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MICROLOCAL HYPOELLIPTICITY OF LINEAR PARTIAL DIFFERENTIAL OPERATORS WITH GENERALIZED FUNCTIONS AS COEFFICIENTS

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Abstract. We investigate microlocal properties of partial differential operators with generalized functions as coefficients. The main result is an extension of a corresponding (microlocalized) distribution theoretic result on operators with smooth hypoelliptic symbols. Methodological novelties and technical refinements appear embedded into classical strategies of proof in order to cope with most delicate interferences by non-smooth lower order terms. We include simplified conditions which are applicable in special cases of interest.

1. Introduction

Consider the linear partial differential equation

$$Pu = f,$$

where u, f, as well as the coefficients of P are generalized functions. We investigate the general question of how to deduce information on the microlocal regularity of u from knowledge of the wave front set of f and properties (of the symbol) of P. As a matter of fact, a central issue in this is the identification of appropriate conditions on P which are generally applicable and, at the same time, allow for non-trivial conclusions. If the coefficients of P are smooth functions (or analytic) and u and f are distributions (or ultradistributions or hyperfunctions), then we may draw very detailed information revealed by the many efficient methods from symbolic calculus, functional analysis, and microlocal analysis (cf. [8, 13, 19, 21, 23]). We take the freedom to refer to this, by now well-established, distributional setting as the 'classical case'. The present paper is an exploration into a realm beyond, namely in the direction where generalized functions may appear among the coefficients of the operator P (for example, to model jump discontinuities or even more complex irregular properties of physical parameters). Since we are concerned here with generally applicable information this puts us into facing multiplication of distributions and thus obliges us to place our analysis in the framework of algebras of generalized functions, specifically Colombeau-type theories. A prominent new challenge in this extended context is a more intricate mechanism to keep control over the influence of non-smooth lower order terms. This is an observation stated

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explicitly in [12] where results addressing this issue in special cases were obtained. It is the purpose of this paper to extend these results in substantial ways.

Subsection 1.1 serves as a review of the classical results which are to be extended in the current paper. In Subsection 1.2 we give a brief account of the basics of the modern theory of generalized functions in the sense of Colombeau. In particular, notions of regularity in this setting are discussed in detail, also including a new result on regular Colombeau functions with slow scale bounds. Section 2 constructs the "basic technical layer" of the main proof in the paper. It also provides an interface with "higher levels" of analytic conditions on the operators in terms of two key assumptions. Section 3 presents the main result, the proof of which utilizes the deductive structure laid out before. In Section 4 we give three types of simplified conditions, which yield regularity results for certain special classes of operators with non-smooth symbols. One of these is illustrated in the example of an acoustic wave equation with bounded, but otherwise possibly highly irregular (e.g. discontinuous), coefficients.

1.1. **General regularity results for partial differential operators with smooth coefficients.** To set the stage, we recall known, "classical" results relevant to the subsequent generalizations. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $T^*(\Omega) \setminus 0 := \Omega \times (\mathbb{R}^n \setminus \{0\})$ be the cotangent space over Ω with the zero section removed. Consider a partial differential operator (PDO) P of order m with smooth coefficients, given by $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$, where $a_\alpha \in \mathrm{C}^\infty(\Omega)$, and with principal part $P_m(x,D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha$. The (full) symbol of P is $P(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ and its principal symbol $P_m(x,\xi)$ is the symbol of P_m. Both are interpreted as smooth functions on $T^*(\Omega)$.

The notion of *wave front set* of a distribution $u \in \mathcal{D}'(\Omega)$ was introduced in [6]. We recall the elementary definition (due to [6, Proposition 2.5.5]; see also [8, Chapter 8]) of the wave front set of $\mathrm{WF}(u) \subseteq T^*(\Omega) \setminus 0$: The distribution u is microlocally regular at $(x,\xi) \in T^*(\Omega) \setminus 0$, i.e., $(x,\xi) \notin \mathrm{WF}(u)$, if there exists a function $\varphi \in \mathcal{D}(\Omega)$ with $\varphi(x) \neq 0$ such that the Fourier transform of φu is rapidly decreasing in a conic neighborhood of the direction ξ.

Non-characteristic regularity. Recall that actions of PDOs (with smooth coefficients) on distributions do not enlarge singular supports. More precisely, PDOs are *microlocal* operators, which means that for any $u \in \mathcal{D}'(\Omega)$ we have

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u). \tag{1}$$

The maximum failure of the reverse inclusion (in the general case) can be captured by the "geometry of the leading order terms" of the operator. Recall that $\mathrm{Char}(P) := P_m^{-1}(0) = \{(x,\xi) \in T^*(\Omega) \setminus 0 \mid P_m(x,\xi) = 0\}$ is the characteristic set of P. It is a conic (with respect to the cotangent variable) closed subset of $T^*(\Omega) \setminus 0$ and, together with $\mathrm{WF}(Pu)$, gives a general upper bound on $\mathrm{WF}(u)$. This is the content of Hörmander's theorem on noncharacteristic regularity (cf. [8, Theorem 8.3.1]).

Theorem 1.1. *For any $u \in \mathcal{D}'(\Omega)$ we have the inclusion relation*

$$\mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \mathit{Char}(P). \tag{2}$$

The operator $P(x,D)$ is *elliptic* if $\mathrm{Char}(P) = \emptyset$, and *(microlocally) elliptic at* $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$ if $P_m(x,\xi) \neq 0$. Recall that microlocal ellipticity at (x_0,ξ_0) can be stated equivalently in terms of estimates on the full symbol: $\exists$ open set $U \ni x_0$ in Ω, $\exists$ open cone $\Gamma \ni \xi_0$ in $\mathbb{R}^n \setminus 0$, $\exists R > 0$, $C_0 > 0$ $\forall \alpha, \beta \in \mathbb{N}_0^n$, $|\beta| \leq m$ $\exists C_{\alpha\beta} > 0$ such that

$$|P(x,\xi)| \geq C_0\,(1+|\xi|)^m, \tag{3}$$

$$|\partial_x^\alpha \partial_\xi^\beta P(x,\xi)| \leq C_{\alpha\beta}\,|P(x,\xi)|\,(1+|\xi|)^{-|\beta|} \tag{4}$$

for all $(x,\xi) \in U \times \Gamma$ and $|\xi| \geq R$.

Remark 1.2. (i) Recall that an operator $P(x,D)$ is said to be *microhypoelliptic* if

$$\mathrm{WF}(u) = \mathrm{WF}(Pu)$$

for any distribution u, whereas it is *hypoelliptic* if

$$\mathrm{singsupp}(u) = \mathrm{singsupp}(Pu).$$

Clearly, the former property implies the latter. If $P(D)$ is a constant coefficient operator the two notions are equivalent (cf. [8, Theorem 11.1.1]). But already operators with polynomial coefficients can provide examples of differential operators which are hypoelliptic and not microhypoelliptic (cf. [18]). However, it follows from (2) that all elliptic operators are microhypoelliptic. More generally, pseudodifferential operators with hypoelliptic symbols are microhypoelliptic [8, Chapter 22], a result which we will state below in detail for differential operators.

(ii) Any non-elliptic but microhypoelliptic operator shows that the upper bound in (2) may be rather coarse. A simple example is the heat operator with symbol $P(\xi,\tau) = i\tau + |\xi|^2$ with $\mathrm{Char}(P) = \Omega \times \{(0,\tau) \mid \tau \neq 0\}$.

(*Microlocal*) *Microhypoellipticity.* Let $P(x,D)$ be a partial differential operator with smooth coefficients. It is said to have a *hypoelliptic symbol* $P(x,\xi)$ if the conditions (5)-(6) below are satisfied on all of $\Lambda = T^*(\Omega)\setminus 0$ ([8, Definition 22.1.1]).

Theorem 1.3. *Let $(x_0,\xi_0) \in T^*(\Omega)\setminus 0$, $U \ni x_0$ open, $\Gamma \ni \xi_0$ open conic (in $\mathbb{R}^n\setminus 0$), $m_0 \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$, with the following property: $\forall K \Subset U$ $\exists R > 0$ $\exists C_0 > 0$ $\forall \alpha, \beta \in \mathbb{N}_0^n$, $|\beta| \leq m$ $\exists C_{\alpha\beta} > 0$ such that*

$$|P(x,\xi)| \geq C_0\,(1+|\xi|)^{m_0}, \tag{5}$$

$$|\partial_x^\alpha \partial_\xi^\beta P(x,\xi)| \leq C_{\alpha\beta}\,|P(x,\xi)|\,(1+|\xi|)^{-\rho|\beta|+\delta|\alpha|} \tag{6}$$

for all $(x,\xi) \in K \times \Gamma$ with $|\xi| > R$. Then we have, with $\Lambda = U \times \Gamma$, for any $u \in \mathcal{D}'(\Omega)$

$$\Lambda \cap \mathrm{WF}(u) = \Lambda \cap \mathrm{WF}(Pu). \tag{7}$$

(Cf. [8, Theorem 22.1.4] for the global microhypoellipticity, and [14, Theorem 3.3.6] for the microlocal statement in the above form.)

Corollary 1.4. *Let $M(P)$ be the union of all open conic subsets $\Lambda \subseteq T^*(\Omega)\setminus 0$, where conditions (5)-(6) are satisfied (m_0, ρ, and δ may vary with Λ). Then we have for any distribution u on Ω the inclusion*

$$\mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup M(P)^c \tag{8}$$

(the set theoretic complement taken in $T^*(\Omega)\setminus 0$).

Remark 1.5. If $M(P) = T^*(\Omega) \setminus 0$, i.e., P has a hypoelliptic symbol, then P is microhypoelliptic, but the converse does not hold (cf. [8, Section 22.2] on generalized Kolmogorov equations). This is in contrast to the constant coefficient case where hypoelliptic symbols are in one-to-one correspondence with hypoelliptic operators (cf. [8, Chapter 11]). Furthermore, in the latter case the set $M(P)$ can be determined by simple algebraic-geometric conditions (cf. [7, p.15]). In general, we have $M(P)^c \subseteq \mathrm{Char}(P)$ since (3)-(4) imply (5)-(6). The inclusion can be strict, as the example of the heat operator with $M(P)^c = \emptyset$ shows. Therefore, (8) is a refinement of (2).

1.2. **Partial differential operators on algebras of generalized functions.**

1.2.1. *Colombeau algebras.* The paper is placed in the framework of algebras of generalized functions introduced by Colombeau in [1, 2]. Here we shall fix the notation and discuss a number of known as well as new properties pertinent to Colombeau generalized functions. As a general reference we recommend [5].

Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. To simplify the notation, we shall write $(u_\varepsilon)_\varepsilon$ in place of $(u_\varepsilon)_{\varepsilon\in(0,1]}$ throughout. We single out the following subalgebras:

Moderate families, denoted by $\mathcal{E}_{\mathrm{M}}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \exists p \geq 0: \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\,. \tag{9}$$

Null families, denoted by $\mathcal{N}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \forall q \geq 0: \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{q}) \text{ as } \varepsilon \to 0\,. \tag{10}$$

In other words, moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null families vanish faster than any power of ε in the same situation. The null families form a differential ideal in the collection of moderate families. The *Colombeau algebra* is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_{\mathrm{M}}(\Omega)/\mathcal{N}(\Omega)\,.$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the *special Colombeau algebra* in [5, Definition 1.2.2], where the notation $\mathcal{G}^s(\Omega)$ has been employed. However, as we will not use other variants of the algebra, we drop the superscript s in the sequel.

Restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives in the obvious way. One can show (see [5, Theorem 1.2.4]) that $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. Thus the support of a generalized function $u \in \mathcal{G}(\Omega)$ is well defined as the complement of the largest open set on which u vanishes. The subalgebra of compactly supported Colombeau generalized functions will be denoted by $\mathcal{G}_{\mathrm{c}}(\Omega)$.

The space of compactly supported distributions is embedded in $\mathcal{G}(\Omega)$ by convolution:

$$\iota : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega),\ \iota(w) = [((w * \varphi_\varepsilon)|_\Omega)_\varepsilon]\,,$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\,(x/\varepsilon) \tag{11}$$

is obtained by scaling a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. Here and henceforth the bracket notation [.] is used to denote the equivalence class in $\mathcal{G}(\Omega)$. By the sheaf property, this can be extended in a unique way to an embedding of the space of distributions $\mathcal{D}'(\Omega)$, and this embedding commutes with derivatives.

One of the main features of the Colombeau construction is the fact that this embedding renders $\mathrm{C}^\infty(\Omega)$ a faithful subalgebra. (This property is optimal and cannot be improved to $\mathrm{C}^k(\Omega)$ for finite k, in view of Schwartz' impossibility result [22].) In fact, given $f \in \mathrm{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant embedding $\sigma(f) =$ class of $[(\varepsilon, x) \mapsto f(x)]$. Then the important equality $\iota(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$. For a discussion of the overall properties of the Colombeau algebra, we refer to the literature (e.g. [2, 5, 17, 20]).

Colombeau generalized numbers $\widetilde{\mathbb{C}}$ can be defined as the Colombeau algebra $\mathcal{G}(\mathbb{R}^0)$, or alternatively as the ring of constants in $\mathcal{G}(\mathbb{R}^n)$. $\widetilde{\mathbb{C}}$ forms a ring, but not a field. Concerning invertibility in $\widetilde{\mathbb{C}}$, we have the following result (see [5, Theorems 1.2.38 and 1.2.39]):

Let r be an element of $\widetilde{\mathbb{R}}$ or $\widetilde{\mathbb{C}}$. Then

> r is invertible if and only if
> there exists some representative $(r_\varepsilon)_\varepsilon$ and an $m \in \mathbb{N}$ with $|r_\varepsilon| \geq \varepsilon^m$ for sufficiently small $\varepsilon > 0$, if and only if
> r is not a zero divisor.

Concerning invertibility of Colombeau generalized functions, we may state (see [5, Theorem 1.2.5]):

Let $u \in \mathcal{G}(\Omega)$. Then

> u possesses a multiplicative inverse if and only if
> there exists some representative $(u_\varepsilon)_\varepsilon$ such that for every compact set $K \subset \Omega$, there is $m \in \mathbb{N}$ with $\inf_{x\in K} |u_\varepsilon(x)| \geq \varepsilon^m$ for sufficiently small $\varepsilon > 0$.

In order to be able to speak about symbols of differential operators, we shall need the notion of a polynomial with generalized coefficients. The most straightforward definition is to consider a generalized polynomial of degree m as a member

$$\sum_{|\gamma|\leq m} a_\gamma \xi^\gamma \in \mathcal{G}_m[\xi]$$

of the space of polynomials of degree m in the indeterminate $\xi = (\xi_1, \ldots, \xi_n)$, with coefficients in $\mathcal{G} = \mathcal{G}(\Omega)$. Alternatively, we can and will view $\mathcal{G}_m[\xi]$ as the factor space

$$\mathcal{G}_m[\xi] = \mathcal{E}_{\mathrm{M},m}[\xi]/\mathcal{N}_m[\xi] \tag{12}$$

of families of polynomials of degree m with moderate coefficients modulo those with null coefficients. In this interpretation, generalized polynomials $P(x, \xi)$ are represented by families

$$(P_\varepsilon(x,\xi))_\varepsilon = \Big(\sum_{|\gamma|\leq m} a_{\varepsilon\gamma}(x)\xi^\gamma\Big)_\varepsilon .$$

Sometimes it will also be useful to regard polynomials as polynomial functions and hence as elements of $\mathcal{G}(\Omega \times \mathbb{R}^n)$. Important special cases are the polynomials with regular coefficients, $\mathcal{G}_m^\infty[\xi]$, and with constant generalized coefficients, $\widetilde{\mathbb{C}}_m[\xi]$. The union of the spaces of polynomials of degree m are the rings of polynomials

$\mathcal{G}[\xi], \mathcal{G}^\infty[\xi]$, and $\widetilde{\mathbb{C}}[\xi]$. Letting $D = (-i\partial_1, \ldots, -i\partial_n)$, a differential operator $P(x,D)$ with coefficients in $\mathcal{G}(\Omega)$ is simply an element of $\mathcal{G}[D]$.

1.2.2. *Regularity of Colombeau functions.* We recall a few notions and present a new result about regularity of Colombeau generalized functions. In this setting, the notion is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \Subset \Omega \, \exists p \geq 0 \, \forall \alpha \in \mathbb{N}_0^n : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \text{ as } \varepsilon \to 0 . \tag{13}$$

Observe the change of quantifiers with respect to formula (9); locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [17, Theorem 25.2])

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathrm{C}^\infty(\Omega) .$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathrm{C}^\infty(\Omega)$ does in the setting of distributions.

The concept of *microlocal regularity* of a Colombeau function follows the classical idea of employing additional spectral information on the singularity from the (Fourier) frequency domain (cf. [9, 11, 15]). It refines $\mathcal{G}^\infty$-regularity in the sense that the projection of the (generalized) wave front set into the base space equals the (generalized) singular support. We recall the definition of the generalized wave front set. First, $u \in \mathcal{G}(\Omega)$ is said to be *microlocally regular* at $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ if (for a representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{\mathrm{M}}(\Omega)$) there is an open neighborhood U of x_0 and a conic neighborhood Γ of ξ_0 such that for all $\varphi \in \mathcal{D}(U)$ we have that $\mathcal{F}(\varphi u)$ is rapidly decreasing in Γ; this means that $\exists N \in \mathbb{R} \; \forall l \in \mathbb{N}_0 \; \exists C > 0 \; \exists \varepsilon_0 > 0$:

$$|(\varphi u_\varepsilon)\widehat{\ }(\xi)| \leq C\varepsilon^{-N}(1 + |\xi|)^{-l} \qquad \forall \xi \in \Gamma, \forall \varepsilon \in (0, \varepsilon_0). \tag{14}$$

(We denote by $\widehat{\ }$ the Fourier transform on test functions and by $\mathcal{F}(\varphi u)$ the corresponding generalized Fourier transform of the compactly supported Colombeau function φu.) The *generalized wave front set* of u, denoted by $\mathrm{WF}_g(u)$, is defined as the complement (in $T^*(\Omega) \setminus 0$) of the set of pairs (x_0, ξ_0), where u is microlocally regular.

Let us recall a recently introduced notion which turns out to be crucial in the context of regularity theory (cf. [12]), namely *slow scale nets*. By this, we mean a moderate net of complex numbers $r = (r_\varepsilon)_\varepsilon \in \widetilde{\mathbb{C}}_M$ with the following property: $\forall t \geq 0$, $\exists \varepsilon_t > 0$ such that $|r_\varepsilon|^t \leq \varepsilon^{-1}$ for all $\varepsilon \in (0, \varepsilon_t)$. Equivalently, defining the *order of* r by $\kappa(r) = \sup\{q \in \mathbb{R} : \exists \varepsilon_q \, \exists C_q > 0 : |r_\varepsilon| \leq C_q \varepsilon^q , \forall \varepsilon \in (0, \varepsilon_q)\}$, r is a slow scale net if (and only if) it has order $\kappa(r) \geq 0$. We refer to [12, Section 2] for a detailed discussion of further properties.

An interesting property of the elements of $\mathcal{G}^\infty(\Omega)$ is that their representatives are never bounded, unless all derivatives are slow scale. More precisely, we have the following result; we denote by $\mathcal{E}^\infty_{\mathrm{M}}(\Omega)$ the nets of smooth functions satisfying the estimates (13).

Proposition 1.6. *Let* $(u_\varepsilon)_\varepsilon \in \mathcal{E}^\infty_{\mathrm{M}}(\Omega), K \Subset \Omega$, *and assume that* $\sup_{x \in K} |u_\varepsilon(x)|$ *is slow scale. Then* $\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)|$ *is slow scale for all* $\alpha \in \mathbb{N}_0^n$.

Proof. We can find a bounded open set $\omega \subset \Omega$ which is of cone type and contains K; in fact, we can take ω as a finite union of n-dimensional intervals with positive distance to the boundary of Ω. We use interpolation theory for the Sobolev spaces $H^\ell(\omega)$ to observe that $(k, \ell \geq 0)$

$$\big[L^2(\omega), H^{k+\ell}(\omega)\big]_{\ell/(k+\ell)} = H^\ell(\omega)\,,$$

and for $v \in H^\ell(\omega)$,

$$||v||_{H^\ell} \leq C_{k,\ell}||v||_{L^2}^{1-\ell/(k+\ell)}||v||_{H^{k+\ell}}^{\ell/(k+\ell)}\,;$$

see e.g. [24, 2.4.2, 4.3.1]. In particular,

$$||v||_{H^\ell}^{k+\ell} \leq C_{k,\ell}||v||_{L^2}^{k}||v||_{H^{k+\ell}}^{\ell}\,. \tag{15}$$

Now assume that $\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|$ is not slow scale for some $\alpha, |\alpha| \geq 1$. Then its order is less than zero, so there is $p > 0$ such that $\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| > \varepsilon^{-p}$ for a certain subsequence of $\varepsilon \to 0$. By Sobolev's embedding theorem, it follows that

$$||u_\varepsilon||_{H^\ell(\omega)} > C\varepsilon^{-p}$$

as well, provided $\ell > |\alpha| + n/2$, with some constant $C > 0$. On the other hand,

$$||u_\varepsilon||_{L^2(\omega)} \leq C_\omega||u_\varepsilon||_{L^\infty(\omega)} =: r_\varepsilon$$

with $(r_\varepsilon)_\varepsilon$ slow scale. Inserting this in (15), we get

$$\big(C\varepsilon^{-p}\big)^{k+\ell} < C_{k,\ell}r_\varepsilon^k||u_\varepsilon||_{H^{k+\ell}(\omega)}^{\ell}\,, \forall k \geq 0\,.$$

Thus

$$||u_\varepsilon||_{H^{k+\ell}(\omega)} > D_{k,\ell}\varepsilon^{-p(1+k/\ell)}r_\varepsilon^{-k/\ell},$$

still for a subsequence as $\varepsilon \to 0$. Given $k \geq 0$, we have that $r_\varepsilon^{-k/\ell}\varepsilon^{-p} > 1$ for sufficiently small $\varepsilon > 0$. Thus we get that, given $k \geq 0$,

$$||u_\varepsilon||_{H^{k+\ell}(\omega)} > D_{k,\ell}\varepsilon^{-pk/\ell}$$

for a subsequence of $\varepsilon \to 0$. Since

$$||u_\varepsilon||_{W^{k+\ell,\infty}(\omega)} \geq C_\omega||u_\varepsilon||_{H^{k+\ell}(\omega)}$$

and k is arbitrary, this contradicts $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{\mathrm{M}}^\infty(\Omega)$ and proves the claim. □

Corollary 1.7. *Let $u \in \mathcal{G}^\infty(\Omega), K \Subset \Omega$ and assume that for some representative, $\sup_{x\in K}|u_\varepsilon(x)|$ is bounded as $\varepsilon \to 0$. Then all derivatives of u_ε are slow scale on K.*

In particular, a locally bounded, regular generalized function has the property that none of its derivatives can have a strictly negative order on any compact set.

Example 1.8. Polynomials of degree ≥ 1 in the variable x/ε are elements of $\mathcal{E}_{\mathrm{M}}(\Omega)$ which are not slow scale and attain negative orders in their derivatives. On the other hand, typical examples of bounded generalized functions are provided by regularizations of the Heaviside function: Let $\varphi \in \mathcal{S}(\mathbb{R})$ with integral one and define the mollifier φ_{r_ε} as in (11). Then $\big(H * \varphi_{r_\varepsilon}\big)_\varepsilon$ belongs to $\mathcal{E}_{\mathrm{M}}^\infty(\mathbb{R})$ if and only if $(1/r_\varepsilon)_\varepsilon$ is slow scale.

The focus of the current work is on microlocalization of the notion of hypoellipticity for PDOs with Colombeau functions as coefficients, as it was introduced in [12]. Operators that enjoy the property

$$\big[u \in \mathcal{G}(\Omega)\,,\ f \in \mathcal{G}^\infty(\Omega) \text{ and } P(x,D)u = f \text{ in } \mathcal{G}(\Omega)\big] \Longrightarrow u \in \mathcal{G}^\infty(\Omega)$$

on every open subset $\Omega \subset \mathbb{R}^n$ are called $\mathcal{G}^\infty$-*hypoelliptic*. General results on global elliptic regularity for operators with generalized constant coefficients as well as on microlocal regularity for certain first-order operators were obtained in [12]. It is also a source for a variety of examples illustrating the above as well as related notions. Recent related research in Colombeau regularity theory has shown progress in a diversity of directions, including such topics as pseudodifferential operators with generalized symbols and case studies in microlocal analysis of non-linear singularity propagation [4, 10, 11, 15].

2. The basic scheme: Deducing regularity of the solution from approximative solutions of the adjoint equation

The proof of (2) in [8, Theorem 8.3.1] is based on an idea to use approximate solutions of the adjoint equation (with a particular right-hand side) in deducing regularity of a solution to the original PDE. This approach is elementary in the sense that it does not rely on pseudodifferential operator machinery, though basic techniques naturally appear in it at an embryonic stage. We adapt this procedure to $\mathcal{G}(\Omega)$ in the constructions carried out below. A closely related path was already taken up in [3], and the current exposition partly serves to correct and improve the results stated there.

Informal description of the underlying idea. We briefly sketch the strategy of the "classical" proof, i.e., in the context of operators with smooth coefficients and distributional solutions. Put $f = Pu$ and let $\varphi \in \mathcal{D}$, $\varphi(x_0) = 1$. To deduce that $(x_0, \xi_0) \notin \mathrm{WF}(u)$ one has to estimate

$$(u\varphi)\widehat{\ }(\xi) = \langle u, \varphi\, e^{-i\xi\cdot}\rangle \tag{16}$$

for ξ varying in a conic neighborhood of x_0. If the adjoint equation

$$ {}^tP(\psi\, e^{-i\xi\cdot}) = \varphi\, e^{-i\xi\cdot} \tag{17}$$

were solvable for some $\psi \in \mathcal{D}$ (with $\mathrm{supp}(\psi)$ close to $\mathrm{supp}(\varphi)$), then (16) could be rewritten as

$$\langle u, {}^tP(\psi e^{-i\xi\cdot})\rangle = \langle f, \psi e^{-i\xi\cdot}\rangle = (f\psi)\widehat{\ }(\xi).$$

Under the assumption $(x_0, \xi_0) \notin \mathrm{WF}(f)$ this would prove the claim and establish a relation of the type (2).

Skeleton of the microlocal regularity proof. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ and put $f_\varepsilon = P_\varepsilon u_\varepsilon$, $f = [(f_\varepsilon)_\varepsilon]$. Assume $(x_0, \xi_0) \notin \mathrm{WF}_g(f)$ and let $U \times \Gamma$ be an open conic neighborhood of (x_0, ξ_0) such that for any $\varphi \in \mathcal{D}(U)$ with $\varphi(x_0) = 1$ the Fourier transform $\mathcal{F}(\varphi f)$ is rapidly decreasing in Γ. This means that $\exists M \in \mathbb{N}_0$ $\forall l \in \mathbb{N}_0$ $\exists C > 0$, $0 < \varepsilon_0 < 1$ such that

$$|(\varphi f_\varepsilon)\widehat{\ }(\xi)| \le C\varepsilon^{-M}(1 + |\xi|)^{-l} \qquad \forall \xi \in \Gamma, 0 < \varepsilon < \varepsilon_0. \tag{18}$$

We want to show that

$$(\varphi u_\varepsilon)\widehat{\ }(\xi) = \int u_\varepsilon(x)\varphi(x)e^{-i\xi x}\, dx \tag{19}$$

is rapidly decreasing in some conic neighborhood $\Gamma_0 \subseteq \Gamma$ of ξ_0 when certain conditions on the family of symbols $P_\varepsilon(x, \xi)$ ($\varepsilon \in (0, 1]$) are met in $\mathrm{supp}(\varphi) \times \Gamma$.

Note that it suffices to establish an estimate of the form (18) whenever $|\xi| \ge r_\varepsilon$, where $r_\varepsilon > 0$ is of slow scale and may depend on N (we refer to a corresponding remark in [12, Section 6]).

We postpone the detailed discussion of various conditions on P_ε suiting the same proof skeleton until the following section. For the moment, we will instead state general assumptions tailored directly towards the adjoint operator method and investigate later on how they can be met in certain circumstances.

The following lemma provides the basic algebraic mechanism in the construction of approximate solutions of the adjoint equation. We re-investigate and state the classical computations here in all details in order to prepare for the close examination of the interplay of ξ-order and ε-asymptotics required later on. One may think of the part of the operator A in it to be played by P_ε (or $P_{\varepsilon,m}$) and Q corresponding to the adjoint ${}^tP_\varepsilon$.

Lemma 2.1. *Let $A(x,D)$ and $Q(x,D)$ be partial differential operators of order m on an open subset U of $\mathbb{R}^n$. Assume that $A(x,\xi) \neq 0$ for all $(x,\xi) \in U \times \Gamma$. Then for any $w \in \mathrm{C}^\infty(U)$ the following equation holds (on $U \times \Gamma$):*

$$Q(x,D)\Big(\frac{e^{-i\xi x}w(x)}{A(x,\xi)}\Big) = e^{-i\xi x}\big(w(x) - R(\xi;x,D)w(x)\big), \tag{20}$$

where $R(\xi;x,D) = -\sum_{|\beta|\le m} r_\beta(x,\xi) D_x^\beta$ is a partial differential operator of order at most m with coefficients given by

$$r_0(x,\xi) = \frac{Q(x,-\xi) - A(x,\xi)}{A(x,\xi)} + \sum_{1\le|\gamma|\le m} \frac{1}{\gamma!}\,\partial_\xi^\gamma Q(x,-\xi)\, D_x^\gamma\big(\frac{1}{A(x,\xi)}\big), \tag{21}$$

$$r_\beta(x,\xi) = \sum_{|\gamma|\le m-|\beta|} \frac{1}{\beta!\gamma!}\,\partial_\xi^{\beta+\gamma} Q(x,-\xi)\, D_x^\gamma\big(\frac{1}{A(x,\xi)}\big), \qquad |\beta| \ge 1. \tag{22}$$

Proof. Using the fact that $\partial_\xi^\beta Q(x,D)(e^{-i\xi x}) = e^{-i\xi x}\partial_\xi^\beta Q(x,-\xi)$ and by repeated application of the Hörmander-Leibniz formula ([8, (1.1.10)]) we have

$$\begin{aligned}
Q(x,D)\Big(\frac{e^{-i\xi x}w(x)}{A(x,\xi)}\Big) &= \sum_{|\beta|\le m} \frac{1}{\beta!}\,\partial_\xi^\beta Q(x,D)\big(\frac{e^{-i\xi x}}{A(x,\xi)}\big)\, D_x^\beta w(x) \\
&= Q(x,D)\big(\frac{e^{-i\xi x}}{A(x,\xi)}\big)\, w(x) + \sum_{1\le|\beta|\le m} \frac{1}{\beta!}\,\partial_\xi^\beta Q(x,D)\big(\frac{e^{-i\xi x}}{A(x,\xi)}\big) x\, D_x^\beta w(x) \\
&= e^{-i\xi x}\Big(\sum_{|\gamma|\le m} \frac{1}{\gamma!}\,\partial_\xi^\gamma Q(x,-\xi)\, D_x^\gamma\big(\frac{1}{A(x,\xi)}\big)\, w(x) \\
&\quad + \sum_{1\le|\beta|\le m}\ \sum_{|\gamma|\le m-|\beta|} \frac{1}{\beta!\gamma!}\,\partial_\xi^{\beta+\gamma} Q(x,-\xi)\, D_x^\gamma\big(\frac{1}{A(x,\xi)}\big)\, D_x^\beta w(x)\Big).
\end{aligned}$$

Here, (22) can be read off directly from the second (double) sum and (21) follows by separating the term corresponding to $|\gamma| = 0$, $Q(x,-\xi)/A(x,\xi)$, and adding and subtracting $A(x,\xi)w(x)/A(x,\xi)$. □

Formula (20) is the basis for solving the adjoint equation (17) approximately. For this, we apply the lemma to $A(x,D) = P_\varepsilon(x,D)$ and $Q(x,D) = {}^tP_\varepsilon(x,D)$, where ε is fixed but arbitrary in some interval $(0,\varepsilon_0)$, $x \in U \subseteq \Omega$ open, independent of ε, and $\xi \in \Gamma_\varepsilon = \Gamma \cap \{|\eta| \ge r_\varepsilon\}$, where the cone Γ is independent of ε and r_ε is of slow scale.

This defines a corresponding family $R_\varepsilon(\xi;x,D)$ ($0 < \varepsilon < \varepsilon_0$, $\xi \in \Gamma_\varepsilon$) of differential operators on U, with coefficients given by corresponding parametrized versions of (21)-(22). Observe that $Q_m(x,-\xi) = P_{\varepsilon,m}(x,\xi)$ and hence $Q(x,-\xi) - A(x,\xi) = {}^tP_\varepsilon(x,-\xi) - P_\varepsilon(x,\xi)$ is a polynomial of order at most $m-1$ with respect to ξ.

The second ingredient is the choice of w, which has to be linked with φ and will also depend on the parameters ξ and ε, as well as on an approximation order $N \in \mathbb{N}$. We express this in the notation $w_\varepsilon^N(x,\xi)$. If we define

$$w_\varepsilon^N(x,\xi) = \sum_{k=0}^{N-1} R_\varepsilon(\xi;x,D)^k \varphi(x), \tag{23}$$

then the expression $w_\varepsilon^N - R_\varepsilon w_\varepsilon^N$ appearing on the right-hand side of (20) is a telescope sum and reduces to $\varphi - R_\varepsilon^N \varphi$. Therefore, equation (20) yields in this case

$${}^tP_\varepsilon(x,D)\big(\frac{w_\varepsilon^N(x,\xi)}{P_\varepsilon(x,\xi)} e^{-i\xi x}\big) = e^{-i\xi x} \cdot \varphi(x) - e^{-i\xi x} \cdot R_\varepsilon(\xi;x,D)^N \varphi(x) \tag{24}$$

which is as close as we get to the informal requirement of (17).

Equation (24) suggests that

$$\psi_\varepsilon^N := w_\varepsilon^N / P_\varepsilon \tag{25}$$

will give a reasonable approximate solution in terms of decrease properties with respect to ξ (while keeping control over the ε-growth), if the operator family $(R_\varepsilon)_\varepsilon$ satisfies corresponding estimates. Note that each $R_\varepsilon(\xi;x,D)$ is a differential operator of order at most m with coefficients depending smoothly on $x \in U$ and being rational functions of $\xi \in \Gamma$.

Assumption 1. There is $M_1 \in \mathbb{N}_0$ and $\tau > 0$ with the property that $\forall N \in \mathbb{N}$ $\exists C > 0$, $0 < \varepsilon_1 \le \varepsilon_0$ and $\exists r_\varepsilon > 0$ of slow scale such that

$$|R_\varepsilon(\xi;x,D)^N \varphi(x)| \le C\varepsilon^{-M_1}(1+|\xi|)^{-N\tau} \tag{26}$$

for all $x \in \mathrm{supp}(\varphi)$, $\xi \in \Gamma$ with $|\xi| \ge r_\varepsilon$, $\varepsilon \in (0,\varepsilon_1)$.

We deduce from (19) and (24) that

$$\begin{aligned}(\varphi u_\varepsilon)\widehat{\ }(\xi) = \int & P_\varepsilon(x,D)u_\varepsilon(x) \cdot \psi_\varepsilon^N(x,\xi)\, e^{-i\xi x}\, dx \\ & + \int u_\varepsilon(x) e^{-i\xi x} \cdot R_\varepsilon(\xi;x,D)^N \varphi(x)\, dx =: J_\varepsilon^N(\xi) + I_\varepsilon^N(\xi).\end{aligned} \tag{27}$$

Lemma 2.2. *If $(R_\varepsilon)_\varepsilon$ satisfies Assumption 1, then there exists $N_1 \in \mathbb{N}_0$ such that for all N the integral $I_\varepsilon^N(\xi)$ satisfies the following estimate: $\exists C > 0$ $\exists \varepsilon_2 > 0$*

$$|I_\varepsilon^N(\xi)| \le C\varepsilon^{-N_1}\,(1+|\xi|)^{-N\tau} \tag{28}$$

for all $x \in U$, $\xi \in \Gamma$ with $|\xi| \ge r_\varepsilon$, $0 < \varepsilon < \varepsilon_2$.

Proof. Let $p \in \mathbb{N}_0$ and $\varepsilon_2 \le \varepsilon_1$ be sufficiently small so that $|u_\varepsilon(x)| \le \varepsilon^{-p}$ uniformly for $x \in \mathrm{supp}(\varphi)$ when $0 < \varepsilon < \varepsilon_2$. Then

$$|I_\varepsilon^N(\xi)| \le C_\varphi\, \varepsilon^{-M_1-p}\,(1+|\xi|)^{-N\tau} \tag{29}$$

for all $x \in U$, $\xi \in \Gamma$ with $|\xi| \ge r_\varepsilon$, $0 < \varepsilon < \varepsilon_2$, where C_φ is the product of the constant in (26) and the measure of $\mathrm{supp}(\varphi)$. ∎

Note that since N_1 is independent of $N \in \mathbb{N}$, (28) can be used in proving rapid decrease in (27) once J_ε^N was shown to be rapidly decreasing. We observe that

$$J_\varepsilon^N(\xi) = \int f_\varepsilon(x)\, \psi_\varepsilon^N(x,\xi)\, e^{-i\xi x}\, dx = \mathcal{F}_{x\to\nu}\big(f_\varepsilon(x)\psi_\varepsilon^N(x,\xi)\big)\,|_{\nu=\xi}, \tag{30}$$

where the notation $\mathcal{F}_{x\to\nu}(\ldots)\,|_{\nu=\xi}$ emphasizes that the Fourier transform (in the x variable) is carried out at fixed parameter values ξ, ε, and N, of its functional argument and then evaluated at Fourier variable ν set equal to the parameter ξ.

Intuitively, rapid decrease of $J_\varepsilon^N(\xi)$ would follow if we could replace the family $\psi_\varepsilon^N(.,\xi)$ by a single test function ψ with $\psi(x_0) = 1$. Note that for all $\varepsilon \in (0,1]$, $\xi \in \Gamma$, $|\xi| \geq r_\varepsilon$, and $N \in \mathbb{N}$,

$$\operatorname{supp}(\psi_\varepsilon^N(.,\xi)) \subseteq \operatorname{supp}(\varphi). \tag{31}$$

The following condition specifies a regularity property of the family $\psi_\varepsilon^N(.,\xi)$ which will finally yield rapid decrease of $J_\varepsilon^N(\xi)$.

Assumption 2. The family $\psi_\varepsilon^N(.,\xi)$ ($\xi \in \Gamma$, $|\xi| \geq r_\varepsilon$, $N \in \mathbb{N}$, $\varepsilon \in (0,1]$) satisfies the following regularity condition: there is $M \in \mathbb{N}_0$, $0 \leq \delta < 1$, and $\tau_0 \in \mathbb{R}$ with the property that $\forall \alpha \in \mathbb{N}_0^n$ $\forall N \in \mathbb{N}$ $\exists C > 0, 0 < \varepsilon_0 < 1$ and $\exists r_\varepsilon > 0$ of slow scale such that

$$|\partial_x^\alpha \psi_\varepsilon^N(x,\xi)| \leq C\varepsilon^{-M}\,(1+|\xi|)^{\delta|\alpha|+\tau_0} \tag{32}$$

for all $x \in \operatorname{supp}(\varphi)$, $\xi \in \Gamma$ with $|\xi| \geq r_\varepsilon$, $\varepsilon \in (0,\varepsilon_0)$.

Lemma 2.3. *Under Assumption* 2 *let* $\Gamma_0 \subset \Gamma \cup \{0\}$ *be a closed conic neighborhood of* ξ_0. *Then* $J_\varepsilon^N(\xi)$ *is rapidly decreasing when* $\xi \in \Gamma_0$.

Proof. (This is a modified variant of a similar proof in [12], the main difference being that in the present case the coupling of the parameters ξ and ε cannot be compensated for simply by homogeneity arguments.) Let $\chi \in \mathcal{D}(U)$ such that $\chi = 1$ on $\operatorname{supp}(\varphi) \supseteq \operatorname{supp}(\psi_\varepsilon^N)$ and let $\xi \in \Gamma$, $|\xi| \geq r_\varepsilon$. We have

$$|J_\varepsilon^N(\xi)| = |\mathcal{F}_{x\to\nu}\big((\chi f_\varepsilon)(x)\,\psi_\varepsilon^N(x,\xi)\big)\,|_{\nu=\xi} = \frac{1}{(2\pi)^n}|\big(g_\varepsilon^N(.,\xi) * \widehat{(\chi f_\varepsilon)}\big)(\xi)|,$$

where $g_\varepsilon^N(\eta,\xi) := \mathcal{F}(\psi_\varepsilon^N(.,\xi))(\eta)$. This implies

$$\begin{aligned}(2\pi)^n|J_\varepsilon^N(\xi)| &\leq \int |g_\varepsilon^N(\xi-\eta,\xi)||\widehat{(\chi f_\varepsilon)}(\eta)|\,d\eta \\ &= \int_{\eta\in\Gamma} |g_\varepsilon^N(\xi-\eta,\xi)||\widehat{(\chi f_\varepsilon)}(\eta)|\,d\eta + \int_{\eta\in\Gamma^c} |g_\varepsilon^N(\xi-\eta,\xi)||\widehat{(\chi f_\varepsilon)}(\eta)|\,d\eta \\ &=: J_{1,\varepsilon}^N(\xi) + J_{2,\varepsilon}^N(\xi).\end{aligned}$$

Using Assumption 2 we derive estimates on $|g_\varepsilon^N(\zeta,\xi)|$ as follows. For $\alpha \in \mathbb{N}_0^n$ arbitrary we have

$$\begin{aligned}|\zeta^\alpha g_\varepsilon^N(\zeta,\xi)| &= |\mathcal{F}(D_x^\alpha \psi_\varepsilon^N(.,\xi))(\zeta)| \\ &\leq \|D_x^\alpha \psi_\varepsilon^N(.,\xi)\|_{\mathrm{L}^1} \leq C_\varphi \|D_x^\alpha \psi_\varepsilon^N(.,\xi)\|_{\mathrm{L}^\infty} \leq CC_\varphi \varepsilon^{-M}(1+|\xi|)^{\delta|\alpha|+\tau_0},\end{aligned}$$

where M is independent of α, $\varepsilon < \varepsilon_1$ as in (32), and $\zeta \in \mathbb{R}^n$ arbitrary. Hence we have shown that for all $l \in \mathbb{N}_0$ $\exists C > 0$ $\exists \varepsilon_1 > 0$ such that

$$|g_\varepsilon^N(\zeta,\xi)| \leq C\varepsilon^{-M}(1+|\zeta|)^{-l}(1+|\xi|)^{\delta l+\tau_0} \tag{33}$$

for all $\zeta \in \mathbb{R}^n$, $\varepsilon < \varepsilon_1$.

In estimating the integrand in $J^N_{1,\varepsilon}(\xi)$ we use (33), the fact that $\widehat{(\chi f_\varepsilon)}$ is rapidly decreasing in Γ, and apply Peetre's inequality to obtain the following: there is M_1, M_2 such that $\forall l, k \in \mathbb{N}_0$

$$\begin{aligned}
&|g^N_\varepsilon(\xi-\eta,\xi)||\widehat{(\chi f_\varepsilon)}(\eta)| \\
&\qquad \le C_1\varepsilon^{-M_1}(1+|\xi|^2)^{(\tau_0+\delta k)/2}(1+|\xi-\eta|^2)^{-k/2}\varepsilon^{-M_2}(1+|\eta|^2)^{-l/2} \\
&\qquad\qquad \le C'\varepsilon^{-M_1-M_2}(1+|\xi|^2)^{(\tau_0-(1-\delta)k)/2}(1+|\eta|^2)^{(k-l)/2}
\end{aligned}$$

for suitable constants, ε sufficiently small, and $\eta \in \Gamma$. If we require $l > k+n$, then we may conclude that for arbitrary $k \in \mathbb{N}_0$

$$J^N_{1,\varepsilon}(\xi) \le C\varepsilon^{-M_1-M_2}(1+|\xi|)^{-(1-\delta)k+\tau_0},$$

where $M_1 + M_2$ is independent of k and ε sufficiently small.

For a similar estimate of $J^N_{2,\varepsilon}(\xi)$ we first note that $\widehat{(\chi f_\varepsilon)}(\eta)$ is temperate in the following sense. There is M_3 and $C > 0$, $\varepsilon_3 > 0$ such that

$$|\widehat{(\chi f_\varepsilon)}(\eta)| \le C\varepsilon^{-M_3}(1+|\eta|^2)^{M_3/2}$$

for all $\eta \in \mathbb{R}^n$ and $0 < \varepsilon < \varepsilon_3$. Applying this and (33), with $l+k$ instead of l, we obtain the following bound on the integrand in $J^N_{2,\varepsilon}(\xi)$:

$$\begin{aligned}
&|g^N_\varepsilon(\xi,\xi-\eta)||\widehat{(\chi f_\varepsilon)}(\eta)| \\
&\le C\varepsilon^{-M_1-M_3}(1+|\xi|^2)^{(\tau_0+\delta(k+l))/2}(1+|\xi-\eta|^2)^{-k/2}(1+|\xi-\eta|^2)^{-l/2}(1+|\eta|^2)^{M_3/2}.
\end{aligned}$$

By Peetre's inequality, $(1+|\xi-\eta|^2)^{-k/2} \le 2^k(1+|\xi|^2)^{-k/2}(1+|\eta|^2)^{k/2}$. Furthermore, one can find $d > 0$ (resp. $d' > 0$) such that $\xi \in \Gamma_0$ and $\eta \in \Gamma^c$ implies $|\xi-\eta| \ge d|\eta|$ (resp. $|\xi-\eta| \ge d'|\xi|$) (cf. [8, proof of Lemma 8.1.1]). Therefore, we can write $(1+|\xi|^2)^{\delta l/2} \le C'(1+|\xi-\eta|^2)^{\delta l/2}$ and $(1+|\xi-\eta|^2)^{-(1-\delta)l/2} \le C(1+|\eta|^2)^{-(1-\delta)l/2}$, showing that the integrand is bounded by some constant times

$$\varepsilon^{-M_1-M_3}(1+|\xi|)^{\tau_0-(1-\delta)k}(1+|\eta|)^{k+M_3-(1-\delta)l}$$

with M_1 and M_3 independent of k, l and ε sufficiently small. Requiring $(1-\delta)l > k+M_3+n$ yields

$$J^N_{2,\varepsilon}(\xi) \le C\varepsilon^{-M_1-M_3}(1+|\xi|)^{\tau_0-(1-\delta)k}.$$

Hence we have proved rapid decrease of $J^N_\varepsilon(\xi)$. □

To summarize the preceding discussion, Lemmas 2.2 and 2.3 imply the following result.

Proposition 2.4. *Let $P(x,D)$ be a partial differential operator with coefficients in $\mathcal{G}(\Omega)$, represented by the family $(P_\varepsilon(x,D))_\varepsilon$, and $(R_\varepsilon)_\varepsilon$ be constructed according to Lemma* 2.1 *(with $A = P_\varepsilon$ and $Q = {}^tP_\varepsilon$). Let $u \in \mathcal{G}(\Omega)$ and assume that $(x_0,\xi_0) \notin \mathrm{WF}_g(Pu)$ with $U \times \Gamma$, φ as in* (18)*. Let $(\psi^N_\varepsilon)_{\varepsilon,N}$ be defined by* (25)*. If $(R_\varepsilon)_\varepsilon$ and $(\psi^N_\varepsilon)_{\varepsilon,N}$ satisfy Assumptions* 1 *and* 2*, then $(x_0,\xi_0) \notin \mathrm{WF}_g(u)$.*

In the remainder of this paper we investigate various possibilities of conditions on the operator family $(P_\varepsilon)_\varepsilon$, or its coefficients, such that the crucial Assumptions 1 and 2 are guaranteed. In all these cases Proposition 2.4 will allow us to deduce microlocal regularity properties of a Colombeau solution to the equation $Pu = f$.

3. Microlocal hypoellipticity conditions

Throughout this section let $(P_\varepsilon)_\varepsilon$ be a family of linear partial differential operators whose coefficients, $(a^\varepsilon_\alpha)_\varepsilon$, are representatives of generalized functions in $\mathcal{G}(\Omega)$. Denote by P the corresponding operator on $\mathcal{G}(\Omega)$, mapping $[(u_\varepsilon)_\varepsilon]$ into $[(P_\varepsilon u_\varepsilon)_\varepsilon]$.

Lemma 3.1. *Let P have coefficients in $\mathcal{G}^\infty(\Omega)$. Then for any $u \in \mathcal{G}(\Omega)$*

$$\mathrm{WF}_g(Pu) \subseteq \mathrm{WF}_g(u). \tag{34}$$

Proof. $\mathrm{WF}_g(D^\alpha u) \subset \mathrm{WF}_g(u)$ is clear from the properties of the Fourier transform. Furthermore, if $a \in \mathcal{G}^\infty$, then $\mathrm{WF}_g(au) \subseteq \mathrm{WF}_g(u)$ is a special case of [11, Theorem 3.1]. □

Theorem 3.2. *Let P be a partial differential operator of order m with coefficients in $\mathcal{G}(\Omega)$. Let $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ with an open conic neighborhood $U \times \Gamma$ and $m_0 \in \mathbb{R}$, $0 \le \delta < \rho \le 1$, such that the following hypotheses are satisfied for any compact subset $K \Subset U$:*

(i) *$\exists q > 0$ $\exists r_\varepsilon > 0$ of slow scale and $\exists \varepsilon_0 > 0$ such that*

$$|P_\varepsilon(x,\xi)| \ge \varepsilon^q\,(1 + |\xi|)^{m_0} \tag{35}$$

for all $(x,\xi) \in K \times \Gamma$, $|\xi| \ge r_\varepsilon$, and $0 < \varepsilon < \varepsilon_0$.

(ii) *$\forall \alpha \in \mathbb{N}_0^n$ $\exists s^\alpha_\varepsilon, r^\alpha_\varepsilon > 0$ of slow scale and $\varepsilon_\alpha > 0$ such that for all $\beta \in \mathbb{N}_0^n$ with $0 \le |\beta| \le m$*

$$|\partial^\alpha_x \partial^\beta_\xi P_\varepsilon(x,\xi)| \le s^\alpha_\varepsilon\,|P_\varepsilon(x,\xi)|\,(1 + |\xi|)^{\delta|\alpha| - \rho|\beta|} \tag{36}$$

for all $(x,\xi) \in K \times \Gamma$, $|\xi| \ge r^\alpha_\varepsilon$, and $0 < \varepsilon < \varepsilon_\alpha$.

Then we have, with $\Lambda = U \times \Gamma$, for any $u \in \mathcal{G}(\Omega)$

$$\Lambda \cap \mathrm{WF}_g(u) = \Lambda \cap \mathrm{WF}_g(Pu). \tag{37}$$

Remark 3.3. Note that condition (36) implies that all coefficients of P are $\mathcal{G}^\infty$ on U. Indeed, by the moderateness of $P_\varepsilon(x,\xi)$ and the fact that $s^\alpha_\varepsilon \le \varepsilon^{-1}$, one obtains uniform ε-growth for all derivatives of P_ε. Then one makes use of the polynomial structure with respect to ξ to first extract each highest order coefficient separately (i.e., $|\beta| = m$) and directly deduces its regularity. Finally proceeding successively to lower orders, i.e., $|\beta| = m - 1$ and so on, each of the coefficients appears as the only one of the current order with additional linear combinations of higher order coefficients. Thus the regularity follows.

The following statement is an immediate consequence, restating (37) as an inclusion relation.

Corollary 3.4. *Let $M_g(P)$ be the union of all open conic subsets $\Lambda \subseteq T^*(\Omega) \setminus 0$ where P satisfies* (35)-(36). *Then the following inclusion relation holds for any generalized function $u \in \mathcal{G}(\Omega)$:*

$$\mathrm{WF}_g(u) \subseteq \mathrm{WF}_g(Pu) \cup M_g(P)^c. \tag{38}$$

Proof of Theorem 3.2. We have to show that the families of operators R_ε and functions $\psi^N_\varepsilon(.,\xi)$ constructed in the previous section satisfy Assumptions 1 and 2. Then the assertion follows from Proposition 2.4. Note that (35) guarantees that $P_\varepsilon(x,\xi)$ is staying away from zero in the regions considered and hence the constructions according to Lemma 2.1 are well defined.

Equations (21)-(22) define the coefficients, r_0^ε and r_β^ε, of R_ε and show that we have to give appropriate bounds of the generic factors which appear in the coefficients of powers of the operator R_ε:

$$\partial_x^\alpha\Big(\frac{{}^tP_\varepsilon(x,-\xi)-P_\varepsilon(x,\xi)}{P_\varepsilon(x,\xi)}\Big), \tag{39}$$

$$\partial_x^\alpha\Big(\partial_\xi^{\gamma+\beta}\big({}^tP_\varepsilon(x,-\xi)\big)D_x^\gamma\big(\frac{1}{P_\varepsilon(x,\xi)}\big)\Big). \tag{40}$$

Step 1: For each $K \Subset \Omega$ and $\alpha,\beta,\gamma \in \mathbb{N}_0^n$ arbitrary $\exists S_\varepsilon^{\alpha,\gamma+\beta}, p_\varepsilon^{\alpha,\gamma+\beta} > 0$ of slow scale and $\mu_{\alpha,\gamma+\beta} > 0$ such that

$$|\partial_x^\alpha\partial_\xi^{\gamma+\beta}\big({}^tP_\varepsilon(x,-\xi)\big)| \leq S_\varepsilon^{\alpha,\gamma+\beta}\,|P_\varepsilon(x,\xi)|\,(1+|\xi|)^{-\rho|\gamma+\beta|+\delta|\alpha|} \tag{41}$$

for all $(x,\xi) \in K\times\Gamma$, $|\xi| \geq p_\varepsilon^{\alpha,\gamma+\beta}$, $0<\varepsilon<\mu_{\alpha,\gamma+\beta}$.

To see this we use the symbol expansion

$$ {}^tP_\varepsilon(x,\eta) = \sum_{0\leq|\sigma|\leq m}\frac{(-1)^{|\sigma|}}{\sigma!}\big(\partial_\xi^\sigma\partial_x^\sigma P_\varepsilon\big)(x,-\eta)$$

which gives

$$\partial_x^\alpha\partial_\xi^{\gamma+\beta}\big({}^tP_\varepsilon(x,-\xi)\big) = \sum_{0\leq|\sigma|\leq m-|\gamma+\beta|}\frac{(-1)^{|\sigma|}}{\sigma!}\big(\partial_\xi^{\sigma+\gamma+\beta}\partial_x^{\sigma+\alpha}P_\varepsilon\big)(x,\xi).$$

Applying (36) term by term and choosing a slow scale net $S_\varepsilon^{\alpha,\gamma+\beta}$ which dominates all appearing constants and slow scale factors, as well as choosing $p_\varepsilon^{\alpha,\gamma+\beta}$ to be the maximum of the occurring radii $r_\varepsilon^{\sigma+\alpha}$, the assertion (41) is immediate.

Step 2: For each $K \Subset \Omega$ and $\lambda \in \mathbb{N}_0^n$ arbitrary $\exists T_\varepsilon^\lambda, t_\varepsilon^\lambda > 0$ of slow scale and $\nu_\lambda > 0$ such that

$$|D_x^\lambda\big(\frac{1}{P_\varepsilon(x,\xi)}\big)| \leq T_\varepsilon^\lambda\,|\frac{1}{P_\varepsilon(x,\xi)}|\,(1+|\xi|)^{\delta|\lambda|} \tag{42}$$

for all $(x,\xi) \in K\times\Gamma$, $|\xi| \geq t_\varepsilon^\lambda$, $0<\varepsilon<\nu_\lambda$.

The assertion is trivial if $|\lambda|=0$, so we assume $|\lambda|\geq 1$ and proceed by induction. Differentiating the equality $1 = P_\varepsilon/P_\varepsilon$ we obtain, by Leibniz' rule,

$$0 = P_\varepsilon\cdot\partial_x^\lambda\big(\frac{1}{P_\varepsilon}\big) + \sum_{\substack{\sigma\leq\lambda\\ 0\leq|\sigma|<|\lambda|}}\binom{\lambda}{\sigma}\partial_x^{\lambda-\sigma}P_\varepsilon\cdot\partial_x^\sigma\big(\frac{1}{P_\varepsilon}\big).$$

Here, (42) is applicable to each term in the sum over σ and combination with (36) yields

$$|D_x^\lambda\big(\frac{1}{P_\varepsilon(x,\xi)}\big)| \leq \sum_{\substack{\sigma\leq\lambda\\ 0\leq|\sigma|<|\lambda|}}\binom{\lambda}{\sigma}s_\varepsilon^{\lambda-\sigma}T_\varepsilon^\sigma\,(1+|\xi|)^{\delta(|\lambda-\sigma|+|\sigma|)}$$

when $(x,\xi) \in K\times\Gamma$, $|\xi| \geq \max_{\lambda-\sigma}(r_\varepsilon^{\lambda-\sigma})$, and ε sufficiently small. From this we see that T_ε^λ, t_ε^λ, and ν_λ can be chosen appropriately under a finite number of conditions so that (42) can be satisfied.

Step 3: For each $K \Subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ $\exists c_\varepsilon^\alpha, a_\varepsilon^\alpha > 0$ of slow scale and $\mu_\alpha > 0$ such that for all $1\leq|\beta|\leq m$

$$|\partial_x^\alpha r_\beta^\varepsilon(x,\xi)| \leq c_\varepsilon^\alpha\,(1+|\xi|)^{-\rho|\beta|+\delta|\alpha|} \tag{43}$$

for all $(x,\xi) \in K\times\Gamma$, $|\xi| \geq a_\varepsilon^\alpha$, $0<\varepsilon<\mu_\alpha$.

We have to estimate terms (40) when $\beta \neq 0$, which according to Leibniz' rule are linear combinations of terms ($\sigma \leq \alpha$)

$$\partial_x^\sigma \partial_\xi^{\gamma+\beta}\big({}^tP_\varepsilon(x,-\xi)\big) \cdot D_x^{\alpha-\sigma+\gamma}\big(\frac{1}{P_\varepsilon(x,\xi)}\big).$$

Combining (41) and (42) gives upper bounds, for $|\xi|$ larger than some slow scale radius, of the form of some slow scale net times $(1+|\xi|)^{-\rho|\gamma+\beta|+\delta|\sigma|+\delta|\alpha-\sigma+\gamma|}$ which has exponent $-(\rho-\delta)|\gamma|-\rho|\beta|+\delta|\alpha|$ and proves the assertion since $\rho > \delta$. (The appropriate slow scale nets are chosen, for each α, subject to finitely many conditions, and this may be done uniformly over $1 \leq |\beta| \leq m$.)

Step 4: For each $K \Subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ $\exists d_\varepsilon^\alpha, b_\varepsilon^\alpha > 0$ of slow scale and $\nu_\alpha > 0$ such that

$$|\partial_x^\alpha r_0^\varepsilon(x,\xi)| \leq d_\varepsilon^\alpha\,(1+|\xi|)^{-(\rho-\delta)+\delta|\alpha|} \tag{44}$$

for all $(x,\xi) \in K \times \Gamma$, $|\xi| \geq b_\varepsilon^\alpha$, $0 < \varepsilon < \nu_\alpha$.

We have to find bounds on (40) when $\beta = 0$ but $|\gamma| \geq 1$ and on (39). The first case is done as in Step 3 and yields as upper bound a slow scale net times $(1+|\xi|)^{-(\rho-\delta)+\delta|\alpha|}$ since now $|\gamma| \geq 1$. The term (39) is a linear combination of terms ($\lambda \leq \alpha$)

$$\partial_x^\lambda\big({}^tP_\varepsilon(x,-\xi) - P_\varepsilon(x,\xi)\big) \cdot D_x^{\alpha-\lambda}\big(\frac{1}{P_\varepsilon(x,\xi)}\big).$$

Here, a bound on the second factor has the usual slow scale data coming with $(1+|\xi|)^{\delta|\alpha-\lambda|}/|P_\varepsilon(x,\xi)|$ according to (42). By the symbol expansion of ${}^tP_\varepsilon(x,-\xi)$ the factor on the left is seen to be a linear combination of the following terms with $|\sigma| \geq 1$

$$\partial_\xi^\sigma \partial_x^{\lambda+\sigma} P_\varepsilon(x,\xi).$$

We obtain upper bounds with slow scale radii and factors times

$$|P_\varepsilon(x,\xi)|(1+|\xi|)^{-(\rho-\delta)|\sigma|+\delta|\lambda|}.$$

Hence this yields, apart from similar slow scale data, a common bound

$$(1+|\xi|)^{-(\rho-\delta)+\delta|\lambda|}$$

for the first factor in the product above since $|\sigma| \geq 1$. Multiplication of the bounds on both factors finally gives the asserted upper bound. (We note once more that all required slow scale nets can be chosen subject to finitely many conditions at fixed α.)

Step 5: $R_\varepsilon(\xi; x, D)$ satisfies Assumption 1 with $\tau = \rho - \delta$ and $M_1 = 1$.

We prove this by induction. If $N = 1$ it follows directly from (43) and (44) (set $\alpha = 0$ in both equations) that, with some slow scale net s_ε^0,

$$|R_\varepsilon(\xi; x, D)\varphi(x)| \leq s_\varepsilon^0\,(1+|\xi|)^{-(\rho-\delta)}$$

when $(x,\xi) \in \operatorname{supp}(\varphi) \times \Gamma$, $|\xi|$ larger than some slow scale radius, and ε sufficiently small. Assume that for $j = 1, \ldots, N$ we have, with some slow scale nets s_ε^j, the induction hypothesis

$$|R_\varepsilon(\xi; x, D)^j \varphi(x)| \leq s_\varepsilon^j\,(1+|\xi|)^{-j\tau} \tag{45}$$

under similar conditions as above on x, ξ, and ε. We let a term $r_\beta^\varepsilon D_x^\beta$ $(0 \leq |\beta| \leq m)$ act on $R_\varepsilon^N \varphi$ from the left. Any derivative D_x^λ, $\lambda \leq \beta$, falling on derivatives of r_σ^ε or φ, raises, according to (43) and (44), an overall upper bound at most by some slow scale factor times $(1+|\xi|)^{\delta|\beta|}$. Again by (43) and (44), the additional factor r_β^ε then brings in another slow scale net times $(1+|\xi|)^{-\rho|\beta|}$, if $\beta \neq 0$, or $(1+|\xi|)^{-(\rho-\delta)}$, if $\beta = 0$. In any case, the new ξ-exponents, added at this stage, sum up to at least $-(\rho-\delta)$ and all slow scale factors and radii can be chosen subject to finitely many conditions when $N+1$ is fixed. Combining this with the bounds on the terms in $R^j_\varepsilon\varphi$ we obtain estimate (45) with $N+1$ instead of j. In particular, we observe that all appearing slow scale factors can be compensated for by $1/\varepsilon$, yielding the assertion.

Step 6: $\psi^N_\varepsilon(x,\xi)$ satisfies Assumption 2 with $M = q+1$, δ from (36), and $\tau_0 = -m_0 - \tau$.

The case $N = 1$ is trivial, hence we assume $N \geq 2$. Recalling (25) we rewrite $\partial^\alpha_x(\psi^N_\varepsilon(x,\xi))$ as linear combination of the following terms ($\sigma \leq \alpha$)

$$\partial^\sigma_x(w^N_\varepsilon(x,\xi)) \cdot \partial^{\alpha-\sigma}_x(\frac{1}{P_\varepsilon(x,\xi)}).$$

Thanks to (42) and (35) the second factor has a bound $\varepsilon^{-q}T^{\alpha-\sigma}_\varepsilon(1+|\xi|)^{\delta(|\alpha-\sigma|)-m_0}$, on the usual domains for x and ξ when ε is small.

Considering (23) we can argue in a similar way as in Step 5 that ∂^σ_x acting on any term $R^k_\varepsilon\varphi$ ($k = 1,\dots,N$) raises the ξ-power in its overall upper bound by at most $\delta|\sigma|$. Together with the bound (from (45)) of the form slow scale times $(1+|\xi|)^{-\tau}$ of $\sum_{1\leq k\leq N}|R^k_\varepsilon\varphi|$ we obtain all in all the upper bound, for some slow scale net s^α_ε,

$$|\partial^\alpha_x(\psi^N_\varepsilon(x,\xi))| \leq \varepsilon^{-q}s^\alpha_\varepsilon(1+|\xi|)^{-m_0-(N-1)\tau+\delta|\alpha|}$$

when $x \in \mathrm{supp}(\varphi)$, $\xi \in \Gamma$ with $|\xi|$ above some slow scale radius, and ε sufficiently small. (Here, the choices of appropriate slow scale nets are restricted by finitely many conditions at fixed α.) □

4. Some special cases and applications

WH-Ellipticity with slow scales. Slightly generalizing a notion from [12], an operator P with Colombeau coefficients on Ω is said to be *WH-elliptic with slow scales* if on any compact subset $K \Subset \Omega$ the following is valid:

(i) $\exists q > 0\ \exists r_\varepsilon > 0$ of slow scale and $\exists \varepsilon_0 > 0$ such that

$$|P_\varepsilon(x,\xi)| \geq \varepsilon^q\,(1+|\xi|)^m \tag{46}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$, $|\xi| \geq r_\varepsilon$, and $0 < \varepsilon < \varepsilon_0$.

(ii) $\forall \alpha \in \mathbb{N}^n_0\ \exists s^\alpha_\varepsilon, r^\alpha_\varepsilon > 0$ of slow scale and $\varepsilon_\alpha > 0$ such that for all $\beta \in \mathbb{N}^n_0$ with $0 \leq |\beta| \leq m$

$$|\partial^\alpha_x\partial^\beta_\xi P_\varepsilon(x,\xi)| \leq s^\alpha_\varepsilon\,|P_\varepsilon(x,\xi)|\,(1+|\xi|)^{-|\beta|} \tag{47}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$, $|\xi| \geq r^\alpha_\varepsilon$, and $0 < \varepsilon < \varepsilon_\alpha$.

The following consequence of Theorem 3.2 is an elliptic regularity result (cf. [12] for related results in the constant coefficient case).

Corollary 4.1. *Let P be WH-elliptic with slow scales. Then for all $u \in \mathcal{G}(\Omega)$*

$$\mathrm{WF}_g(u) = \mathrm{WF}_g(Pu). \tag{48}$$

First-order operators with slow scale coefficients. Here it is possible to obtain microlocal regularity from estimates of the principal part over conic regions. Operators of this type were considered earlier in [10, 11]. Let $P_\varepsilon(x,\xi) = \sum_{j=1}^n a_j^\varepsilon(x)\xi_j + a_0^\varepsilon(x)$ with a_k^ε $(k = 0,\dots,n)$ having slow scale ε-growth on compact sets in each derivative. Let $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$ and assume that $U \times \Gamma$ is an open neighborhood where the following holds: for each $K \Subset \Omega$ there are $s_\varepsilon, r_\varepsilon > 0$ of slow scale and $\varepsilon_0 > 0$ such that

$$|P_{\varepsilon,1}(x,\xi)| \geq \frac{1}{s_\varepsilon}(1+|\xi|) \tag{49}$$

for all $(x,\xi) \in K \times \Gamma$, $|\xi| \geq r_\varepsilon$, and $0 < \varepsilon < \varepsilon_0$.

Proposition 4.2. *Let P be a first-order operator with variable slow scale Colombeau coefficients and $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$ be such that property* (49) *holds. Then for any $u \in \mathcal{G}(\Omega)$, $(x_0,\xi_0) \notin \mathrm{WF}_g(Pu)$ implies $(x_0,\xi_0) \notin \mathrm{WF}_g(u)$.*

Proof. We show that P satisfies (35)-(36) (with $m_0 = 1$, $\delta = 0$, and $\rho = 1$) when $(x,\xi) \in K \times \Gamma$. First, using the slow scale property of a_0^ε and (49) we obtain, with some slow scale net s_ε^0,

$$|P_\varepsilon(x,\xi)| \geq |P_{\varepsilon,1}(x,\xi)| - |a_0^\varepsilon(x)| \geq \frac{1+|\xi|}{s_\varepsilon} - s_\varepsilon^0 \geq \frac{1+|\xi|}{2s_\varepsilon}$$

if $|\xi| \geq 2s_\varepsilon s_\varepsilon^0$ and ε sufficiently small. This is (35).

There are only first order non-trivial ξ-derivatives, and we have with some slow scale net s_ε^j

$$|\partial_{\xi_j} P_\varepsilon(x,\xi)| = |a_j^\varepsilon(x)| \leq s_\varepsilon^j = \frac{s_\varepsilon^j 2s_\varepsilon(1+|\xi|)}{(1+|\xi|)2s_\varepsilon} \leq 2s_\varepsilon^j s_\varepsilon |P_\varepsilon(x,\xi)|(1+|\xi|)^{-1},$$

where we have used the above estimate on $|P_\varepsilon(x,\xi)|$, and assume that $|\xi|$ is larger than some slow scale radius and ε small. Finally, the estimate of the x-derivatives is also straightforward. Let $\alpha \in \mathbb{N}_0^n$ and assume that $|\partial_x^\alpha a_k^\varepsilon(x)| \leq s_k^\varepsilon$ (slow scale) when $x \in K$. Then we have

$$\begin{aligned}|\partial_x^\alpha P_\varepsilon(x,\xi)| &\leq \sum_{j=1}^n |\partial_x^\alpha a_j^\varepsilon(x)||\xi| + |\partial_x^\alpha a_0^\varepsilon(x)| \\ &\qquad\qquad \leq C(\max_{k=0,\dots,n} s_\varepsilon^k)\,(1+|\xi|) \leq s_\varepsilon'\,|P_\varepsilon(x,\xi)|,\end{aligned}$$

where s_ε' can be chosen to be $2Cs_\varepsilon \max s_\varepsilon^k$ and the usual assumptions on $|\xi|$ and ε are in effect. □

Conditions on the principal part. Let P be an operator of order m with Colombeau coefficients $a_\beta^\varepsilon(x)$. Let $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$ and with an open conic neighborhood $U \times \Gamma$ on which the following holds: For all $K \Subset U$

(i) $\exists s_\varepsilon, r_\varepsilon > 0$ of slow scale $\exists \varepsilon_0 > 0$ such that

$$|P_{\varepsilon,m}(x,\xi)| \geq \frac{1}{s_\varepsilon} \sum_{|\alpha|=m} |a_\alpha^\varepsilon(x)| \cdot (1+|\xi|)^m \tag{50}$$

when $(x,\xi) \in K \times \Gamma$, $|\xi| \geq r_\varepsilon$, and $0 < \varepsilon < \varepsilon_0$;

(ii) $\forall \gamma \in \mathbb{N}_0^n$ $\exists s_\varepsilon^\gamma, r_\varepsilon^\gamma > 0$ of slow scale and $\exists \varepsilon_\gamma > 0$ such that for all $\beta \in \mathbb{N}_0^n$, $0 \leq |\beta| \leq m$

$$|\partial_x^\gamma a_\beta^\varepsilon(x)| \leq s_\varepsilon^\gamma \sum_{|\alpha|=m} |a_\alpha^\varepsilon(x)| \tag{51}$$

when $(x, \xi) \in K \times \Gamma$, $|\xi| \geq r_\varepsilon^\gamma$, and $0 < \varepsilon < \varepsilon_\gamma$.

Lemma 4.3. *Hypotheses* (50) *and* (51) *imply* (36).

Proof. Denote $b_m^\varepsilon(x) = \sum_{|\alpha|=m} |a_\alpha^\varepsilon(x)|$. Since

$$\partial_x^\gamma \partial_\xi^\beta P_\varepsilon(x, \xi) = \sum_{\beta \leq \sigma, |\sigma| \leq m} \partial_x^\gamma a_\sigma^\varepsilon(x) \frac{\sigma!}{(\sigma - \beta)!} \xi^{\sigma - \beta}$$

we obtain

$$|\partial_x^\gamma \partial_\xi^\beta P_\varepsilon(x, \xi)| \leq C|\xi|^{m-|\beta|} \sum_{\beta \leq \sigma, |\sigma| \leq m} |\partial_x^\gamma a_\sigma^\varepsilon(x)| \leq C'\, s_\varepsilon^\gamma\, b_m^\varepsilon(x)\, (1 + |\xi|)^{m-|\beta|}.$$

On the other hand,

$$\begin{aligned} |P_\varepsilon(x, \xi)| &\geq |P_{\varepsilon,m}(x, \xi)| - |\xi|^{m-1} \sum_{|\alpha| \leq m-1} |a_\alpha^\varepsilon(x)| \\ &\geq b_m^\varepsilon(x)\, C\, (1 + |\xi|)^m \left(\frac{1}{s_\varepsilon} - \frac{s_\varepsilon^0}{1 + |\xi|}\right) \geq \frac{C}{2 s_\varepsilon} (1 + |\xi|)^m\, b_m^\varepsilon(x) \end{aligned} \tag{52}$$

if $|\xi| \geq 2 s_\varepsilon s_\varepsilon^0$. Combining the two estimates above yields (36). □

Lemma 4.4. *Assume that one of the following equivalent conditions holds:*
(i) *for all $L \Subset \Omega$ there is $p \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that*

$$\inf_{x \in L} \sum_{|\alpha|=m} |a_\alpha^\varepsilon(x)| \geq \varepsilon^p \qquad 0 < \varepsilon < \varepsilon_1, \tag{53}$$

that is, $\sum_{|\alpha|=m} |a_\alpha|$ is invertible in $\mathcal{G}(U)$;
(ii) $\sum_{|\alpha|=m} |a_\alpha|^2$ *is invertible in $\mathcal{G}(U)$.*
Then hypotheses (50) *and* (51) *imply* (35) *as well.*

Proof. That (i) and (ii) are equivalent follows from the equivalence of the l^1- and the l^2-norm on $\mathbb{C}^m$. By [5, Theorem 1.2.5] there is $p \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that $b_m^\varepsilon(x) = \sum_{|\alpha|=m} |a_\alpha^\varepsilon(x)| \geq \varepsilon^p$ for all $x \in L$, $0 < \varepsilon < \varepsilon_1$. Now (35) follows directly from (52). □

We summarize the above results in the following statement.

Proposition 4.5. *Let P be a partial differential operator with Colombeau coefficients on Ω. Assume that $\sum_{|\alpha|=m} |a_\alpha|^2$ is invertible in $\mathcal{G}$ on U and let $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ and* (50)-(51) *be satisfied in some open conic neighborhood $U \times \Gamma$. Then P satisfies hypotheses* (35)-(36) *in the same region. In particular, we have the microlocal regularity property* (37) *for each $u \in \mathcal{G}(\Omega)$.*

Remark 4.6. (i) The invertibility assumption in Proposition 4.5 cannot be dropped in general. For example, consider the zero divisor $c = [(c_\varepsilon)_\varepsilon] \in \tilde{\mathbb{C}}$, defined by $c_\varepsilon = 0$, if $1/\varepsilon \in \mathbb{N}$, and $c_\varepsilon = i$ otherwise. Then the operator with symbol $P_\varepsilon(\xi) = c_\varepsilon \xi$ satisfies (50)-(51) on all of $\mathbb{R} \times \mathbb{R} \setminus 0$ but $Pu = 0$ admits any non-regular solution of the following kind: let u_ε equal a representative of some element in $\mathcal{G}(\mathbb{R}) \setminus \mathcal{G}^\infty(\mathbb{R})$ if $1/\varepsilon \in \mathbb{N}$, and $u_\varepsilon = 0$ otherwise. Note that P is not hypoelliptic and does not satisfy (35).

(ii) On the other hand, the invertibility of $\sum_{|\alpha|=m} |a_\alpha|^2$ is not necessary for hypoellipticity of an operator. Neither is it necessary for (the stronger) conditions (35)-(36) to hold. For example, consider $P_\varepsilon(\xi) = a_\varepsilon \xi + b_\varepsilon$, where $a_\varepsilon = 0$ if $1/\varepsilon \in \mathbb{N}$, $a_\varepsilon = 1$ otherwise, and $b_\varepsilon = 1 - a_\varepsilon$ for all ε. Then one easily verifies that (35)-(36) hold (e.g., with $m_0 = 0$, $q = 0$, $\varepsilon_0 = 0$ $r_\varepsilon = 2$, $\delta = 0$, $\rho = 1$, $s_\varepsilon = 3$), whereas the principal part coefficient $[(a_\varepsilon)_\varepsilon]$ is not invertible. Furthermore, condition (51) fails to hold for P while (50) is trivially satisfied.

(iii) The operator with symbol $P_\varepsilon(\xi) = \varepsilon\xi + i$ satisfies (35)-(36) but not (51) (since $1 = |i| \not\lesssim s_\varepsilon^1 \varepsilon$). However, note that in this example the invertibility assumption on the principal part coefficient is met. (Again, the estimate (50) is trivial.)

Example 4.7. In this example we consider the situation of a hyperbolic operator with discontinuous coefficients. Such operators arise e.g. in acoustic wave propagation in a medium with irregularly changing properties. Let ρ be the density and let c be the sound speed of the medium. The pressure (perturbation) p solves the equation $P(p) = 0$, where

$$P = \partial_t^2 - c(x)^2 \rho(x) \partial_x \big(\frac{1}{\rho(x)} \partial_x \big). \tag{54}$$

A typical assumption on the medium properties is that both ρ and c are (time independent and) measurable functions varying between strictly positive bounds (but allow, e.g., for jump discontinuities). To illustrate our theory, we interpret the coefficients ρ and c as elements of $\mathcal{G}(\mathbb{R})$ with representatives satisfying

$$0 < \gamma_0 \le c_\varepsilon(x) \le \gamma_1, \qquad 0 < r_0 \le \rho_\varepsilon(x) \le r_1$$

for $x \in \mathbb{R}$ and $\varepsilon \in (0,1]$. In the setting of $\mathcal{G}(\mathbb{R}^2)$, the equation $P(p) = 0$ can be uniquely solved even with generalized functions as initial data (see [16]), and thus can model propagation of strong disturbances even in media with highly complex structure.

In such circumstances the regions of regularity of the solution provide valuable information. While global propagation of regularity for the constant coefficient case of (54) was dealt with in [12], we are now able to address the general case here. A representative of the operator (54) with coefficients in $\mathcal{G}(\mathbb{R}^2)$ as above is given by $P_\varepsilon(x,t,\partial_x,\partial_t) = \partial_t^2 - c_\varepsilon(x)^2 \rho_\varepsilon(x) \partial_x \big(\rho_\varepsilon(x)^{-1} \partial_x\big)$ with symbol

$$P_\varepsilon(x,t,\xi,\tau) = -\tau^2 + c_\varepsilon(x)^2 \xi^2 - i c_\varepsilon(x)^2 \frac{\rho_\varepsilon'(x)}{\rho_\varepsilon(x)} \xi.$$

We are going to identify regions of microhypoellipticity for P, i.e., the set $M_g(P)$ introduced in Corollary 3.4. First, note that $U \times \Gamma \subseteq M_g(P)$ implies that $(\rho \otimes 1) \mid_U$ and $(c \otimes 1) \mid_U$ are $\mathcal{G}^\infty$ by Remark 3.3. Hence, denoting by

$$S_g(c,\rho) := \big(\operatorname{singsupp}_g(c) \cup \operatorname{singsupp}_g(\rho)\big) \times \mathbb{R}$$

the union of the singular supports of $c \otimes 1$ and $\rho \otimes 1$, we have

$$S_g(c,\rho) \times \mathbb{R}^2 \setminus \{0\} \subseteq M_g(P)^c.$$

Let U be open in $\mathbb{R}^2$ such that $U \cap S_g(c,\rho) = \emptyset$ (i.e., the coefficients of P are $\mathcal{G}^\infty$ on U). We observe that in the regions of $\mathcal{G}^\infty$-regularity, the coefficients actually satisfy the stronger property of having slow scale growth in each derivative. This follows from the boundedness of c and ρ by Corollary 1.7. Together with the (constant) positive lower bound it yields that $1/\rho$ has the same properties there. It follows that P_ε, restricted to $U \times \mathbb{R}^2$, is of the structure

$$P_\varepsilon(x,t,\xi,\tau) = -\tau^2 + c_\varepsilon(x)^2\xi^2 - ib_\varepsilon(x)\xi,$$

where $b = [(b_\varepsilon)_\varepsilon]$ is real and of slow scale in each derivative.

Let $0 < \theta < \gamma_0$ and define the open cone Γ_θ in $\mathbb{R}^2 \setminus \{0\}$ by the conditions $|\tau| < (\gamma_0 - \theta)|\xi|$ or $|\tau| > (\gamma_1 + \theta)|\xi|$. We will show that P is microhypoelliptic on $U \times \Gamma_\theta$.

In fact, we can apply Proposition 4.5 with the principal part $P_{\varepsilon,2}(x,t,\xi,\tau) = -\tau^2 + c_\varepsilon(x)^2\xi^2$. Clearly $1 + c_\varepsilon(x)^2 \geq 1 + \gamma_0^2$ is invertible; furthermore, all estimates required in (51) are then trivially satisfied due to the slow scale properties of the coefficients. It remains to check (50). The two conditions defining Γ_θ yield immediately that $|P_{\varepsilon,2}(x,t,\xi,\tau)| \geq (\gamma_0^2 - (\gamma_0 - \theta)^2)(\xi^2 + \tau^2/(\gamma_0 - \theta)^2)/2$, resp. $|P_{\varepsilon,2}(x,t,\xi,\tau)| \geq ((\gamma_1 + \theta)^2 - \gamma_1^2)(\tau^2 + \xi^2(\gamma_1 + \theta)^2)/2$; therefore, $|P_{\varepsilon,2}(x,t,\xi,\tau)| \geq d_\theta(1 + \xi^2 + \tau^2)$ when $|(\xi,\tau)| \geq r_\theta$, for suitable positive constants d_θ and r_θ. On the other hand, $1 + c_\varepsilon^2(x) \leq 1 + \gamma_1^2$ hence (50) follows easily.

Since θ was arbitrary in the interval $(0,\gamma_0)$ we obtain, letting $\theta \to 0$, that

$$S_g(c,\rho)^c \times W^c \subseteq M_g(P),$$

where $W := \{(\xi,\tau) \in \mathbb{R}^2 \mid \gamma_0|\xi| \leq |\tau| \leq \gamma_1|\xi|\}$. Taking complements and summarizing we have shown that

$$S_g(c,\rho) \times \mathbb{R}^2 \setminus \{0\} \subseteq M_g(P)^c \subseteq \big(S_g(c,\rho) \times \mathbb{R}^2 \setminus \{0\}\big) \cup \big(S_g(c,\rho)^c \times W\big).$$

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