A microscopic theory of Curzon-Ahlborn heat engine

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The Curzon-Ahlborn (CA) efficiency, as the efficiency at the maximum power (EMP) of the endoreversible Carnot engine, has a significant impact on finite-time thermodynamics. However, the CA engine model is based on many assumptions. In the past few decades, although a lot of efforts have been made, a microscopic theory of the CA engine is still lacking. By adopting the method of the stochastic differential equation of energy, we formulate a microscopic theory of the CA engine realized with an underdamped Brownian particle in a class of non-harmonic potentials. This theory gives microscopic interpretation of all assumptions made by Curzon and Ahlborn, and thus puts the results about CA engine on a solid foundation. Also, based on this theory, we obtain analytical expressions of the power and the efficiency statistics for the Brownian CA engine. Our research brings new perspectives to experimental studies of finite-time microscopic heat engines featured with fluctuations.

Introduction.—For practical heat engines, not only the efficiency but also the power characterizes the performance. Optimizing the power and the efficiency of heatengine cycles is one of the goals in the study of finite-time thermodynamics [1, 2]. Compared to the Carnot efficiency achieved in infinite time [3], the efficiency at the maximum power (EMP) attracts a lot of attention in the studies of finite-time heat engines. The early studies [4–11] of the endoreversible Carnot engine concluded that the EMP is the well-known Curzon-Ahlborn (CA) efficiency

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}},\tag{1}$$

where T_C and T_H denote the temperatures of the cold and the hot heat baths. The CA efficiency is in a similar form to the Carnot efficiency, and is only relevant to the temperatures of the two heat baths and is independent of any other characteristic of the heat engine. The CA efficiency aroused a lot of attention and led to many following-up researches (see for example Refs. [8, 12–14]). It has become a paradigmatic result in studies dealing with thermodynamic optimization in the framework of finite-time and stochastic thermodynamics. In addition, the CA efficiency is relevant to many practical thermal machines [15].

The CA efficiency as the EMP has also been derived in some different setups [16]. For example, in Ref. [17] it is shown that CA efficiency is a result which can be obtained in the well-founded linear irreversible thermodynamics. In Ref. [15], the CA efficiency, as the EMP, is derived in the symmetric low-dissipation regime, where the irreversible entropy production is assumed inversely proportional to the period of a cycle. Such a $1/\tau$ -scaling has been recently verified in the experiment of finitetime isothermal compression of dry air [18, 19]. These studies confirm the validity of the CA efficiency as the EMP in many heat engine models. Meanwhile, in some other models, the EMP deviates from the CA efficiency [13, 15, 17, 20–42], probably due to different circumstances of these models.

The original derivation of the CA efficiency [4–7] is based on a lot of assumptions, such as the endoreversible assumption and the assumption of constant temperature difference. But how reliable are those assumptions made by Curzon and Ahlborn remains unclarified due to the lack of a microscopic theory of the CA engine. Also, due to this lack, the control scheme of the work parameter which is essential to construct the optimal cycle can not be determined (except for the harmonic potential [14]), neither can the work and heat statistics as well as the fluctuation theorems of the CA engine. In the past few decades, a lot of efforts have been made to seek a microscopic interpretation of the CA engine, but were unsuccessful.

In this Letter, we fill this long-standing gap by realizing the CA engine with an underdamped Brownian particle [14, 43–51] in a time-dependent potential as the working substance. By adopting the method of stochastic differential equation of energy [52, 53], we give microscopic interpretation of all assumptions made by Curzon and Ahlborn, including the endoreversibility, Newton's cooling law and the constant temperature difference. Thus we lay a solid foundation for the CA engine model. Furthermore, this microscopic theory allows us to determine the control scheme of the work parameter of the Brownian CA engine and study the fluctuations of the power and the efficiency of the Brownian CA engine. Our study demonstrates that when downsizing the working substance to a single Brownian particle, results about the average power and efficiency of the CA engine remain valid, but the fluctuations become prominent.

The model.—The working substance of the engine is modeled as a Brownian particle [45, 47, 54– 65] constrained in a controllable potential $\mathcal{U}(x,t) = k(t)x^{2n}/(2n)$ with the control parameter k(t) and a positive integer n. In the isothermal expansion (compression) process, the control parameter k(t) is varied when the engine is in contact with the heat bath at temperature $T_{\rm b}$. The motion of the particle with mass m is governed by the complete Langevin equation

$$\ddot{x} + \gamma \dot{x} + k(t)x^{2n-1} = \frac{1}{m}\xi(t),$$
 (2)

where the random force $\xi(t)$ represents a Gaussian white noise satisfying $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2m\gamma T_{\rm b}\delta(t - t')$ with the friction coefficient γ . Throughout the text the Boltzmann constant is set to be $k_B = 1$.

We consider the highly underdamped regime $\tau_p \ll \gamma^{-1}$ and slow external driving $\tau_p \ll k/\dot{k}$, where τ_p is the period of the unperturbed motion of the particle, e.g., $\tau_p = 2\pi \sqrt{m/k}$ for a harmonic oscillator (n = 1). Under these two conditions, the variation of the stochastic energy of the particle within a period is relatively small, which allows us to study the dynamics of the stochastic energy $E = m\dot{x}^2/2 + \mathcal{U}(x, t)$. Based on Ito's lemma and Virial theorem, the equation of motion is expressed as the stochastic differential equation of the energy E [52, 53]

$$dE = \frac{\dot{\lambda}}{\lambda} E dt - \Gamma \left(E - \frac{f_n T_{\rm b}}{2} \right) dt + \sqrt{2\Gamma T_{\rm b} E} dB_t. \quad (3)$$

where the work parameter is rewritten into $\lambda(t) = k(t)^{1/(n+1)}$ with the increment dB_t of the Wiener process, the effective friction coefficient $\Gamma = 2n\gamma/(n+1)$, and the effective degrees of freedom $f_n = 1 + 1/n$. The increment of trajectory work for this system is

$$dW = \frac{\dot{\lambda}}{\lambda} E dt. \tag{4}$$

The trajectory heat is obtained from the first law of thermodynamics as

$$dQ = -\Gamma\left(E - \frac{f_n T_{\rm b}}{2}\right)dt + \sqrt{2\Gamma T_{\rm b}E}dB_t.$$
 (5)

The Fokker-Planck equation associated with Eq. (3) can be solved explicitly [53]. During the dynamical evolution process, the system remains in a Maxwell-Boltzmann distribution in the energy space and thus can be described by an effective temperature $\theta(t)$ as

$$P(E,t) = \frac{e^{-E/\theta(t)}}{\Gamma(f_n/2)} \frac{E^{f_n/2-1}}{\theta(t)^{f_n/2}},$$
(6)

where $\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy$ is gamma function. Notice that Eq. (6) leads to the endoreversibility, which is usually assumed in previous studies relevant to the CA engine, but is derived as a consequence of the equation of motion in our setup. The ensemble average of the energy is $\langle E(t) \rangle = f_n \theta(t)/2$ with the effective temperature $\theta(t)$ governed by

$$\dot{\theta}(t) = \frac{\lambda}{\lambda} \theta(t) - \Gamma[\theta(t) - T_{\rm b}].$$
(7)

We emphasize that the l.h.s. of Eq. (7) corresponds to the time derivative of the average energy up to a factor $f_n/2$. The two terms on the r.h.s. correspond to the average work flux and heat flux, respectively. The average heat flux satisfies Newton's cooling law, which is also derived as a consequence of the equation of motion. The effective friction coefficient Γ , as a cooling rate, is independent of the work parameter λ . We would like to point out that a similar equation of motion for the effective temperature $\theta(t)$ has been obtained previously for the ideal gas as the working substance [12]. However, their derivation relies on several assumptions, for example, the equation of state of ideal gas, the phenomenological Newton's cooling law and the endoreversible assumption. On the contrary, the results presented here are all derived from the microscopic dynamics, and are capable of describing microscopic systems featured with fluctuations.

Realization of Curzon-Ahlborn engine based on a Brownian particle.—With the model introduced above, we study the EMP of such a microscopic Brownian engine and formulate a microscopic theory of the CA engine. To construct a finite-time Carnot cycle, two heat baths at different temperatures T_i , i = H, C are required in the hot and the cold isothermal processes. The (effective) friction coefficients γ_i (Γ_i) may be different in the two processes. Based on Curzon and Ahlborn's derivation [7], we summarize the preconditions of the CA engine as follows

- (i) **Endoreversibility** [66]. The state of the working substance of the engine can be described by an effective temperature.
- (ii) Newton's cooling law (or linear heat transfer law). The heat flux between the working substance and the heat bath is proportional to the temperature difference.
- (iii) Constant temperature difference. During the isothermal expansion (compression) process, the effective temperature of the working substance remains at a constant value θ_H (θ_C) different from that of the heat bath T_H (T_C).
- (iv) **Internal reversible Carnot cycle.** All irreversibilities are associated with the heat exchange between the working substance and the heat baths

while the adiabatic processes remain reversible [16]. The heat engine operates like a reversible Carnot engine between two virtual heat baths at temperatures θ_C and θ_H , respectively.

(v) Constant heat capacity [67] and cooling rate [68]. Both the heat capacity of the working substance and the cooling rate are independent of the temperature and the work parameters.

According to Eqs. (6) and (7), our setup fulfills requirements (i) and (ii). In supplementary material, we prove that the optimal cycle corresponding to the maximum power is exactly the CA cycle satisfying preconditions (iii) and (iv). Precondition (v) is guaranteed by the generalized equipartition theorem and the highly underdamped condition.

Based on the equation of motion of the effective temperature (Eq. (7)) and the definition of trajectory work and heat, we can optimize the average power of the finitetime Brownian engine. In Fig. 1(a), we plot the cycle diagram of the Brownian CA engine. In order to construct a closed CA cycle, the values of work parameter at the end of each process in Fig. 1(a) satisfy

$$\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_4} = r. \tag{8}$$

The four processes of the CA cycle are illustrated as follows (see supplementary material),

(I) Isothermal compression. The working substance is in contact with the cold heat bath. Initiated from λ_1 at t = 0, the work parameter is varied exponentially with time $\lambda(t) = \lambda_1 (\lambda_2/\lambda_1)^{t/\tau_C}$, where τ_C is the duration of the process. From the protocol, it can be found that the effective temperature of the working substance remains at a constant during the process

$$\theta_C = \frac{\tau_C \Gamma_C}{\tau_C \Gamma_C - \ln r} T_C. \tag{9}$$

The heat released to the cold heat bath during the isothermal compression process is

$$-\langle Q_C \rangle = \frac{f_n}{2} \Gamma_C (\theta_C - T_C) \tau_C. \tag{10}$$

(II) Adiabatic compression. The work parameter $\lambda(t)$ is quenched instantaneously from λ_2 to λ_3 with the effective temperature $\theta(t)$ changing from θ_C to θ_H accordingly. When the timescale of varying the work parameter is much shorter than that of the heat dissipation, Eq. (7) becomes $\dot{\theta}(t) = \dot{\lambda}\theta(t)/\lambda$, which leads to $\theta_H/\theta_C = \lambda_3/\lambda_2$.

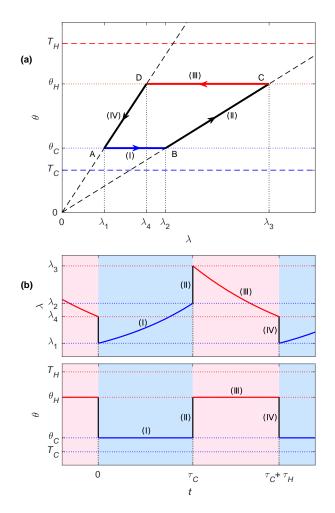


Figure 1. CA cycle based on a Brownian particle. (a) The cycle diagram in the space of the work parameter λ and the effective temperature θ . In the isothermal expansion (compression) process, the working substance remains at a constant effective temperature $\theta_H(\theta_C)$, which is different from the temperature $T_H(T_C)$ of the hot (cold) heat bath. In the two adiabatic processes, the effective temperature θ of the working substance is proportional to the work parameter λ . (b) The control scheme of the work parameter $\lambda(t)$ and the evolution of the effective temperature $\theta(t)$ in a finite-time cycle. The work parameter λ is varied exponentially with time in an isothermal process, and is quenched abruptly in an adiabatic process.

(III) Isothermal expansion. The working substance is in contact with the hot heat bath. The work parameter is varied exponentially with time $\lambda(t) = \lambda_3(\lambda_4/\lambda_3)^{(t-\tau_C)/\tau_H}$, and the working substance remains at a constant effective temperature

$$\theta_H = \frac{\tau_H \Gamma_H}{\tau_H \Gamma_H + \ln r} T_H, \qquad (11)$$

where τ_H is the duration of the process. The heat absorbed from the hot heat bath during the isothermal expansion process is

$$\langle Q_H \rangle = \frac{f_n}{2} \Gamma_H (T_H - \theta_H) \tau_H.$$
 (12)

(IV) Adiabatic expansion. Finally, the work parameter is quenched from λ_4 to the initial value λ_1 instantaneously with the effective temperature changing from θ_H to θ_C accordingly, satisfying $\theta_H/\theta_C = \lambda_4/\lambda_1$.

The control scheme of the work parameter λ and the evolution of the effective temperature θ in a finite-time cycle are illustrated in Fig. 1(b).

Combing Eqs. (9)-(12), it is straightforward to verify the precondition (iv) that the entropy change of the working substance after a cycle is zero $\Delta S = \langle Q_H \rangle / \theta_H + \langle Q_C \rangle / \theta_C = 0$. Therefore, the microscopic dynamics of the model, together with the explicit control scheme $\lambda(t)$, constitutes a microscopic theory of the CA engine.

The net work of a full cycle is $-\langle W \rangle = \langle Q_H \rangle + \langle Q_C \rangle$, and the average power and the average efficiency follow as $\overline{P} := -\langle W \rangle / (\tau_H + \tau_C)$ and $\eta := -\langle W \rangle / \langle Q_H \rangle$, which are explicitly

$$\overline{P} = \frac{f_n \ln r}{2(\tau_H + \tau_C)} \left(\frac{\tau_H \Gamma_H T_H}{\tau_H \Gamma_H + \ln r} - \frac{\tau_C \Gamma_C T_C}{\tau_C \Gamma_C - \ln r} \right), \quad (13)$$

and

$$\eta = 1 - \frac{1 + (\tau_H \Gamma_H)^{-1} \ln r}{1 - (\tau_C \Gamma_C)^{-1} \ln r} \frac{T_C}{T_H}.$$
 (14)

In order to achieve the maximum power, we first fix r and optimize the power over τ_H and τ_C . The maximum power is obtained as (see supplementary material)

$$\overline{P}_{\max} = \frac{f_n \Gamma_C \Gamma_H \left(\sqrt{T_H} - \sqrt{T_C}\right)^2}{2 \left(\sqrt{\Gamma_H} + \sqrt{\Gamma_C}\right)^2},$$
(15)

with the corresponding optimal duration of the two isothermal processes

$$\tau_H^{\max} = \frac{\ln r \left(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C}\right)}{\Gamma_H \sqrt{\Gamma_C} \left(\sqrt{T_H} - \sqrt{T_C}\right)},$$

$$\tau_C^{\max} = \frac{\ln r \left(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C}\right)}{\Gamma_C \sqrt{\Gamma_H} \left(\sqrt{T_H} - \sqrt{T_C}\right)}.$$
 (16)

Please note that \overline{P}_{\max} is independent of r. Hence \overline{P}_{\max} is also the global maximum power. It is straightforward to see that the EMP of the Brownian engine in the highly underdamped regime is the CA efficiency

$$\eta_{\rm EMP} = 1 - \sqrt{\frac{T_C}{T_H}} = \eta_{CA},\tag{17}$$

as we expect. Based on Eqs. (14) and (15), we derive the trade-off relation between power and efficiency in supplementary material. Compared to previous studies [25, 69–73], our trade-off relation is tight and is shown to be reachable with the explicit control scheme of the work parameter $\lambda(t)$. It is worth mentioning that Ref. [74] obtains the same tight trade-off relation, and Ref. [14] obtains the tight trade-off relation as well as the control scheme for the harmonic potential. As a generalization, our results are valid for a Brownian particle in a class of non-harmonic potentials.

Generating function of work and heat in a finite-time isothermal process.—For a microscopic Brownian engine, average values are insufficient to characterize the performance. Fluctuations are non-negligible [75]. To evaluate the performance of a finite-time heat engine, we need to quantify the extracted work and heat absorbed from the hot bath in one heat-engine cycle. For the dynamics described by the above model, we can derive the analytical results of the joint generating function of work and heat $I(u,s) \coloneqq \langle e^{uQ+sW} \rangle$ by generalizing the techniques used in Ref. [76]. The result is

$$I(u,s) = \left[\frac{1+u\tilde{\psi}(\tau)}{1+u\tilde{\psi}_0}\right]^{\frac{f_n}{2}} e^{\frac{fn(s-u)}{2}\int_0^{\tau}\frac{\dot{\lambda}(t)}{\lambda(t)}\psi(t)dt}, \quad (18)$$

where τ is the time duration of the process, and the temperature-like variable $\psi(t)$ satisfies

$$\frac{d\psi}{dt} = \frac{\dot{\lambda}}{\lambda}\psi - \Gamma(\psi - T_{\rm b}) + (s - u)\frac{\dot{\lambda}}{\lambda}\psi^2.$$
 (19)

with the initial condition $\psi(0) = \tilde{\psi}_0/(1 + u\tilde{\psi}_0)$. The initial value $\tilde{\psi}_0$ is either set as the initial temperature θ_0 or obtained from the previous process. A shifted temperature-like variable is defined as $\tilde{\psi}(t) \coloneqq \psi(t)/[1 - u\psi(t)]$, whose value $\tilde{\psi}(\tau)$ at the end of this process is used as the initial value $\tilde{\psi}_0$ of the subsequent process. Detailed derivations to Eqs. (18) and (19) and the analytical expression of the joint generating function I(u, s)are left in supplementary material.

Statistics of power and efficiency of the Brownian CA engine.—Based on the microscopic theory, especially the joint generating function of work and heat I(u, s) and the control scheme $\lambda(t)$ of the full cycle, we can further study the fluctuations of the power and the efficiency [75–80] together with the fluctuation theorems [81, 82] of the finite-time Brownian Carnot engine. Specifically, we calculate the distribution p(P) of the fluctuating power $P := -W/(\tau_C + \tau_H)$ and the distribution $p(\zeta)$ of the fluctuating efficiency $\zeta := -(W + \eta Q_H)/\langle Q_H \rangle$ from the generating function $I_{\text{cycle}}(u_H, u_C, s) = \langle e^{u_H Q_H + u_C Q_C + sW} \rangle$ (see supplementary material) of a whole cycle. Please note that, instead of $-W/Q_H$, we define ζ as the fluctuating efficiency which characterizes the deviation from the average efficiency η defined above. It is straightforward to see that $\langle \zeta \rangle = 0$.

As a special case, we plot the distributions of the power and the efficiency of the Brownian CA engine in Fig. 2, where $\eta = \eta_{CA}$, $\tau_C = \tau_C^{\max}$, $\tau_H = \tau_H^{\max}$. Due to the fluctuation, the power can be negative or much larger than the average power. Similarly, the efficiency can be negative or larger than Carnot efficiency (even larger than unity). From the analytical results of the joint generating function $I_{cycle}(u_H, u_C, s)$, we can show the tendency of the distributions when we increase the duration of the cycle $\tau_t \coloneqq \tau_C^{\max} + \tau_H^{\max}$ by increasing $\ln r$

$$\operatorname{Var}(P) \approx \frac{4\overline{P}_{\max}^2}{f_n \eta_{CA}^2} \frac{\left[(1 - \eta_{CA})^2 + 1/\delta\right] (1 + \delta)}{\sqrt{\Gamma_C \Gamma_H}} \frac{1}{\tau_t}, \quad (20)$$

$$\operatorname{Var}(\zeta) \approx \frac{4(1 - \eta_{CA})^2}{f_n \sqrt{\Gamma_C \Gamma_H}} \frac{(1 + \delta)^2}{\delta} \frac{1}{\tau_{\mathrm{t}}},\tag{21}$$

where $\delta = \sqrt{\Gamma_H T_H / (\Gamma_C T_C)}$. For both the power and the efficiency, their variances decrease inversely with τ_t .

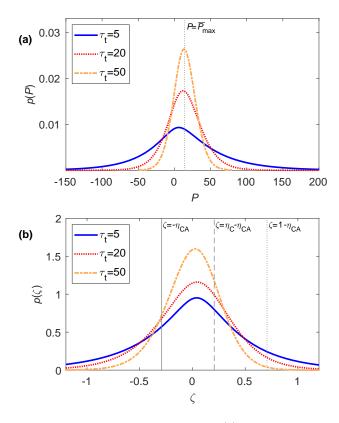


Figure 2. Distribution of the power (a) and the efficiency (b) of a Brownian CA engine for three different periods $\tau_t = 5, 20, 50$. (a) The vertical dotted line indicates the average power. (b) The vertical solid, dashed and dotted lines correspond to efficiency $\eta = 0, \eta = \eta_C$ (Carnot efficiency) and $\eta = 1$ respectively. Here we have chosen $T_C = 300, T_H = 600,$ $\Gamma_C = 1, \Gamma_H = 1.2.$

Summary and discussion.—In this Letter, we realize the Curzon-Ahlborn heat engine with a Brownian particle in the highly underdamped regime. By adopting the method of stochastic differential equation of energy, we formulate a microscopic theory of the CA engine based on this model. This theory gives microscopic interpretation of all assumptions of the CA engine model including the endoreversibility, Newton's cooling law and the constant temperature difference. Hence, we lay a solid foundation for the CA engine.

From this microscopic theory, the explicit control scheme $\lambda(t)$ of the CA engine can be uniquely determined, which leads to the maximum power of the Brownian engine. The control scheme associated with the maximum power for any given efficiency can be obtained based on the microscopic theory. In addition, we calculate the generating function of work and heat, and obtain the analytical results of statistics of the power and the efficiency together with the fluctuation theorems of the Brownian CA engine. These quantitative results about the CA engine bring important insights to the studies of finite-time thermodynamics beyond the low-dissipation regime [15, 25, 70, 71, 73, 83, 84]. For example, results about the average power and efficiency of the CA engine remain valid when downsizing the working substance to a single Brownian particle, but fluctuations become prominent. Our study will shed new light on the experimental explorations about finite-time Brownian engine, and may inspire future studies about the design of nanomachines with higher power and efficiency.

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- [1] B. Andresen, Angew. Chem. Int. Ed. 50, 2690 (2011).
- [2] R. Berry, P. Salamon, and B. Andresen, Entropy 22, 908 (2020).
- [3] H. Callen, Thermodynamics and an introduction to thermostatistics (Wiley, New York, 1985).
- [4] J. Yvon, in First Geneva Conf. Proc. UN (1955).
- [5] P. Chambadal, Les Centrales Nuclaires (Armand Colin, France, 1957).
- [6] I. Novikov, J. Nucl. Eng. 7, 125 (1958).
- [7] F. L. Curzon and B. Ahlborn, Am. J. Phys 43, 22 (1975).
- [8] M. H. Rubin, Phys. Rev. A **19**, 1272 (1979).
- [9] M. Moreau and Y. Pomeau, Eur. Phys. J. Special Topics 224, 769 (2015).
- [10] H. Ouerdane, Y. Apertet, C. Goupil, and P. Lecoeur, Eur. Phys. J. Special Topics 224, 839 (2015).
- [11] M. Feidt, Entropy **19**, 369 (2017).
- [12] M. H. Rubin, Phys. Rev. A 19, 1277 (1979).
- [13] T. Schmiedl and U. Seifert, EPL (Europhysics Letters) 81, 20003 (2007).
- [14] A. Dechant, N. Kiesel, and E. Lutz, Europhys. Lett. 119, 50003 (2017).
- [15] M. Esposito, R. Kawai, K. Lindenberg, and C. V.

Broeck, Phys. Rev. Lett. 105, 150603 (2010).

- [16] A. Calvo Hernández, J. Roco, A. Medina, and N. Sánchez-Salas, Rev. Mex. Fís. 60, 384 (2014).
- [17] C. Van den Broeck, Phys. Rev. Lett. **95**, 190602 (2005).
- [18] Y.-H. Ma, R.-X. Zhai, J. Chen, C. P. Sun, and H. Dong, Phys. Rev. Lett. **125**, 210601 (2020).
- [19] J.-F. Chen, C. P. Sun, and H. Dong, Phys. Rev. E 104, 034117 (2021).
- [20] Z. C. Tu, J. Phys. A: Math. Theor. 41, 312003 (2008).
- [21] Y. Izumida and K. Okuda, Europhys. Lett. 83, 60003 (2008).
- [22] J. Wang and J. He, Phys. Rev. E 86, 051112 (2012).
- [23] V. Cavina, A. Mari, and V. Giovannetti, Phys. Rev. Lett. **119**, 050601 (2017).
- [24] Y.-H. Ma, D. Xu, H. Dong, and C.-P. Sun, Phys. Rev. E 98, 022133 (2018).
- [25] P. Abiuso and M. Perarnau-Llobet, Phys. Rev. Lett. 124, 110606 (2020).
- [26] O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz, Phys. Rev. Lett. 109, 203006 (2012).
- [27] B. J. de Cisneros and A. C. Hernández, Phys. Rev. Lett. 98, 130602 (2007).
- [28] Y. Izumida and K. Okuda, Phys. Rev. E 80, 021121 (2009).
- [29] N. Sánchez-Salas, L. López-Palacios, S. Velasco, and A. C. Hernández, Phys. Rev. E 82, 051101 (2010).
- [30] K. Nakamura, J. Matrasulov, and Y. Izumida, Phys. Rev. E 102, 012129 (2020).
- [31] R. Fu, A. Taghvaei, Y. Chen, and T. T. Georgiou, (2020), arXiv:2001.00979 [math.OC].
- [32] B. H. Lavenda, Am. J. Phys. **75**, 169 (2007).
- [33] Z.-C. Tu, Front. Phys. 16, 33202 (2020).
- [34] M. Bauer, K. Brandner, and U. Seifert, Phys. Rev. E 93, 042112 (2016).
- [35] J. Chen, J. Phys. D: Appl. Phys. 27, 1144 (1994).
- [36] J.-F. Chen, C.-P. Sun, and H. Dong, Phys. Rev. E 100, 062140 (2019).
- [37] R. Dann, R. Kosloff, and P. Salamon, Entropy 22, 1255 (2020).
- [38] S. Deffner, Entropy **20**, 875 (2018).
- [39] Z. Smith, P. S. Pal, and S. Deffner, J. Non-Equilib. Thermodyn. 45, 305 (2020).
- [40] M. V. S. Bonança, J. Stat. Mech.: Theory Exp. 2019, 123203 (2019).
- [41] Z.-C. Tu, Chin. Phys. B **21**, 020513 (2012).
- [42] L. Chen, Z. Ding, and F. Sun, J. Non-Equilib. Thermodyn. 36, 155 (2011).
- [43] T. Hondou and K. Sekimoto, Phys. Rev. E 62, 6021 (2000).
- [44] V. Blickle and C. Bechinger, Nat. Phys. 8, 143 (2011).
- [45] I. A. Martínez, É. Roldán, L. Dinis, D. Petrov, J. M. R. Parrondo, and R. A. Rica, Nat. Phys. **12**, 67 (2015).
- [46] V. Holubec and R. Marathe, Phys. Rev. E 102, 060101 (2020).
- [47] J. Gieseler, R. Quidant, C. Dellago, and L. Novotny, Nat. Nanotechnology 9, 358 (2014).
- [48] Z. Gong, Y. Lan, and H. Quan, Phys. Rev. Lett. 117, 180603 (2016).
- [49] C. Kwon, J. D. Noh, and H. Park, Phys. Rev. E 88, 062102 (2013).
- [50] A. Gomez-Marin, T. Schmiedl, and U. Seifert, J. Chem.

Phys. **129**, 024114 (2008).

- [51] A. Celani, S. Bo, R. Eichhorn, and E. Aurell, Phys. Rev. Lett. 109, 260603 (2012).
- [52] D. S. P. Salazar and S. A. Lira, J. Phys. A: Math. Theor. 49, 465001 (2016).
- [53] D. S. P. Salazar and S. A. Lira, Phys. Rev. E 99, 062119 (2019).
- [54] Y. Jun, M. Gavrilov, and J. Bechhoefer, Phys. Rev. Lett. 113, 190601 (2014).
- [55] I. A. Martínez, A. Petrosyan, D. Guéry-Odelin, E. Trizac, and S. Ciliberto, Nat. Phys. 12, 843 (2016).
- [56] I. A. Martínez, É. Roldán, L. Dinis, and R. A. Rica, Soft Matter 13, 22 (2017).
- [57] T. M. Hoang, R. Pan, J. Ahn, J. Bang, H. Quan, and T. Li, Phys. Rev. Lett. **120**, 080602 (2018).
- [58] J. A. C. Albay, P.-Y. Lai, and Y. Jun, Appl. Phys. Lett. 116, 103706 (2020).
- [59] Z. C. Tu, Phys. Rev. E 89, 052148 (2014).
- [60] G. Li, H. T. Quan, and Z. C. Tu, Phys. Rev. E 96, 012144 (2017).
- [61] G. Li and Z. C. Tu, Phys. Rev. E 100, 012127 (2019).
- [62] G. Paneru, D. Y. Lee, T. Tlusty, and H. K. Pak, Phys. Rev. Lett. **120**, 020601 (2018).
- [63] T. Speck, J. Phys. A: Math. Theor. 44, 305001 (2011).
- [64] T. Schmiedl and U. Seifert, Phys. Rev. Lett. 98, 108301 (2007).
- [65] S. Rana, P. Pal, A. Saha, and A. Jayannavar, Phys. Rev. E 90, 042146 (2014).
- [66] K. H. Hoffmann, J. M. Burzler, and S. Schubert, J. Non-Equilib. Thermodyn. 22, 311 (1997).
- [67] R. S. Johal and A. M. Jayannavar, Resonance 26, 211 (2021).
- [68] Y. Apertet, H. Ouerdane, C. Goupil, and P. Lecoeur, Phys. Rev. E 96, 022119 (2017).
- [69] N. Shiraishi, K. Saito, and H. Tasaki, Phys. Rev. Lett. 117, 190601 (2016).
- [70] V. Holubec and A. Ryabov, J. Stat. Mech: Theory Exp. 2016, 073204 (2016).
- [71] J. Gonzalez-Ayala, A. C. Hernández, and J. M. M. Roco, Phys. Rev. E 95, 022131 (2017).
- [72] P. Pietzonka and U. Seifert, Phys. Rev. Lett. 120, 190602 (2018).
- [73] Y.-H. Ma, D. Xu, H. Dong, and C.-P. Sun, Phys. Rev. E 98, 042112 (2018).
- [74] L. Chen and Z. Yan, J. Chem. Phys. 90, 3740 (1989).
- [75] V. Holubec and A. Ryabov, (2021), arXiv:2108.00172 [cond-mat.stat-mech].
- [76] D. S. P. Salazar, Phys. Rev. E 101, 030101 (2020).
- [77] V. Holubec, J. Stat. Mech.: Theory Exp. 2014, P05022 (2014).
- [78] T. Denzler and E. Lutz, New J. Phys. 23, 075003 (2021).
- [79] T. Denzler and E. Lutz, Phys. Rev. Research 2, 032062 (2020).
- [80] T. Denzler and E. Lutz, Phys. Rev. Research 3, L032041 (2021).
- [81] N. Sinitsyn, J. Phys. A: Math. Theor. 44, 405001 (2011).
- [82] U. Seifert, Rep. Prog. Phys. 75, 126001 (2012).
- [83] J. Gonzalez-Ayala, J. Guo, A. Medina, J. Roco, and A. C. Hernández, Phys. Rev. Lett. **124**, 050603 (2020).
- [84] Y.-H. Ma, C. P. Sun, and H. Dong, (2020), arXiv:2012.08748 [quant-ph].

Supplementary Materials: A microscopic theory of Curzon-Ahlborn heat engine

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The supplementary materials are organized as follows. In Sec. I, we construct and optimize the finite-time Carnot engine with an underdamped Brownian particle as the working substance, and obtain the power and the efficiency of the finite-time Brownian engine. In Sec. II, we derive the trade-off relation between the power and the efficiency for the finite-time Brownian Carnot engine. In Sec. III, we calculate the joint generating function of work and heat, and verify the fluctuation theorem for heat engines. We also derive the generating function for the power and the efficiency and discuss their variances for the Brownian CA engine.

I. CONSTRUCTION AND OPTIMIZATION OF A FINITE-TIME BROWNIAN ENGINE

We attempt to construct and optimize a finite-time Brownian Carnot engine with the microscopic model proposed in the main text. A Carnot cycle consists of two isothermal processes and two adiabatic processes. From Ref. [1, 2], we know that for a given duration of time and the initial and the final values of the work parameter of an isothermal expansion (compression) process, the protocol $\lambda(t)$ that maximizes the work output (minimizes the work input) is an exponential function of time with additional instantaneous jumps at the beginning and the end. The effective temperature of the working substance is kept at a constant except for the sudden jump processes. The jump-exponential-jump form of the optimal work protocol $\lambda(t)$ and the constant effective temperature are general properties of the optimal control, and are independent of specific settings of the control time and the initial and final values of the work parameter, as long as we maximize the work output.

We model the adiabatic process to be an instantaneous jump of the work parameter λ , i.e., the process is so fast that no heat is exchanged. Then, the equation of motion of the effective temperature by Eq. (7) is reduced to

$$\dot{\theta}(t) = \frac{\lambda(t)}{\lambda(t)} \theta(t).$$
(S1)

After doing integral on both sides of Eq. (S1), we obtain the relation between the initial (final) value of the effective temperature θ_i (θ_f) and the work parameter λ_i (λ_f)

$$\frac{\theta_i}{\theta_f} = \frac{\lambda_i}{\lambda_f}.$$
(S2)

The work input in an adiabatic process is equal to the energy change

$$W = E_f - E_i, \ \langle W \rangle = \frac{f_n}{2} (\theta_f - \theta_i), \tag{S3}$$

since there is no heat exchange.

To form a closed cycle, we need to connect the two isothermal processes with the two adiabatic processes. The additional jumps and the adiabatic processes (also instantaneous jumps) merge together. The control scheme $\lambda(t)$ consists of two exponential protocols and two instantaneous jump protocols (Fig. 1). It is natural to describe the cycle with the work parameter at four "corners" in the $\lambda - \theta$ diagram (Fig. 1), and the duration $\tau_H(\tau_C)$ of two exponential protocols. Hereafter, we also use "isothermal process" to denote the exponential protocol, and "adiabatic processes" to denote the sudden jump protocol.

We formally write down the four processes as follows.

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(I) Isothermal compression. Starting from λ_1 at t = 0, the control scheme of the work parameter is $\lambda(t) = \lambda_1 (\lambda_2/\lambda_1)^{t/\tau_C}$, where τ_C is the duration of the process. For such an exponential protocol, the effective temperature of the working substance remains at a constant

$$\theta_C = \frac{\tau_C \Gamma_C}{\tau_C \Gamma_C - \ln r} T_C. \tag{S4}$$

with the quantity $r = \lambda_2/\lambda_1$ defined in Eq. (8) of the main text. The heat released to the cold heat bath and the work input in this process is

$$-\langle Q_C \rangle = \langle W_1 \rangle = \frac{f_n}{2} \Gamma_C (\theta_C - T_C) \tau_C.$$
(S5)

(II) Adiabatic compression. The work parameter is quenched suddenly from λ_2 to λ_3 with the effective temperature jumping from θ_C to θ_H accordingly, satisfying $\theta_H/\theta_C = \lambda_3/\lambda_2$. Work output in this adiabatic compression process is

$$-\langle W_2 \rangle = -\frac{f_n}{2} (\theta_H - \theta_C). \tag{S6}$$

(III) Isothermal expansion. Similar to process (I), the work parameter is varied as $\lambda(t) = \lambda_3 (\lambda_4/\lambda_3)^{(t-\tau_C)/\tau_H}$ with the duration τ_H of this process. The effective temperature of the working substance, as a constant, is explicitly

$$\theta_H = \frac{\tau_H \Gamma_H}{\tau_H \Gamma_H + \ln r} T_H. \tag{S7}$$

The heat absorbed from the hot bath and the work output in this process is

$$\langle Q_H \rangle = -\langle W_3 \rangle = \frac{f_n}{2} \Gamma_H (T_H - \theta_H) \tau_H.$$
 (S8)

(IV) Adiabatic expansion. Finally, the work parameter is quenched from λ_4 to the initial value λ_1 abruptly. The values of the work parameter satisfy $\theta_H/\theta_C = \lambda_4/\lambda_1$, which ensures a closed cycle. The work output in this adiabatic process is

$$-\langle W_4 \rangle = -\frac{f_n}{2} (\theta_C - \theta_H). \tag{S9}$$

The work output of a cycle is equal to

$$-\langle W \rangle \coloneqq -\sum_{j=1}^{4} \langle W_j \rangle$$
$$= \frac{f_n}{2} \left(\frac{\tau_H \Gamma_H \ln r}{\tau_H \Gamma_H + \ln r} T_H - \frac{\tau_C \Gamma_C \ln r}{\tau_C \Gamma_C - \ln r} T_C \right).$$
(S10)

The average power and efficiency of this finite-time Brownian Carnot engine are

$$\overline{P} \coloneqq \frac{-\langle W \rangle}{\tau_H + \tau_C} = \frac{f_n}{2(\tau_H + \tau_C)} \left(\frac{\tau_H \Gamma_H \ln r}{\tau_H \Gamma_H + \ln r} T_H - \frac{\tau_C \Gamma_C \ln r}{\tau_C \Gamma_C - \ln r} T_C \right),$$
(S11)

and

$$\eta \coloneqq -\frac{\langle W \rangle}{\langle Q_H \rangle} = 1 - \frac{1 + (\tau_H \Gamma_H)^{-1} \ln r}{1 - (\tau_C \Gamma_C)^{-1} \ln r} \frac{T_C}{T_H}.$$
(S12)

To maximize the power, we choose a fixed r and optimize \overline{P} with respect to τ_H and τ_C . By solving equations $\partial \overline{P}/\partial \tau_H = 0$ and $\partial \overline{P}/\partial \tau_C = 0$, we obtain the optimal duration of the two isothermal processes

$$\tau_H^{\max} = \frac{\ln r \left(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C}\right)}{\Gamma_H \sqrt{\Gamma_C} \left(\sqrt{T_H} - \sqrt{T_C}\right)},$$

$$\tau_C^{\max} = \frac{\ln r \left(\sqrt{\Gamma_H T_H} + \sqrt{\Gamma_C T_C}\right)}{\Gamma_C \sqrt{\Gamma_H} \left(\sqrt{T_H} - \sqrt{T_C}\right)}.$$
(S13)

Substituting Eq. (S13) into Eq. (S11), we find the maximum power with the fixed r is

$$\overline{P}_{\max} = \frac{f_n \Gamma_C \Gamma_H \left(\sqrt{T_H} - \sqrt{T_C}\right)^2}{2 \left(\sqrt{\Gamma_H} + \sqrt{\Gamma_C}\right)^2}.$$
(S14)

This result, however, is independent of r, and is the global maximum of power. Similarly substituting Eq.(S13) into Eq. (S12), the efficiency at the maximum power is obtained

$$\eta_{\rm EMP} = 1 - \sqrt{\frac{T_C}{T_H}},\tag{S15}$$

which is exactly the CA efficiency.

II. THE TRADE-OFF RELATION BETWEEN POWER AND EFFICIENCY

The trade-off relation between power and efficiency, namely the maximum power under a given efficiency, has recently attracted much attention in finite-time thermodynamics. Various trade-off relations have been found under different circumstances [3–8]. Nevertheless, these relations either are obtained based on the low-dissipation approximation, or are very loose constraints.

It is worth pointing out that in 1989, a tight trade-off relation between power and efficiency has been obtained for the endoreversible Carnot cycle with the assumption of Newton's cooling law [9],

$$\overline{P}^* \propto \frac{\eta \left(\eta_{\rm C} - \eta\right)}{1 - \eta}.\tag{S16}$$

However, due to the lack of microscopic theory in their study, they are unable to determine the control scheme associated with the maximum power for a given efficiency.

Here we derive the control scheme $\lambda(t)$ associated with the above tight trade-off relation between power and efficiency for the finite-time Brownian Carnot engine. With the expressions of the power and the efficiency, we introduce the function

$$L = \frac{f_n}{2(\tau_H + \tau_C)} \left(\frac{\tau_H \Gamma_H \ln r}{\tau_H \Gamma_H + \ln r} T_H - \frac{\tau_C \Gamma_C \ln r}{\tau_C \Gamma_C - \ln r} T_C \right) + \mu \left(1 - \frac{1 + (\tau_H \Gamma_H)^{-1} \ln r}{1 - (\tau_C \Gamma_C)^{-1} \ln r} \frac{T_C}{T_H} - \eta \right), \tag{S17}$$

where μ is the Lagrange multiplier to include the given efficiency as the constraint. The maximum power for the given efficiency η is obtained from

$$\frac{\partial L}{\partial \tau_H} = 0, \tag{S18}$$

$$\frac{\partial L}{\partial \tau_C} = 0,\tag{S19}$$

$$\frac{\partial L}{\partial \mu} = 0. \tag{S20}$$

The solution to the above equations are

$$\tau_H^* = \frac{\ln r}{\sqrt{\Gamma_H} \left(\eta_{\rm C} - \eta\right)} \left(\frac{1 - \eta}{\sqrt{\Gamma_C}} + \frac{1 - \eta_{\rm C}}{\sqrt{\Gamma_H}}\right),\tag{S21}$$

$$\tau_C^* = \frac{\ln r}{\sqrt{\Gamma_C} \left(\eta_C - \eta\right)} \left(\frac{1 - \eta}{\sqrt{\Gamma_C}} + \frac{1 - \eta_C}{\sqrt{\Gamma_H}}\right),\tag{S22}$$

$$\mu^* = \frac{f_n \Gamma_C \Gamma_H T_H \left[(2 - \eta)\eta - \eta_C \right]}{2(1 - \eta)^2 (\sqrt{\Gamma_H} + \sqrt{\Gamma_C})^2}.$$
(S23)

Since we have adopted the exponential protocol, the control scheme $\lambda(t)$ of the whole cycle for the CA engine is

$$\lambda(t) = \begin{cases} \lambda_1 (\lambda_2 / \lambda_1)^{t/\tau_C^*} & 0 < t < \tau_C^* \\ \lambda_3 (\lambda_4 / \lambda_3)^{(t-\tau_C^*)/\tau_H^*} & \tau_C^* < t < \tau_H^* \end{cases}.$$
(S24)

Substituting τ_H^* and τ_C^* into the expression of the power, we obtain the maximum power for a given efficiency η as

$$\overline{P}^* = \frac{f_n T_H \Gamma_C \Gamma_H}{2 \left(\sqrt{\Gamma_C} + \sqrt{\Gamma_H}\right)^2} \frac{\eta(\eta_C - \eta)}{(1 - \eta)},\tag{S25}$$

which agrees with the trade-off relation obtained in Ref. [9]. Notice that Eq. (S25) is independent of the compression ratio r, and is also irrelevant to the symmetry ($\Gamma_H = \Gamma_C$) of dissipation.

We emphasize that the dependence of the maximum power on the efficiency is always in the factorized form, which is universal and independent of the cooling rate $\Gamma_H(\Gamma_C)$ in the two isothermal processes. In comparison with Ref. [3], the trade-off relation (S25) is tighter and this upper bound of the power can be achieved with the explicit control scheme.

III. CALCULATION OF THE GENERATING FUNCTION

Based on the method proposed in Ref. [10], we derive the joint generating function of work and heat for both the finite-time driving process (finite-time isothermal process). The system is coupled to only one heat bath with the exchanged heat Q[X], and the work parameter of the system is tuned with the performed work W[X]. In stochastic thermodynamics, both heat and work is defined on the trajectory $X = \{E(t)|0 \le t \le \tau\}$ [10, 11]. The joint generating function of work and heat is defined as the path integral in the trajectory space

$$I(u,s) \coloneqq \left\langle e^{uQ+sW} \right\rangle \tag{S26}$$

$$= \int D[X]p[X]e^{uQ[X]+sW[X]},$$
(S27)

where p[X] denotes the probability of a given trajectory X.

The method [10] to be applied requires that

(I) The system always obeys a Maxwell-Boltzmann distribution in the energy space described by an effective temperature θ , whose evolution is governed by

$$\dot{\theta} = \Phi_t(\theta),$$
 (S28)

Here, $\Phi_t(\theta)$ contains the contribution of the work. When performed work performed on the system, the temperature of the system increases. If no modulation is performed, $\Phi_t(\theta) = -\Gamma(\theta - T_b)$ is Newton's law of cooling with a constant cooling rate Γ . T_b is the temperature of the bath.

(II) The increment in work is proportional to the stochastic energy of the system,

$$dW = \alpha(t)E(t)dt, \tag{S29}$$

where α_t is determined by the control protocol, and is chosen as $\alpha_t = \lambda/\lambda$ relating to the work parameter $\lambda = \lambda(t)$; E = E(t) is the stochastic energy of the system affected by both the heat exchange and the work performed. As a result, $\Phi_t(\theta)$ is explicitly

$$\Phi_t(\theta) = -\Gamma(\theta - T_{\rm b}) + \alpha_t \theta. \tag{S30}$$

(III) The structure (shape) of the partition function does not depend on λ . Generally, the partition function $Z_{\lambda}(\beta)$ should depend on the work parameter λ . The inverse temperature is $\beta = 1/\theta$, where we set $k_{\rm B} = 1$ for convenience. For example, for a quantum harmonic oscillator, the frequency determines energy-level spacing which affects the $Z_{\lambda}(\beta)$. But here, we require $Z_{\lambda}(\beta)$ to be in the same form of λ which means

$$Z_{\lambda}(\beta) = g(\lambda) \times Z(\beta), \tag{S31}$$

where $g(\lambda)$ is a function of the work parameter, and $Z(\beta)$ characterize the form of the energy density.

All conditions are satisfied in the current microscopic model. The probability distribution of the energy is

$$p(E|\theta) = \frac{E^{\frac{f_n}{2} - 1} e^{-\beta E}}{Z(\beta)}.$$
 (S32)

The partition function is $Z(\beta) = \int_0^\infty E^{f_n/2-1} e^{-\beta E} dE = \beta^{-f_n/2} \Gamma(f_n/2)[12]$, which is also the normalization factor in the energy space. The average energy is determined by

$$\langle E \rangle = -\frac{\partial \ln \left[Z(\beta) \right]}{\partial \beta} = \frac{f_n}{2} \theta.$$
(S33)

Namely, the heat capacity of the working substance is $f_n/2$, which remains a constant.

Along a given trajectory $X = \{E(t) | 0 \le t \le \tau\}$, the increment of work at time t is

$$dW = \alpha_t E dt, \tag{S34}$$

and the increment of heat is obtained inversely from the first law

$$dQ = dE - \alpha_t E dt. \tag{S35}$$

A. Discretization to calculate the joint generating function of work and heat

We discretize the dynamics with $t_j = j\epsilon$, j = 0, 1, ..., N, N + 1 and $\epsilon = \tau/(N + 1)$, the thermal distribution of the system is $p(E_j|\theta_j)$ with the inverse temperature $\beta_j = 1/\theta_j$ at t_j . The work parameter $\lambda(t)$ remains a constant during each time slice $t_j < t < t_{j+1}$, and is quenched at the moment t_j . As a result, the work $W_j = \ln[\lambda(t_j^+)/\lambda(t_j^-)]E_j \approx \alpha_j E_j \epsilon$ is performed at each moment t_j with the neglected exchanged heat, while the heat $Q_j = E_{j+1} - E_j - \alpha_j E_j \epsilon$ is generated during each time slice. The evolution between every adjacent moment t_j and t_{j+1} can be described by the transition probability $\mathcal{R}_{j+1}^j = \mathcal{R}(E_{j+1}, t_{j+1}; E_j, t_j)$ (also named as the propagator). To ensure the condition (I), the propagator \mathcal{R}_{j+1}^j should maps a local equilibrium state to another local equilibrium state

$$\int dE_j p(E_j | \theta_j) \mathcal{R}_{j+1}^j = p(E_{j+1} | \theta_{j+1}).$$
(S36)

The temperature of the next step is

$$\theta_{j+1} = \theta_j + \Phi_j(\theta_j)\epsilon, \tag{S37}$$

where we denote $\Phi_i(\cdot) = \Phi_{t_i}(\cdot)$ for simplicity

We calculate the joint generating function I(u, s) as follows. We rewrite the joint generating function with the N + 1 discrete steps

$$I(u,s) = \int \prod_{j=0}^{N+1} dE_j \prod_{j=0}^{N} \left(\mathcal{R}_{j+1}^j\right) p(E_0|\tilde{\psi}_0) \prod_{j=0}^{N} e^{uQ_j + sW_j},$$
(S38)

where the initial value $\tilde{\psi}_0$ is a temperature-like quantity. We use a different notation ψ to distinguish with the temperature θ . The meaning of "tilde" will be clarified later [by Eq. (S63)]. If there is no previous process, $\tilde{\psi}_0 = \theta_0$ is the local temperature of the system. Otherwise, the initial value $\tilde{\psi}_0$ is obtained from the previous process.

From the first law, we can rewrite

$$uQ_{j} + sW_{j} = (s - u)\alpha_{j}E_{j}\epsilon + u(E_{j+1} - E_{j}).$$
(S39)

We then consider the first step

$$\mathcal{R}_1^0 p(E_0|\tilde{\psi}_0) e^{(s-u)W_0 + u(E_1 - E_0)} \tag{S40}$$

$$=\mathcal{R}_{1}^{0} \frac{e^{-E_{0}/\psi_{0}}}{Z(1/\tilde{\psi}_{0})} e^{(s-u)\epsilon\alpha_{0}E_{0}+u(E_{1}-E_{0})}$$
(S41)

$$=\mathcal{R}_{1}^{0}\frac{e^{-E_{0}/\psi_{0}}}{Z(1/\tilde{\psi}_{0})}e^{uE_{1}},$$
(S42)

where ψ_0 is

$$\psi_0 = 1/[1/\psi_0 - (s-u)\epsilon\alpha_0 + u].$$
(S43)

After doing the integral over E_0 , we obtain

$$e^{uE_1} \int dE_0 \mathcal{R}_1^0 \frac{e^{-E_0/\psi_0}}{Z(1/\tilde{\psi}_0)}$$
 (S44)

$$=e^{uE_1}\frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)}\int dE_0 \mathcal{R}_1^0 \frac{e^{-E_0/\psi_0}}{Z(1/\psi_0)}$$
(S45)

$$=e^{uE_1}\frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)}\frac{e^{-E_1/\varphi_1}}{Z(1/\varphi_1)},$$
(S46)

where the intermediate variable φ_1 , according to Eq. (S37), is obtained as

$$\varphi_1 = \psi_0 + \Phi_0(\psi_0)\epsilon \tag{S47}$$

$$=\psi_0 - \Gamma(\psi_0 - T_{\rm b})\epsilon + \alpha_0\psi_0\epsilon.$$
(S48)

We next calculate the second step with the integral over ${\cal E}_1$

$$\frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)} \int dE_1 \mathcal{R}_2^1 \frac{e^{-E_1/\varphi_1} e^{uE_1}}{Z(1/\varphi_1)} e^{(s-u)W_1 + u(E_2 - E_1)}$$
(S49)

$$=\frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)}e^{uE_2}\int dE_1 \mathcal{R}_2^1 \frac{e^{-\overline{[1/\varphi_1 - (s-u)\alpha_1\epsilon]}E_1}}{Z(1/\varphi_1)}$$
(S50)

$$=\frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)}\frac{Z(1/\psi_1)}{Z(1/\varphi_1)}\frac{e^{uE_2}e^{-E_2/\varphi_2}}{Z(1/\varphi_2)}.$$
(S51)

In Eq. (S50), we have set

$$\frac{1}{\psi_1} = \frac{1}{\varphi_1} - (s - u)\alpha_1\epsilon.$$
(S52)

It is similarly to do the integral over $E_2, E_3, \dots E_N$ and the result is

$$\int \prod_{j=0}^{N} dE_j \prod_{j=0}^{N} \left(\mathcal{R}_{j+1}^j\right) p(E_0|\tilde{\psi}_0) \prod_{j=0}^{N} e^{uQ_j + sW_j}$$
(S53)

$$= \frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)} \frac{Z(1/\psi_1)}{Z(1/\varphi_1)} \cdots \frac{Z(1/\psi_N)}{Z(1/\varphi_N)} \frac{e^{-(1/\varphi_{N+1}-u)E_{N+1}}}{Z(1/\varphi_{N+1})}.$$
(S54)

We integrate over E_{N+1} and obtain

$$I(u,s) = \frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)} \frac{Z(1/\psi_1)}{Z(1/\varphi_1)} \cdots \frac{Z(1/\psi_N)}{Z(1/\varphi_N)} \frac{Z(1/\tilde{\psi}_{N+1})}{Z(1/\varphi_{N+1})},$$
(S55)

where

$$\tilde{\psi}_{N+1} = \frac{1}{1/\varphi_{N+1} - u}.$$
(S56)

The reduction formulas are

$$\varphi_n = \psi_{n-1} + \Phi_{n-1}(\psi_{n-1})\epsilon \tag{S57}$$

$$\psi_n = \frac{1}{1/\varphi_n - (s-u)\alpha_n \epsilon},\tag{S58}$$

which lead to

$$\frac{\psi_n - \psi_{n-1}}{\epsilon} = \Phi_{n-1}(\psi_{n-1}) + (s-u)\alpha_n\psi_{n-1}^2.$$
(S59)

The continuum limit is

$$\frac{d\psi}{dt} = \Phi_t(\psi) + (s-u)\alpha_t\psi^2.$$
(S60)

The initial condition is obtained from Eq. (S43) as

$$\psi(0) = \frac{\tilde{\psi}_0}{1 + u\tilde{\psi}_0}.$$
(S61)

At the end $t = \tau$, Eq. (S56) leads to the final condition

$$\tilde{\psi}(\tau) = \frac{\psi(\tau)}{1 - u\psi(\tau)}.$$
(S62)

We can define a shifted temperature-like variable $\tilde{\psi}(t)$ as

$$\tilde{\psi}(t) \coloneqq \frac{\psi(t)}{1 - u\psi(t)}.$$
(S63)

The corresponding differential equation is

$$\frac{d\tilde{\psi}}{dt} = (1+u\tilde{\psi})^2 \Phi_t \left(\frac{\tilde{\psi}}{1+u\tilde{\psi}}\right) + (s-u)\alpha_t \tilde{\psi}^2, \tag{S64}$$

and the initial value is $\tilde{\psi}(0) = \tilde{\psi}_0$.

The joint generating function Eq. (S55) becomes

$$\ln I(u,s) = \ln \frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)} + \sum_{n=1}^N \ln \frac{Z(1/\psi_n)}{Z(1/\varphi_n)} + \ln \frac{Z(1/\tilde{\psi}_{N+1})}{Z(1/\varphi_{N+1})}.$$
(S65)

The first and the third terms are explicitly

$$\ln \frac{Z(1/\psi_0)}{Z(1/\tilde{\psi}_0)} \approx \ln \frac{Z(1/\tilde{\psi}_0 + u)}{Z(1/\tilde{\psi}_0)}$$
(S66)

$$= -\frac{f_n}{2}\ln\left(1+u\tilde{\psi}_0\right),\tag{S67}$$

 $\quad \text{and} \quad$

$$\ln \frac{Z(1/\tilde{\psi}_{N+1})}{Z(1/\varphi_{N+1})} \approx \ln \frac{Z(1/\psi_{N+1}-u)}{Z(1/\psi_{N+1})}$$
(S68)

$$= -\frac{f_n}{2}\ln\left(1 - u\psi(\tau)\right) \tag{S69}$$

$$=\frac{f_n}{2}\ln\left(1+u\tilde{\psi}(\tau)\right).$$
(S70)

The second term is explicitly

$$\sum_{n=1}^{N} \ln \frac{Z(1/\psi_n)}{Z(1/\varphi_n)} = \sum_{n=1}^{N} \ln \frac{Z(1/\psi_n)}{Z(1/\psi_n + (s-u)\alpha_n \epsilon)}$$
(S71)

$$\approx -\sum_{n=1}^{N} (s-u)\alpha_n \frac{Z'(1/\psi_n)}{Z(1/\psi_n)} \epsilon$$
(S72)

$$= -\int_{0}^{\tau} (s-u)\alpha_{t} \frac{Z'(1/\psi(t))}{Z(1/\psi(t))} dt$$
(S73)

$$= (s-u) \int_0^\tau \alpha_t \left\langle E \right\rangle (\psi(t)) dt, \tag{S74}$$

where the average internal energy is

$$\langle E \rangle \left(\theta \right) = -\frac{\partial \ln Z}{\partial \beta} \Big|_{\beta = 1/\theta} = \frac{f_n}{2} \theta.$$
 (S75)

The final result of the joint generating function is

$$\ln I(u,s) = \frac{f_n}{2} \left[\ln \left(\frac{1 + u\tilde{\psi}(\tau)}{1 + u\tilde{\psi}_0} \right) + (s - u) \int_0^\tau \alpha_t \psi(t) dt \right].$$
(S76)

Notice that the temperature-like variable $\psi(t)$ is solved from the differential equation (S60) with the initial condition (S61) and the final condition (S62).

As follows, we discuss the joint generating function for the sudden jump process and the exponential protocol process. These two processes are used to form the finite-time Carnot cycle.

B. Sudden jump process

In a sudden jump process, work is performed with the neglected heat exchange. The work parameter is quenched from λ^- to λ^+ with temperature-like quantity changed from $\tilde{\psi}^-$ to $\tilde{\psi}^+$. No exchanged heat is produced in such a sudden process, and we can set u = 0. Therefore, $\tilde{\psi}$ becomes the same as ψ in this process. The differential equation (S64) [or Eq. (S60)] becomes

$$\frac{d\tilde{\psi}}{dt} = \alpha_t \tilde{\psi} + s \alpha_t \tilde{\psi}^2, \tag{S77}$$

The change of the energy comes from the work

$$d\tilde{\psi} = \tilde{\psi} \left(1 + s\tilde{\psi} \right) \frac{d\lambda}{\lambda},\tag{S78}$$

and $\tilde{\psi}$ becomes a function of λ in this process, explicitly as

$$\frac{\tilde{\psi}}{1+\tilde{\psi}s} = \frac{\lambda}{\lambda^{-}} \frac{\tilde{\psi}^{-}}{1+\tilde{\psi}^{-}s}.$$
(S79)

Thus, the final value $\tilde{\psi}^+$ is associated with $\tilde{\psi}^-$ as

$$\frac{\tilde{\psi}^+}{1+\tilde{\psi}^+s} = \frac{\lambda^+}{\lambda^-} \frac{\tilde{\psi}^-}{1+\tilde{\psi}^-s}.$$
(S80)

Notice that $\tilde{\psi}^+$ serves as the initial value $\tilde{\psi}_0$ of the next process. If there is a previous process, $\tilde{\psi}^-$ is substituted with $\tilde{\psi}(\tau)$ of the previous process. The joint generating function of the sudden jump process is

$$I_{\text{jump}}(u,s) = \left[1 + \left(1 - \frac{\lambda^+}{\lambda^-}\right)\tilde{\psi}^- s\right]^{-\frac{f_n}{2}}.$$
(S81)

C. Exponential protocol process

Equation (S60) becomes a Ricatti equation (time-independent) for a exponential process $\alpha_t = \alpha = \text{const}$, and is explicitly

$$\dot{\psi} = -\Gamma(\psi - T_{\rm b}) + \alpha\psi + (s - u)\alpha\psi^2.$$
(S82)

We solve the above differential equation with the initial condition $\psi(0) = \tilde{\psi}_0/(1+u\tilde{\psi}_0)$. The solution to Eq. (S82) is

$$\psi(t) = \frac{\Gamma - \alpha + \sqrt{D(s-u)} \tan\left(\frac{c_1+t}{2}\sqrt{D(s-u)}\right)}{2\alpha(s-u)},\tag{S83}$$

where

$$D(x) = 4\alpha\Gamma T_{\rm b}x - (\Gamma - \alpha)^2.$$
(S84)

The joint generating function of the process is

$$\ln I_{\exp}(u,s) = \frac{f_n}{2} \left[\ln \left(\frac{1+u\tilde{\psi}(\tau)}{1+u\tilde{\psi}_0} \right) + (s-u)\alpha \int_0^\tau \psi(t)dt \right].$$
(S85)

The initial condition gives

$$\psi(0) = \frac{\Gamma - \alpha + \sqrt{D(s-u)} \tan\left(\frac{c_1}{2}\sqrt{D(s-u)}\right)}{2\alpha(s-u)},\tag{S86}$$

which determine the constant c_1 as

$$\tan\left(\frac{c_1}{2}\sqrt{D(s-u)}\right) = \frac{2\alpha(s-u)\psi(0) - (\Gamma - \alpha)}{\sqrt{D(s-u)}}.$$
(S87)

The integral in Eq. (S85) is rewritten as

$$(s-u)\alpha \int_0^\tau \psi(t)dt = \frac{\Gamma - \alpha}{2}\tau + \ln\left[\frac{\cos\left(\frac{c_1}{2}\sqrt{D(s-u)}\right)}{\cos\left(\frac{c_1+t}{2}\sqrt{D(s-u)}\right)}\right]$$
(S88)

$$=\frac{\Gamma-\alpha}{2}\tau-\ln\left[\cos\left(\frac{t}{2}\sqrt{D(s-u)}\right)-\frac{2\alpha(s-u)\psi(0)-(\Gamma-\alpha)}{\sqrt{D(s-u)}}\sin\left(\frac{t}{2}\sqrt{D(s-u)}\right)\right].$$
 (S89)

We obtain the joint generating function as

$$I_{\exp}(u,s) = \left\{ \frac{\exp\left[(\Gamma - \alpha)\tau/2\right]\left[1 + u\tilde{\psi}(\tau)\right]/\left[1 + u\tilde{\psi}_{0}\right]}{\cos\left(\frac{t}{2}\sqrt{D(s-u)}\right) - \frac{2\alpha(s-u)\psi(0) - (\Gamma - \alpha)}{\sqrt{D(s-u)}}\sin\left(\frac{t}{2}\sqrt{D(s-u)}\right)} \right\}^{\frac{f_{n}}{2}}.$$
(S90)

If there is a previous process, we need to substitute ψ_0 as the final value $\psi(\tau)$ of the previous process.

D. Joint generating function of the finite-time Carnot cycle

Now we calculate the joint generating function of the finite-time Carnot cycle. The heat is absorbed from the hot bath and released to the cold, namely, $Q_H > 0$ and $Q_C < 0$. The joint generating function $I(u_H, u_C, s)$ of work and heat for a whole finite-time cycle is

$$I(u_H, u_C, s) = \int D[X] p[X] e^{uQ_H[X] + u_C Q_C[X] + sW[X]}.$$
(S91)

Notice that the system is in contact with only one heat bath at a time. We start from A point (Fig. 1 in main text) and calculate the joint generating function of a whole cycle.

1. Process I Isothermal compression

We define the compression ratio as $r = \lambda_2/\lambda_1$. In this process, the work parameter is varied exponentially with the time

$$\lambda(t) = \lambda_1 r^{\frac{t}{\tau_C}}.$$
(S92)

The quantity α of this process is explicitly $\alpha_C = \ln r / \tau_C$. The initial temperature of the working substance is (which is also the initial condition)

$$\tilde{\psi}_0 = \theta_C = \frac{\Gamma_C}{\Gamma_C - \alpha_C} T_C.$$
(S93)

If we consider the heat engine runs for many cycles, $\tilde{\psi}_0$ takes the final value of $\tilde{\psi}(\tau_t)$ of the previous cycle. According to Eq. (S90), the joint generating function of this process is

$$I_{\rm I}(u_H, u_C, s) = \left\{ \frac{\exp\left[(\Gamma_C - \alpha_C)\tau_C/2\right]\left[1 + u_C\tilde{\psi}(\tau_C^-)\right]/\left[1 + u_C\tilde{\psi}_0\right]}{\cos\left(\Omega_C\tau_C\right) - \frac{\alpha_C(s - u_C)\psi(0) - (\Gamma_C - \alpha_C)/2}{\Omega_C}\sin\left(\Omega_C\tau_C\right)} \right\}^{\frac{j\pi}{2}},\tag{S94}$$

where the initial value is $\psi(0) = \tilde{\psi}_0/(1 + u_C \tilde{\psi}_0)$, τ_C^- represents the moment of the end of process I, and the frequency Ω_C is

$$\Omega_C = \frac{\sqrt{D_C(s - u_C)}}{2} = \sqrt{\alpha_C \Gamma_C T_C(s - u_C) - (\Gamma_C - \alpha_C)^2/4}.$$
(S95)

Notice that the expression $I_{\rm I}$ does not contain u_H . But if there exist a previous cycle, $\tilde{\psi}_0$ also relies on u_H . Therefore, we always write $I(u_H, u_C, s)$ for all the processes in a finite-time heat engine cycle.

2. Process II Adiabatic compression.

At the end of process I, the temperature-like quantity is $\tilde{\psi}(\tau_C)$, which is plugged as the initial value for the process II as a sudden quench of the work parameter λ from λ_2 to λ_3 at the moment $t = \tau_C$. Notice that the ratio of the work parameter in this process satisfy $\lambda_3/\lambda_2 = \theta_H/\theta_C$. According to Eq. (S80), the temperature-like quantity after the quench $\tilde{\psi}^+_{\tau_C}$ is associated with $\tilde{\psi}^-_{\tau_C} = \tilde{\psi}(\tau^-_C)$ as

$$\tilde{\psi}_{\tau_C}^+ = \frac{\lambda_3 \tilde{\psi}(\tau_C^-)}{\lambda_2 + s(\lambda_2 - \lambda_3) \tilde{\psi}(\tau_C^-)}$$
(S96)
$$= \frac{\tilde{\psi}(\tau_C^-)}{\tilde{\psi}(\tau_C^-)}.$$
(S97)

$$=\frac{\psi(\tau_C^-)}{1-\eta-s\eta\tilde{\psi}(\tau_C^-)},\tag{S97}$$

where the efficiency of the finite-time Carnot cycle is $\eta = 1 - \theta_C/\theta_H$. According to Eq. (S81), the joint generating function of process II is

$$I_{\rm II}(u_H, u_C, s) = \left[1 - \frac{s\eta}{1 - \eta} \tilde{\psi}(\tau_C^-)\right]^{-\frac{I_R}{2}}.$$
 (S98)

Notice that the value of the temperature-like quantity $\tilde{\psi}(\tau_C^-)$ depends on u_C and s.

3. Process III Isothermal expansion

The work parameter is varied exponentially with the time

$$\lambda(t) = \lambda_3 r^{-\frac{t-\tau_C}{\tau_H}}.$$
(S99)

The quantity α of this process is explicitly $\alpha_H = -\ln r/\tau_H$. The initial condition of this process is

$$\tilde{\psi}(\tau_C^+) = \tilde{\psi}_{\tau_C}^+,\tag{S100}$$

or

$$\psi(\tau_C^+) = \frac{\tilde{\psi}_{\tau_C}^+}{1 + u_H \tilde{\psi}_{\tau_C}^+}.$$
(S101)

The adiabatic process is completed suddenly at the moment $t = \tau_C$, and we use τ_C^+ to indicate the beginning of process III. Equation (S90) gives the joint generating function of this process

$$I_{\rm III}(u_H, u_C, s) = \left\{ \frac{\exp\left[(\Gamma_H - \alpha_H)\tau_H/2\right] \frac{1 + u_H \tilde{\psi}(\tau_C + \tau_H^-)}{1 + u_H \tilde{\psi}_{\tau_C}^+}}{\cos\left(\Omega_H \tau_H\right) - \frac{\alpha_H (s - u_H)\psi(\tau_C^+) - (\Gamma_H - \alpha_H)/2}{\Omega_H} \sin\left(\Omega_H \tau_H\right)} \right\}^{\frac{f_H}{2}},$$
(S102)

where $\tau_C + \tau_H^-$ represents the moment of the end of process III, and the frequency Ω_H is

$$\Omega_H = \frac{1}{2}\sqrt{D_H(s - u_H)} = \sqrt{\alpha_H \Gamma_H T_H(s - u_H) - (\Gamma_H - \alpha_H)^2/4}.$$
(S103)

4. Process IV Adiabatic expansion

The last adiabatic is a sudden quench with the work parameter λ changing from λ_4 to λ_1 at the moment $t = \tau_C + \tau_H$. The ratio of the work parameter satisfies $\lambda_1/\lambda_4 = \theta_C/\theta_H$. According to Eq. (S80), the temperature-like quantity after the quench $\tilde{\psi}^+_{\tau_C+\tau_H}$ is associated with $\tilde{\psi}^-_{\tau_C+\tau_H} = \psi(\tau_C + \tau_H)$ as

$$\tilde{\psi}^+_{\tau_C+\tau_H} = \frac{\lambda_1 \tilde{\psi}(\tau_C + \tau_H^-)}{\lambda_4 + s(\lambda_4 - \lambda_1)\tilde{\psi}(\tau_C + \tau_H^-)}$$
(S104)

$$=\frac{(1-\eta)\psi(\tau_C+\tau_H^-)}{1+s\eta\tilde{\psi}(\tau_C+\tau_H^-)},\tag{S105}$$

According to Eq. (S81), the joint generating function of process IV is

$$I_{\rm IV}(u_H, u_C, s) = \left[1 + s\eta \tilde{\psi}(\tau_C + \tau_H^-)\right]^{-\frac{f_n}{2}}.$$
 (S106)

5. Joint generating function of a whole cycle

The joint generating function $I_{cycle}(u_H, u_C, s)$ of a whole cycle is

$$I_{\text{cycle}} = I_{\text{I}} \times I_{\text{II}} \times I_{\text{III}} \times I_{\text{IV}}$$

$$= \left\{ \frac{e^{(\Gamma_{C}\tau_{C} + \Gamma_{H}\tau_{H})/2} \left[1 - \frac{s\eta}{1 - \eta} \tilde{\psi}(\tau_{C}^{-})\right]^{-1} \left[1 + s\eta \tilde{\psi}(\tau_{C} + \tau_{H}^{-})\right]^{-1} \frac{1 + u_{C}\tilde{\psi}(\tau_{C}^{-})}{1 + u_{C}\tilde{\psi}_{0}} \frac{1 + u_{H}\tilde{\psi}(\tau_{C} + \tau_{H}^{-})}{1 + u_{H}\tilde{\psi}_{\tau_{C}}^{+}}}}{\left[\cos\left(\Omega_{C}\tau_{C}\right) - \frac{\alpha_{C}(s - u_{C})\psi(0) - (\Gamma_{C} - \alpha_{C})/2}{\Omega_{C}} \sin\left(\Omega_{C}\tau_{C}\right)\right] \left[\cos\left(\Omega_{H}\tau_{H}\right) - \frac{\alpha_{H}(s - u_{H})\psi(\tau_{C}^{+}) - (\Gamma_{H} - \alpha_{H})/2}{\Omega_{H}} \sin\left(\Omega_{H}\tau_{H}\right)\right]} \right\}^{\frac{f_{n}}{2}}}$$
(S107)

The evolution of $\psi(t)$ is governed by

$$\dot{\psi} = \begin{cases} -\Gamma_C(\psi - T_C) + \alpha_C \psi + (s - u_C)\alpha_C \psi^2, & 0 < t < \tau_C, \\ -\Gamma_H(\psi - T_H) + \alpha_H \psi + (s - u_H)\alpha_H \psi^2, & \tau_C < t < \tau_H. \end{cases}$$
(S109)

The initial $\psi(0)$ and the connecting conditions $\psi(\tau_C^+)$ are

$$\psi(0) = \frac{\tilde{\psi}_0}{1 + u_C \tilde{\psi}_0} \tag{S110}$$

$$\psi(\tau_C^+) = \frac{\tilde{\psi}_{\tau_C}^+}{1 + u_H \tilde{\psi}_{\tau_C}^+}.$$
(S111)

where $\tilde{\psi}_{\tau_C}^+$ is associated with $\tilde{\psi}(\tau_C^-)$ through process II. As follows, we give the generating functions for work and efficiency respectively.

6. Generating function for work

By setting $u_C = 0$ and $u_H = 0$, the generating function for work is $I_{\text{cycle}}^W(s) = I_{\text{cycle}}(0, 0, s)$, explicitly as

$$I_{\text{cycle}}^{W}(s) = \left\{ \frac{e^{(\Gamma_{C}\tau_{C} + \Gamma_{H}\tau_{H})/2} \left[1 - \frac{s\eta}{1 - \eta}\psi(\tau_{C}^{-})\right]^{-1} \left[1 + s\eta\psi(\tau_{C} + \tau_{H}^{-})\right]^{-1}}{\left[\cos\left(\Omega_{C}\tau_{C}\right) - \frac{\alpha_{C}s\psi(0) - (\Gamma_{C} - \alpha_{C})/2}{\Omega_{C}}\sin\left(\Omega_{C}\tau_{C}\right)\right] \left[\cos\left(\Omega_{H}\tau_{H}\right) - \frac{\alpha_{H}s\psi(\tau_{C}^{+}) - (\Gamma_{H} - \alpha_{H})/2}{\Omega_{H}}\sin\left(\Omega_{H}\tau_{H}\right)\right]}\right\}^{\frac{f_{n}}{2}}$$
(S112)

with $\Omega_C = \sqrt{\alpha_C \Gamma_C T_C s - (\Gamma_C - \alpha_C)^2/4}$ and $\Omega_H = \sqrt{\alpha_H \Gamma_H T_H s - (\Gamma_H - \alpha_H)^2/4}$. Notice that Eq. (S63) indicates $\tilde{\psi}(t) = \psi(t)$ in this situation.

7. Generating function for efficiency

By setting $u_H = s\eta$ and $u_C = 0$, the generating function for efficiency is $I_{\text{cycle}}^{\zeta}(s) = I_{\text{cycle}}(s\eta, 0, s)$, explicitly as

$$I_{\text{cycle}}^{\zeta}(s) = \left\{ \frac{e^{(\Gamma_C \tau_C + \Gamma_H \tau_H)/2} \left[1 - \frac{s\eta}{1 - \eta} \tilde{\psi}_{\tau_C}^-\right]^{-1} \left[1 + s\eta \tilde{\psi}_{\tau_C}^+\right]^{-1}}{\left[\cos\left(\Omega_C \tau_C\right) - \frac{\alpha_C s\psi(0) - (\Gamma_C - \alpha_C)/2}{\Omega_C} \sin\left(\Omega_C \tau_C\right)\right] \left[\cos\left(\Omega_H \tau_H\right) - \frac{\alpha_H (s - s\eta)\psi(\tau_C^+) - (\Gamma_H - \alpha_H)/2}{\Omega_H} \sin\left(\Omega_H \tau_H\right)\right]}\right\}^{\frac{f_{R}}{2}}$$

$$(S113)$$

with $\Omega_C = \sqrt{\alpha_C \Gamma_C T_C s} - (\Gamma_C - \alpha_C)^2 / 4$ and $\Omega_H = \sqrt{\alpha_H \Gamma_H T_H (s - u_H)} - (\Gamma_H - \alpha_H)^2 / 4$.. Since $u_C = 0$, $\psi(t) = \psi(t)$ during process I ($0 < t < \tau_C$). Plugging Eq. (S97) into Eq. (S113), we obtain

$$I_{\text{cycle}}^{\zeta}(s) = \left\{ \frac{e^{(\Gamma_C \tau_C + \Gamma_H \tau_H)/2}}{\left[\cos\left(\Omega_C \tau_C\right) - \frac{\alpha_C s \psi(0) - (\Gamma_C - \alpha_C)/2}{\Omega_C} \sin\left(\Omega_C \tau_C\right)\right] \left[\cos\left(\Omega_H \tau_H\right) - \frac{\alpha_H (s - s\eta)\psi(\tau_C^+) - (\Gamma_H - \alpha_H)/2}{\Omega_H} \sin\left(\Omega_H \tau_H\right)\right]}{\left(\text{S114}\right)} \right\}^{\frac{j_n}{2}}$$

E. Self-consistency check of the joint generating function: fluctuation theorem

The fluctuation theorem for heat engines [13] provides a microscopic understanding of Carnot's theorem. For the model presented here which never reaches thermal equilibrium with the heat bath, the fluctuation theorem takes a different form. In the following, we derive the fluctuation theorem for our finite-time Brownian Carnot engine.

Suppose the engine operates between a hot bath at temperature T_H and a cold bath at temperature T_C . Moreover, the the working substance of the engine obeys the same Maxwell-Boltzmann distribution at an effective temperature θ_C at both the beginning and the end of the cycle (point A in Fig. 1(a)). The ratio of the probability of a trajectory X to that of its time reversal trajectory \tilde{X} under the time reversed protocol is related to the entropy change of the system and the heat functional of the forward trajectory [14],

$$\frac{p[X]}{\tilde{p}[\tilde{X}]} = \exp(\Delta s^{bath} + \Delta s^{sys}) = \exp\left(-\frac{Q_H}{T_H} - \frac{Q_C}{T_C}\right)\frac{\rho_i(x_i)}{\rho_f(x_f)} = \exp\left(-\frac{Q_H}{T_H} - \frac{Q_C}{T_C} + \frac{E_f - E_i}{\theta_C}\right),\tag{S115}$$

where $Q_{H,C}$ denote the trajectory heat transferred from the bath at temperature $T_{H,C}$ to the system, $\rho_{i,f}$ refer to the initial and the final probability distribution of the system, and $E_{i,f}$ denote the initial and the final stochastic energy of the system in a cycle. According to the first law of thermodynamics along a single trajectory, the stochastic energy change comprises the trajectory work and heat,

$$E_f - E_i = W + Q_H + Q_C.$$
 (S116)

Accordingly, we can rewrite Eq. (S115) into

$$p[X] \exp\left[Q_H\left(\frac{1}{T_H} - \frac{1}{\theta_C}\right) + Q_C\left(\frac{1}{T_C} - \frac{1}{\theta_C}\right) - \frac{W}{\theta_C}\right] = \tilde{p}[\tilde{X}].$$
(S117)

By taking ensemble average, we obtain the following fluctuation theorem

$$\left\langle \exp\left[Q_H\left(\frac{1}{T_H} - \frac{1}{\theta_C}\right) + Q_C\left(\frac{1}{T_C} - \frac{1}{\theta_C}\right) - \frac{W}{\theta_C}\right] \right\rangle = 1.$$
(S118)

By choosing the beginning and the ending of a closed cycle at point C [see Fig. 1(a)], we find another fluctuation theorem

$$\left\langle \exp\left[Q_H\left(\frac{1}{T_H} - \frac{1}{\theta_H}\right) + Q_C\left(\frac{1}{T_C} - \frac{1}{\theta_H}\right) - \frac{W}{\theta_H}\right]\right\rangle' = 1,\tag{S119}$$

where the prime indicates the average is over a different ensemble of trajectories from the former one. With Jensen's inequality $\exp \langle x \rangle \leq \langle \exp(x) \rangle$, Carnot's theorem can be derived as a corollary of both fluctuation theorems

$$\eta = -\frac{\langle W \rangle}{\langle Q_H \rangle} \le 1 - \frac{T_C}{T_H} = \eta_C, \tag{S120}$$

where we have used the conservation of the energy for a periodic steady closed cycle $\langle W \rangle + \langle Q_H \rangle + \langle Q_C \rangle = \langle E_f \rangle - \langle E_i \rangle = 0.$

Since the fluctuation theorem (S118) can also be written as $I_{\text{cycle}}(1/T_H - 1/\theta_C, 1/T_C - 1/\theta_C, -1/\theta_C) = 1$, we can use it to check the validity of the analytical expression of the joint generating function $I_{\text{cycle}}(u_H, u_C, s)$. In this case, the initial condition is $\tilde{\psi}_0 = \theta_C$, and a full cycle can be decomposed into four processes as in Fig. 1. The analytical expressions of the generating function associated with these processes have been obtained previously. For a specific set of parameters $u_H = 1/T_H - 1/\theta_C$, $u_C = 1/T_C - 1/\theta_C$, $s = -1/\theta_C$, the auxiliary quantity $\psi(t)$ in the cycle can be obtained by solving Eq. (S109)

$$\psi(t) = \begin{cases} T_C & 0 < t < \tau_C \\ T_H & \tau_C < t < \tau_C + \tau_H, \end{cases}$$
(S121)

and the shifted temperature-like variable is $\tilde{\psi}(t) \equiv \theta_C$ throughout the whole cycle. Ω_C and Ω_H are explicitly $\Omega_C = i(\Gamma_C + \alpha_C)/2$ and $\Omega_H = i(\Gamma_H + \alpha_H)/2$. Therefore, the joint generating function of each process becomes

$$I_{\rm I} = e^{-\frac{f_n}{2}\alpha_C\tau_C} \tag{S122}$$

$$I_{\rm II} = (1 - \eta)^{\frac{f_n}{2}} \tag{S123}$$

$$I_{\rm III} = e^{-\frac{f_n}{2}\alpha_H \tau_H} \tag{S124}$$

$$I_{\rm IV} = (1 - \eta)^{-\frac{f_n}{2}}.$$
 (S125)

Thus, we confirm the fluctuation theorem $I_{\text{cycle}}(1/T_H - 1/\theta_C, 1/T_C - 1/\theta_C, -1/\theta_C) = 1$ for heat engines.

F. Variances of the power and the efficiency

The derivatives of the generating function give the moments of work distribution. The average work is

$$\langle W \rangle = \frac{\partial}{\partial s} \left\langle e^{sW} \right\rangle|_{s=0} = \frac{f_n}{2} (\theta_H - \theta_C) \ln r, \qquad (S126)$$

which can also be obtained from Eqs.(4,7) in the main text. The fluctuation of the work output in a cycle can also be calculated from the generating function

$$\left\langle \Delta W^2 \right\rangle = \left\langle W^2 \right\rangle - \left\langle W \right\rangle^2 = \left[\frac{\partial^2}{\partial s^2} \left\langle e^{sW} \right\rangle - \left(\frac{\partial}{\partial s} \left\langle e^{sW} \right\rangle \right)^2 \right]|_{s=0}.$$
(S127)

The fluctuation of the power is defined as

$$\operatorname{Var}(P) \coloneqq \frac{\left\langle \Delta W^2 \right\rangle}{(\tau_C + \tau_H)^2},\tag{S128}$$

and the relative fluctuation of the power is given by

$$\frac{\operatorname{Var}(P)}{\overline{P}^{2}} = \frac{\left\langle \Delta W^{2} \right\rangle}{\left\langle W \right\rangle^{2}} \\
= \frac{4}{f_{n}} \left[\frac{1}{\kappa_{C} \tau_{C}} \left(\frac{1-\eta}{\eta} \right)^{2} + \frac{1}{\kappa_{H} \tau_{H}} \frac{1}{\eta^{2}} \right] + \frac{4}{f_{n}} \left[\frac{1-\mathrm{e}^{-\kappa_{H} \tau_{H}}}{\kappa_{H}^{2} \tau_{H}^{2}} \left(\frac{\kappa_{H}^{2} \tau_{H}^{2}}{(\ln r)^{2}} - \frac{1}{\eta^{2}} \right) - \frac{1-\mathrm{e}^{-\kappa_{C} \tau_{C}}}{\kappa_{C}^{2} \tau_{C}^{2}} \left(\frac{1-\eta}{\eta} \right)^{2} \right] \\
+ \frac{4}{f_{n}} \frac{(1-\mathrm{e}^{-\kappa_{C} \tau_{C}})(1-\mathrm{e}^{-\kappa_{H} \tau_{H}})}{\kappa_{C} \tau_{C} \kappa_{H} \tau_{H}} \left(\frac{\kappa_{H} \tau_{H}}{\ln r} - \frac{1}{\eta} \right) \frac{1-\eta}{\eta}, \tag{S129}$$

where $\kappa_C = \Gamma_C - \ln r / \tau_C$ and $\kappa_H = \Gamma_H - \ln r / \tau_H$. The fluctuation of efficiency is given by

$$\operatorname{Var}(\zeta) \coloneqq \frac{\left\langle (W + \eta Q_H)^2 \right\rangle}{\langle Q_H \rangle^2} = \frac{4(1 - \eta)^2}{f_n} \left[\frac{1}{\kappa_C \tau_C} + \frac{1}{\kappa_H \tau_H} \right] - \frac{4(1 - \eta)^2}{f_n} \left[\frac{1 - \mathrm{e}^{-\kappa_C \tau_C}}{\kappa_C^2 \tau_C^2} + \frac{1 - \mathrm{e}^{-\kappa_H \tau_H}}{\kappa_H^2 \tau_H^2} + -\frac{(1 - \mathrm{e}^{-\kappa_C \tau_C})(1 - \mathrm{e}^{-\kappa_H \tau_H})}{\kappa_C \tau_C \kappa_H \tau_H} \right].$$
(S130)

In order to achieve the maximum power, we should choose $\tau_C = \tau_C^{\text{max}}$ and $\tau_H = \tau_H^{\text{max}}$ according to Eq. (S13) to realize the CA engine. As a result, the efficiency η should be replaced by η_{CA} .

Finally, we would like to analyze the scaling of the variance of the power and the efficiency for the CA engine. When $\kappa_C \tau_C^{\max}, \kappa_H \tau_H^{\max} \gg 1$, both variances $\operatorname{Var}(P)$ and $\operatorname{Var}(\zeta)$ decrease inversely with the period of the cycle $\tau_{\rm t} \coloneqq \tau_{C}^{\rm max} + \tau_{H}^{\rm max}$ (note that for the CA engine, $\ln r$ is related to $\tau_{\rm t}$ by Eq. (S13)), that is

$$\operatorname{Var}(P) = \left\{ \frac{4}{f_n \eta_{CA}^2} \frac{\left[(1 - \eta_{CA})^2 + 1/\delta \right] (1 + \delta)}{\sqrt{\Gamma_C \Gamma_H}} \frac{1}{\tau_t} + \mathcal{O}(\tau_t^{-2}) \right\} \overline{P}_{\max}^2,$$
(S131)

and

$$\operatorname{Var}(\zeta) = \frac{4(1 - \eta_{CA})^2}{f_n \sqrt{\Gamma_C \Gamma_H}} \frac{(1 + \delta)^2}{\delta} \frac{1}{\tau_t} + \mathcal{O}(\tau_t^{-2}),$$
(S132)

where $\delta = \sqrt{\Gamma_H T_H / (\Gamma_C T_C)}$.

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- [1] D. S. P. Salazar and S. A. Lira, Phys. Rev. E 99, 062119 (2019).
- [2] M. H. Rubin, Phys. Rev. A 19, 1277 (1979).
- [3] N. Shiraishi, K. Saito, and H. Tasaki, Phys. Rev. Lett. 117, 190601 (2016).
- [4] V. Holubec and A. Ryabov, J. Stat. Mech: Theory Exp. **2016**, 073204 (2016).
- [5] J. Gonzalez-Ayala, A. C. Hernández, and J. M. M. Roco, Phys. Rev. E 95, 022131 (2017).
- [6] P. Pietzonka and U. Seifert, Phys. Rev. Lett. **120**, 190602 (2018).
- [7] Y.-H. Ma, D. Xu, H. Dong, and C.-P. Sun, Phys. Rev. E 98, 042112 (2018).
- [8] P. Abiuso and M. Perarnau-Llobet, Phys. Rev. Lett. 124, 110606 (2020).
- [9] L. Chen and Z. Yan, J. Chem. Phys. 90, 3740 (1989).
- [10] D. S. P. Salazar, Phys. Rev. E 101, 030101 (2020).
- [11] K. Sekimoto, Prog. Theor. Phys. Suppl. **130**, 17 (1998). [12] $\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy$ is the Gamma function.
- [13] N. Sinitsyn, J. Phys. A: Math. Theor. 44, 405001 (2011).
- [14] U. Seifert, Rep. Prog. Phys. 75, 126001 (2012).