hbar,z}$ vanishes then $\mathrm{Re}\, p_{\hbar,z}$ does not vanish, i.e. that on the orthocomplement of the span of $\frac{d\tau}{\tau}$ the metric should have the opposite sign as that of $\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$. For a Lorentzian metric this is only possible if $\frac{d\tau}{\tau}$ is time-like (note that $\varpi + \mathrm{Re}\, z\,\frac{d\tau}{\tau}$ spans the whole fiber of the b-cotangent bundle as $\mathrm{Re}\, z$ and $\varpi \in \Pi$ vary), when, however, this is automatically the case, namely the metric is negative definite on this orthocomplement.

Furthermore, for z real, non-zero, the characteristic set of $p_{\hbar,z}$ cannot intersect the hypersurface $\langle \varpi + \mathrm{Re}\, z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = 0$, for G is negative definite on covectors satisfying this equality, so if the intersection were non-empty, $\varpi + \mathrm{Re}\, z\frac{d\tau}{\tau}$ would vanish there, which cannot happen for $\varpi \in \Pi$ since $\mathrm{Re}\, z \neq 0$ by assumption. Correspondingly, we can divide the semiclassical characteristic set in two parts by

$$\Sigma_{\hbar,\pm} \cap T^*X = \{\varpi \in \Sigma_\hbar \cap T^*X : \pm\langle \varpi + \mathrm{Re}\, z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G > 0\}; \tag{3.14}$$

note that by the definiteness of the quadratic form on this hypersurface, in fact this separation holds on the fiber-compactified bundle, $\overline{T}^* X$. In general, one of the 'components' $\Sigma_{\hbar,\pm}$ may be empty. However, in any case, when $\mathrm{Im}\, z \geq 0$, the sign of the imaginary part of $p_{\hbar,z}$ on $\Sigma_{\hbar,\pm}$ is given by $\pm\,\mathrm{Im}\, p_{\hbar,z} \geq 0$, as needed for the propagation of estimates: in $\Sigma_{\hbar,+}$ we can propagate estimates backwards, in $\Sigma_{\hbar,-}$ we can propagate estimates forward. For $\mathrm{Im}\, z \leq 0$, the direction of propagation is reversed.

Moreover, for $m \in \bar{M}$, and with Π denoting the 'spatial' hyperplane in the real cotangent bundle, ${}^{\mathrm{b}}T^*_m\bar{M}$, the Lorentzian nature of G means that for z real and non-zero, the intersection of $\Pi + z\frac{d\tau}{\tau}$ with the zero-set of G in ${}^{\mathrm{b}}T^*_q\bar{M}$, i.e. the characteristic set, has two components if $G|_\Pi$ is Lorentzian, and one component if it is negative definite (i.e. Riemannian, up to the sign). Further, in the second case, on the only component $\langle \varpi + \operatorname{Re} z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ and $\langle \operatorname{Re} z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ have the same sign, so only $\Sigma_{\hbar,\operatorname{sgn}(\operatorname{Re} z)}$ can enter the elliptic region.

In summary, we have shown that *if* $\frac{d\tau}{\tau}$ *is time-like* then the assumptions on imaginary part of $p_{\hbar,z}$ in Section 2 are automatically satisfied in the Lorentzian setting. Thus, these need not be checked individually in specific cases.

It is useful to note the following explicit calculation regarding the time-like character of $\frac{d\tau}{\tau}$ if we are given a Lorentzian b-metric g that, with respect to some local boundary defining function $\tilde\tau$ and local product decomposition $U \times [0,\delta)_{\tilde\tau}$ of $\bar M$ near $U \subset X$ open, is of the form $G = (\tilde\tau\partial_{\tilde\tau})^2 - \tilde G$ on $U \times [0,\delta)_{\tilde\tau}$, $\tilde G$ a Riemannian metric on U. In this case, if we define $\tau = \tilde\tau e^{\phi}$, ϕ a function on X, so $\frac{d\tau}{\tau} = \frac{d\tilde\tau}{\tilde\tau} + d\phi$, then

$$\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = \langle \frac{d\tilde\tau}{\tilde\tau}, \frac{d\tilde\tau}{\tilde\tau}\rangle_G - \langle d\phi, d\phi\rangle_G = 1 - \langle d\phi, d\phi\rangle_{\tilde G},$$

so $\frac{d\tau}{\tau}$ is time-like if $|d\phi|_{\tilde G} < 1$. Note that the effect of such a coordinate change on the Mellin transform of the normal operator of $\Box_g$ is conjugation by $e^{-\imath\sigma\phi}$ since $\tilde\tau^{-\imath\sigma} = \tau^{-\imath\sigma} e^{\imath\sigma\phi}$. Such a coordinate change is useful when G has a product structure on $U \times [0,\delta)_{\tilde\tau}$, but $\tilde\tau$ is only a local boundary defining function on $U \times [0,\delta)_{\tilde\tau}$ (the product structure might not extend smoothly beyond U), in which case it is useful to see if one can conserve the time-like nature of $\frac{d\tilde\tau}{\tilde\tau}$ for a global boundary defining function. This is directly relevant for the study of conformally compact spaces in Subsection 4.9.

Finally, we remark that if $\frac{d\tau}{\tau}$ is time-like and f is supported in $\tau < \tau_0$, the forward problem for the wave equation,

$$\Box_g u = f,\ u|_{\tau > \tau_0} = 0,$$

is uniquely solvable on $\bar M \setminus X$ near X by standard energy estimates (a priori, without more structure, with no estimates on growth at X), see e.g. [31, Chapter 23]. Applying Lemma 3.1 in this case, assuming that the normal operator family has the structure stated there, gives another solution, which must be equal to u by uniqueness. Thus, u has the expansion stated in the lemma.

4. De Sitter space and conformally compact spaces

4.1. **De Sitter space as a symmetric space.** Rather than starting with the static picture of de Sitter space, we consider it as a Lorentzian symmetric space. We follow the treatment of [51] and [39]. De Sitter space M is given by the hyperboloid

$$z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1 \text{ in } \mathbb{R}^{n+1}$$

equipped with the pull-back g of the Minkowski metric

$$dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2.$$

Introducing polar coordinates (R,θ) in $(z_1,\ldots,z_n)$, so

$$R = \sqrt{z_1^2 + \ldots + z_n^2} = \sqrt{1 + z_{n+1}^2},\ \theta = R^{-1}(z_1,\ldots,z_n) \in \mathbb{S}^{n-1},\ \tilde\tau = z_{n+1},$$

the hyperboloid can be identified with $\mathbb{R}_{\tilde\tau} \times \mathbb{S}^{n-1}_\theta$ with the Lorentzian metric

$$g = \frac{d\tilde\tau^2}{\tilde\tau^2 + 1} - (\tilde\tau^2 + 1)\, d\theta^2,$$

where $d\theta^2$ is the standard Riemannian metric on the sphere. For $\tilde{\tau} > 1$, set $x = \tilde{\tau}^{-1}$, so the metric becomes

$$g = \frac{(1+x^2)^{-1}\,dx^2 - (1+x^2)\,d\theta^2}{x^2}.$$

An analogous formula holds for $\tilde{\tau} < -1$, so compactifying the real line to an interval $[0,1]_T$, with $T = x = \tilde{\tau}^{-1}$ for $x < \frac{1}{4}$ (i.e. $\tilde{\tau} > 4$), say, and $T = 1 - |\tilde{\tau}|^{-1}$, $\tilde{\tau} < -4$, gives a compactification, $\hat{M}$, of de Sitter space on which the metric is conformal to a non-degenerate Lorentz metric. There is natural generalization, to *asymptotically de Sitter-like spaces* $\hat{M}$, which are diffeomorphic to compactifications $[0,1]_T \times Y$ of $\mathbb{R}_{\tilde{\tau}} \times Y$, where Y is a compact manifold without boundary, and $\hat{M}$ is equipped with a Lorentz metric on its interior which is conformal to a Lorentz metric smooth up to the boundary. These space-times are Lorentzian analogues of the much-studied conformally compact (Riemannian) spaces. On this class of space-times the solutions of the Klein-Gordon equation were analyzed by Vasy in [51], and were shown to have simple asymptotics analogous to those for generalized eigenfunctions on conformally compact manifolds.

Theorem. *(*[51, Theorem 1.1.]*) Set* $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. *If* $s_+(\lambda) - s_-(\lambda) \notin \mathbb{N}$, *any solution* u *of the Cauchy problem for* $\Box - \lambda$ *with* $\mathcal{C}^\infty$ *initial data at* $\tilde{\tau} = 0$ *is of the form*

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-,\ \ v_\pm \in \mathcal{C}^\infty(\hat{M}).$$

If $s_+(\lambda) - s_-(\lambda)$ *is an integer, the same conclusion holds if* $v_- \in \mathcal{C}^\infty(\hat{M})$ *is replaced by* $v_- = \mathcal{C}^\infty(\hat{M}) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(\hat{M})$.

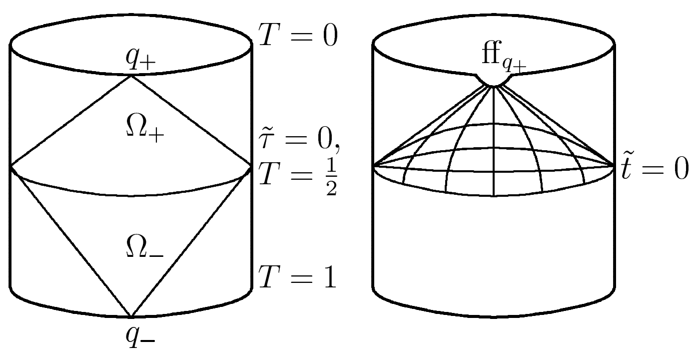

FIGURE 3. On the left, the compactification of de Sitter space with the backward light cone from $q_+ = (1,0,0,0)$ and forward light cone from $q_- = (-1,0,0,0)$ are shown. Ω_+, resp. Ω_-, denotes the intersection of these light cones with $\tilde{\tau} > 0$, resp. $\tilde{\tau} < 0$. On the right, the blow up of de Sitter space at q_+ is shown. The interior of the light cone inside the front face ff_{q_+} can be identified with the spatial part of the static model of de Sitter space. The spatial and temporal coordinate lines for the static model are also shown.

One important feature of asymptotically de Sitter spaces is the following: a conformal factor, such as x^{-2} above, does not change the image of null-geodesics, only reparameterizes them. More precisely, recall that null-geodesics are merely projections to M of null-bicharacteristics of the metric function in T^*M. Since $p \mapsto \mathsf{H}_p$ is a derivation, $ap \mapsto a\mathsf{H}_p + p\mathsf{H}_a$, which is $a\mathsf{H}_p$ on the characteristic set of p. Thus, the null-geodesics of de Sitter space are the same (up to reparameterization) as those of the metric

$$(1+x^2)^{-1}\,dx^2 - (1+x^2)\,d\theta^2$$

which is smooth on the compact space $\hat{M}$.

4.2. **The static model of a part of de Sitter space.** The simple structure of de Sitter metric (and to some extent of the asymptotically de Sitter-like metrics) can be hidden by blowing up certain submanifolds of the boundary of $\hat{M}$. In particular, the *static model* of de Sitter space arises by singling out a point on $\mathbb{S}^{n-1}_\theta$, e.g. $q_0 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Note that $(\theta_2, \dots, \theta_n) \in \mathbb{R}^{n-1}$ are local coordinates on $\mathbb{S}^{n-1}$ near q_0. Now consider the intersection of the backward light cone from q_0 considered as a point q_+ at future infinity, i.e. where $T = 0$, and the forward light cone from q_0 considered as a point q_- at past infinity, i.e. where $T = 1$. These intersect the equator $T = 1/2$ (here $\tilde{\tau} = 0$) in the same set, and together form a 'diamond', $\hat{\Omega}$, with a conic singularity at q_+ and q_-. Explicitly $\hat{\Omega}$ is given by $z_2^2 + \dots + z_n^2 \leq 1$ inside the hyperboloid. If q_+, q_- are blown up, as well as the corner $\partial\Omega \cap \{\tilde{\tau} = 0\}$, i.e. where the light cones intersect $\tilde{\tau} = 0$ in $\hat{\Omega}$, we obtain a manifold $\bar{M}$, which can be blown down to (i.e. is a blow up of) the space-time product $[0, 1] \times \mathbb{B}^{n-1}$, with $\mathbb{B}^{n-1} = \{Z \in \mathbb{R}^{n-1} : |Z| \leq 1\}$ on which the Lorentz metric has a time-translation invariant warped product form. Namely, first considering the interior Ω of $\hat{\Omega}$ we introduce the global (in Ω) standard static coordinates $(\tilde{t}, Z)$, given by (with the expressions involving x valid near $T = 0$)

$$\begin{aligned}(\mathbb{B}^{n-1})^\circ \ni Z = (z_2, \dots, z_n) &= x^{-1}\sqrt{1+x^2}(\theta_2, \dots, \theta_n),\\ \sinh \tilde{t} = \frac{z_{n+1}}{\sqrt{z_1^2 - z_{n+1}^2}} &= (x^2 - (1+x^2)(\theta_2^2 + \dots + \theta_n^2))^{-1/2}.\end{aligned}$$

It is convenient to rewrite these as well in terms of polar coordinates in Z (valid away from $Z = 0$):

$$\begin{aligned}r &= \sqrt{z_2^2 + \dots + z_n^2} = \sqrt{1 + z_{n+1}^2 - z_1^2} = x^{-1}\sqrt{1+x^2}\sqrt{\theta_2^2 + \dots + \theta_n^2},\\ \sinh \tilde{t} &= \frac{z_{n+1}}{\sqrt{z_1^2 - z_{n+1}^2}} = (x^2 - (1+x^2)(\theta_2^2 + \dots + \theta_n^2))^{-1/2} = x^{-1}(1-r^2)^{-1/2},\\ \omega &= r^{-1}(z_2, \dots, z_n) = (\theta_2^2 + \dots + \theta_n^2)^{-1/2}(\theta_2, \dots, \theta_n) \in \mathbb{S}^{n-2}.\end{aligned}$$

In these coordinates the metric becomes

$$(1 - r^2)\,d\tilde{t}^2 - (1-r^2)^{-1}dr^2 - r^2\,d\omega^2, \tag{4.1}$$

which is a special case of the de Sitter-Schwarzschild metrics with vanishing mass, $M = 0$, and cosmological constant $\Lambda = 3$, see Section 6. Correspondingly, the dual metric is

$$(1 - r^2)^{-1}\partial_{\tilde{t}}^2 - (1 - r^2)\partial_r^2 - r^{-2}\partial_\omega^2. \tag{4.2}$$

We also rewrite this in terms of coordinates valid at the origin, namely $Y = r\omega$:

$$(1 - |Y|^2)^{-1}\partial_{\tilde{t}}^2 + (\sum_{j=1}^{n-1} Y_j \partial_{Y_j})^2 - \sum_{j=1}^{n-1} \partial_{Y_j}^2. \tag{4.3}$$

4.3. **Blow-up of the static model.** We have already seen that de Sitter space has a smooth conformal compactification; the singularities in the metric of the form (4.1) at $r = 1$ must thus be artificial. On the other hand, the metric is already well-behaved for $r < 1$ bounded away from 1, so we want the coordinate change to be smooth there — this means smoothness in valid coordinates (Y above) at the origin as well. This singularity can be removed by a blow-up on an appropriate compactification. We phrase this at first in a way that is closely related to our treatment of Kerr-de Sitter space, and the Kerr-star coordinates. So let

$$t = \tilde{t} + h(r),\ h(r) = -\frac{1}{2}\log\mu,\ \mu = 1 - r^2.$$

Note that h is smooth at the origin, A key feature of this change of coordinates is

$$h'(r) = -\frac{r}{\mu} = -\frac{1}{\mu} + \frac{1}{1+r},$$

which is $-\mu^{-1}$ near $r = 1$ modulo terms smooth at $r = 1$. Other coordinate changes with this property would also work. Let

$$\tau = e^{-t} = \frac{e^{-\tilde{t}}}{\mu^{1/2}}.$$

Thus, if we compactify static space-time as $\mathbb{B}^{n-1}_{r\omega} \times [0,1]_T$, with $T = \tau$ for say $t > 4$, then this procedure amounts to blowing up $T = 0$, $\mu = 0$ parabolically. (If we used $\tau = e^{-2t}$, everything would go through, except there would be many additional factors of 2; then the blow-up would be homogeneous, i.e. spherical.) Then the dual metric becomes

$$-\mu\partial_r^2 - 2r\partial_r\tau\partial_\tau + \tau^2\partial_\tau^2 - r^{-2}\partial_\omega^2,$$

or

$$-4r^2\mu\partial_\mu^2 + 4r^2\tau\partial_\tau\partial_\mu + \tau^2\partial_\tau^2 - r^{-2}\partial_\omega^2,$$

which is a non-degenerate Lorentzian b-metric[19] on $\mathbb{R}^{n-1}_{r\omega} \times [0,1)_\tau$, i.e. it extends smoothly and non-degenerately across the 'event horizon', $r = 1$. Note that in coordinates valid near $r = 0$ this becomes

$$(\sum_j Y_j\partial_{Y_j})^2 - 2(\sum_j Y_j\partial_{Y_j})\tau\partial_\tau + \tau^2\partial_\tau^2 - \sum_j \partial_{Y_j}^2 = (\tau\partial_\tau - \sum_j Y_j\partial_{Y_j})^2 - \sum_j \partial_{Y_j}^2.$$

In slightly different notation, this agrees with the symbol of [51, Equation (7.3)].

We could have used other equivalent local coordinates; for instance replaced $e^{-\tilde{t}}$ by $(\sinh\tilde{t})^{-1}$, in which case the coordinates (r,τ,ω) we obtained are replaced by

$$r,\ \rho = (\sinh\tilde{t})^{-1}/(1-r^2)^{1/2} = x,\ \omega. \tag{4.4}$$

As expected, in these coordinates the metric would still be a smooth and non-degenerate b-metric. These coordinates also show that Kerr-star-type coordinates are smooth in the interior of the front face on the blow-up of our conformal compactification of de Sitter space at q_+. [20] In summary we have reproved (modulo a few details):

[19] See Section 3 for a quick introduction to b-geometry and further references.

[20] If we had worked with e^{-2t} instead of e^{-t} above, we would obtain x^2 as the defining function of the temporal face, rather than x.

Lemma 4.1. *(See [39, Lemma 2.1] for a complete version.) The lift of $\hat{\Omega}$ to the blow up $[\hat{M}; q_+, q_-]$ is a $\mathcal{C}^\infty$ manifold with corners, $\bar{\Omega}$. Moreover, near the front faces $\mathrm{ff}_{q_\pm}$, $\bar{\Omega}$ is naturally diffeomorphic to a neighborhood of the temporal faces $\mathrm{tf}_\pm$ in the $\mathcal{C}^\infty$ manifold with corners obtained from $[0,1]_T \times \mathbb{B}^{n-1}$ by blowing up $\{0\} \times \partial\mathbb{B}^{n-1}$ and $\{1\} \times \partial\mathbb{B}^{n-1}$ in the parabolic manner indicated in (4.4); here $\mathrm{tf}_\pm$ are the lifts of $\{0\} \times \partial\mathbb{B}^{n-1}$ and $\{1\} \times \partial\mathbb{B}^{n-1}$.*

It is worthwhile comparing the de Sitter space wave asymptotics of [51],

$$u = x^{n-1} v_+ + v_-,\ v_+ \in \mathcal{C}^\infty(\hat{M}),\ v_- \in \mathcal{C}^\infty(\hat{M}) + x^{n-1}(\log x)\mathcal{C}^\infty(\hat{M}), \tag{4.5}$$

with our main result, Theorem 1.4. The fact that the coefficients in the de Sitter expansion are $\mathcal{C}^\infty$ on $\hat{M}$ means that on $\bar{M}$, the leading terms are constant. Thus, (4.5) implies (and is much stronger than) the statement that u decays to a constant on $\bar{M}$ at an exponential rate.

4.4. **D'Alembertian and its Mellin transform.** Consider the d'Alembertian, $\Box_g$, whose principal symbol, including subprincipal terms, is given by the metric function. Thus, writing b-covectors as

$$\xi\, d\mu + \sigma\, \frac{d\tau}{\tau} + \eta\, d\omega,$$

we have

$$G = \sigma_{\mathrm{b},2}(\Box) = -4r^2\mu\xi^2 + 4r^2\sigma\xi + \sigma^2 - r^{-2}|\eta|^2, \tag{4.6}$$

with $|\eta|^2_\omega$ denoting the dual metric function on the sphere. Note that there is a polar coordinate singularity at $r = 0$; this is resolved by using actually valid coordinates $Y = r\omega$ on $\mathbb{R}^{n-1}$ near the origin; writing b-covectors as

$$\sigma\, \frac{d\tau}{\tau} + \zeta\, dY,$$

we have

$$\begin{gathered} G = \sigma_{\mathrm{b},2}(\Box) = (Y\cdot\zeta)^2 - 2(Y\cdot\zeta)\sigma + \sigma^2 - |\zeta|^2 = (Y\cdot\zeta - \sigma)^2 - |\zeta|^2, \\ Y\cdot\zeta = \sum_j Y_j\cdot\zeta_j,\ |\zeta|^2 = \sum_j \zeta_j^2. \end{gathered} \tag{4.7}$$

Since there are no interesting phenomena at the origin, we may ignore this point below.

Via conjugation by the (inverse) Mellin transform, see Subsection 3.1, we obtain a family of operators P_σ depending on σ on $\mathbb{R}^{n-1}_{r\omega}$ with both principal and high energy ($|\sigma| \to \infty$) symbol given by (4.6). Thus, the principal symbol of $P_\sigma \in \mathrm{Diff}^2(\mathbb{R}^{n-1})$, including in the high energy sense ($\sigma \to \infty$), is

$$\begin{aligned} p_{\mathrm{full}} &= -4r^2\mu\xi^2 + 4r^2\sigma\xi + \sigma^2 - r^{-2}|\eta|^2_\omega \\ &= (Y\cdot\zeta)^2 - 2(Y\cdot\zeta)\sigma + \sigma^2 - |\zeta|^2 = (Y\cdot\zeta - \sigma)^2 - |\zeta|^2. \end{aligned} \tag{4.8}$$

The Hamilton vector field is

$$\begin{aligned} \mathsf{H}_{p_{\mathrm{full}}} &= 4r^2(-2\mu\xi + \sigma)\partial_\mu - r^{-2}\mathsf{H}_{|\eta|^2_\omega} - (4(1-2r^2)\xi^2 - 4\sigma\xi - r^{-4}|\eta|^2_\omega)\partial_\xi \\ &= 2(Y\cdot\zeta - \sigma)(Y\cdot\partial_Y - \zeta\cdot\partial_\zeta) - 2\zeta\cdot\partial_Y, \end{aligned} \tag{4.9}$$

with $\zeta\cdot\partial_Y=\sum\zeta_j\partial_{Y_j}$, etc. Thus, in the standard 'classical' sense, which effectively means letting $\sigma=0$, the principal symbol is

$$\begin{aligned} p=\sigma_2(P_\sigma)&=-4r^2\mu\xi^2-r^{-2}|\eta|^2_\omega\\ &=(Y\cdot\zeta)^2-|\zeta|^2, \end{aligned}\tag{4.10}$$

while the Hamilton vector field is

$$\begin{aligned} \mathsf{H}_p&=-8r^2\mu\xi\partial_\mu-r^{-2}\mathsf{H}_{|\eta|^2_\omega}-(4(1-2r^2)\xi^2-r^{-4}|\eta|^2_\omega)\partial_\xi\\ &=2(Y\cdot\zeta)(Y\cdot\partial_Y-\zeta\cdot\partial_\zeta)-2\zeta\cdot\partial_Y, \end{aligned}\tag{4.11}$$

Moreover, the imaginary part of the subprincipal symbol, given by the principal symbol of $\frac{1}{2\imath}(P_\sigma-P_\sigma^*)$, is

$$\sigma_1(\frac{1}{2\imath}(P_\sigma-P_\sigma^*))=4r^2(\operatorname{Im}\sigma)\xi=-2(Y\cdot\zeta)\operatorname{Im}\sigma.$$

When comparing these with [51, Section 7], it is important to keep in mind that what is denoted by σ there (which we refer to as $\tilde\sigma$ here to avoid confusion) is $\imath\sigma$ here corresponding to the Mellin transform, which is a decomposition in terms of $\tau^{\imath\sigma}\sim x^{\imath\sigma}$, being replaced by weights $x^{\tilde\sigma}$ in [51, Equation (7.4)].

One important feature of this operator is that

$$N^*\{\mu=0\}=\{(\mu,\omega,\xi,\eta):\ \mu=0,\ \eta=0\}$$

is invariant under the classical flow (i.e. effectively letting $\sigma=0$). Let

$$N^*S\setminus o=\Lambda_+\cup\Lambda_-,\qquad \Lambda_\pm=N^*S\cap\{\pm\xi>0\},\qquad S=\{\mu=0\}.$$

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*\mathbb{R}^{n-1}$. Next we analyze the flow at $\Lambda_\pm$. First,

$$\mathsf{H}_p|\eta|^2_\omega=0\tag{4.12}$$

and

$$\mathsf{H}_p\mu=-8r^2\mu\xi=-8\xi\mu+a\mu^2\xi\tag{4.13}$$

with a being $\mathcal{C}^\infty$ in T^*X, and homogeneous of degree 0. While, in the spirit of linearizations, we used an expression in (4.13) that is linear in the coordinates whose vanishing defines N^*S, the key point is that μ is an elliptic multiple of p in a linearization sense, so one can simply use $\hat p=p/|\xi|^2$ (which is homogeneous of degree 0, like μ), in its place.

It is convenient to rehomogenize (4.12) in terms of $\hat\eta=\eta/|\xi|$. To phrase this more invariantly, consider the fiber-compactification $\overline{T}^*\mathbb{R}^{n-1}$ of $T^*\mathbb{R}^{n-1}$, see Subsection 2.2. On this space, the classical principal symbol, p, is (essentially) a function on $\partial\overline{T}^*\mathbb{R}^{n-1}=S^*\mathbb{R}^{n-1}$. Then at fiber infinity near N^*S, we can take $(|\xi|^{-1},\hat\eta)$ as coordinates on the fibers of the cotangent bundle, with $\tilde\rho=|\xi|^{-1}$ defining S^*X in $\overline{T}^*X$. Then $|\xi|^{-1}\mathsf{H}_p$ is a $\mathcal{C}^\infty$ vector field in this region and

$$|\xi|^{-1}\mathsf{H}_p|\hat\eta|^2=|\hat\eta|^2\mathsf{H}_p|\xi|^{-1}=-4(\operatorname{sgn}\xi)|\hat\eta|^2+\tilde a,\tag{4.14}$$

where $\tilde a$ vanishes cubically at N^*S, i.e. (2.3) holds. In similar notation we have

$$\begin{aligned} \mathsf{H}_p|\xi|^{-1}&=-4\operatorname{sgn}(\xi)+\tilde a',\\ |\xi|^{-1}\mathsf{H}_p\mu&=-8(\operatorname{sgn}\xi)\mu. \end{aligned}\tag{4.15}$$

with $\tilde a'$ smooth (indeed, homogeneous degree zero without the compactification) vanishing at N^*S. As the vanishing of $\hat\eta,|\xi|^{-1}$ and μ defines ∂N^*S, we conclude that $L_-=\partial\Lambda_-$ is a source, while $L_+=\partial\Lambda_+$ is a sink, in the sense that

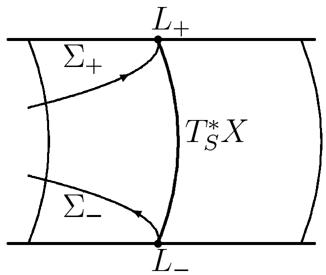

FIGURE 4. The cotangent bundle near the event horizon $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view. The boundary of the fiber compactificaton is the cosphere bundle $S^*\mathbb{R}^{n-1}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \leq 0$, only meeting $S^*_S\mathbb{R}^{n-1}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^*\mathbb{R}^{n-1}$, for $z = h^{-1}\sigma > 0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu > 0$. This is reversed for $z < 0$.

all nearby bicharacteristics (in fact, including semiclassical (null)bicharacteristics, since $\mathsf{H}_p|\xi|^{-1}$ contains the additional information needed) converge to $L_\pm$ as the parameter along the bicharacteristic goes to $\pm\infty$. In particular, the quadratic defining function of $L_\pm$ given by

$$\rho_0 = \widehat{\widehat{p}} + \hat{p}^2, \text{ where } \hat{p} = |\xi|^{-2}p,\ \widehat{\widehat{p}} = |\hat{\eta}|^2,$$

satisfies (2.4).

The imaginary part of the subprincipal symbol at $L_\pm$ is given by

$$(4\,\mathrm{sgn}(\xi))\,\mathrm{Im}\,\sigma|\xi|;$$

here $(4\,\mathrm{sgn}(\xi))$ is pulled out due to (4.15), namely its size relative to $\mathsf{H}_p|\xi|^{-1}$ matters, with a change of sign, see Subsection 2.2, thus (2.5)-(2.6) hold. This corresponds to the fact[21] that $(\mu \pm \imath 0)^{\imath\sigma}$, which are Lagrangian distributions associated to $\Lambda_\pm$, solve the PDE modulo an error that is two orders lower than what one might a priori expect, i.e. $P_\sigma(\mu \pm \imath 0)^{\imath\sigma} \in (\mu \pm i0)^{\imath\sigma}\mathcal{C}^\infty(\mathbb{R}^{n-1})$. Note that P_σ is second order, so one should lose two orders a priori; the characteristic nature of $\Lambda_\pm$ reduces the loss to 1, and the particular choice of exponent eliminates the loss. This has much in common with $e^{\imath\lambda/x}x^{(n-1)/2}$ being an approximate solution in asymptotically Euclidean scattering. The precise situation for Kerr-de Sitter space is more delicate as the Hamilton vector field does not vanish at $L_\pm$, but this[22] is irrelevant for our estimates: only a quantitative version of the source/sink statements and the imaginary part of the subprincipal symbol are relevant.

While $(\mu \pm \imath 0)^{\imath\sigma}$ is singular regardless of σ apart from integer coincidences (when this should be corrected anyway), it is interesting to note that for $\mathrm{Im}\,\sigma > 0$ this is not bounded at $\mu = 0$, while for $\mathrm{Im}\,\sigma < 0$ it vanishes there. This is interesting because if one reformulates the problem as one in $\mu \geq 0$, as was done for instance by Sá Barreto and Zworski [46], and later by Melrose, Sá Barreto and Vasy [39] for de Sitter-Schwarzschild space then one obtains an operator that is essentially (up to a

[21]This needs the analogous statement for full subprincipal symbol, not only its imaginary part.

[22]This would be relevant for a full Lagrangian analysis, as done e.g. in [38], or in a somewhat different, and more complicated, context by Hassell, Melrose and Vasy in [29, 30].

conjugation and a weight, see below) the Laplacian on an asymptotically hyperbolic space at energy $\sigma^2+\frac{(n-2)^2}{4}$ — more precisely its normal operator (which encodes its behavior near $\mu=0$) is a multiple of that of the hyperbolic Laplacian. Then the growth/decay behavior corresponds to the usual scattering theory phenomena, but in our approach smooth extendability across $\mu=0$ is the distinguishing feature of the solutions we want, not growth/decay. See Remark 4.5 for more details.

4.5. **Global behavior of the characteristic set.** First remark that $\langle\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle_G=1>0$, so $\frac{d\tau}{\tau}$ is time-like. Correspondingly all the results of Subsection 3.2 apply. In particular, (3.14) gives that the characteristic set is divided into two components with $\Lambda_\pm$ in different components. It is easy to make this explicit: points with $\xi=0$, or equivalently $Y\cdot\zeta=0$, cannot lie in the characteristic set. Thus,

$$\Sigma_\pm=\Sigma\cap\{\pm\xi>0\}=\Sigma\cap\{\mp(Y\cdot\zeta)>0\}.$$

While it is not important here since the characteristic set in $\mu\geq 0$ is localized at N^*S, hence one has a similar localization for nearby μ, for global purposes (which we do not need here), we point out that $\mathsf{H}_p\mu=-8r^2\mu\xi$. Since $\xi\neq 0$ on Σ, and in Σ, $r=1$ can only happen at N^*S, i.e. only at the radial set, the $\mathcal{C}^\infty$ function μ provides a negative global escape function which is increasing on Σ_+, decreasing on Σ_-. Correspondingly, bicharacteristics in Σ_+ travel from infinity to L_+, while in Σ_- they travel from L_- to infinity.

4.6. **High energy, or semiclassical, asymptotics.** We are also interested in the high energy behavior, as $|\sigma|\to\infty$. For the associated semiclassical problem one obtains a family of operators

$$P_{h,z}=h^2P_{h^{-1}z},$$

with $h=|\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. Then the semiclassical principal symbol $p_{\hbar,z}$ of $P_{h,z}$ is a function on $T^*\mathbb{R}^{n-1}$. As in Section 2, we are interested in $\operatorname{Im} z\geq -Ch$, which corresponds to $\operatorname{Im}\sigma\geq -C$. It is sometimes convenient to think of $p_{\hbar,z}$, and its rescaled Hamilton vector field, as objects on $\overline{T}^*\mathbb{R}^{n-1}$. Thus,

$$\begin{aligned}p_{\hbar,z}=\sigma_{2,h}(P_{h,z})&=-4r^2\mu\xi^2+4r^2z\xi+z^2-r^{-2}|\eta|^2_\omega\\&=(Y\cdot\zeta)^2-2(Y\cdot\zeta)z+z^2-|\zeta|^2=(Y\cdot\zeta-z)^2-|\zeta|^2.\end{aligned}\tag{4.16}$$

We make the general discussion of Subsection 3.2 explicit. First,

$$\operatorname{Im} p_{\hbar,z}=2\operatorname{Im} z(2r^2\xi+\operatorname{Re} z)=-2\operatorname{Im} z(Y\cdot\zeta-\operatorname{Re} z).\tag{4.17}$$

In particular, for z non-real, $\operatorname{Im} p_{\hbar,z}=0$ implies $2r^2\xi+\operatorname{Re} z=0$, i.e. $Y\cdot\zeta-\operatorname{Re} z=0$, which means that $\operatorname{Re} p_{\hbar,z}$ is

$$-r^{-2}(\operatorname{Re} z)^2-(\operatorname{Im} z)^2-r^{-2}|\eta|^2_\omega=-(\operatorname{Im} z)^2-|\zeta|^2<0,\tag{4.18}$$

i.e. $p_{\hbar,z}$ is semiclassically elliptic on $T^*\mathbb{R}^{n-1}$, but *not* at fiber infinity, i.e. at $S^*\mathbb{R}^{n-1}$ (standard ellipticity is lost only in $r\geq 1$, of course). Explicitly, if we introduce for instance

$$(\mu,\omega,\nu,\hat\eta),\qquad \nu=|\xi|^{-1},\ \hat\eta=\eta/|\xi|,$$

as valid projective coordinates in a (large!) neighborhood of $L_\pm$ in $\overline{T}^*\mathbb{R}^{n-1}$, then

$$\nu^2p_{\hbar,z}=-4r^2\mu+4r^2(\operatorname{sgn}\xi)z\nu+z^2\nu^2-r^{-2}|\hat\eta|^2_\omega$$

so

$$\nu^2 \operatorname{Im} p_{\hbar,z} = 4r^2(\operatorname{sgn}\xi)\nu \operatorname{Im} z + 2\nu^2 \operatorname{Re} z \operatorname{Im} z$$

which automatically vanishes at $\nu = 0$, i.e. at $S^*\mathbb{R}^{n-1}$. Thus, for σ large and pure imaginary, the semiclassical problem adds no complexity to the 'classical' quantum problem, but of course it does not simplify it. In fact, we need somewhat more information at the characteristic set, which is thus at $\nu = 0$ when $\operatorname{Im} z$ is bounded away from 0:

$$\nu \text{ small, } \operatorname{Im} z \geq 0 \Rightarrow (\operatorname{sgn}\xi) \operatorname{Im} p_{\hbar,z} \geq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm},$$
$$\nu \text{ small, } \operatorname{Im} z \leq 0 \Rightarrow (\operatorname{sgn}\xi) \operatorname{Im} p_{\hbar,z} \leq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm},$$

which, as we have seen, means that for $P_{\hbar,z}$ with $\operatorname{Im} z > 0$ one can propagate estimates forwards along the bicharacteristics where $\xi < 0$ (in particular, away from L_-, as the latter is a source) and backwards where $\xi > 0$ (in particular, away from L_+, as the latter is a sink), while for $P^*_{\hbar,z}$ the directions are reversed. The directions are also reversed if $\operatorname{Im} z$ switches sign. This is important because it gives invertibility for $z = \imath$ (corresponding to $\operatorname{Im}\sigma$ large positive, i.e. the physical halfplane), but does not give invertibility for $z = -\imath$ negative.

We now return to the claim that even semiclassically, for z almost real[23], the characteristic set can be divided into two components $\Sigma_{\hbar,\pm}$, with $L_\pm$ in different components. As explained in Subsection 3.2 the vanishing of the factor following $\operatorname{Im} z$ in (4.17) gives a hypersurface that separates $\Sigma_\hbar$ into two parts; this can be easily checked also by a direct computation. Concretely, this is the hypersurface given by

$$0 = 2r^2\xi + \operatorname{Re} z = -(Y \cdot \zeta - \operatorname{Re} z), \tag{4.19}$$

and so

$$\Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\mp(Y \cdot \zeta - \operatorname{Re} z) > 0\}.$$

We finally need more information about the global semiclassical dynamics. Here all null-bicharacteristics go to either L_+ in the forward direction or to L_- in the backward direction, and escape to infinity in the other direction. Rather than proving this at once, which depends on the global non-trapping structure on $\mathbb{R}^{n-1}$, we first give an argument that is local near the event horizon, and suffices for the extension discussed below for asymptotically hyperbolic spaces.

As stated above, first, we are only concerned about semiclassical dynamics in $\mu > \mu_0$, where $\mu_0 < 0$ might be close to 0. To analyze this, we observe that the semiclassical Hamilton vector field is

$$\begin{aligned} \mathsf{H}_{p_{\hbar,z}} &= 4r^2(-2\mu\xi + z)\partial_\mu - r^{-2}\mathsf{H}_{|\eta|^2_\omega} - (4(1-2r^2)\xi^2 - 4z\xi - r^{-4}|\eta|^2_\omega)\partial_\xi \\ &= 2(Y \cdot \zeta - z)(Y \cdot \partial_Y - \zeta \cdot \partial_\zeta) - 2\zeta \cdot \partial_Y; \end{aligned} \tag{4.20}$$

here we are concerned about z real. Thus,

$$\mathsf{H}_{p_{\hbar,z}}(Y \cdot \zeta) = -2|\zeta|^2,$$

and $\zeta = 0$ implies $p_{\hbar,z} = z^2$, so $\mathsf{H}_{p_{\hbar,z}}(Y \cdot \zeta)$ has a negative upper bound on the characteristic set in compact subsets of $T^*\{r < 1\}$; note that the characteristic set is compact in $T^*\{r \leq r_0\}$ if $r_0 < 1$ by standard ellipticity. Thus, bicharacteristics have to leave $\{r \leq r_0\}$ for $r_0 < 1$ in both the forward and backward direction (as $Y \cdot \zeta$ is bounded over this region on the characteristic set). We already know the dynamics

[23] So the operator is not semiclassically elliptic on $T^*\mathbb{R}^{n-1}$; as mentioned above, for $\operatorname{Im} z$ uniformly bounded away from $\mathbb{R}$, we have ellipticity in $T^*\mathbb{R}^{n-1}$.

near $L_\pm$, which is the only place where the characteristic set intersects $S^*_S\mathbb{R}^{n-1}$, namely L_+ is a sink and L_- is a source. Now, at $\mu = 0$, $\mathsf{H}_{p_{\hbar,z}}\mu = z$, which is positive when $z > 0$, so bicharacteristics can only cross $\mu = 0$ in the inward direction. In view of our preceding observations, thus, once a bicharacteristic crossed $\mu = 0$, it has to tend to L_+. As bicharacteristics in a neighborhood of L_+ (even in $\mu < 0$) tend to L_+ since L_+ is a sink, it follows that in $\Sigma_{\hbar,+}$ the same is true in $\mu > \mu_0$ for some $\mu_0 < 0$. On the other hand, in a neighborhood of L_- all bicharacteristics emanate from L_- (but cannot cross into $\mu > 0$ by our observations), so leave $\mu > \mu_0$ in the forward direction. These are all the relevant features of the bicharacteristic flow for our purposes as we shall place a complex absorbing potential near $\mu = \mu_0$ in the next subsection.

However, it is easy to see the global claim by noting that $\mathsf{H}_{p_{\hbar,z}}\mu = 4r^2(-2\mu\xi+z)$, and this cannot vanish on $\Sigma_\hbar$ in $\mu < 0$, since where it vanishes, a simple calculation gives $p_{\hbar,z} = 4\mu\xi^2 - r^{-2}|\eta|^2$. Thus, $\mathsf{H}_{p_{\hbar,z}}\mu$ has a constant sign on $\Sigma_{\hbar,\pm}$ in $\mu < 0$, so combined with the observation above that all bicharacteristics escape to $\mu = \mu_0$ in the appropriate direction, it shows that all bicharacteristics in fact escape to infinity in that direction[24].

In fact, for applications, it is also useful to remark that for $\alpha \in T^*X$,

$$0 < \mu(\alpha) < 1,\ p_{\hbar,z}(\alpha) = 0 \text{ and } (\mathsf{H}_{p_{\hbar,z}}\mu)(\alpha) = 0 \Rightarrow (\mathsf{H}^2_{p_{\hbar,z}}\mu)(\alpha) < 0. \tag{4.22}$$

Indeed, as $\mathsf{H}_{p_{\hbar,z}}\mu = 4r^2(-2\mu\xi + z)$, the hypotheses imply $z = 2\mu\xi$ and $\mathsf{H}^2_{p_{\hbar,z}}\mu = -8r^2\mu\mathsf{H}_{p_{\hbar,z}}\xi$, so we only need to show that $\mathsf{H}_{p_{\hbar,z}}\xi > 0$ at these points. Since

$$\mathsf{H}_{p_{\hbar,z}}\xi = -4(1-2r^2)\xi^2 + 4z\xi + r^{-4}|\eta|^2_\omega = 4\xi^2 + r^{-4}|\eta|^2_\omega = 4r^{-2}\xi^2,$$

where the second equality uses $\mathsf{H}_{p_{\hbar,z}}\mu = 0$ and the third uses that in addition $p_{\hbar,z} = 0$, this follows from $2\mu\xi = z \neq 0$, so $\xi \neq 0$. Thus, μ can be used for gluing constructions as in [15].

4.7. **Complex absorption.** The final step of fitting P_σ into our general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. We thus consider a compact manifold without boundary X for which $X_{\mu_0} = \{\mu > \mu_0\}$, $\mu_0 < 0$, say, is identified as an open subset with smooth boundary; it is convenient to take X to be the double[25] of X_{μ_0}.

---

[24] There is in fact a not too complicated global escape function, e.g.

$$f = \frac{2Y\cdot\zeta - \operatorname{Re} z}{2\sqrt{1+|Y|^2}(Y\cdot\zeta - \operatorname{Re} z)} = \frac{2Y\cdot\hat\zeta - \operatorname{Re} z|\zeta|^{-1}}{2\sqrt{1+|Y|^2}(Y\cdot\hat\zeta - \operatorname{Re} z|\zeta|^{-1})},$$

which is a smooth function on the characteristic set in $T^*\mathbb{R}^{n-1}$ as $Y\cdot\zeta \neq \operatorname{Re} z$ there; further, it extends smoothly to the characteristic set in $\overline{T}^*\mathbb{R}^{n-1}$ away from $L_\pm$ since $\sqrt{1+|Y|^2}(Y\cdot\hat\zeta - \operatorname{Re} z|\zeta|^{-1})$ vanishes only there near $S^*\mathbb{R}^{n-1}$ (where these are valid coordinates), at which it has conic points. This function arises in a straightforward manner when one reduces Minkowski space, $\mathbb{R}^n = \mathbb{R}^{n-1}_{z'} \times \mathbb{R}_t$ with metric g_0, to the boundary of its radial compactification, as described in Section 5, and uses the natural escape function

$$\tilde f = \frac{tt^* - z'(z')^*}{t^*\sqrt{t^2+|z'|^2}} \tag{4.21}$$

there; here t^* is the dual variable of t and $(z')^*$ of z', outside the origin.

[25] In fact, in the de Sitter context, this essentially means moving to the boundary of n-dimensional Minkowski space, where our $(n-1)$-dimensional model is the 'upper hemisphere', see Section 5. Thus, doubling over means working with the whole boundary, but putting an absorbing operator near the equator, corresponding to the usual Cauchy hypersurface in Minkowski

It is convenient to separate the 'classical' (i.e. quantum!) and 'semiclassical' problems, for in the former setting trapping does not matter, while in the latter it does.

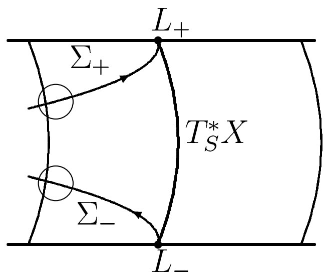

FIGURE 5. The cotangent bundle near the event horizon $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view, as in Figure 4. The circles on the left show the support of q; it has opposite signs on the two disks corresponding to the opposite directions of propagation relative to the Hamilton vector field.

We then introduce a complex absorbing operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with principal symbol q, such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and such that $p \pm \imath q$ is elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that the $\mp q \geq 0$ on $\Sigma_\pm$. Having done this, we extend P_σ and Q_σ to X in such a way that $p \pm \imath q$ are elliptic near $X \setminus X_{\mu_0}$; the region we added is thus irrelevant. In particular, as the event horizon is characteristic for the wave equation, the solution in the exterior of the event horizons is *unaffected* by thus modifying P_σ, i.e. working with P_σ and $P_\sigma - \imath Q_\sigma$ is equivalent for this purpose.

An alternative to this extension would be simply adding a boundary at $\mu = \mu_0$; this is easy to do since this is a space-like hypersurface, but this is slightly unpleasant from the point of view of microlocal analysis as one has to work on a manifold with boundary (though as mentioned this is easily done, see Remark 2.5).

For the semiclassical problem, when z is almost real (i.e. none of this is important when $\operatorname{Im} z$ is bounded away from 0) we need to increase the requirements on Q_σ. We need in addition, in the semiclassical notation, semiclassical ellipticity near $\mu = \mu_0$, i.e. that $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that the $\mp q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. Again, we extend P_σ and Q_σ to X in such a way that $p \pm \imath q$ and $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near $X \setminus X_{\mu_0}$; the region we added is thus irrelevant.

4.8. **More general metrics.** If the operator is replaced by one on a neighborhood of $Y_y \times (-\delta, \delta)_\mu$ with full principal symbol (including high energy terms)

$$-4(1 + a_1)\mu\xi^2 + 4(1 + a_2)\sigma\xi + (1 + a_3)\sigma^2 - |\eta|^2_h, \tag{4.23}$$

and h a family of Riemannian metrics on Y depending smoothly on μ, a_j vanishing at $\mu = 0$, then the local behavior of this operator P_σ near the 'event horizon' $Y \times \{0\}$ is exactly as in the de Sitter setting. If we start with a compact manifold X_0 with boundary Y and a neighborhood of the boundary identified with $Y \times [0, \delta)_\mu$ with the operator of the form above, and which is elliptic in X_0 (we only need to assume this away from $Y \times [0, \delta/2)$, say), including in the non-real high energy sense (i.e. for

space, and solving from the radial points at both the future and past light cones towards the equator — this would be impossible without the complex absorption.

z away from $\mathbb{R}$ when $\sigma = h^{-1}z$) then we can extend the operator smoothly to one on X_{μ_0}, $\mu_0 = -\delta$, which enjoys all the properties above, except semiclassical non-trapping. If we assume that X_0° is non-trapping in the usual sense, the semiclassical non-trapping property also follows. In addition, for $\mu > 0$ sufficiently small, (4.22) also holds since η is small when $\mathsf{H}_{p_{\hbar,z}}\mu = 0$ and $p_{\hbar,z} = 0$, for the former gives $z = 2(1+a_2)^{-1}(1+a_1)\mu\xi$, and then the latter gives

$$4(1+a_1)\left(1+\frac{(1+a_1)(1+a_3)}{(1+a_2)^2}\mu\right)\mu\xi^2 = |\eta|_h^2,$$

so the contribution of $|\eta|_h^2$ to $\mathsf{H}_{p_{\hbar,z}}\xi$, which can be large elsewhere even at $\mu = 0$, is actually small.

4.9. **Results.** The preceding subsections show that for the Mellin transform of $\Box_g$ on n-dimensional de Sitter space, all the hypotheses needed in Section 2 are satisfied, thus analogues of the results stated for Kerr-de Sitter space in the introduction, Theorems 1.1-1.4, hold. It is important to keep in mind, however, that there is no trapping to remove, so Theorem 1.1 applies with Q_σ supported outside the event horizon, and one does not need gluing or the result of Wunsch and Zworski [58]. In particular, Theorem 1.3 holds with arbitrary C', without the logarithmic or polynomial loss. As already mentioned when discussing [51, Theorem 1.1] at the beginning of this section, this is weaker than the result of [51, Theorem 1.1], since there one has smooth asymptotics without a blow-up of a boundary point[26].

We now reinterpret our results on the Mellin transform side in terms of $(n-1)$-dimensional hyperbolic space. Let $\mathbb{B}^{n-1}_{1/2}$ be $\mathbb{B}^{n-1} = \{r \leq 1\}$ with $\nu = \sqrt{\mu}$ added to the smooth structure. For the purposes of the discussion below, we identify the interior $\{r < 1\}$ of $\mathbb{B}^{n-1}_{1/2}$ with the Poincaré ball model of hyperbolic $(n-1)$-space $(\mathbb{H}^{n-1}, g_{\mathbb{H}^{n-1}})$. Using polar coordinates around the origin, let $\cosh\rho = \nu^{-1}$, ρ is the distance from the origin. The Laplacian on $\mathbb{H}^{n-1}$ in these coordinates is

$$\Delta_{\mathbb{H}^{n-1}} = D_\rho^2 - \imath(n-2)\coth\rho\, D_\rho + (\sinh\rho)^{-2}\Delta_\omega.$$

It is shown in [51, Lemma 7.10] that in $r < 1$, and with s be such that $2s = \imath\sigma - \frac{n}{2}$,

$$\begin{aligned}(1-r^2)^{-s}P_\sigma(1-r^2)^s &= \nu^{\frac{n}{2}-\imath\sigma}P_\sigma\nu^{\imath\sigma-\frac{n}{2}}\\ &= -\nu^{-1}\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2\frac{n(n-2)}{4}\right)\nu^{-1}\\ &= -\cosh\rho\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - (\cosh\rho)^{-2}\frac{n(n-2)}{4}\right)\cosh\rho.\end{aligned} \tag{4.24}$$

We thus deduce:

Proposition 4.2. *The inverse $\mathcal{R}(\sigma)$ of*

$$\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - (\cosh\rho)^{-2}\frac{n(n-2)}{4}$$

[26] Note that our methods work equally well for asymptotically de Sitter spaces in the sense of [51]; after the blow up, the boundary metric is 'frozen' at the point that is blown up, hence the induced problem at the front face is the same as for the de Sitter metric with asymptotics given by this 'frozen' metric.

has a meromorphic continuation from $\operatorname{Im}\sigma > 0$ *to* $\mathbb{C}$ *with poles with finite rank residues as a map* $\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1}) \to \mathcal{C}^{-\infty}(\mathbb{B}^{n-1})$*, and with non-trapping estimates in every strip* $|\operatorname{Im}\sigma| < C$*,* $|\operatorname{Re}\sigma| \gg 0$*:* $s > \frac{1}{2} + C$*,*

(4.25)
$$\|(\cosh\rho)^{(n-2)/2-\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(\mathbb{B}^{n-1})} \leq C|\sigma|^{-1}\|(\cosh\rho)^{(n+2)/2-\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(\mathbb{B}^{n-1})},$$

where the Sobolev spaces are those on $\mathbb{B}^{n-1}$ *(rather than* $\mathbb{B}^{n-1}_{1/2}$*). If* $\operatorname{supp} f \subset (\mathbb{B}^{n-1})^\circ$*, the* $s-1$ *norm on* f *can be replaced by the* $s-2$ *norm.*

The same conclusion holds for small even $\mathcal{C}^\infty$ *perturbations, vanishing at* $\partial\mathbb{B}^{n-1}_{1/2}$*, of* $g_{\mathbb{H}^{n-1}}$ *in the class of conformally compact metrics, or the addition of (not necessarily small)* $V \in \mu\mathcal{C}^\infty(\mathbb{B}^{n-1})$*.*

Proof. By self-adjointness and positivity of $\Delta_{\mathbb{H}^{n-1}}$,

$$\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2\frac{n(n-2)}{4}\right)u = f \in \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$$

has a unique solution $u = \mathcal{R}(\sigma)f \in L^2(\mathbb{B}^{n-1}_{1/2}, |dg_{\mathbb{H}^{n-1}}|)$ when when $\operatorname{Im}\sigma \gg 0$. On the other hand, let $\tilde f_0 = \nu^{\imath\sigma-n/2}\nu^{-1}f$ in $r \leq 1$, and $\tilde f_0$ still vanishes to infinite order at $r = 1$. Let $\tilde f$ be an arbitrary smooth extension of $\tilde f_0$ to the compact manifold X on which $P_\sigma - \imath Q_\sigma$ is defined. Let $\tilde u = (P_\sigma - \imath Q_\sigma)^{-1}\tilde f$, with $(P_\sigma - \imath Q_\sigma)^{-1}$ given by our results in Section 2; this satisfies $(P_\sigma - \imath Q_\sigma)\tilde u = \tilde f$ and $\tilde u \in \mathcal{C}^\infty(X)$. Thus, $u' = \nu^{-\imath\sigma+n/2}\nu^{-1}\tilde u|_{r<1}$ satisfies $u' \in \nu^{(n-2)/2-\imath\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$, and

$$\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2\frac{n(n-2)}{4}\right)u' = f$$

by (4.24) (as Q_σ is supported in $r > 1$). Since $u' \in L^2(\mathbb{B}^{n-1}, |dg_{\mathbb{H}^{n-1}}|)$ for $\operatorname{Im}\sigma > 0$, by the aforementioned uniqueness, $u = u'$.

To make the extension from $\mathbb{B}^{n-1}$ to X more systematic, let $E_s : H^s(\mathbb{B}^{n-1}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(\mathbb{B}^{n-1})$ the restriction map. Then, as we have just seen, for $f \in \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$,

$$\mathcal{R}(\sigma)f = \nu^{-\imath\sigma+n/2}\nu^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}\nu^{\imath\sigma-n/2}\nu^{-1}f. \tag{4.26}$$

Thus, the first half of the proposition (including the non-trapping estimate) follows immediately from the results of Section 2. Note also that this proves that every pole of $\mathcal{R}(\sigma)$ is a pole of $(P_\sigma - \imath Q_\sigma)^{-1}$ (for otherwise (4.26) would show $\mathcal{R}(\sigma)$ does not have a pole either), but it is possible for $(P_\sigma - \imath Q_\sigma)^{-1}$ to have poles which are not poles of $\mathcal{R}(\sigma)$. However, in the latter case, the Laurent coefficients of $(P_\sigma - \imath Q_\sigma)^{-1}$ would be annihilated by multiplication by R_s from the left, i.e. the resonant states (which are smooth) would be supported in $\mu \leq 0$, in particular vanish to infinite order at $\mu = 0$.

In fact, a stronger statement can be made: by a calculation completely analogous to what we just performed, we can easily see that in $\mu < 0$, P_σ is a conjugate (times a power of μ) of a Klein-Gordon-type operator on $(n-1)$-dimensional de Sitter space with $\mu = 0$ being the boundary (i.e. where time goes to infinity). Thus, if σ is not a pole of $\mathcal{R}(\sigma)$ and $(P_\sigma - \imath Q_\sigma)\tilde u = 0$ then one would have a solution u of this Klein-Gordon-type equation near $\mu = 0$, i.e. infinity, that rapidly vanishes at infinity. It is shown in [51, Proposition 5.3] by a Carleman-type estimate that this cannot happen; although there $\sigma^2 \in \mathbb{R}$ is assumed, the argument given there goes through almost verbatim in general. Thus, if Q_σ is supported in $\mu < c$, $c < 0$, then $\tilde{u}$ is also supported in $\mu < c$. This argument can be iterated for Laurent coefficients of higher order poles; their range (which is finite dimensional) contains only functions supported in $\mu < c$.

We now turn to the perturbation. After the conjugation, division by $\mu^{1/2}$ from both sides, elements of $V \in \mu\mathcal{C}^\infty(\mathbb{B}^{n-1})$ can be extended to become elements of $\mathcal{C}^\infty(\mathbb{R}^{n-1})$, and they do not affect any of the structures discussed in Section 2, so the results automatically go through. Operators of the form $x^2 L$, $L \in \mathrm{Diff}_{\mathrm{b,even}}(\mathbb{B}^{n-1}_{1/2})$, i.e. with even coefficients with respect to the local product structure, become elements of $\mathrm{Diff}_{\mathrm{b}}(\mathbb{B}^{n-1})$ after conjugation and division by $\mu^{1/2}$ from both sides. Hence, they can be smoothly extended across $\partial\mathbb{B}^{n-1}$, and they do not affect either the principal or the subprincipal symbol at $L_\pm$ in the classical sense. They do, however, affect the classical symbol elsewhere and the semiclassical symbol everywhere, thus the semiclassical Hamilton flow, but under the smallness assumption the required properties are preserved, since the dynamics is non-degenerate (the rescaled Hamilton vector field on $\overline{T}^*\mathbb{R}^{n-1}$ does not vanish) away from the radial points. □

Without the non-trapping estimate, this is a special case of a result of Mazzeo and Melrose [36], with improvements by Guillarmou [28]. The point is that first, we do not need the machinery of the zero calculus here, and second, the analogous result holds true on arbitary asymptotically hyperbolic spaces, with the non-trapping estimates holding under dynamical assumptions (namely, no trapping). The poles were actually computed in [51, Section 7] using special algebraic properties, within the Mazzeo-Melrose framework; however, given the Fredholm properties our methods here give, the rest of the algebraic computation in [51] go through. Indeed, the results are stable under perturbations[27], provided they fit into the framework after conjugation and the weights. In the context of the perturbations (so that the asymptotically hyperbolic structure is preserved) though with evenness conditions relaxed, the non-trapping estimate is almost the same as in [40], where it is shown by a parametrix construction; here the estimates are slightly stronger.

In fact, by the discussion of Subsection 4.8, we deduce a more general result, which in particular, for even metrics, generalizes the results of Mazzeo and Melrose [36], Guillarmou [28], and adds high-energy non-trapping estimates under non-degeneracy assumptions. It also adds the semiclassically outgoing property which is useful for resolvent gluing, including for proving non-trapping bounds microlocally away from trapping, provided the latter is mild, as shown by Datchev and Vasy [15, 16].

Theorem 4.3. *Suppose that (X_0, g_0) is an $(n-1)$-dimensional manifold with boundary with an even conformally compact metric and boundary defining function x. Let $X_{0,\mathrm{even}}$ denote the even version of X_0, i.e. with the boundary defining function replaced by its square with respect to a decomposition in which g_0 is even. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-2}{2}\right)^2 - \sigma^2,$$

[27]Though of course the resonances vary with the perturbation, in the same manner as they would vary when perturbing any other Fredholm problem.

written as $\mathcal{R}(\sigma) : L^2 \to L^2$*, has a meromorphic continuation from* $\operatorname{Im}\sigma \gg 0$ *to* $\mathbb{C}$*,*

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition (X_0, g_0) *is non-trapping, then non-trapping estimates hold in every strip* $|\operatorname{Im}\sigma| < C$*,* $|\operatorname{Re}\sigma| \gg 0$*: for* $s > \frac{1}{2} + C$*,*

$$\|x^{-(n-2)/2+\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+2)/2+\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}. \tag{4.27}$$

If f *is supported in* X_0°*, the* $s-1$ *norm on* f *can be replaced by the* $s-2$ *norm.*

If instead $\Delta_{g_0} - \sigma^2$ *satisfies mild trapping assumptions with order* $\varkappa$ *estimates in a* C_0*-strip, see Definition 2.16, then the mild trapping estimates hold, with* $|\sigma|^{\varkappa-1}$ *replacing* $|\sigma|^{-1}$ *on the right hand side of* (4.27)*, as long as* $C \leq C_0$*.*

Furthermore, for $\operatorname{Re} z > 0$*,* $\operatorname{Im} z = \mathcal{O}(h)$*, the resolvent* $\mathcal{R}(h^{-1}z)$ *is* semiclassically outgoing with a loss of h^{-1} *in the sense that if* f *has compact support in* X_0°*,* $\alpha \in T^*X$ *is in the semiclassical characteristic set and if* $\mathrm{WF}_h^{s-1}(f)$ *is disjoint from the backward bicharacteristic from* α*, then* $\alpha \notin \mathrm{WF}_h^s(h^{-1}\mathcal{R}(h^{-1}z)f)$*.*

We remark that although in order to go through without changes, our methods require the evenness property, it is not hard to deduce more restricted results without this. Essentially one would have operators with coefficients that have a conormal singularity at the event horizon; as long as this is sufficiently mild relative to what is required for the analysis, it does not affect the results. The problems arise for the analytic continuation, when one needs strong function spaces (H^s with s large); these are not preserved when one multiplies by the singular coefficients.

Proof. Suppose that g_0 is an even asymptotically hyperbolic metric. Then we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 + h}{x^2} \tag{4.28}$$

there, where h is an even family of metrics; it is convenient to take x to be a globally defined boundary defining function. Then

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-2+x^2\gamma)(xD_x) + x^2\Delta_h, \tag{4.29}$$

with γ even. Changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n-2+\mu\gamma)(\mu D_\mu) + \mu\Delta_h, \tag{4.30}$$

Now we conjugate by $\mu^{-\imath\sigma/2+n/4}$ to obtain

$$\begin{aligned}
&\mu^{\imath\sigma/2-n/4}(\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+n/4}\\
&= 4(\mu D_\mu - \sigma/2 - \imath n/4)^2 + 2\imath(n-2+\mu\gamma)(\mu D_\mu - \sigma/2 - \imath n/4)\\
&\qquad\qquad + \mu\Delta_h - \frac{(n-2)^2}{4} - \sigma^2\\
&= 4(\mu D_\mu)^2 - 4\sigma(\mu D_\mu) + \mu\Delta_h - 4\imath(\mu D_\mu) + 2\imath\sigma - 1 + 2\imath\mu\gamma(\mu D_\mu - \sigma/2 - \imath n/4).
\end{aligned}$$

Next we multiply by $\mu^{-1/2}$ from both sides to obtain

$$\begin{aligned}&\mu^{-1/2}\mu^{\imath\sigma/2-n/4}(\Delta_{g_0}-\frac{(n-2)^2}{4}-\sigma^2)\mu^{-\imath\sigma/2+n/4}\mu^{-1/2}\\&=4\mu D_\mu^2-\mu^{-1}-4\sigma D_\mu-2\imath\sigma\mu^{-1}+\Delta_h-4\imath D_\mu+2\mu^{-1}+2\imath\sigma\mu^{-1}-\mu^{-1}\\&\qquad+2\imath\gamma(\mu D_\mu-\sigma/2-\imath(n-2)/4)\\&=4\mu D_\mu^2-4\sigma D_\mu+\Delta_h-4\imath D_\mu+2\imath\gamma(\mu D_\mu-\sigma/2-\imath(n-2)/4).\end{aligned}\tag{4.31}$$

This is certainly in $\mathrm{Diff}^2(X)$, and for σ (almost) real, is equivalent to the form we want via conjugation by a smooth function, with exponent depending on σ. The latter would make no difference even semiclassically in the real regime as it is conjugation by an elliptic semiclassical FIO. However, in the non-real regime (where we would like ellipticity) it does; the present operator is not semiclassically elliptic at the zero section. So finally we conjugate by $(1+\mu)^{\imath\sigma/4}$ to obtain

$$4\mu D_\mu^2-4\sigma D_\mu-\sigma^2+\Delta_h-4\imath D_\mu+2\imath\gamma(\mu D_\mu-\sigma/2-\imath(n-2)/4)\tag{4.32}$$

modulo terms that can be absorbed into the error terms in the *negative* of operators in the class (4.23).

We still need to check that μ can be appropriately chosen in the interior away from the region of validity of the product decomposition (4.28) (where we had no requirements so far on μ). This only matters for semiclassical purposes, and (being smooth and non-zero in the interior) the factor $\mu^{-1/2}$ multiplying from both sides does not affect any of the relevant properties (semiclassical ellipticity and possible non-trapping properties), so can be ignored — the same is true for σ independent powers of μ.

To do so, it is useful to think of $(\tilde\tau\partial_{\tilde\tau})^2-G_0$, G_0 the dual metric of g_0, as a Lorentzian b-metric on $X_0^\circ\times[0,\infty)_{\tilde\tau}$. From this perspective, we want to introduce a new boundary defining function $\tau=\tilde\tau e^\phi$, with our σ the b-dual variable of τ and ϕ a function on X_0, i.e. with our τ already given, at least near $\mu=0$, i.e. ϕ already fixed there, namely $e^\phi=\mu^{1/2}(1+\mu)^{-1/4}$. Recall from the end of Subsection 3.2 that such a change of variables amounts to a conjugation on the Mellin transform side by $e^{-\imath\sigma\phi}$. Further, properties of the Mellin transform are preserved provided $\frac{d\tau}{\tau}$ is globally time-like, which, as noted at the end of Subsection 3.2, is satisfied if $|d\phi|_{G_0}<1$. But, reading off the dual metric from the principal symbol of (4.30),

$$\frac14\left|d(\log\mu-\frac12\log(1+\mu))\right|^2_{G_0}=\left(1-\frac{\mu}{2(1+\mu)}\right)^2<1$$

for $\mu>0$, with a strict bound as long as μ is bounded away from 0. Correspondingly, $\mu^{1/2}(1+\mu)^{-1/4}$ can be extended to a function e^ϕ on all of X_0 so that $\frac{d\tau}{\tau}$ is time-like, and we may even require that ϕ is constant on a fixed (but arbitrarily large) compact subset of X_0°. Then, after conjugation by $e^{-\imath\sigma\phi}$ all of the semiclassical requirements of Section 2 are satisfied. Naturally, the semiclassical properties could be easily checked directly for the conjugate of $\Delta_{g_0}-\sigma^2$ by the so-extended μ.

Thus, all of the results of Section 2 apply. The only part that needs some explanation is the direction of propagation for the semiclassically outgoing condition. For $\operatorname{Re} z>0$, as in the de Sitter case, null-bicharacteristics in X_0° must go to L_+, hence lie in $\Sigma_{\hbar,+}$. Theorem 2.15 states *backward* propagation of regularity for the operator considered there. However, the operator we just constructed is the negative of the class considered in (4.23), and under changing the sign of the operator, the

Hamilton vector field also changes direction, so semiclassical estimates (or WF_h) indeed propagate in the forward direction. □

Remark 4.4. We note that if the dual metric G_1 on X_0 is of the form $\kappa^2 G_0$, G_0 the dual of g_0 as in (4.28), then

$$\Delta_{G_1} - \kappa^2 \frac{(n-2)^2}{4} - \sigma^2 = \kappa^2 (\Delta_{G_0} - \frac{(n-2)^2}{4} - (\sigma/\kappa)^2).$$

Thus, with μ as above, and with $\tilde{P}_\sigma$ the conjugate of $\Delta_{G_0} - \frac{(n-2)^2}{4} - (\sigma/\kappa)^2$, of the form (4.32) (modulo error terms as described there) then with $e^\phi = \mu^{1/(2\kappa)}(1+\mu)^{-1/(4\kappa)}$ extended into the interior of X_0 as above, we have

$$\mu^{-1/2}\mu^{n/4}e^{\imath\sigma\phi}(\Delta_{g_1} - \kappa^2 \frac{(n-2)^2}{4} - \sigma^2)e^{-\imath\sigma\phi}\mu^{n/4}\mu^{-1/2} = \kappa^2 \tilde{P}_{\sigma/\kappa}.$$

Now, $P_\sigma = \kappa^2 \tilde{P}_{\sigma/\kappa}$ still satisfies all the assumptions of Section 2, thus directly conjugation by $e^{-\imath\sigma\phi}$ and multiplication from both sides by $\mu^{-1/2}$ gives an operator to which the results of Section 2 apply. This is relevant because if we have an asymptotically hyperbolic manifold with ends of different sectional curvature, the manifold fits into the general framework directly, including the semiclassical estimates[28]. A particular example is de Sitter-Schwarzschild space, on which resonances and wave propagation were analyzed from this asymptotically hyperbolic perspective in [46, 5, 39]; this is a special case of the Kerr-de Sitter family discussed in Section 6. The stability of estimates for operators such as P_σ under small smooth, in the b-sense, perturbations of the coefficients of the associated d'Alembertian means that all the properties of de Sitter-Schwarzschild obtained by this method are also valid for Kerr-de Sitter with sufficiently small angular momentum. However, working directly with Kerr-de Sitter space, and showing that it satisfies the assumptions of Section 2 on its own, gives a better result; we accomplish this in Section 6.

Remark 4.5. We now return to our previous remarks regarding the fact that our solution disallows the conormal singularities $(\mu \pm i0)^{\imath\sigma}$ from the perspective of conformally compact spaces of dimension $n-1$. The two indicial roots on these spaces[29] correspond to the asymptotics $\mu^{\pm\imath\sigma/2+(n-2)/4}$ in $\mu > 0$. Thus for the operator

$$\mu^{-1/2}\mu^{\imath\sigma/2-n/4}(\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+n/4}\mu^{-1/2},$$

or indeed P_σ, they correspond to

$$\left(\mu^{-\imath\sigma/2+n/4}\mu^{-1/2}\right)^{-1}\mu^{\pm\imath\sigma/2+(n-2)/4} = \mu^{\imath\sigma/2\pm\imath\sigma/2}.$$

Here the indicial root $\mu^0 = 1$ corresponds to the smooth solutions we construct for P_σ, while $\mu^{\imath\sigma}$ corresponds to the conormal behavior we rule out. Back to the original Laplacian, thus, $\mu^{-\imath\sigma/2+(n-2)/4}$ is the allowed asymptotics and $\mu^{\imath\sigma/2+(n-2)/4}$ is the disallowed one. Notice that $\operatorname{Re}\imath\sigma = -\operatorname{Im}\sigma$, so the disallowed solution is growing at $\mu = 0$ relative to the allowed one, as expected in the physical half plane, and the behavior reverses when $\operatorname{Im}\sigma < 0$. Thus, in the original asymptotically hyperbolic picture one has to distinguish two different rates of growths, whose relative size

[28]For 'classical' results, the interior is automatically irrelevant.

[29]Note that $\mu = x^2$.

changes. On the other hand, in our approach, we rule out the singular solution and allow the non-singular (smooth one), so there is no change in behavior at all for the analytic continuation.

Remark 4.6. For *even* asymptotically de Sitter metrics on an $(n-1)$-dimensional manifold X_0' with boundary, the methods for asymptotically hyperbolic spaces work, except $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma$ switch roles, which does not affect Fredholm properties, see Remark 2.7. Again, evenness means that we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 - h}{x^2} \tag{4.33}$$

there, where h is an even family of Riemannian metrics; as above, we take x to be a globally defined boundary defining function. Then with $\tilde\mu = x^2$, so $\tilde\mu > 0$ is the Lorentzian region, $\overline{\sigma}$ in place of σ (recalling that our aim is to get to $P_\sigma^* + \imath Q_\sigma$) the above calculations for $\Box_{g_0} - \frac{(n-2)^2}{4} - \overline{\sigma}^2$ in place of $\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2$ leading to (4.31) all go through with μ replaced by $\tilde\mu$, σ replaced by $\overline{\sigma}$ and Δ_h replaced by $-\Delta_h$. Letting $\mu = -\tilde\mu$, and conjugating by $(1+\mu)^{\imath\overline{\sigma}/4}$ as above, yields

$$-4\mu D_\mu^2 + 4\overline{\sigma} D_\mu + \overline{\sigma}^2 - \Delta_h + 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \overline{\sigma}/2 - \imath(n-2)/4), \tag{4.34}$$

modulo terms that can be absorbed into the error terms in operators in the class (4.23), i.e. this is indeed of the form $P_\sigma^* + \imath Q_\sigma$ in the framework of Subsection 4.8, at least near $\tilde\mu = 0$. If now X_0' is extended to a manifold without boundary in such a way that in $\tilde\mu < 0$, i.e. $\mu > 0$, one has a classically elliptic, semiclassically either non-trapping or mildly trapping problem, then all the results of Section 2 are applicable.

5. Minkowski space

Perhaps our simplest example is Minkowski space $M = \mathbb{R}^n$ with the metric

$$g_0 = dz_n^2 - dz_1^2 - \ldots - dz_{n-1}^2.$$

Also, let $\hat M = \overline{\mathbb{R}^n}$ be the radial (or geodesic) compactification of space-time, see [38, Section 1]; thus $\hat M$ is the n-ball, with boundary $X = \mathbb{S}^{n-1}$. Writing $z' = (z_1, \ldots, z_{n-1}) = r\omega$ in terms of Euclidean product coordinates, and $t = z_n$, local coordinates on $\hat M$ in $|z'| > \epsilon |z_n|$, $\epsilon > 0$, are given by

$$s = \frac{t}{r},\ \rho = r^{-1},\ \omega, \tag{5.1}$$

while in $|z_n| > \epsilon|z'|$, by

$$\tilde\rho = |t|^{-1},\ Z = \frac{z'}{|t|}. \tag{5.2}$$

Note that in the overlap, the curves given by Z constant are the same as those given by s, ω constant, but the actual defining function of the boundary we used, namely $\tilde\rho$ vs. ρ, differs, and does so by a factor which is constant on each fiber. For some purposes it is useful to fix a global boundary defining function, such as $\hat\rho = (r^2+t^2)^{-1/2}$. We remark that if one takes a Mellin transform of functions supported near infinity along these curves, and uses conjugation by the Mellin transform to obtain families of operators on $X = \partial\hat M$, the effect of changing the boundary defining function in this manner is conjugation by a non-vanishing factor which does not affect most relevant properties of the induced operator on the boundary, so one can use local defining functions when convenient.

The metric g_0 is a Lorentzian scattering metric in the sense of Melrose [38] (where, however, only the Riemannian case was discussed) in that it is a symmetric non-degenerate bilinear form on the scattering tangent bundle of $\hat{M}$ of Lorentzian signature. This would be the appropriate locus of analysis of the Klein-Gordon operator, $\Box_{g_0}-\lambda$ for $\lambda > 0$, but for $\lambda = 0$ the scattering problem becomes degenerate at the zero section of the scattering cotangent bundle at infinity. However, one can convert $\Box_{g_0}$ to a non-degenerate b-operator on $\hat{M}$: it is of the form $\hat{\rho}^2\tilde{P}$, $\tilde{P} \in \mathrm{Diff}^2_{\mathrm{b}}(X)$, where $\hat{\rho}$ is a defining function of the boundary. In fact, following Wang [57], we consider (taking into account the different notation for dimension)

$$\begin{aligned}\rho^{-2}\rho^{-(n-2)/2}\Box_{g_0}\rho^{(n-2)/2} &= \Box_{\tilde{g}_0} + \frac{(n-2)(n-4)}{4};\\ \tilde{G}_0 &= (1-s^2)\partial_s^2 - 2(s\partial_s)(\rho\partial_\rho) - (\rho\partial_\rho)^2 - \partial_\omega^2,\end{aligned} \tag{5.3}$$

with $\tilde{G}_0$ being the dual metric of $\tilde{g}_0$. Again, this ρ is not a globally valid defining function, but changing to another one does not change the properties we need[30] where this is a valid defining function. It is then a straightforward calculation that the induced operator on the boundary is

$$P'_\sigma = D_s(1-s^2)D_s - \sigma(sD_s + D_s s) - \sigma^2 - \Delta_\omega + \frac{(n-2)(n-4)}{4},$$

In the other coordinate region, where $\tilde{\rho}$ is a valid defining function, and $t > 0$, it is even easier to compute

$$\Box_{g_0} = \tilde{\rho}^2\left((\tilde{\rho}D_{\tilde{\rho}})^2 + 2(\tilde{\rho}D_{\tilde{\rho}})ZD_Z + (ZD_Z)^2 - \Delta_Z - \imath(\tilde{\rho}D_{\tilde{\rho}}) - \imath ZD_Z\right), \tag{5.4}$$

so after Mellin transforming $\tilde{\rho}^{-2}\Box$, we obtain

$$\tilde{P}_\sigma = (\sigma - \imath/2)^2 + \frac{1}{4} + 2(\sigma - \imath/2)ZD_Z + (ZD_Z)^2 - \Delta_Z.$$

Conjugation by $\tilde{\rho}^{(n-2)/2}$ simply replaces σ by $\sigma - \imath\frac{n-2}{2}$, yielding that the Mellin transform P_σ of $\tilde{\rho}^{-(n-2)/2}\tilde{\rho}^{-2}\Box_{g_0}\tilde{\rho}^{(n-2)/2}$ is

$$\begin{aligned}P_\sigma &= (\sigma - \imath(n-1)/2)^2 + + \frac{1}{4} + 2(\sigma - \imath(n-1)/2)ZD_Z + (ZD_Z)^2 - \Delta_Z\\ &= (ZD_Z + \sigma - \imath(n-1)/2)^2 + \frac{1}{4} - \Delta_Z.\end{aligned} \tag{5.5}$$

Note that P_σ and P'_σ are not the same operator in different coordinates; they are related by a σ-dependent conjugation. The operator P_σ in (5.5) is *almost* exactly the operator arising from de Sitter space on the front face, see the displayed equation after [51, Equation 7.4] (the σ in [51, Equation 7.4] is $\imath\sigma$ in our notation as already remarked in Section 4), with the only change that our σ would need to be replaced by $-\sigma$, and we need to add $\frac{(n-1)^2}{4} - \frac{1}{4}$ to our operator. (Since replacing $t > 0$ by $t < 0$ in the region we consider reverses the sign when relating D_ρ and D_t, the signs would agree with those from the discussion after [51, Equation 7.4] at the backward light cone.) However, we need to think of this as the *adjoint* of an operator of the type we considered in Section 4 up to Remark 4.6, or after [51, Equation 7.4] due

[30] Only when $\mathrm{Im}\,\sigma \to \infty$ can such a change matter.

to the way we need to propagate estimates. (This is explained below.) Thus, we think of P_σ as the adjoint (with respect to $|dZ|$) of

$$\begin{aligned} P_\sigma^* &= (ZD_Z + \overline{\sigma} - \imath(n-1)/2)^2 + \frac{1}{4} - \Delta_Z \\ &= (\overline{\sigma} - \imath(n-1)/2)^2 + 2(\overline{\sigma} - \imath(n-1)/2)ZD_Z + (ZD_Z)^2 + \frac{1}{4} - \Delta_Z \\ &= (ZD_Z + \overline{\sigma} - \imath(n-1))(ZD_Z + \overline{\sigma}) - \Delta_Z + \frac{1}{4} - \frac{(n-1)^2}{4}, \end{aligned}$$

which is like the de Sitter operator after [51, Equation 7.4], except, denoting σ of that paper by $\check{\sigma}$, $\imath\check{\sigma} = \overline{\sigma}$, and we need to take $\lambda = \frac{(n-1)^2}{4} - \frac{1}{4}$ in [51, Equation 7.4]. Thus, all of the analysis of Section 4 applies.

In particular, note that P_σ is elliptic inside the light cone, where $s > 1$, and hyperbolic outside the light cone, where $s < 1$. It follows from Subsection 4.9 that P_σ is a conjugate of the hyperbolic Laplacian plus a potential (decaying quadratically in the usual conformally compact sense) inside the light cones[31], and of the Klein-Gordon operator plus a potential on de Sitter space outside the light cones: with $\nu = (1 - |Z|^2)^{1/2} = (\cosh \rho_{\mathbb{H}^{n-1}})^{-1}$,

$$\begin{aligned} &\nu^{\frac{n}{2} - \imath\sigma} P_\sigma \nu^{\imath\sigma - \frac{n}{2}} \\ &= -\nu^{-1} \left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \frac{(n-2)^2}{4} - \nu^2 \frac{n(n-2) - (n-1)^2}{4} \right) \nu^{-1} \\ &= -\cosh \rho_{\mathbb{H}^{n-1}} \left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \frac{(n-2)^2}{4} + \frac{1}{4} (\cosh \rho_{\mathbb{H}^{n-1}})^{-2} \right) \cosh \rho_{\mathbb{H}^{n-1}}. \end{aligned}$$

We remark that in terms of dynamics on ${}^{\mathrm{b}}S^*\hat{M}$, as discussed in Subsection 3.1, there is a sign difference in the normal to the boundary component of the Hamilton vector field (normal in the b-sense, only), so in terms of the full b-dynamics (rather than normal family dynamics) the radial points here are sources/sinks, unlike the saddle points in the de Sitter case. This is closely related to the appearance of adjoints in the Minkowski problem (as compared to the de Sitter one).

This immediately assures that not only the wave equation on Minkowski space fits into our framework, wave propagation on it is stable under small smooth perturbation in $\mathrm{Diff}_{\mathrm{b}}^2(X)$ of $\hat{\rho}^2 \Box_{g_0}$.

Further, it is shown in [51, Corollary 7.18] that the problem for P_σ^* is invertible in the interior of hyperbolic space, but with the behavior that corresponds to our more global point of view at the boundary, unless

$$-\imath\overline{\sigma} = \check{\sigma} \in -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda} - \mathbb{N} = -\frac{n-1}{2} \pm \frac{1}{2} - \mathbb{N} = -\frac{n-2}{2} - \mathbb{N},$$

i.e.

$$\sigma \in -\imath\left(\frac{n-2}{2} + \mathbb{N}\right). \tag{5.6}$$

Recall also from the proof of Proposition 4.2 that $(P_\sigma - \imath Q_\sigma)^{-1}$ may have additional poles as compared to the resolvent of the asymptotically hyperbolic model, but the resonant states would vanish in a neighborhood of the event horizon and the elliptic region — with the vanishing valid in a large region, denoted by $\mu > c$, $c < 0$,

[31] As pointed out to the author by Gunther Uhlmann, this means that the Klein model of hyperbolic space is the one induced by the Minkowski boundary reduction.

there, depending on the support of Q_σ. For $(P_\sigma^*+\imath Q_\sigma)^{-1}$ the resonant states corresponding to these additional poles may have support in the elliptic region, but their coefficients are given by pairing with resonant states of $(P_\sigma-\imath Q_\sigma)^{-1}$. Thus, if f vanishes in $\mu<c$ then $(P_\sigma^*+\imath Q_\sigma)^{-1}f$ only has the poles given by the asymptotically hyperbolic model.

To recapitulate, P_σ is of the form described in Section 2, at least if we restrict away from the backward light cone[32]. To be more precise, for the forward problem for the wave equation, the *adjoint* of the operator P_σ we need to study satisfies the properties in Section 2, i.e. singularities are propagated towards the radial points at the forward light cone, which means that our solution lies in the 'bad' dual spaces – of course, these are just the singularities corresponding to the radiation field of Friedlander [25], see also [45], which is singular on the radial compactification of Minkowski space. However, by elliptic regularity or microlocal propagation of singularities, we of course automatically have estimates in better spaces away from the boundary of the light cone. We also need a complex absorbing potential supported, say, near $s=-1/2$ in the coordinates (5.1). If we wanted to, we could instead add a boundary at $s=-1/2$, or indeed at $s=0$ (which would give the standard Cauchy problem), see Remark 2.5. By standard uniqueness results based on energy estimates, this does not affect the solution in $s>0$, say, when the forcing f vanishes in $s<0$ and we want the solution u to vanish there as well.

We thus deduce from Lemma 3.1 and the analysis of Section 2:

Theorem 5.1. *Let K be a compact subset of the interior of the light cone at infinity on $\hat M$. Suppose that g is a Lorentzian scattering metric and $\hat\rho^2\Box_g$ is sufficiently close to $\hat\rho^2\Box_{g_0}$ in $\mathrm{Diff}^2_{\mathrm{b}}(\hat M)$, with n the dimension of $\hat M$. Then solutions of the wave equation $\Box_g u=f$ vanishing in $t<0$ and $f\in\dot{\mathcal C}^\infty(\hat M)=\mathcal S(\mathbb R^n)$ have a polyhomogeneous asymptotic expansion in the sense of* [42] *in K of the form $\sim\sum_j\sum_{k\le m_j}a_{jk}\hat\rho^{\delta_j}(\log|\hat\rho|)^k$, with a_{jk} in $\mathcal C^\infty$, and with*

$$\delta_j=\imath\sigma_j+\frac{n-2}{2},$$

with σ_j being a point of non-invertibility of P_σ on the appropriate function spaces. On Minkowski space, the exponents are given by

$$\delta_j=\imath(-\imath\frac{n-2}{2}-\imath j)+\frac{n-2}{2}=n-2+j,\ j\in\mathbb N,$$

and they depend continuously on the perturbation if one perturbs the metric. A distributional version holds globally.

For polyhomogeneous f the analogous conclusion holds, except that one has to add to the set of exponents (index set) the index set of f, increased by 2 (corresponding factoring out $\hat\rho^2$ in (5.4)*), in the sense of extended unions* [42, Section 5.18]*.*

Remark 5.2. Here a compact K is required since we allow drastic perturbations that may change where the light cone hits infinity. If one imposes more structure, so that the light cone at infinity is preserved, one can get more precise results.

[32] The latter is only done to avoid combining for the same operator the estimates we state below for an operator P_σ and its adjoint; as follows from the remark above regarding the sign of σ, for the operator here, the microlocal picture near the backward light cone is like that for the P_σ considered in Section 2, and near the forward light cone like that for P_σ^*. It is thus fine to include both the backward and the forward light cones; we just end up with a combination of the problem we study here and its adjoint, and with function spaces much like in [38, 54].

As usual, the smallness of the perturbation is only relevant to the extent that rough properties of the global dynamics and the local dynamics at the radial points are preserved (so the analysis is only impacted via dynamics). There are no size restrictions on perturbations if one keeps the relevant features of the dynamics.

In a different class of spaces, namely asymptotically conic Riemannian spaces, analogous and more precise results exist for the induced product wave equation, see especially the work of Guillarmou, Hassell and Sikora [27]; the decay rate in their work is the same in *odd* dimensional space-time (i.e. even dimensional space). In terms of space-time, these spaces look like a blow-up of the 'north and south poles' $Z = 0$ of Minkowski space, with product type structure in terms of space time, but general smooth dependence on ω (with the sphere in ω replaceable by another compact manifold). In that paper a parametrix is constructed for Δ_g at all energies by combining a series of preceding papers. Their conclusion in even dimensional space-time is one order better; this is presumably the result of a global (as opposed to local, via complex absorbing potentials near, say, $t/r = -1/2$) cancellation. It is a very interesting question whether our analysis can be extended to non-product versions of their setting.

Note that for the Mellin transform of $\Box_{g_0}$ one can perform a more detailed analysis, giving Lagrangian regularity at the light cone, with high energy control. This would be preserved for other metrics that preserve the light cone at infinity to sufficiently high order. The result is an expansion on the $\hat{M}$ blown up at the boundary of the light cone, with the singularities corresponding to the Friedlander radiation field. However, in this relatively basic paper we do not pursue this further.

6. The Kerr-de Sitter metric

6.1. **The basic geometry.** We now give a brief description of the Kerr-de Sitter metric on

$$\begin{aligned} &M_\delta = X_\delta \times [0,\infty)_\tau,\ X_\delta = (r_- - \delta, r_+ + \delta)_r \times \mathbb{S}^2, \\ &X_+ = (r_-, r_+)_r \times \mathbb{S}^2,\ X_- = \big((r_- - \delta, r_+ + \delta)_r \setminus [r_-, r_+]_r\big) \times \mathbb{S}^2, \end{aligned}$$

where $r_\pm$ are specified later. We refer the reader to the excellent treatments of the geometry by Dafermos and Rodnianski [13, 14] and Tataru and Tohaneanu [49, 48] for details, and Dyatlov's paper [20] for the set-up and most of the notation we adopt.

Away from the north and south poles $q_\pm$ we use spherical coordinates (θ, ϕ) on $\mathbb{S}^2$:

$$\mathbb{S}^2 \setminus \{q_+, q_-\} = (0,\pi)_\theta \times \mathbb{S}^1_\phi.$$

Thus, away from $[0,\infty)_{\tilde{\mu}} \times [0,\infty)_\tau \times \{q_+, q_-\}$, the Kerr-de Sitter space-time is

$$(r_- - \delta, r_+ + \delta)_r \times [0,\infty)_\tau \times (0,\pi)_\theta \times \mathbb{S}^1_\phi$$

with the metric we specify momentarily.

The Kerr-de Sitter metric has a very similar microlocal structure at the event horizon to de Sitter space. Rather than specifying the metric g, we specify the dual metric; it is

$$\begin{aligned} G = -\rho^{-2}\Big(&\tilde{\mu}\partial_r^2 + \frac{(1+\gamma)^2}{\kappa \sin^2\theta}(\alpha \sin^2\theta \partial_{\tilde{t}} + \partial_{\tilde{\phi}})^2 + \kappa \partial_\theta^2 \\ &- \frac{(1+\gamma)^2}{\tilde{\mu}}((r^2+\alpha^2)\partial_{\tilde{t}} + \alpha\partial_{\tilde{\phi}})^2\Big) \end{aligned} \tag{6.1}$$

with r_s, Λ, α constants, $r_s, \Lambda \geq 0$,

$$\rho^2 = r^2 + \alpha^2 \cos^2\theta,$$
$$\tilde\mu = (r^2+\alpha^2)(1-\frac{\Lambda r^2}{3}) - r_s r,$$
$$\kappa = 1+\gamma\cos^2\theta,$$
$$\gamma = \frac{\Lambda\alpha^2}{3}.$$

While G is defined for all values of the parameters r_s, Λ, α, with $r_s, \Lambda \geq 0$, we make further restrictions. Note that under the rescaling

$$r' = \sqrt{\Lambda} r,\ \tilde t' = \sqrt{\Lambda}\tilde t,\ r'_s = \sqrt{\Lambda} r_s,\ \alpha' = \sqrt{\Lambda}\alpha,\ \Lambda' = 1,$$

$\Lambda^{-1}G$ would have the same form, but with all the unprimed variables replaced by the primed ones. Thus, effectively, the general case $\Lambda > 0$ is reduced to $\Lambda = 1$.

Our first assumption is that $\tilde\mu(r) = 0$ has two positive roots $r = r_\pm$, $r_+ > r_-$, with

$$F_\pm = \mp\frac{\partial\tilde\mu}{\partial r}|_{r=r_\pm} > 0; \tag{6.2}$$

r_+ is the de Sitter end, r_- is the Kerr end. Since $\tilde\mu$ is a quartic polynomial, is > 0 at $r = 0$ if $|\alpha| > 0$, and goes to $-\infty$ at $\pm\infty$, it can have at most 3 positive roots; the derivative requirements imply that these three positive roots exist, and $r_\pm$ are the larger two of these. If $\alpha = 0$, (6.2) is satisfied if and only if $0 < \frac{9}{4}r_s^2\Lambda < 1$. Indeed, if (6.2) is satisfied, $\frac{\partial}{\partial r}(r^{-4}\tilde\mu)$ must have a zero between r_- and r_+, where $\tilde\mu$ must be positive; $\frac{\partial}{\partial r}(r^{-4}\tilde\mu) = 0$ gives $r = \frac{3}{2}r_s$, and then $\tilde\mu(r) > 0$ gives $1 > \frac{9}{4}r_s^2\Lambda$. Conversely, if $0 < \frac{9}{4}r_s^2\Lambda < 1$, then the cubic polynomial $r^{-1}\tilde\mu = r - \frac{\Lambda}{3}r^3 - r_s$ is negative at 0 and at $+\infty$, and thus will have exactly two positive roots if it is positive at one point, which is the case at $r = \frac{3}{2}r_s$. Indeed, note that $r^{-4}\tilde\mu = r^{-2} - \frac{\Lambda}{3} - r_s r^{-3}$ is a cubic polynomial in r^{-1}, and $\partial_r(r^{-4}\tilde\mu) = -2r^{-3}\left(1-\frac{3r_s}{2r}\right)$, so $r^{-4}\tilde\mu$ has a non-degenerate critical point at $r = \frac{3}{2}r_s$, and if $0 \leq \frac{9}{4}r_s^2\Lambda < 1$, then the value of $\tilde\mu$ at this critical point is positive. Correspondingly, for small α (depending on $\frac{9}{4}r_s^2\Lambda$, but with uniform estimates in compact subintervals of $(0,1)$), $r_\pm$ satisfying (6.2) still exist.

We next note that for α not necessarily zero, if (6.2) is satisfied then $\frac{d^2\tilde\mu}{dr^2} = 2 - \frac{2}{3}\Lambda\alpha^2 - 4\Lambda r^2$ must have a positive zero, so we need

$$0 \leq \gamma = \frac{\Lambda\alpha^2}{3} < 1, \tag{6.3}$$

i.e. (6.2) implies (6.3).

Physically, Λ is the cosmological constant, $r_s = 2M$ the Schwarzschild radius, with M being the mass of the black hole, α the angular momentum. Thus, de Sitter-Schwarzschild space is the particular case with $\alpha = 0$, while further de Sitter space is the case when $r_s = 0$ in which limit r_- goes to the origin and simply 'disappears', and Schwarzschild space is the case when $\Lambda = 0$, in which case r_+ goes to infinity, and 'disappears', creating an asymptotically Euclidean end. On the other hand, Kerr is the special case $\Lambda = 0$, with again $r_+ \to \infty$, so the structure near the event horizon is unaffected, but the de Sitter end is replaced by a different, asymptotically Euclidean, end. One should note, however, that of the limits $\Lambda \to 0$, $\alpha \to 0$ and $r_s \to 0$, the only non-degenerate one is $\alpha \to 0$; in both other cases the geometry changes drastically corresponding to the disappearance of the de Sitter, resp. the black hole, ends. Thus, arguably, from a purely mathematical point of view, de Sitter-Schwarzschild space-time is the most natural limiting case. Perhaps the best way to follow this section then is to keep de Sitter-Schwarzschild space in mind. Since our methods are stable, this automatically gives the case of small α; of course working directly with α gives better results.

In fact, from the point of view of our setup, all the relevant features are symbolic, including dependence on the Hamiltonian dynamics. Thus, the only not completely straightforward part in showing that our abstract hypotheses are satisfied is the semi-global study of dynamics. The dynamics of the rescaled Hamilton flow depends smoothly on α, so it is automatically well-behaved for finite times for small α if it is such for $\alpha = 0$; here rescaling is understood on the fiber-radially compactified cotangent bundle $\overline{T}^* X_\delta$ (so that one has a smooth dynamical system whose only non-compactness comes from that of the base variables). The only place where dynamics matters for unbounded times are critical points or trapped orbits of the Hamilton vector field. In $S^*X_\delta = \partial\overline{T}^*X_\delta$, one can analyze the structure easily for all α, and show that for a specific range of α, given below implicitly by (6.12), the only critical points/trapping is at fiber-infinity SN^*Y of the conormal bundle of the event horizon Y. We also analyze the semiclassical dynamics (away from $S^*X_\delta = \partial\overline{T}^*X_\delta$) directly for α satisfying (6.22), which allows α to be comparable to r_s. We show that in this range of α (subject to (6.2) and (6.12)), the only trapping is hyperbolic trapping, which was analyzed by Wunsch and Zworski [58]; further, we also show that the trapping is normally hyperbolic for small α, and is thus structurally stable then.

In summary, apart from the full analysis of semiclassical dynamics, we work with arbitrary α for which (6.2) and (6.12) holds, which are both natural constraints, since it is straightforward to check the requirements of Section 2 in this generality. Even in the semiclassical setting, we work under the relatively large α bound, (6.22), to show hyperbolicity of the trapping, and it is only for normal hyperbolicity that we deal with (unspecified) small α.

We now put the metric (6.1) into a form needed for the analysis. Since the metric is not smooth b-type in terms of $r, \theta, \tilde\phi, e^{-\tilde t}$, in order to eliminate the $\tilde\mu^{-1}$ terms we let

$$t = \tilde t + h(r),\ \ \phi = \tilde\phi + P(r)$$

with

$$h'(r) = \mp\frac{1+\gamma}{\tilde\mu}(r^2+\alpha^2) \mp c, \qquad P'(r) = \mp\frac{1+\gamma}{\tilde\mu}\alpha \tag{6.4}$$

near $r_\pm$. Here $c = c(r)$ is a smooth function of r (unlike $\tilde\mu^{-1}$!), that is to be specified. One also needs to specify the behavior in $\tilde\mu > 0$ bounded away from 0, much like we did so in the asymptotically hyperbolic setting; this affects semiclassical ellipticity for σ away from the reals as well as semiclassical propagation there. We at first focus on the 'classical' problem, however, for which the choice of c is irrelevant. Then the dual metric becomes

$$\begin{aligned} G = -\rho^{-2}\Big(&\tilde\mu\big(\partial_r \mp c\partial_t\big)^2 \mp 2(1+\gamma)(r^2+\alpha^2)\big(\partial_r \mp c\partial_t\big)\partial_t \\ &\mp 2(1+\gamma)\alpha\big(\partial_r \mp c\partial_t\big)\partial_\phi + \kappa\partial_\theta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(\alpha\sin^2\theta\partial_t + \partial_\phi)^2\Big).\end{aligned}$$

We write $\tau = e^{-t}$, so $-\tau\partial_\tau = \partial_t$, and b-covectors as

$$\xi\,dr + \sigma\,\frac{d\tau}{\tau} + \eta\,d\theta + \zeta\,d\phi,$$

so

$$\begin{aligned}\rho^2 G = &-\tilde\mu\big(\xi \pm c\sigma\big)^2 \mp 2(1+\gamma)(r^2+\alpha^2)\big(\xi \pm c\sigma\big)\sigma\\ &\pm 2(1+\gamma)\alpha\big(\xi \pm c\sigma\big)\zeta - \kappa\eta^2 - \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-\alpha\sin^2\theta\sigma+\zeta)^2.\end{aligned}$$

Note that the sign of ξ here is the *opposite* of the sign in our de Sitter discussion in Section 4 where it was the dual variable (thus the symbol of D_μ) of μ, which is $r^{-2}\tilde\mu$ in the present notation, since $\frac{d\tilde\mu}{dr} < 0$ at the de Sitter end, $r = r_+$.

A straightforward calculation shows $\det g = (\det G)^{-1} = -(1+\gamma)^4\rho^4\sin^2\theta$, so apart from the usual polar coordinate singularity at $\theta = 0, \pi$, which is an artifact of the spherical coordinates and is discussed below, we see at once that g is a smooth Lorentzian b-metric. In particular, it is non-degenerate, so the d'Alembertian $\Box_g = d^*d$ is a well-defined b-operator, and

$$\sigma_{\mathrm{b},2}(\rho^2\Box_g) = \rho^2 G.$$

Factoring out ρ^2 does not affect any of the statements below but simplifies some formulae, see Footnote 9 and Footnote 12 for general statements; one could also work with G directly.

6.2. **The 'spatial' problem: the Mellin transform.** The Mellin transform, P_σ, of $\rho^2\Box_g$ has the same principal symbol, including in the high energy sense,

$$\begin{aligned}p_{\mathrm{full}} = \sigma_{\mathrm{full}}(P_\sigma) = &-\tilde\mu\big(\xi \pm c\sigma\big)^2 \mp 2(1+\gamma)(r^2+\alpha^2)\big(\xi \pm c\sigma\big)\sigma\\ &\pm 2(1+\gamma)\alpha\big(\xi \pm c\sigma\big)\zeta - \tilde p_{\mathrm{full}}\end{aligned} \tag{6.5}$$

with

$$\tilde p_{\mathrm{full}} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-\alpha\sin^2\theta\sigma+\zeta)^2,$$

so $\tilde p_{\mathrm{full}} \geq 0$ for real σ. Thus,

$$\begin{aligned}\mathsf{H}_{p_{\mathrm{full}}} = &\Big(-2\tilde\mu\big(\xi \pm c\sigma\big) \mp 2(1+\gamma)(r^2+\alpha^2)\sigma \pm 2(1+\gamma)\alpha\zeta\Big)\partial_r\\ &-\Big(-\frac{\partial\tilde\mu}{\partial r}\big(\xi \pm c\sigma\big)^2 \mp 4r(1+\gamma)\sigma\big(\xi \pm c\sigma\big) \pm \frac{\partial c}{\partial r}\tilde c\sigma\Big)\partial_\xi\\ &\pm 2(1+\gamma)\alpha\big(\xi \pm c\sigma\big)\partial_\phi - \mathsf{H}_{\tilde p_{\mathrm{full}}},\\ \tilde c = &-2\tilde\mu\big(\xi \pm c\sigma\big) \mp 2(1+\gamma)(r^2+\alpha^2)\sigma \pm 2(1+\gamma)\alpha\zeta.\end{aligned}$$

To deal with q_+ given by $\theta = 0$ (q_- being similar), let

$$y = \sin\theta\sin\phi,\ z = \sin\theta\cos\phi,\ \text{so}\ \cos^2\theta = 1-(y^2+z^2).$$

We can then perform a similar calculation yielding that if λ is the dual variable to y and ν is the dual variable to z then

$$\zeta = z\lambda - y\nu$$

and

$$\begin{aligned}\tilde p_{\mathrm{full}} &= (1+\gamma\cos^2\theta)^{-1}\Big((1+\gamma)^2(\lambda^2+\nu^2)+\tilde p''\Big) + \tilde p^\sharp_{\mathrm{full}},\\ \tilde p^\sharp_{\mathrm{full}} &= (1+\gamma\cos^2\theta)^{-1}(1+\gamma)^2(2\alpha\sin^2\theta\sigma-\zeta)\alpha\sigma,\end{aligned}$$

with $\tilde{p}''$ smooth and vanishing quadratically at the origin. Correspondingly, by (6.5), P_σ is indeed smooth at $q_\pm$. Thus, one can perform all symbol calculations away from $q_\pm$, since the results will extend smoothly to $q_\pm$, and correspondingly from now on we do not emphasize these two poles.

In the sense of 'classical' microlocal analysis, we thus have:

$$\begin{aligned} p &= \sigma_2(P_\sigma) = -\tilde{\mu}\xi^2 \pm 2(1+\gamma)\alpha\xi\zeta - \tilde{p}, \qquad \tilde{p} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}\zeta^2 \geq 0, \\ \mathsf{H}_p &= \Big(-2\tilde{\mu}\xi \pm 2(1+\gamma)\alpha\zeta\Big)\partial_r \pm 2(1+\gamma)\alpha\xi\partial_\phi + \frac{\partial\tilde{\mu}}{\partial r}\xi^2\partial_\xi - \mathsf{H}_{\tilde{p}}. \end{aligned} \tag{6.6}$$

6.3. **Microlocal geometry of Kerr-de Sitter space-time.** As already stated in Section 2, it is often convenient to consider the fiber-radial compactification $\overline{T}^* X_\delta$ of the cotangent bundle T^*X_δ, with S^*X_δ considered as the boundary at fiber-infinity of $\overline{T}^* X_\delta$.

We let

$$\Lambda_+ = N^*\{\tilde{\mu} = 0\} \cap \{\mp\xi > 0\},\ \Lambda_- = N^*\{\tilde{\mu} = 0\} \cap \{\pm\xi > 0\},$$

with the sign inside the braces corresponding to that of $r_\pm$. This is consistent with our definition of $\Lambda_\pm$ in the de Sitter case. We let $L_\pm = \partial\Lambda_\pm \subset S^*X_\delta$. Since $\Lambda_+ \cup \Lambda_-$ is given by $\eta = \zeta = 0$, $\tilde{\mu} = 0$, $\Lambda_\pm$ are preserved by the *classical* dynamics (i.e. with $\sigma = 0$). Note that the special structure of $\tilde{p}$ is irrelevant for the purposes of this observation; only the quadratic vanishing at $L_\pm$ matters. Even for other local aspects of analysis, considered below, the only relevant part[33], is that $\mathsf{H}_p\tilde{p}$ vanishes cubically at $L_\pm$, which in some sense reflects the behavior of the linearization of $\tilde{p}$.

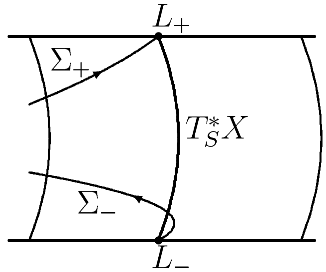

FIGURE 6. The cotangent bundle near the event horizon $S = \{\tilde{\mu} = 0\}$. It is drawn in a fiber-radially compactified view. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. The characteristic set crosses the event horizon on both components; here the part near L_+ is hidden from view. The projection of this region to the base space is the ergoregion. Semiclassically, i.e. the interior of $\overline{T}^* X_\delta$, for $z = h^{-1}\sigma > 0$, only $\Sigma_{\hbar,+}$ can enter $\tilde{\mu} > \alpha^2$, see the paragraph after (6.14).

To analyze the dynamics near $L_\pm$ on the characteristic set, starting with the classical dynamics, we note that

$$\mathsf{H}_{\tilde{p}} r = 0,\ \mathsf{H}_{\tilde{p}}\xi = 0,\ \mathsf{H}_p\zeta = 0,\ \mathsf{H}_{\tilde{p}}\tilde{p} = 0,\ \mathsf{H}_p\tilde{p} = 0;$$

[33] This could be relaxed: quadratic behavior with small leading term would be fine as well; quadratic behavior follows from H_p being tangent to $\Lambda_\pm$; smallness is needed so that $\mathsf{H}_p|\xi|^{-1}$ can be used to dominate this in terms of homogeneous dynamics, so that the dynamical character of $L_\pm$ (sink/source) is as desired.

note that $\mathsf{H}_p\tilde{p} = 0$ and $\mathsf{H}_p\zeta = 0$ correspond to the integrability of the Hamiltonian dynamical system; these were observed by Carter [8] in the Kerr setting. Furthermore,

$$\mathsf{H}_p|\xi|^{-1}|_{S^*X_\delta} = -(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}, \tag{6.7}$$

so at $\partial N^*\{\tilde{\mu} = 0\}$ it is given by $\pm(\operatorname{sgn}\xi)\digamma_\pm$, which is bounded away from 0. We note that

$$|\xi|^{-1}\mathsf{H}_p(|\xi|^{-2}\tilde{p})|_{S^*X_\delta} = -2(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}\tilde{p}|\xi|^{-2}. \tag{6.8}$$

Since $\tilde{p} = 0$ and $\tilde{\mu} \neq 0$ implies $\xi = 0$ on the classical characteristic set (i.e. when we take $\sigma = 0$), which cannot happen on S^*X (we are away from the zero section!), this shows that the Hamilton vector field is non-radial except possibly at $\Lambda_\pm$. Moreover,

$$\mathsf{H}_p\left(\tilde{\mu} \mp 2(1+\gamma)\alpha\frac{\zeta}{\xi}\right)|_{S^*X_\delta} = -2|\xi|\left(\tilde{\mu} \mp 2(1+\gamma)\alpha\frac{\zeta}{\xi}\right)(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r};$$

as usual, this corresponds to $\hat{p} = |\xi|^{-2}p$ at $L_\pm$. Finally, the imaginary part of the subprincipal symbol at $L_\pm$ is

$$\begin{aligned}&((\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r})(\beta_\pm \operatorname{Im}\sigma)|\xi|, \text{ where}\\ &\beta_\pm = \mp 2\left(\frac{d\tilde{\mu}}{dr}\right)^{-1}(1+\gamma)(r^2+\alpha^2)|_{r=r_\pm} = 2\digamma_\pm^{-1}(1+\gamma)(r_\pm^2+\alpha^2) > 0;\end{aligned} \tag{6.9}$$

here $(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}$ was factored out in view of (6.7), (2.5) and (2.3).

Thus, L_+ is a sink, L_- a source. Furthermore, in the classical sense, $\xi = 0$ is disjoint from the characteristic set in the region of validity of the form (6.5) of the operator, as well as at the poles of the sphere (i.e. the only issue is when r is farther from $r_\pm$), so the characteristic set has two components there with $L_\pm$ lying in different components. We note that that as $\gamma < 1$, $\kappa\sin^2\theta = \sin^2\theta(1+\gamma) - \gamma\sin^4\theta$ has its maximum at $\theta = \pi/2$, where it is 1. Since on the characteristic set

$$\alpha^2\xi^2 + (1+\gamma)^2\zeta^2 \geq \pm 2(1+\gamma)\alpha\xi\zeta = \tilde{p} + \tilde{\mu}\xi^2 \geq \eta^2 + (1+\gamma)^2\zeta^2 + \tilde{\mu}\xi^2 \tag{6.10}$$

and $\xi \neq 0$, we conclude that

$$\tilde{\mu} \leq \alpha^2 \tag{6.11}$$

there, so this form of the operator remains valid, and the characteristic set can indeed be divided into two components, separating $L_\pm$.

Next, we note that if α is so large that at $r = r_0$ with $\frac{d\tilde{\mu}}{dr}(r_0) = 0$, one has $\tilde{\mu}(r_0) = \alpha^2$, then letting $\eta_0 = 0$, $\theta_0 = \frac{\pi}{2}$, $\xi_0 \neq 0$, $\zeta_0 = \pm\frac{\alpha}{1+\gamma}\xi_0$, the bicharacteristics through $(r_0, \theta_0, \phi_0, \xi_0, \eta_0, \zeta_0)$ are stationary for any ϕ_0, so the operator is classically trapping in the strong sense that not only is the Hamilton vector field radial, but it vanishes. Since such vanishing means that weights cannot give positivity in positive commutator estimates, see Section 2, it is natural to impose the restriction on α that

$$r_0 \in (r_+, r_-),\ \frac{d\tilde{\mu}}{dr}(r_0) = 0 \Rightarrow \alpha^2 < \tilde{\mu}(r_0). \tag{6.12}$$

Under this assumption, by (6.11), the ergoregions from the two ends do not intersect.

Finally, we show that bicharacteristics leave the region $\tilde{\mu} > \tilde{\mu}_0$, where $\tilde{\mu}_0 < 0$ is such that $\frac{d\tilde{\mu}}{dr}$ is bounded away from 0 on $[\tilde{\mu}_0, (1+\epsilon)\alpha^2]_{\tilde{\mu}}$ for some $\epsilon > 0$, which completes checking the hypotheses in the classical sense. Note that by (6.2) and (6.12) such $\tilde{\mu}_0$ and ϵ exists. To see this, we use $\tilde{p}$ to measure the size of the characteristic set over points in the base. Using $ab \leq (1+\epsilon)a^2 + b^2/(1+\epsilon)$ and $\kappa \sin^2\theta \leq 1$, we note that on the characteristic set

$$(1+\epsilon)\alpha^2\xi^2 + \frac{(1+\gamma)^2}{1+\epsilon}\zeta^2 \geq \tilde{p} + \tilde{\mu}\xi^2 \geq \frac{\epsilon}{1+\epsilon}\tilde{p} + \frac{(1+\gamma)^2}{1+\epsilon}\zeta^2 + \tilde{\mu}\xi^2,$$

so

$$((1+\epsilon)\alpha^2 - \tilde{\mu}) \geq \frac{\epsilon}{1+\epsilon}|\xi|^{-2}\tilde{p},$$

where now both sides are homogeneous of degree zero, or equivalently functions on S^*X_δ. Note that $\tilde{p} = 0$ implies that $\xi \neq 0$ on S^*X_δ, so our formulae make sense. By (6.8), using that $\frac{\partial\tilde{\mu}}{\partial r}$ is bounded away from 0, $|\xi|^{-2}\tilde{p}$ is growing exponentially in the forward/backward direction along the flow as long as the flow remains in a region $\tilde{\mu} \geq \tilde{\mu}_0$, where the form of the operator is valid (which is automatic in this region, as farther on 'our side' of the event horizon, X_+, where the form of the operator is not valid, it is elliptic), which shows that the bicharacteristics have to leave this region. As noted already, this proves that the operator fits into our framework in the classical sense.

6.4. **Semiclassical behavior.** The semiclassical principal symbol is

(6.13)

$$p_{\hbar,z} = -\tilde{\mu}\big(\xi \pm cz\big)^2 \mp 2(1+\gamma)(r^2+\alpha^2)\big(\xi \pm cz\big)z \pm 2(1+\gamma)\alpha\big(\xi \pm cz\big)\zeta - \tilde{p}_{\hbar,z}$$

with

$$\tilde{p}_{\hbar,z} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-\alpha\sin^2\theta z + \zeta)^2.$$

Recall now that $M_\delta = X_\delta \times [0,\infty)_\tau$, and that, due to Section 3.2, we need to choose c in our definition of τ so that $\frac{d\tau}{\tau}$ is time-like with respect to G. But

$$\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = -\tilde{\mu}c^2 - 2c(1+\gamma)(r^2+\alpha^2) - \frac{\alpha^2(1+\gamma)^2\sin^2\theta}{\kappa},$$

and as this must be positive for all θ, we need to arrange that

$$\tilde{\mu}c^2 + 2c(1+\gamma)(r^2+\alpha^2) + \alpha^2(1+\gamma)^2 < 0, \tag{6.14}$$

and this in turn suffices. Note that $c = -\tilde{\mu}^{-1}(1+\gamma)(r^2+\alpha^2)$ automatically satisfies this in $\tilde{\mu} > 0$; this would correspond to undoing our change of coordinates in (6.4) (which is harmless away from $\tilde{\mu} = 0$, but of course c needs to be smooth at $\tilde{\mu} = 0$). At $\tilde{\mu} = 0$, (6.14) gives a (negative) upper bound for c; for $\tilde{\mu} > 0$ we have an interval of possible values of c; for $\tilde{\mu} < 0$ large negative values of c always work. Thus, we may choose a smooth function c such that (6.14) is satisfied everywhere, and we may further arrange that $c = -\tilde{\mu}^{-1}(1+\gamma)(r^2+\alpha^2)$ for $\tilde{\mu} > \tilde{\mu}_1$ where $\tilde{\mu}_1$ is an arbitrary positive constant; in this case, as already discussed, $p_{\hbar,z}$ is semiclassically elliptic when $\operatorname{Im} z \neq 0$.

Note also that, as discussed in Subsection 3.2, there is only one component of the characteristic set in $\tilde{\mu} > \alpha^2$ by (6.11), namely $\Sigma_{\hbar,\operatorname{sgn}(\operatorname{Re} z)}$.

It remains to discuss trapping. Note that the dynamics depends continuously on α, with $\alpha = 0$ being the de Sitter-Schwarzschild case, when there is no trapping near the event horizon, so the same holds for Kerr-de Sitter with slow rotation. Below we first describe the dynamics in de Sitter-Schwarzschild space explicitly, and then, in (6.22), give an explicit range of α in which the non-trapping dynamical assumption, apart from hyperbolic trapping, is satisfied.

First, on de Sitter-Schwarzschild space, recalling that c is irrelevant for the dynamics for real z, we may take $c = 0$ (i.e. otherwise we would simply change this calculation by the effect of a symplectomorphism, corresponding to a conjugation, which we note does not affect the 'base' variables on the cotangent bundle). Then

$$p_{\hbar,z} = -\tilde{\mu}\xi^2 \mp 2r^2\xi z - \tilde{p}_{\hbar,z}, \qquad \tilde{p}_{\hbar,z} = \eta^2 + \frac{\zeta^2}{\sin^2\theta},$$

so

$$\mathsf{H}_{p_{\hbar,z}} = -2(\tilde{\mu}\xi \pm r^2 z)\partial_r + \left(\frac{\partial\tilde{\mu}}{\partial r}\xi^2 \pm 4rz\xi\right)\partial_\xi - \mathsf{H}_{\tilde{p}_{\hbar,z}},$$

hence $\mathsf{H}_{p_{\hbar,z}} r = -2(\tilde{\mu}\xi \pm r^2 z)$, and so $\mathsf{H}_{p_{\hbar,z}} r = 0$ implies $\mp z = r^{-2}\tilde{\mu}\xi$. We first note that $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish in T^*X_δ in $\tilde{\mu} \leq 0$ (though it can vanish at fiber infinity at $L_\pm$) since (for $z \neq 0$)

$$\tilde{\mu} \leq 0 \text{ and } \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \tilde{\mu}\xi \neq 0 \text{ and } p_{\hbar,z} = \tilde{\mu}\xi^2 - \tilde{p}_{\hbar,z} < 0. \tag{6.15}$$

It remains to consider $\mathsf{H}_{p_{\hbar,z}} r = 0$ in $\tilde{\mu} > 0$. At such a point

$$\mathsf{H}^2_{p_{\hbar,z}} r = -2\tilde{\mu}\mathsf{H}_{p_{\hbar,z}}\xi = -2\tilde{\mu}\xi^2\left(\frac{\partial\tilde{\mu}}{\partial r} - 4r^{-1}\tilde{\mu}\right) = -2\tilde{\mu}\xi^2 r^4\frac{\partial(r^{-4}\tilde{\mu})}{\partial r},$$

so as $\mp z = r^{-2}\tilde{\mu}\xi$, so $\xi \neq 0$, by the discussion after (6.2),

$$\tilde{\mu} > 0,\ \pm(r - \frac{3}{2}r_s) > 0,\ \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \pm\mathsf{H}^2_{p_{\hbar,z}} r > 0,$$

and thus the gluing hypotheses of [15] are satisfied arbitrarily close to[34] $r = \frac{3}{2}r_s$. Furthermore, as $p_{\hbar,z} = -\tilde{\mu}^{-1}(\tilde{\mu}\xi \pm r^2 z)^2 + r^4 z^2 - \tilde{p}_{\hbar,z}$, if $r = \frac{3}{2}r_s$, $\mathsf{H}_{p_{\hbar,z}} r = 0$ and $p_{\hbar,z} = 0$ then $\tilde{p}_{\hbar,z} = r^4 z^2$, so with

$$\Gamma_z = \{r = \frac{3}{2}r_s,\ \tilde{\mu}\xi \pm r^2 z = 0,\ \tilde{p}_{\hbar,z} = r^4 z^2\},$$

we have

$$p_{\hbar,z}(\varpi) = 0,\ \tilde{\mu}(\varpi) > 0,\ \varpi \notin \Gamma_z,\ (\mathsf{H}_{p_{\hbar,z}} r)(\varpi) = 0 \Rightarrow (\pm\mathsf{H}^2_{p_{\hbar,z}} r)(\varpi) > 0, \tag{6.16}$$

with $\pm$ corresponding to whether $r > \frac{3}{2}r_s$ or $r < \frac{3}{2}r_s$. In particular, taking into account (6.15), r gives rise to an escape function in $T^*X_\delta \setminus \Gamma_z$ as discussed in Footnote 16, and Γ_z is the only possible trapping. (In this statement we ignore fiber infinity.) Correspondingly, if one regards a compact interval I in $(r_-, \frac{3}{2}r_s)$, or $(\frac{3}{2}r_s, r_+)$ as the gluing region, for sufficiently small α, for $r \in I$, $\mathsf{H}_{p_{\hbar,z}} r = 0$ still implies $\pm\mathsf{H}^2_{p_{\hbar,z}} r > 0$, and [15] is applicable. If instead one works with compact subsets of $\{\tilde{\mu} > 0\} \setminus \Gamma_z$, one has non-trapping dynamics for α small.

Since in [58, Section 2] Wunsch and Zworski only check normal hyperbolicity in Kerr space-times with sufficiently small angular momentum, in order to use their general results for normally hyperbolic trapped sets, we need to check that Kerr-de Sitter space-times are still normally hyperbolic. For this, with small α, we follow

[34] Or far from, in $\tilde{\mu} > 0$.

[58, Section 2], and note that for $\alpha = 0$ the linearization of the flow at Γ in the normal variables $r - \frac{3}{2}r_s$ and $\tilde\mu\xi \pm r^2 z$ is

$$\begin{bmatrix} r-\frac{3}{2}r_s \\ \tilde\mu\xi \pm r^2 z\end{bmatrix}' = \begin{bmatrix} 0 & -2(\frac{3}{2}r_s)^4 z^2 \tilde\mu|^{-1}_{r=\frac{3}{2}r_s} \\ -2 & 0 \end{bmatrix} \begin{bmatrix} r-\frac{3}{2}r_s \\ \tilde\mu\xi \pm r^2 z\end{bmatrix} + \mathcal{O}((r-\frac{3}{2}r_s)^2 + (\tilde\mu\xi \pm r^2 z)^2),$$

so the eigenvalues of the linearization are $\lambda = \pm 3\sqrt{3} r_s z \big(1 - \frac{9}{4}\Lambda r_s^2\big)^{-1/2}$, in agreement with the result of [58] when $\Lambda = 0$. The rest of the arguments concerning the flow in [58, Section 2] go through. In particular, when analyzing the flow *within* $\Gamma = \cup_{z>0}\Gamma_z$, the pull backs of both dp and $d\zeta$ are *exactly* as in the Schwarzschild setting (unlike the normal dynamics, which has different eigenvalues), so the arguments of [58, Proof of Proposition 2.1] go through unchanged, giving normal hyperbolicity for small α by the structural stability.

We now check the hyperbolic nature of trapping for larger values of α. With $c = 0$, as above,

$$p_{\hbar,z} = -\tilde\mu\xi^2 \mp 2(1+\gamma)\big((r^2+\alpha^2)z - \alpha\zeta\big)\xi - \tilde p_{\hbar.z},$$

and in the region $\tilde\mu > 0$ this can be rewritten as

$$p_{\hbar,z} = -\tilde\mu\left(\xi \pm \frac{1+\gamma}{\tilde\mu}\big((r^2+\alpha^2)z - \alpha\zeta\big)\right)^2 + \frac{(1+\gamma)^2}{\tilde\mu}\big((r^2+\alpha^2)z - \alpha\zeta\big)^2 - \tilde p_{\hbar.z};$$

note that the first term would be just $-\tilde\mu\xi^2$ in the original coordinates (6.1) which are valid in $\tilde\mu > 0$. Thus,

$$\mathsf{H}_{p_{\hbar.z}} r = -2\left(\tilde\mu\xi \pm (1+\gamma)\big((r^2+\alpha^2)z - \alpha\zeta\big)\right). \tag{6.17}$$

Correspondingly,

$$\tilde\mu \leq 0,\ \mathsf{H}_{p,\hbar,z} r = 0 \Rightarrow p_{\hbar,z} = \tilde\mu\xi^2 - \tilde p_{\hbar.z} \leq 0,$$

and equality on the right hand side implies $\zeta = \alpha\sin^2\theta z$, so $\mathsf{H}_{p_{\hbar,z}} r = (r^2 + \alpha^2\cos^2\theta)z > 0$, a contradiction, showing that in $\tilde\mu \leq 0$, $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish on the characteristic set.

We now turn to $\tilde\mu > 0$, where

$$\mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \mathsf{H}^2_{p_{\hbar,z}} r = -2\tilde\mu \mathsf{H}_{p_{\hbar,z}}\xi = 2\tilde\mu(1+\gamma)^2 \frac{\partial}{\partial r}\left(\tilde\mu^{-1}\big((r^2+\alpha^2)z - \alpha\zeta\big)^2\right).$$

Thus, we are interested in critical points of

$$F = \tilde\mu^{-1}\big((r^2+\alpha^2)z - \alpha\zeta\big)^2,$$

and whether these are non-degenerate. We remark that

$$\begin{aligned} \mathsf{H}_{p_{\hbar,z}} r = 0 \text{ and } (r^2+\alpha^2)z - \alpha\zeta = 0 \text{ and } p_{\hbar,z} = 0 &\Rightarrow \xi = 0 \text{ and } \tilde p_{\hbar,z} = 0; \\ \tilde p_{\hbar,z} = 0 \text{ and } (r^2+\alpha^2)z - \alpha\zeta = 0 \text{ and } p_{\hbar,z} = 0 &\Rightarrow (r^2+\alpha^2\cos^2\theta)z = 0; \end{aligned}$$

which is a contradiction, so $(r^2+\alpha^2)z - \alpha\zeta$ does not vanish when $p_{\hbar,z}$ and $\mathsf{H}_{p_{\hbar,z}} r$ do. Note that

$$\frac{\partial F}{\partial r} = -\big((r^2+\alpha^2)z - \alpha\zeta\big)\tilde\mu^{-2} f, \qquad f = \big((r^2+\alpha^2)z - \alpha\zeta\big)\frac{\partial\tilde\mu}{\partial r} - 4r\tilde\mu z, \tag{6.18}$$

so

$$\frac{\partial F}{\partial r} = 0 \text{ and } (r^2+\alpha^2)z - \alpha\zeta \neq 0 \Rightarrow \frac{\partial^2 F}{\partial r^2} = -\big((r^2+\alpha^2)z - \alpha\zeta\big)\tilde\mu^{-2}\frac{\partial f}{\partial r}. \tag{6.19}$$

Also, from (6.18),

$$\tilde\mu>0,\ f=0\Rightarrow \frac{\partial\tilde\mu}{\partial r}\neq 0.$$

Now

$$\frac{\partial f}{\partial r}=\big((r^2+\alpha^2)z-\alpha\zeta\big)\frac{\partial^2\tilde\mu}{\partial r^2}-4\tilde\mu z-2rz\frac{\partial\tilde\mu}{\partial r}, \tag{6.20}$$

and

$$\frac{\partial F}{\partial r}=0\Rightarrow (r^2+\alpha^2)z-\alpha\zeta=\frac{4r\tilde\mu z}{\frac{\partial\tilde\mu}{\partial r}},$$

so substituting into (6.20),

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r}=4r\tilde\mu z\frac{\partial^2\tilde\mu}{\partial r^2}-4z\tilde\mu\frac{\partial\tilde\mu}{\partial r}-2rz\left(\frac{\partial\tilde\mu}{\partial r}\right)^2. \tag{6.21}$$

Thus,

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r}=2z\left(2\tilde\mu\left(r\frac{\partial^2\tilde\mu}{\partial r^2}-3\frac{\partial\tilde\mu}{\partial r}\right)-(r\frac{\partial\tilde\mu}{\partial r}-4\tilde\mu)\frac{\partial\tilde\mu}{\partial r}\right),$$

so taking into account

$$\begin{aligned} r\frac{\partial\tilde\mu}{\partial r}-4\tilde\mu&=-2\left(1-\frac{\Lambda\alpha^2}{3}\right)r^2+3r_sr-4\alpha^2,\\ r\frac{\partial^2\tilde\mu}{\partial r^2}-3\frac{\partial\tilde\mu}{\partial r}&=-4\left(1-\frac{\Lambda\alpha^2}{3}\right)r+3r_s,\\ r\frac{\partial^2\tilde\mu}{\partial r^2}-3\frac{\partial\tilde\mu}{\partial r}&=\frac{2}{r}\left(r\frac{\partial\tilde\mu}{\partial r}-4\tilde\mu\right)-3r_s+\frac{8\alpha^2}{r},\end{aligned}$$

we obtain

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r}=2z\left(-\frac{1}{r}\left(r\frac{\partial\tilde\mu}{\partial r}-4\tilde\mu\right)^2-\frac{2\tilde\mu}{r}(3r_sr-8\alpha^2)\right).$$

We claim that if $|\alpha|<r_s/2$ then $r_->r_s/2$. To see this, note that for $r=r_s/2$,

$$\tilde\mu(r)=\left(\frac{r_s^2}{4}+\alpha^2\right)\left(1-\frac{\Lambda r_s^2}{12}\right)-\frac{r_s^2}{2}<0;$$

since at $\alpha=0$, $r_->r_s/2$, we deduce that $r_->r_s/2$ for $|\alpha|<r_s/2$. Making the slightly stronger assumption,

$$|\alpha|<\frac{\sqrt{3}}{4}r_s, \tag{6.22}$$

we obtain that for $\tilde\mu>0$, $r>r_-$, $3r_sr-8\alpha^2>\frac{3}{2}r_s^2-8\alpha^2>0$, so

$$z\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r}<0.$$

Thus, when $\frac{\partial F}{\partial r}=0$, using (6.19),

$$\frac{\partial^2 F}{\partial r^2}=-\big((r^2+\alpha^2)z-\alpha\zeta\big)\tilde\mu^{-2}\frac{\partial f}{\partial r}=-\frac{4r}{\tilde\mu\left(\frac{\partial\tilde\mu}{\partial r}\right)^2}z\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r}>0, \tag{6.23}$$

so critical points of F are all non-degenerate and are minima. Correspondingly, as $F\to+\infty$ as $\tilde\mu\to 0$ in $\tilde\mu>0$, the critical point r_c of F exists and is unique in (r_-, r_+) (when ζ is fixed), depends smoothly on ζ, and $\frac{\partial F}{\partial r} > 0$ if $r > r_c$, and $\frac{\partial F}{\partial r} < 0$ if $r < r_c$. Thus,

$$\tilde{\mu} > 0,\ \pm(r - r_c) > 0,\ \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \pm \mathsf{H}^2_{p_{\hbar,z}} r > 0,$$

giving the natural generalization of (6.16), allowing the application of the results of [15]. Since $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish in $\tilde{\mu} \leq 0$ (apart from fiber infinity, which is understood already), we conclude that r gives rise to an escape function, as in Footnote 16, away from

$$\Gamma_z = \{\varpi :\ \frac{\partial F}{\partial r}(\varpi) = 0,\ (\mathsf{H}_{p_{\hbar,z}} r)(\varpi) = 0,\ p_{\hbar,z}(\varpi) = 0\},$$

which is a smooth submanifold as the differentials of the defining functions are linearly independent on it in view of (6.23), (6.17), and the definition of $\tilde{p}_{\hbar,z}$ (as the latter is independent of r and ξ).

The linearization of the Hamilton flow at Γ_z is

$$\begin{aligned}&\begin{bmatrix} r - r_c \\ \tilde{\mu}\xi \pm (1+\gamma)\big((r^2+\alpha^2)z - \alpha\zeta\big)\end{bmatrix}' \\ &\qquad = \begin{bmatrix} 0 & -\tilde{\mu}(1+\gamma)^2 \frac{\partial^2 F}{\partial r^2} \\ -2 & 0 \end{bmatrix} \begin{bmatrix} r - r_c \\ \tilde{\mu}\xi \pm (1+\gamma)\big((r^2+\alpha^2)z - \alpha\zeta\big)\end{bmatrix} \\ &\qquad\qquad + \mathcal{O}\Big((r-r_c)^2 + \big(\tilde{\mu}\xi \pm (1+\gamma)\big((r^2+\alpha^2)z - \alpha\zeta\big)\big)^2\Big),\end{aligned}$$

so by (6.23), the linearization is non-degenerate, and is indeed hyperbolic. This suffices for the resolvent estimates of [58] for exact Kerr-de Sitter, but for stability one also needs to check normal hyperbolicity. While it is quite straightforward to check that the only degenerate location is $\eta = 0$, $\theta = \frac{\pi}{2}$, the computation of the Morse-Bott non-degeneracy in the spirit of [58, Proof of Proposition 2.1], where it is done for Kerr spaces with small angular momentum, is rather involved, so we do not pursue this here (for small angular momentum in Kerr-de Sitter space, the de Sitter-Schwarzschild calculation above implies normal hyperbolicity already).

In addition, in view of an overall sign difference between our convention and that of [58] for the operator we are considering, [58] requires the positivity of $z\frac{\partial}{\partial z} p_{\hbar,z}$ for $z \neq 0$. (Note that the notation for z is also different; our z is $1+z$ in the notation of [58], so our z being near 1 corresponds to the z of [58] being near 0.) Unlike the flow, whose behavior is independent of c when z is real, this fact does depend on the choice of c. Note that in the high energy version, this corresponds to the positivity of $\sigma\frac{\partial}{\partial\sigma} p_{\text{full}}$. Now, $p_{\text{full}} = \langle \sigma\,\frac{d\tau}{\tau} + \varpi, \sigma\,\frac{d\tau}{\tau} + \varpi\rangle_G$, with $\varpi \in \Pi$, the 'spatial' hyperplane, identified with T^*X in ${}^{\mathrm{b}}T^*\bar{M}$, so

$$\begin{aligned}\sigma\partial_\sigma p_{\text{full}} &= 2\langle \sigma\,\frac{d\tau}{\tau}, \sigma\,\frac{d\tau}{\tau}\rangle_G + 2\langle \sigma\,\frac{d\tau}{\tau}, \varpi\rangle_G \\ &= \sigma^2\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G + \langle \sigma\,\frac{d\tau}{\tau} + \varpi, \sigma\,\frac{d\tau}{\tau} + \varpi\rangle_G - \langle \varpi, \varpi\rangle_G.\end{aligned}$$

Thus, if non-zero elements of Π are space-like and $\frac{d\tau}{\tau}$ is time-like, $\sigma\partial_\sigma p_{\text{full}} > 0$ for $\sigma \neq 0$ on the characteristic set of p_{full}. If c is such that $c = -\tilde{\mu}^{-1}(1+\gamma)(r^2+\alpha^2)$ near $r = \frac{3}{2} r_s$, which as we mentioned can be arranged, and which corresponds to undoing our change of coordinates in (6.4), then directly from (6.1) both the time-like and space-like statements hold, completing our checking of the hypotheses of [58], and thus their result is applicable for de Sitter-Schwarzschild space-times. As these results are structurally stable, see the proof of [58, Proposition 2.1], the result follows for Kerr-de Sitter spaces with angular momenta satisfying (6.22).

6.5. **Complex absorption.** The final step of fitting P_σ into a general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. This section is *almost completely parallel* to Subsection 4.7 in the de Sitter case; the only change is that absorption needs to be added at the trapped set as well.

We thus consider a compact manifold without boundary X for which X_δ is identified as an open subset with smooth boundary; we can again take X to be the double of X_δ. As in the de Sitter case, we discuss the 'classical' and 'semiclassical' cases separately, for in the former setting trapping does not matter, while in the latter it does.

We then introduce a complex absorbing operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with principal symbol q, such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and such that $p \pm \imath q$ is elliptic near ∂X_δ, i.e. near $\tilde\mu = \tilde\mu_0$, and which satisfies that the $\pm q \geq 0$ on $\Sigma_\mp$. Having done this, we extend P_σ and Q_σ to X in such a way that $p \pm \imath q$ are elliptic near $X \setminus X_\delta$; the region we added is thus irrelevant. In particular, as the event horizon is characteristic for the wave equation, the solution in the exterior of the event horizons is *unaffected* by thus modifying P_σ, i.e. working with P_σ and $P_\sigma - \imath Q_\sigma$ is equivalent for this purpose.

Again, as in de Sitter space, an alternative to this extension would be simply adding a boundary at $\tilde\mu = \tilde\mu_0$; this is easy to do since this is a space-like hypersurface, see Remark 2.5.

For the semiclassical problem, when z is almost real we need to increase the requirements on Q_σ. We need in addition, in the semiclassical notation, semiclassical ellipticity near $\tilde\mu = \tilde\mu_0$, i.e. that $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near ∂X_δ, i.e. near $\tilde\mu = \tilde\mu_0$, and which satisfies that $\pm q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\mp}$. However, we also need semiclassical ellipticity at the trapped set, which is in X_+. To achieve this, we want $q_{\hbar,z}$ elliptic on the trapped set; since this is in $\Sigma_{\hbar,\mathrm{sgn}\,\mathrm{Re}\,z}$, we need $q_{\hbar,z} \leq 0$ there. Again, we extend P_σ and Q_σ to X in such a way that $p \pm \imath q$ and $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near $X \setminus X_\delta$; the region we added is thus irrelevant. For α as in (6.22), the dynamics (away from the radial points) has only the hyperbolic trapping (and for small α, it is normally hyperbolic); however, our results apply more generally, as long as the dynamics has the same non-trapping character (so α might be even larger). Note also that since the trapping is in a compact subset of $X_+ = \{\tilde\mu > 0\}$, one can arrange that Q_σ is the sum of two terms: one supported near the trapping in X_+, the other in $\tilde\mu < 0$; this is useful for relating our construction to that of Dyatlov [20] in the appendix.

This completes the setup. Now all of the results of Section 2 are applicable, proving all the theorems stated in the introduction on Kerr-de Sitter spaces, Theorems 1.1-1.4. Namely, Theorem 1.1 follows from Theorem 2.13, Theorem 1.2 follows from Theorem 2.10, Theorem 1.3 follows from Theorem 2.17. Finally Theorem 1.4 is an immediate consequence of Theorem 1.3 and the Mellin transform lemma, Lemma 3.1, in the Kerr-de Sitter setting, or Proposition 3.3 for general b-perturbations (so $\partial_{\tilde t}$ may no longer be Killing, and the space-time may no longer be stationary), either taking into account that by usual energy estimates the complex absorbing region cannot affect the solution in a neighborhood of $\overline{X_+}$, or simply, for

$\Box_g u = f$, so $P_\sigma \mathcal{M} u = \mathcal{M} f$, so $(P_\sigma - \imath Q_\sigma)\mathcal{M} u = f - \imath Q_\sigma \mathcal{M} u$, and using the last part of Theorem 2.15 to note that $\imath Q_\sigma \mathcal{M} u$ does not affect the asymptotics in X_δ.

Appendix A. Comparison with cutoff resolvent constructions

By Semyon Dyatlov[35]

In this appendix, we will first examine the relation of the resolvent considered in the present paper to the cutoff resolvent for slowly rotating Kerr–de Sitter metric constructed in [20] using separation of variables and complex contour deformation near the event horizons. Then, we will show how to extract information on the resolvent beyond event horizons from information about the cutoff resolvent.

First of all, let us list some notation of [20] along with its analogues in the present paper:

Present paper	[20]	Present paper	[20]
α	a	r_s	$2M_0$
γ	α	$\tilde{\mu}$	Δ_r
κ	Δ_θ	$F_\pm$	$A_\pm$
$\tilde{t}, \tilde{\phi}$	t, φ	t, ϕ	t^*, φ^*
ω	σ	$e^{-i\sigma h(r)} P_\sigma e^{i\sigma h(r)}$	$-P_g(\sigma)$
X_+	M	K_δ	M_K

The difference between $P_g(\omega)$ and P_σ is due to the fact that $P_g(\omega)$ was defined using Fourier transform in the $\tilde{t}$ variable and P_σ is defined using Fourier transform in the variable $t = \tilde{t} + h(r)$. We will henceforth use the notation of the present paper.

We assume that $\delta > 0$ is small and fixed, and α is small depending on δ. Define

$$K_\delta = (r_- + \delta, r_+ - \delta)_r \times \mathbb{S}^2.$$

Then [20, Theorem 2] gives a family of operators

$$R_g(\sigma) : L^2(K_\delta) \to H^2(K_\delta)$$

meromorphic in $\sigma \in \mathbb{C}$ and such that $P_g(\sigma) R_g(\sigma) f = f$ on K_δ for each $f \in L^2(K_\delta)$.

Proposition A.1. *Assume that the complex absorbing operator Q_σ satisfies the assumptions of Section 6.5 in the 'classical' case and furthermore, its Schwartz kernel is supported in $(X \setminus X_+)^2$. Let $R_g(\sigma)$ be the operator constructed in* [20] *and $R(\sigma) = (P_\sigma - iQ_\sigma)^{-1}$ be the operator defined in Theorem 1.2 of the present paper. Then for each $f \in C_0^\infty(K_\delta)$,*

$$-e^{i\sigma h(r)} R_g(\sigma) e^{-i\sigma h(r)} f = R(\sigma) f|_{K_\delta}. \tag{A.1}$$

Proof. The proof follows [20, Proposition 1.2]. Denote by u_1 the left-hand side of (A.1) and by u_2 the right-hand side. Without loss of generality, we may assume that f lies in the kernel $\mathcal{D}'_k$ of the operator $D_\phi - k$, for some $k \in \mathbb{Z}$; in this case, by [20, Theorem 1], u_1 can be extended to the whole X_+ and solves the equation $P_\sigma u_1 = f$ there. Moreover, by [20, Theorem 3], u_1 is smooth up to the event horizons $\{r = r_\pm\}$. Same is true for u_2; therefore, the difference $u = u_1 - u_2$ solves the equation $P_\sigma(u) = 0$ and is smooth up to the event horizons.

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Since both sides of (A.1) are meromorphic, we may further assume that $\operatorname{Im}\sigma > C_e$, where C_e is a large constant. Now, the function $\tilde{u}(t,\cdot) = e^{-it\sigma}u(\cdot)$ solves the wave equation $\Box_g \tilde{u} = 0$ and is smooth up to the event horizons in the coordinate system (t, r, θ, ϕ); therefore, if C_e is large enough, by [20, Proposition 1.1] $\tilde{u}$ cannot grow faster than $\exp(C_e t)$. Therefore, $u = 0$ as required. $\square$

Now, we show how to express the resolvent $R(\sigma)$ on the whole space in terms of the cutoff resolvent $R_g(\sigma)$ and the nontrapping parametrix constructed in the present paper. Let Q_σ be as above, but with the additional assumption of semiclassical ellipticity near ∂X_δ, and Q'_σ be an operator satisfying the assumptions of Section 6.5 in the 'semiclassical' case on the trapped set. Moreover, we require that the semiclassical wavefront set of $|\sigma|^{-2}Q'_\sigma$ be compact and $Q'_\sigma = \chi Q'_\sigma = Q'_\sigma \chi$, where $\chi \in C_0^\infty(K_\delta)$. Such operators exist for α small enough, as the trapped set is compact and located $O(\alpha)$ close to the photon sphere $\{r = 3r_s/2\}$ and thus is far from the event horizons. Denote $R'(\sigma) = (P_\sigma - iQ_\sigma - iQ'_\sigma)^{-1}$; by Theorem 2.13 applied in the case of Section 6.5, for each C_0 there exists a constant σ_0 such that for s large enough, $\operatorname{Im}\sigma > -C_0$, and $|\operatorname{Re}\sigma| > \sigma_0$,

$$\|R'(\sigma)\|_{H^{s-1}_{|\sigma|^{-1}} \to H^s_{|\sigma|^{-1}}} \leq C|\sigma|^{-1}.$$

We now use the identity

$$R(\sigma) = R'(\sigma) - R'(\sigma)(iQ'_\sigma + Q'_\sigma(\chi R(\sigma)\chi)Q'_\sigma)R'(\sigma). \tag{A.2}$$

(To verify it, multiply both sides of the equation by $P_\sigma - iQ_\sigma - iQ'_\sigma$ on the left and on the right.) Combining (A.2) with the fact that for each N, Q'_σ is bounded $H^{-N}_{|\sigma|^{-1}} \to H^N_{|\sigma|^{-1}}$ with norm $O(|\sigma|^2)$, we get for σ not a pole of $\chi R(\sigma)\chi$,

$$\|R(\sigma)\|_{H^{s-1}_{|\sigma|^{-1}} \to H^s_{|\sigma|^{-1}}} \leq C(1 + |\sigma|^2 \|\chi R(\sigma)\chi\|_{L^2(K_\delta) \to L^2(K_\delta)}). \tag{A.3}$$

Also, if σ_0 is a pole of $R(\sigma)$ of algebraic multiplicity j, then we can multiply the identity (A.2) by $(\sigma - \sigma_0)^j$ to get an estimate similar to (A.3) on the function $(\sigma - \sigma_0)^j R(\sigma)$, holomorphic at $\sigma = \sigma_0$.

The discussion above in particular implies that the cutoff resolvent estimates of [5] also hold for the resolvent $R(\sigma)$. Using the Mellin transform, we see that the resonance expansion of [5] is valid for any solution u to the forward time Cauchy problem for the wave equation on the whole M_δ, with initial data in a high enough Sobolev class; the terms of the expansion are defined and the remainder is estimated on the whole M_δ as well.

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