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Microstate counting via Bethe Ansätze in the 4d $\mathcal{N} = 1$ superconformal index

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Abstract: We study the superconfomal index of four-dimensional toric quiver gauge theories using a Bethe Ansatz approach recently applied by Benini and Milan. Relying on a particular set of solutions to the corresponding Bethe Ansatz equations we evaluate the superconformal index in the large N limit, thus avoiding to take any Cardy-like limit. We present explicit results for theories arising as a stack of N D3 branes at the tip of toric Calabi-Yau cones: the conifold theory, the suspended pinch point gauge theory, the first del Pezzo theory and $Y^{p,q}$ quiver gauge theories. For a suitable choice of the chemical potentials of the theory we find agreement with predictions made for the same theories in the Cardy-like limit. However, for other regions of the domain of chemical potentials the superconformal index is modified and consequently the associated black hole entropy receives corrections. We work out explicitly the simple case of the conifold theory.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory

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1 Introduction

The understanding of the quantum microstates responsible for the entropy of black holes has long been one of the central questions in the path to a quantum theory of gravity. In the context of the AdS/CFT correspondence it has recently been shown that the entropy of certain asymptotically AdS_4 black holes admits a microscopic explanation in terms of a topologically twisted field theory [1] (see [2, 3] for reviews with extensive lists of references).

More recently, the question of microstates for asymptotically AdS_5 black holes dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM), which was originally tackled in [4], has been revisited providing a microscopic entropy matching using various approaches. A broader interpretation of localization was successfully put forward in [5] while an analysis of the free-field partition function in a particular limit led to the entropy in [6] (see also [7]). Both these groups relied on a particular Cardy-like limit to evaluate the path integral. Another approach, put forward by Benini and Milan in [8], attacked the superconformal index using a Bethe Ansatz approach developed in [9]. Understanding that the superconformal index can be written as a sum over solutions to Bethe Ansatz equations was demonstrated in [10] based on interesting relations between observables on manifolds of different topologies developed in [11]. One key advantage of the Bethe Ansatz approach is that it does not require taking the Cardy limit and thus opens the door for a more in-depth understanding of the superconformal index. In this brief note we simply generalize the large N results obtained for $\mathcal{N} = 4$ SYM using the Bethe Ansatz approach to a large class of $\mathcal{N} = 1$ 4d supersymmetric field theories.

Other recent studies demonstrating that the Cardy-like limit of the superconformal index of 4d $\mathcal{N} = 4$ SYM accounts for the entropy function, whose Legendre transform corresponds to the entropy of the holographically dual AdS_5 rotating black holes were

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presented in [12, 13]. Such analysis has by now been extended to generic $\mathcal{N}=1$ supersymmetric gauge theories [14, 15] including a particular description specialized to arbitrary $\mathcal{N}=1$ toric quiver gauge theories, observing that the corresponding entropy function can be interpreted in terms of the toric data [16]. These powerful results rest on systematic studies of the Cardy limit developed in, for example, [17–21].

In this note we verify that a class of holonomies of the form $u_i - u_j = \frac{\tau}{N}(i-j)$, used prominently in [8] for the case of $\mathcal{N}=4$ SYM, can be generalized to evaluate the superconformal index of generic $\mathcal{N}=1$ four-dimensional superconformal field theories.

The rest of the note is organized as follows. In section 2 we show that a particular class of holonomies solves the Bethe Ansatz equation for generic 4d $\mathcal{N}=1$ gauge theories and proceed to evaluate the superconformal index in the large N limit. Section 3 works out explicitly the index for a number of superconformal field theories. We find that there is always a way of redefining the chemical potentials suitably, such that the superconformal index obtained reproduces successfully the entropy of the dual AdS_5 black holes upon extremization of its Legendre transform. We also focus on the conifold theory in which the simplicity of the superconformal index allows us to study it for some region of the domain of chemical potentials that can provide a black hole entropy with corrections purely depending on the angular velocity τ. We conclude in section 5.

Note added. After this manuscript was originally submitted to the arxiv we received [22] with a considerable overlap with this work. The authors of [22] perform a more exhaustive analysis of the behavior of the entropy function for different regions in the domain of complex chemical potentials. The present version of this manuscript contains substantial changes with respect to the first two versions appearing in arxiv. We have essentially found that, selecting a set of chemical potentials that ensures an optimal obstruction of cancellations between bosonic and fermionic contributions to the superconformal index, one can always find a region of chemical potentials where the index accounts for the black hole entropy. This resolves an apparent tension between our conclusions and the ones subsequently reported in [22].

2 Bethe Ansatz approach to the superconformal index

In this section we generalize the solutions to the Bethe Ansatz type equations proposed in [8–10] to evaluate the superconformal index of $\mathcal{N}=4$ SYM to generic 4d $\mathcal{N}=1$ supersymmetric gauge theories. For concreteness we will work in the context of toric quiver gauge theories which are naturally decorated with extra global and baryonic symmetries but the results apply more generally to $4d$ $\mathcal{N}=1$ supersymmetric gauge theories.

Consider a generic $\mathcal{N}=1$ theory with semi-simple gauge group G, flavor symmetry G_F and non-anomalous $U(1)_R$ R-symmetry. The matter content of this theory is taken to be n_χ chiral multiplets Φ_a in representations $\mathfrak{R}_a$ of G, with flavor weights ω_a in some representation $\mathfrak{R}_F$ of G_F and superconformal R-charge r_a. Let us start by introducing the following quantities which are related to global fugacities and holonomies in the Cartan of the gauge group:

$$p = e^{2\pi i\tau}, \quad q = e^{2\pi i\sigma}, \quad v_\alpha = e^{2\pi i\xi_\alpha}, \quad z_i = e^{2\pi i u_i} \tag{2.1}$$

and the R-charge chemical potential which is fixed by supersymmetry to:

$$\nu_R = \frac{1}{2}(\tau + \sigma). \tag{2.2}$$

With the above data, the integral representation for the superconformal index can be written as [23, 24]:

$$\mathcal{I}(p,q;v) = \frac{(p;p)_\infty^{\mathrm{rk}(G)}(q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \frac{\prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathfrak{R}_a}\Gamma_e\left[(pq)^{r_a/2} z^{\rho_a} v^{\omega_a}; p, q\right]}{\prod_{\alpha\in\Delta}\Gamma_e(z^\alpha;p,q)} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i}. \tag{2.3}$$

The integration variables z_i parameterize the maximal torus of the gauge group G and the integration contour is the product of $\mathrm{rk}(G)$ unit circles. Following standard notation, ρ_a are the weights of the representation $\mathfrak{R}_a$, α parameterize the roots of G and $|\mathcal{W}_G|$ is the order of the Weyl group. The notation adopted also denotes $z^{\rho_a} \equiv \prod_{i=1}^{\mathrm{rk}(G)} z_i^{\rho_a^i}$ and $v^{\omega_a} = \prod_{\alpha=1}^{\mathrm{rk}(G_F)} v_\alpha^{\omega_a^\alpha}$. The other functions involved in the expression for the superconformal index are the Elliptic Gamma function

$$\Gamma_e(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1}q^{n+1}/z}{1 - p^m q^n z}, \quad |p| < 1, \ |q| < 1, \tag{2.4}$$

and the q-Pochhamer symbol

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n), \quad |q| < 1. \tag{2.5}$$

An interesting result of [9] and [8], based on [10], is to rewrite the above superconformal index in terms of solutions to certain Bethe Ansatz like system of equations taking the generic form of

$$Q_i(u;\xi,\nu_R,\omega) = 1 \ \forall \ i = 1,\ldots,\mathrm{rk}(G) \tag{2.6}$$

where ω is such that $r\tau = s\sigma$ with r and s coprime integer numbers (in practice we will evaluate the equations for $r = s$). Furthermore, the "Bethe Ansatz operator" is defined as:

$$Q_i(u;\xi,\nu_R,\omega) = \prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathfrak{R}_a} P\left(\rho_a(u) + \omega_a(\xi) + r_a\nu_R;\ \omega\right)^{\rho_a^i}, \tag{2.7}$$

where

$$P(u;\omega) = \frac{e^{-\pi i \frac{u^2}{\omega} + \pi i u}}{\theta_0(u;\omega)}. \tag{2.8}$$

Thus,

$$P\left(\rho_a(u) + \omega_a(\xi) + r_a\nu_R;\ \omega\right) = \frac{e^{-\pi i\frac{1}{\omega}(\rho_a(u)+\omega_a(\xi)+r_a\nu_R)^2 + \pi i(\rho_a(u)+\omega_a(\xi)+r_a\nu_R)}}{\theta_0\left(\rho_a(u) + \omega_a(\xi) + r_a\nu_R;\omega\right)}, \tag{2.9}$$

where:

$$\theta_0\left(u;\omega\right)=\left(e^{2\pi iu};e^{2\pi i\omega}\right)_\infty\left(e^{2\pi i(\omega-u)};e^{2\pi i\omega}\right)_\infty. \tag{2.10}$$

Now we would like to evaluate the Bethe Ansatz equations for the case of a toric quiver gauge theory. Toric quiver gauge theories describe the low energy dynamics of a stack of N D3 branes probing the tip of a toric Calabi-Yau singularity; there is by now a vast literature detailing how to construct a supersymmetric field theory given toric data (see, for example, [25, 26]). Consider a toric quiver gauge theory whose gauge group G has $n_{\rm v}$ simple factors (in all the $\mathcal{N}=1$ quiver gauge theories we will deal with, the number of simple factors coincides with the number of vector multiplets). We focus, for concreteness, on the case in which all the gauge group factors are $\mathrm{SU}(N_a)$, a goes from 1 to $n_{\rm v}$, with $N_a=N\ \forall\, a$, the same numerical value for all nodes. In these theories the weight vectors ρ are such that for any bi-fundamental field Φ_{ab} (notice that in the more generic notation used in [9], the index a of Φ_a would now split into ab):

$$\rho_{ij}^{\Phi_{ab}}\left(u\right)\equiv u_{ij}^{ab}\equiv u_i^a-u_j^b. \tag{2.11}$$

Let us now evaluate the operator $P(u;\omega)$ for a generic field Φ_{ab} (when Φ_{ab} transforms in the adjoint representation of G then, in this notation, $a=b$):

$$Q_{i_a}\left(u;\xi,\tau,\sigma,\omega\right)=\prod_{(a,b)}\prod_{j_b}\prod_{\rho_{ij}^{(a,b)}}P\left(u_{i_a}-u_{j_b}+\sum_{l=1}^{d-1}q^l_{(a,b)}\Delta_l+r_{ab}\nu_R\right)^{\rho_{ij}^{(a,b)}}, \tag{2.12}$$

where (a,b) run over all the fields Φ_{ab} for a fixed a and r_{ab} are the R-charges of the fields Φ_{ab}. The $d-1$ fugacities correspond to the flavor symmetries appearing in the generic toric gauge theories that we will study, d is the number of external points of the toric diagram that are related to the quivers defining the theory [16]. If we denote $\langle a,b\rangle\equiv(a,b)|_{\rho_{ij}^{(a,b)}>0}$, which implies:

$$\begin{aligned}Q_{i_a}\left(u;\xi,\tau,\sigma,\omega\right)&=\prod_{\langle a,b\rangle}\prod_{j_b}\prod_{\rho_{ij}^{\langle a,b\rangle}}\left[\frac{P\left(u_{i_a}-u_{j_b}+\sum_{l=1}^{d-1}q^l_{\langle a,b\rangle}\Delta_l+r_{ab}\nu_R\right)}{P\left(u_{j_b}-u_{i_a}+\sum_{l=1}^{d-1}q^l_{\langle b,a\rangle}\Delta_l+r_{ba}\nu_R\right)}\right]^{\rho_{ij}^{\langle a,b\rangle}}\\&=\prod_{\langle a,b\rangle}\prod_{j_b}\prod_{\rho_{ij}^{\langle a,b\rangle}}\left[\frac{e^{-2\pi i(-u_{i_a}+u_{j_b})}\theta_0\left(-u_{j_b}+u_{i_a}+\sum_{l=1}^{d-1}q^l_{\langle a,b\rangle}\Delta_l+r_{ab}\nu_R;\omega\right)}{\theta_0\left(u_{i_a}-u_{j_b}+\sum_{l=1}^{d-1}q^l_{\langle b,a\rangle}\Delta_l+r_{ba}\nu_R;\omega\right)}\right]^{\rho_{ij}^{\langle a,b\rangle}}\\&=e^{-2\pi i\sum_{j_b}(u_{i_a}-u_{j_b})}\prod_{\langle a,b\rangle}\prod_{j_b}\frac{\theta_0\left(-u_{i_a}+u_{j_b}+\sum_{l=1}^{d-1}q^l_{\langle a,b\rangle}\Delta_l+r_{ab}\nu_R;\omega\right)}{\theta_0\left(-u_{j_b}+u_{i_a}+\sum_{l=1}^{d-1}q^l_{\langle b,a\rangle}\Delta_l+r_{ab}\nu_R;\omega\right)}.\end{aligned} \tag{2.13}$$

Let us now introduce a Lagrange multiplier λ_a that accounts for the constraint ensuring the condition $\sum_i u_i^a=0$ [8], with its help, equation (2.13) can be written as:

$$Q_{i_a}\left(u;\xi,\tau,\sigma,\omega\right)=e^{2\pi i\left(\sum_b\lambda_b-\sum_{j_b}u_{ij}^{ab}\right)}\prod_{\langle a,b\rangle}\prod_{j_b}\frac{\theta_0\left(-u_{ij}^{ab}+\sum_{l=1}^{d-1}q^l_{\langle a,b\rangle}\Delta_l+r_{ab}\nu_R;\omega\right)}{\theta_0\left(-u_{ji}^{ba}+\sum_{l=1}^{d-1}q^l_{\langle b,a\rangle}\Delta_l+r_{ba}\nu_R;\omega\right)}, \tag{2.14}$$

where we have denoted $u_{i_a} - u_{j_b} \equiv u_{ij}^{ab}$. Restricting ourselves to the case with $\tau = \sigma$, we would like to propose a set of u_{ij}^{ab} that makes (2.14) equal to 1, thus solving the Bethe Ansatz equation (2.6). It is natural to make an attempt with a direct generalization of the type of solution encountered in [8], namely: $u_{ij}^{ab} = \frac{\tau}{N}(i_a - j_b)$. These solutions appeared first in [27] while evaluating the topologically twisted of 4d $\mathcal{N} = 1$ theories on $T^2 \times S^2$ in the high temperature limit; it was later shown in [28] that such configuration provides an exact solution to the Bethe Ansatz equations.

Consider one generic factor entering in (2.14) for a fixed value of b:

$$
\begin{aligned}
\prod_{j_b} \frac{\theta_0\left(u_{ij}^{ab}+\Delta_{ab};\omega\right)}{\theta_0\left(-u_{ij}^{ab}+\Delta_{ba};\omega\right)}\Big|_{u_{ij}^{ab}=\frac{\tau}{N}(i_a-i_b)} &= \frac{\prod_{k=0}^{i_a-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ab}\right)\times\prod_{k=i_a-N}^{-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ab}\right)}{\prod_{k=0}^{N-i_a}\theta_0\left(\frac{\tau}{N}k+\Delta_{ba}\right)\times\prod_{k=i_a-1}^{-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ba}\right)} \\
\text{with } \Delta_{ab} &\equiv \sum_{l=1}^{d-1} q^l_{\langle a,b\rangle}\Delta_l + r_{ab}\tau \\
\prod_{j_b} \frac{\theta_0\left(u_{ij}^{ab}+\Delta_{ab};\omega\right)}{\theta_0\left(-u_{ij}^{ab}+\Delta_{ba};\omega\right)} &= \frac{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ab}\right)\times\prod_{k=i_a-N}^{-1}\left(-e^{2\pi i\tau\frac{k}{N}}e^{2\pi i\Delta_{ab}}\right)}{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ba}\right)\times\prod_{k=1-i_a}^{-1}\left(-e^{2\pi i\tau\frac{k}{N}}e^{2\pi i\Delta_{ba}}\right)} \\
&= \frac{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ab}\right)}{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ba}\right)} \\
&\quad \times\left(e^{\pi i(-1+\tau)}\right)^{(2i_a-N-1)} e^{-2\pi i[i_a(\Delta_{ab}+\Delta_{ba})-N\Delta_{ab}-\Delta_{ba}]} \\
&= \frac{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ab}\right)}{\prod_{k=0}^{N-1}\theta_0\left(\frac{\tau}{N}k+\Delta_{ba}\right)} \\
&\quad \times e^{2\pi i_a(\tau-1-\Delta_{ba}-\Delta_{ab})} e^{\pi i[(1-\tau)(1+N)+2(\Delta_{ba}+N\Delta_{ab})]} \\
&\equiv F(\Delta_{ab},\Delta_{ba},\tau)e^{2\pi i_a(\tau-1-\Delta_{ba}-\Delta_{ab})}
\end{aligned}
\tag{2.15}
$$

In (2.15) we have used the following properties of the θ_0 function:

$$
\begin{aligned}
\theta_0(u+n+m\tau;\tau) &= -e^{-2\pi m i u-\pi i m\tau(m-1)}\theta_0(u;\tau) \\
\theta_0(u;\tau) = \theta_0(\tau-u;\tau) &= -e^{2\pi i u}\theta_0(-u;\tau),
\end{aligned}
\tag{2.16}
$$

and for the sake of compactness we have absorbed all the factors independent of i_a in the function $F(\Delta_{ab},\Delta_{ba},\tau)$. Inserting (2.15) back into (2.14) leads to multiplying all the results obtained in (2.15) for all n_a values of b connected with a via some field Φ_{ab}:

$$
\begin{aligned}
Q_{i_a}(u;\xi,\tau) &= e^{2\pi i\left(\sum_b\lambda_b-\sum_{j_b}\frac{\tau}{N}(i_a-j_b)\right)}F_a(\tau)e^{2\pi i_a\left[n_a(\tau-1)-\sum_{b=1}^{n_a}(\Delta_{ba}+\Delta_{ab})\right]} \\
\text{where } F_a(\tau) &\equiv \prod_{b=1}^{n_a}F(\Delta_{ab},\Delta_{ba},\tau) \\
\sum_{b=1}^{n_a}(\Delta_{ab}+\Delta_{ba}) &= \sum_{b=1}^{n_a}r_{ab}\tau \\
&\Downarrow \\
Q_{i_a}(u;\xi,\tau) &= e^{2\pi i\left(\sum_b\lambda_b-n_a\frac{\tau}{N}\left(Ni_a-\frac{N(N-1)}{2}\right)\right)}F_a(\tau)e^{2\pi i_a n_a(\tau-1)} \\
&= e^{2\pi i\left(\sum_b\lambda_b-n_a\tau\frac{N-1}{2}\right)}F_a(\tau)e^{-2\pi i_a n_a}.
\end{aligned}
\tag{2.17}
$$

Upon a proper choice for the Lagrange multipliers we can ensure that:

$$Q_{i_a}\left(u;\xi,\tau\right)=e^{-2\pi i_a n_a}=1\blacksquare. \tag{2.18}$$

2.1 Evaluation of the index

The formula for the superconformal index in terms of solutions to the Bethe Ansatz like equations reads [9, 10]:

$$\begin{aligned}\mathcal{I}\left(p,q;v\right)&=\kappa_G\sum_{\hat{u}\in\mathfrak{M}_{\mathrm{BAE}}}\mathcal{Z}_{\mathrm{tot}}\left(\hat{u};\xi,\nu_R,r\omega,s\omega\right)H\left(\hat{u};,\xi,\nu_R,\omega\right)^{-1} \\ \kappa_G&=\frac{\left(p;p\right)_\infty^{\mathrm{rk}(G)}\left(q;q\right)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \\ \mathcal{Z}_{\mathrm{tot}}\left(u;\xi,\nu_R,r\omega,s\omega\right)&=\sum_{\{m_{i_a}\}=1}^{rs}\mathcal{Z}\left(u-m\omega;\xi,\nu_R,r\omega,s\omega\right) \\ \mathcal{Z}\left(u;\xi,\nu_R,r\omega,s\omega\right)&=\frac{\prod_{\Phi_{ab}}\prod_{i_a\neq j_b}\Gamma_e\left(u_{i_a}-u_{j_b}+\Delta_{ab};\tau,\sigma\right)}{\prod_{\alpha\in\Delta}\Gamma_e\left(\alpha\left(u\right);\tau,\sigma\right)} \\ H\left(u;\xi,\nu_R,\omega\right)&=\det\left[\frac{1}{2\pi i}\frac{\partial Q_{i_a}\left(u;,\xi,\nu_R,\omega\right)}{\partial u_{j_b}}\right]_{i_a j_b}.\end{aligned} \tag{2.19}$$

We assume that dominant contributions to the index in the large N limit will come from terms analogous to those dominating the expression obtained in [8] for the $\mathcal{N}=4$ SYM theory. This implies that in order to investigate the large N limit of (2.19), we only need to consider the following term:

$$\begin{aligned}\Gamma_e\left(u_{ij}^{ab}+\Delta_{ab};\tau,\tau\right)&=\frac{e^{-\pi i\mathcal{Q}\left(u_{ij}^{ab}+\Delta_{ab};\tau,\tau\right)}}{\theta_0\left(\frac{u_{ij}^{ab}+\Delta_{ab}}{\tau};-\frac{1}{\tau}\right)}\times\prod_{k=0}^{\infty}\frac{\psi\left(\frac{k+1+u_{ij}^{ab}}{\tau}\right)}{\psi\left(\frac{k-u_{ij}^{ab}-\Delta_{ab}}{\tau}\right)} \\ \mathcal{Q}\left(u;\tau,\sigma\right)&=\frac{u^3}{3\tau\sigma}-\frac{\tau+\sigma-1}{2\tau\sigma}u^2+\frac{\left(\tau+\sigma\right)^2+\tau\sigma-3\left(\tau+\sigma\right)+1}{6\tau\sigma}u+\frac{\left(\tau+\sigma-1\right)\left(\tau+\sigma-\tau\sigma\right)}{12\tau\sigma} \\ \mathcal{Q}\left(u+\Delta;\tau,\tau\right)&=\frac{u^3}{3\tau^2}+u^2\left(\frac{\Delta}{\tau^2}-\frac{2\tau-1}{2\tau^2}\right)+u\left(\frac{1-6\tau+5\tau^2}{6\tau^2}+\frac{\Delta^2}{\tau^2}-\frac{2\tau-1}{\tau^2}\Delta\right) \\ &\quad-\frac{\Delta^2}{2\tau^2}\left(2\tau-1\right)+\frac{\Delta}{6\tau^2}\left(5\tau^2-6\tau+1\right)+\frac{1}{12\tau^2}\left(2\tau-1\right)\left(2\tau-\tau^2\right)+\frac{\Delta^3}{\tau^2}.\end{aligned} \tag{2.20}$$

Note that, the leading contribution coming from the vector multiplets can be obtained from (2.20) by setting $\Delta_{ab}=0$. In the large N limit we can write:

$$\log\mathcal{I}\big|_{\text{large }N}=\sum_{\Phi_{ab}}\sum_{i_a,j_b}\log\Gamma_e\left(u_{ij}^{ab}+\Delta_{ab};\tau,\tau\right)\big|_{\text{large }N}-\sum_{\alpha\in\Delta}\Gamma_e\left(\alpha\cdot u,\tau,\tau\right)\big|_{\text{large }N}. \tag{2.21}$$

As a clarifying example, let us now analyze the case of $\mathcal{N}=4$ SYM theory already studied in [8] and peroform the same calculation using the toric data language of [16]. The corresponding Φ_{ab} are the three chiral fields $\Phi_{1,2,3}$ appearing in the superpotential:

$$W=\mathrm{Tr}\left(\Phi_1\left[\Phi_2,\Phi_3\right]\right), \tag{2.22}$$

with the associated chemical potentials being $\Delta_{1,2,3}$. Accordinng to our definition of the chemical potentials we have that, for the R-charge assignment used in [8]:

$$\begin{aligned}\Delta_{\Phi_1} &= \Delta_1 \\ \Delta_{\Phi_2} &= \Delta_2 \\ \Delta_{\Phi_3} &= 2\tau - \Delta_1 - \Delta_2\end{aligned} \tag{2.23}$$

Using the identity:

$$\Gamma_e\left(\Delta + 2\tau; \tau, \tau\right) = \frac{1}{\Gamma_e\left(-\Delta; \tau, \tau\right)} \tag{2.24}$$

reduces (2.21) to the following expression:

$$\begin{aligned}\log \mathcal{I}\big|_{\text{Large } N} &= \sum_{i,j} \log \Gamma_e\left(u^1_{ij} + \Delta_1; \tau, \tau\right)\big|_{\text{Large } N} + \log \Gamma_e\left(u^2_{ij} + \Delta_2; \tau, \tau\right)\big|_{\text{Large } N} \\ &\quad - \log \Gamma_e\left(u^3_{ij} + \Delta_1 + \Delta_2; \tau, \tau\right)\big|_{\text{Large } N} - \frac{i\pi N^2}{3\tau^2}\tau\left(\tau - \frac{1}{2}\right)(\tau - 1) \\ &= -\frac{i\pi N^2}{3\tau^2}\left([\Delta_1]_\tau - \tau\right)\left([\Delta_1]_\tau - \tau + \frac{1}{2}\right)\left([\Delta_1]_\tau - \tau + 1\right) \\ &\quad - \frac{i\pi N^2}{3\tau^2}\left([\Delta_2]_\tau - \tau\right)\left([\Delta_2]_\tau - \tau + \frac{1}{2}\right)\left([\Delta_2]_\tau - \tau + 1\right) \\ &\quad + \frac{i\pi N^2}{3\tau^2}\left([\Delta_1 + \Delta_2]_\tau - \tau\right)\left([\Delta_1 + \Delta_2]_\tau - \tau + \frac{1}{2}\right)\left([\Delta_1 + \Delta_2]_\tau - \tau + 1\right) \\ &\quad - \frac{i\pi N^2}{3\tau^2}\tau\left(\tau - \frac{1}{2}\right)(\tau - 1),\end{aligned} \tag{2.25}$$

where $[\Delta_{ab}]_\tau$ is defined such that $[\Delta]_\tau = \Delta \mod 1$ [8] and depends on the region withing the domain of complex chemical potentials one is evaluating (for a more detailed description of this function see also [22]). If $|\Delta_{ab}| < 1$, then:

$$\log \mathcal{I}\big|_{\text{Large } N} = -\frac{i\pi N^2}{\tau^2}\Delta_1\Delta_2\left(2\tau - \Delta_1 - \Delta_2\right), \tag{2.26}$$

which is indeed the necessary structure in order for the superconformal index of $\mathcal{N} = 4$ SYM to account for the entropy of the dual AdS$_5$ black hole [8].

Before proceeding to generic toric quiver gauge theories, let us comment on the choice of R-charge assignment, since one might expect a more symmetric one based on a-maximization. We notice that, if one chooses a set of chemical potentials and R-charges as the one used in [16], namely where $r_1 = r_2 = r_3 = \frac{2}{3}$, in contrast with the choice $r_1 = r_2 = 0$, $r_3 = 2$, then the use of identity (2.24) is not directly possible. This means that, if one starts with the data suggested by a-maximization [16] ($r_1 = r_2 = r_3 = \frac{2}{3}$), then (2.26) should be understood in terms of shifted chemical potentials that would permit some of the arguments of the Elliptic Gamma functions in (2.21) to have the structure

$\Delta + 2\tau$ as needed in (2.24). Specifically, we have:

$$\begin{aligned}
\Delta_{\Phi_1} + \frac{2}{3}\tau &= \Delta_1 + \frac{2}{3}\tau = \left(\Delta_1 + \frac{2}{3}\tau\right) + \frac{2}{3}\tau - \frac{2}{3}\tau \to \Delta_1 \\
\Delta_{\Phi_2} + \frac{2}{3}\tau &= \Delta_2 + \frac{2}{3}\tau = \left(\Delta_2 + \frac{2}{3}\tau\right) + \frac{2}{3}\tau - \frac{2}{3}\tau \to \Delta_2 \\
\Delta_{\Phi_3} + \frac{2}{3}\tau &= -\Delta_1 - \Delta_2 + \frac{2}{3}\tau \to -\Delta_1 - \Delta_2 + 2\tau.
\end{aligned} \tag{2.27}$$

We can either interpret this as a suitable redefinition of the chemical potentials wich does not affect the physical R-charge obtained via a-maximization or rather as a computation done directly with the more naive R-charge assignment used in [8]. Let us now explore more generically the consequences of shifting Δ_{ab} in such a way that the arguments of the elliptic Gamma functions in (2.21) look either like Δ or $\Delta + 2\tau$. Suppose we do such a shift obtaining that a certain number, let us call this number n_s, of the total of n_χ chiral fields contributions to (2.21) are of the form $\Delta + 2\tau$. Thus, the leading contribution in N to $\log \mathcal{I}$ takes the form:

$$\begin{aligned}
\log \mathcal{I} = &-\frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} \left([s_{ab}\Delta_{ab}]_\tau - \tau\right) \left([s_{ab}\Delta_{ab}]_\tau - \tau + \frac{1}{2}\right) \left([s_{ab}\Delta_{ab}]_\tau - \tau + 1\right) \\
&- \frac{i\pi N^2}{3\tau^2} \sum_{\mathrm{v}} \tau \left(\tau - \frac{1}{2}\right)(\tau - 1) \\
= &-\frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} \left[[s_{ab}\Delta_{ab}]_\tau \left([s_{ab}\Delta_{ab}]_\tau + \frac{1}{2}\right)\left([s_{ab}\Delta_{ab}]_\tau + 1\right) - 3\tau \left[s_{ab}\Delta_{ab}\right]_\tau^2 \right] \\
&- \frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} \left[3\tau^2 \left[s_{ab}\Delta_{ab}\right]_\tau - 3\tau \left[s_{ab}\Delta_{ab}\right]_\tau\right] \\
&+ \frac{i\pi N^2}{3\tau^2} \left(n_\chi - 2n_s - n_{\mathrm{v}}\right) \tau \left(\tau - \frac{1}{2}\right)(\tau - 1),
\end{aligned} \tag{2.28}$$

where $[\Delta_{ab}]_\tau$ is defined such that $[\Delta]_\tau = \Delta \mod 1$ [8], the sum $\sum_{\mathrm{v}}$ is carried over the n_{v} vector multiplets and n_χ is the number of chiral fields, s_{ab} is 1 if Φ_{ab} effectively has R-charge 0 and -1 if it has $R-$ charge 2 with a new set of chemical potentials. Conservation of U(1) charges implies $\sum_{\Phi_{ab}} [\Delta_{ab}]_\tau = 0$, which allows us to eliminate every linear term in $[\Delta_{ab}]_\tau$ appearing in (2.28), therefore we can write:

$$\begin{aligned}
\log \mathcal{I} = &-\frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} K\left(s_{ab}\left[\Delta_{ab}\right]_\tau, \tau\right) \\
&+ \frac{i\pi N^2}{3\tau^2}\left(n_\chi - 2n_s - n_{\mathrm{v}}\right)\tau\left(\tau - \frac{1}{2}\right)(\tau - 1),
\end{aligned} \tag{2.29}$$

where we have defined

$$K(\Delta, \tau) \equiv [\Delta]_\tau \left([\Delta]_\tau + \frac{1}{2}\right)\left([\Delta]_\tau + 1\right) - 3\tau \left[\Delta\right]_\tau^2 = \frac{1}{2}\left(2\Delta^3 - 3|\Delta|\Delta + \Delta - 6\tau|\Delta|\Delta\right) \tag{2.30}$$

Recalling that:

$$\begin{aligned} [\Delta+1]_\tau &= [\Delta]_\tau\,, \\ [-\Delta]_\tau &= -[\Delta]_\tau - 1 \\ [\Delta+\tau]_\tau &= [\Delta]_\tau + \tau, \end{aligned} \tag{2.31}$$

then (2.30) holds when $|\Delta| < 1$.

Let us now analyze the properties of the function we have obtained. Equation (2.29) is very similar to the one obtained in [16] when analyzed in the Cardy-like limit of the index, however, there is an extra contribution of the form $\frac{i\pi N^2}{3\tau^2}\left(n_\chi - 2n_s - n_{\text{v}}\right)\tau\left(\tau-\frac{1}{2}\right)(\tau-1)$ which is still of order $\mathcal{O}(N^2)$ but sub-leading when $\tau \to 0$. Notice that at this point there is no dependence on the holonomies of the gauge groups since we have already evaluated in the solutions of the Bethe Ansatz equations. We still need to determine if we can find a consistent way of redefining the chemical potentials, thus fixing the value of n_s and s_{ab}. The shifting has to preserve the $R-$ charge of the superpotential which is ensured by the constrain:

$$\sum_{(ab)\in A} \Delta_{ab} = 2\tau, \tag{2.32}$$

where A denotes monomial terms of the superpotential W. Let us call n_F the number of elements in A. Using the fact that for these toric quiver gauge theories each chiral field appears only once in exactly two terms in the superpotential, then (2.32) implies that $2n_s = n_F$.

To gain a better understanding of the implications that shifting the chemical potentials has on the superconformal index of a toric quiver gauge theory let us consider:

$$\mathcal{I} = \text{Tr}_{BPS}\,(-1)^F\, e^{-\beta H} e^{2\pi i 2\tau\left(J+\frac{1}{d}\sum_{I=1}^{d} Q_d\right)} e^{2\pi i \sum_{i=1}^{d-1}\Delta_i(Q_i-Q_d)}, \tag{2.33}$$

where we have used the same basis for the non R-global symmetries used in [16]. Shifting the chemical potentials as

$$\Delta_i \to \Delta_i - \frac{2\tau}{d} \quad \text{with } i = 1,\cdots,d-1 \tag{2.34}$$

allows us to rewrite the index as

$$\mathcal{I} = \text{Tr}_{BPS}\,(-1)^F\, e^{2\pi i\left(\sum_{I=1}^{d}\Delta_I - 2\tau\right)Q_d} e^{-\beta H} e^{2\pi i 2\tau J} e^{2\pi i \sum_{i=1}^{d-1}\Delta_i Q_i}. \tag{2.35}$$

Exploiting the constraint

$$\sum_{I=1}^{d}\Delta_I - 2\tau = \pm 1 \tag{2.36}$$

and identifying $e^{2\pi i Q_d} = (-1)^F$ [16] allows us to express the superconformal index in such a way that bosonic-fermionic cancellations are optimally obstructed [6]:

$$\mathcal{I} = \text{Tr}_{BPS} e^{-\beta H} e^{2\pi i 2\tau J} e^{2\pi i \sum_{i=1}^{d-1}\Delta_i Q_i}. \tag{2.37}$$

The shifting (2.34) which is dictated by the geometry of the toric diagram, in particular by its number of vertices, turns out to be the adequate one in order to reproduce the dual black hole entropy.

Finally, recalling that we are dealing with toric quivers, which can be drawn on a torus providing a polygonalization of the torus [29] , and n_χ is the number of edges n_E of the graph, n_F is associated to the number of faces and n_v to the number of vertices then, the last term in (2.30) vanishes due to the Euler relation, $n_E - n_F - n_\mathrm{v} = 0$:

$$\begin{aligned}\log \mathcal{I} &= -\frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} K\left(s_{ab}\Delta_{ab}, \tau\right) + \frac{i\pi N^2}{3\tau^2}\left(n_\chi - n_F - n_\mathrm{v}\right)\tau\left(\tau - \frac{1}{2}\right)(\tau - 1) \\ &= -\frac{i\pi N^2}{3\tau^2} \sum_{\Phi_{ab}} s_{ab} K\left(s_{ab}\Delta_{ab}, \tau\right).\end{aligned} \tag{2.38}$$

Defining Δ_d such that: $\sum_{I=1}^{d} \Delta_I - 2\tau = -1$ [8], it can be shown that $\log \mathcal{I}$ can be writen as:

$$\log \mathcal{I} = -\frac{i\pi N^2}{6\tau^2} C_{IJK}\Delta_I\Delta_J\Delta_K. \tag{2.39}$$

The coefficients C_{IJK} in (2.39) correspond, as pointed out originally in [30] and later in [16], to the Chern-Simons couplings of the holographic dual gravitational description as elucidated in [31]. In the following section we proceed to evaluate the superconformal index for various models, some of them recently discussed in a similar context in [16], and compare our results with (2.39).

3 The superconformal index of various SCFT's

We will apply our general result (2.39) in various cases in each of which we follow the prescription of charge assignment used in [16]. Indeed, below we will see that in order to obtain (2.39) all the chemical potentials have to be shifted by $-\frac{2\tau}{d}$, exactly like [16]. We will restrict ourselves to the regime of chemical potentials Δ_i of the $d-1$ U(1) global symmetries such that:

$$0 \leq |\Delta_i| \leq \frac{1}{2} \;\; \forall i, \;\; 0 \leq \sum_{i=1}^{d-1} |\Delta_i|, \leq 1, \tag{3.1}$$

which is inside the fundamental domain:

$$\mathrm{Im}\left(-\frac{1}{\tau}\right) > \mathrm{Im}\left(\frac{\sum_{i=1}^{d-1}[\Delta_i]_\tau}{\tau}\right) > 0, \tag{3.2}$$

which in our case will be useful to evaluate the function $K(\Delta, \tau)$ using equation (2.30). The region (3.2) has been highlighted in figure 1 in grey. This regime also coincides with the one in which the existence of a universal saddle point in which all the holonomies vanish according to the analysis carried in [16], can be ensured.

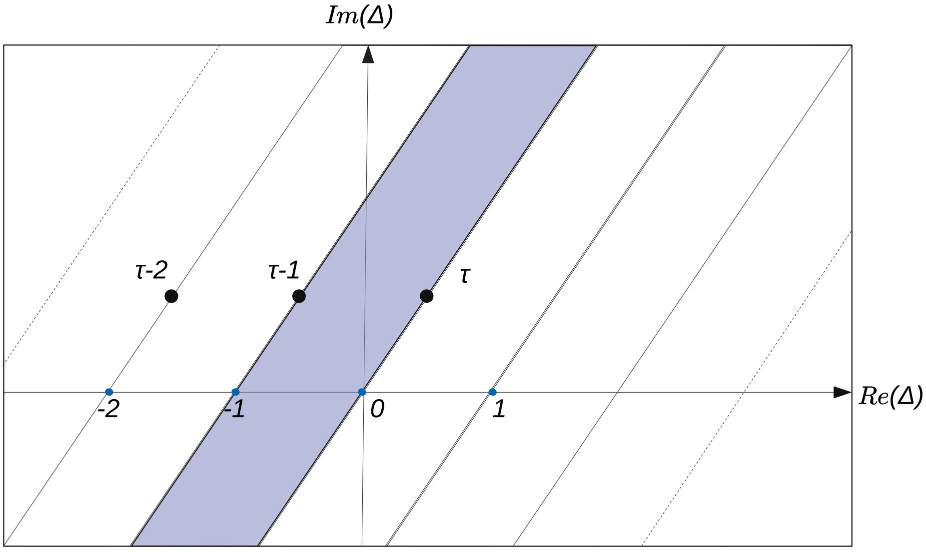
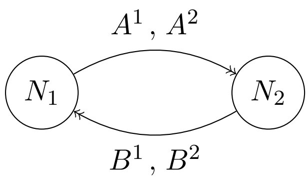


Figure 1. The figure shows the complex plane of chemical potentials for a generic Δ where the region specified by (3.2) is shown in grey.

3.1 The conifold theory

We would like to study the index in the large N limit and thus investigate it beyond the Cardy-like limit. To do so we start with one of the simplest examples of toric quiver gauge theories — the conifold theory [32] whose quiver diagram is given below. We take the ranks of all the gauge groups equal ($N_1 = N_2 = N$) and the sub-index in N_i helps describe the representations of the matter fields:



The superpotential is

$$W \propto \epsilon_{ij}\epsilon_{kl}\mathrm{Tr}\left[A^i B^k A^j B^l\right] . \tag{3.3}$$

The global charges of the conformal field theory are: a $\mathrm{U}(1)_R$ factor, two $\mathrm{SU}(2)$ factors and finally there is a $\mathrm{U}(1)_B$ baryonic symmetry. A fascinating fact about this theory is that it admits a gravity dual in terms of strings in $\mathrm{AdS}_5 \times T^{1,1}$. The isometries of $T^{1,1}$ realize the mesonic symmetries of the field theory in terms of the isometries of $\mathbb{CP}^1 \times \mathbb{CP}^1$; the $\mathrm{U}(1)_B$ baryonic symmetry is associated to the unique non-trivial three-cycle of the geometry. It is worth pointing out that the rotating electrically charged black holes dual to the superconformal index have not yet been constructed on the supergravity side, and that remains an outstanding problem.

We use the basis for the charges suggested by the toric diagram discussed in [16] and we summarize them in the following table:

Field	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$
A_1	$1/2$	1	0	0
A_2	$1/2$	0	0	1
B_1	$1/2$	0	1	0
B_2	$1/2$	-1	-1	-1

After performing the shifting $\Delta_{1,2,3} \to \Delta_{1,2,3} - \frac{\tau}{2}$, we are ready to evaluate equation (2.38):

$$\begin{aligned}\log\mathcal{I} &= -\frac{i\pi N^2}{3\tau^2}\left[K(\Delta_1,\tau)+K(\Delta_2,\tau)+K(\Delta_3,\tau)-K(-(-\Delta_1-\Delta_2-\Delta_3),\tau)\right] \\ &= -\frac{i\pi N^2}{\tau^2}\left[-\Delta_1^2(\Delta_2+\Delta_3)-\Delta_2\Delta_3(1-2\tau+\Delta_2+\Delta_3)-\Delta_1(\Delta_2+\Delta_3)(1-2\tau+\Delta_2+\Delta_3)\right].\end{aligned} \tag{3.4}$$

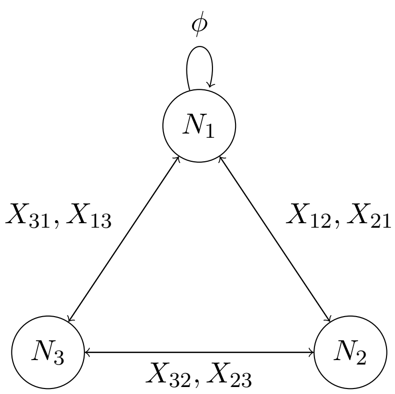
After imposing the condition $\sum_{I=1}^{d}\Delta_I - 2\tau = -1$ yields:

$$\log\mathcal{I} = -\frac{i\pi N^2}{\tau^2}\left[\Delta_2\Delta_3\Delta_4 + \Delta_1\Delta_3\Delta_4 + \Delta_1\Delta_2\Delta_3 + \Delta_1\Delta_2\Delta_4\right] \tag{3.5}$$

We see that $\log\mathcal{I}$ presents the behavior proposed in (2.39).

3.2 The suspended pinch point

The suspended pinch point (SPP) gauge theory corresponds to the near horizon limit of a stack of N $D3$ branes probing the tip of the conical singularity, $x^2y = wz$. The SPP gauge theory is described by the following quiver



All the ranks are taken to be the same with $N_1 = N_2 = N_3 = N$ and the sub-indices are meant to help understand the representation properties of the matter fields. The superpotential is

$$W = \mathrm{Tr}\left[X_{21}X_{12}X_{23}X_{32} - X_{32}X_{23}X_{31}X_{13} + X_{13}X_{31}\phi - X_{12}X_{21}\phi\right]. \tag{3.6}$$

Each X_{ij} transforms in the $\mathbf{N}$ representation of the index i-th node and in the $\overline{\mathbf{N}}$ of the j-th node. The field ϕ transforms in the adjoint representation of the corresponding gauge group. The charge assignment for the $U(1)_R$ and the extra $U(1)_i$ global symmetries can be taken as:

Field	$U(1)_R$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$
ϕ	4/5	1	1	0	0
X_{12}	2/5	0	0	0	1
X_{21}	4/5	−1	−1	0	−1
X_{23}	2/5	0	1	0	0
X_{32}	2/5	1	0	0	0
X_{31}	4/5	−1	−1	−1	0
X_{13}	2/5	0	0	1	0

We shift now the chemical potentials $\Delta_{1,2,3,4} \to \Delta_{1,2,3,4} - \frac{2\tau}{5}$. The next step is to use this information and perform the evaluation (2.38).

$$\log \mathcal{I} = -\frac{i\pi N^2}{3\tau^2}[K(\Delta_1+\Delta_2,\tau)+K(\Delta_4,\tau)-K(-(-\Delta_1-\Delta_2-\Delta_4),\tau) \\ +K(\Delta_2,\tau)+K(\Delta_1,\tau)-K(-(-\Delta_1-\Delta_2-\Delta_3),\tau)+K(\Delta_4,\tau)] \tag{3.7}$$

$$= -\frac{i\pi N^2}{\tau^2}[-\Delta_1^2(\Delta_2+\Delta_3+\Delta_4) \\ +\Delta_1\left((1-2\tau+\Delta_2+\Delta_3)(-\Delta_2-\Delta_3)+(2\tau-1-2\Delta_2)\Delta_4-\Delta_4^2\right) \\ +\Delta_2\left((1-2\tau+\Delta_2)\Delta_3+\Delta_3^2+\Delta_4(1-2\tau+\Delta_2+\Delta_4)\right)]. \tag{3.8}$$

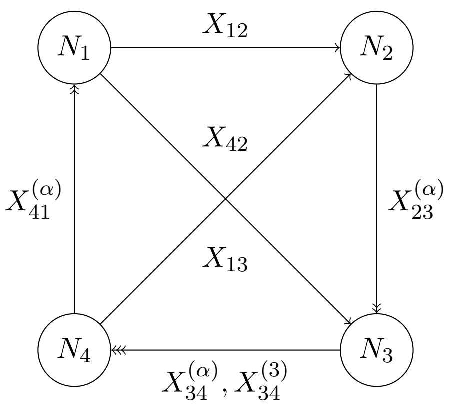
Now we use: $\sum_{I=1}^5 \Delta_I - 2\tau = -1$ we introduce a fifth fugacity Δ_5 that permits us to rewrite (3.7) in the following, more symmetric, way:

$$\log \mathcal{I} = -\frac{i\pi N^2}{\tau^2}[2\Delta_2\Delta_3\Delta_4 + \Delta_2\Delta_3\Delta_5 + \Delta_2\Delta_4\Delta_5 + 2\Delta_1\Delta_3\Delta_4 + \Delta_5\Delta_3\Delta_2 \\ + \Delta_1\Delta_4\Delta_5 + \Delta_1\Delta_2\Delta_3 + \Delta_1\Delta_2\Delta_4 + \Delta_1\Delta_2\Delta_5] \tag{3.9}$$

This result is in agreement with equation (2.39) which is what is expected from toric geometry and reinforces the validity of the analysis of [16] which was limited to the Cardy-Like limit.

3.3 The dP_1 theory

We consider now the theory arising from a stack of N $D3$ branes at the tip of the complex Calabi-Yau cone whose base is the first del Pezzo surface. The quiver associated to this theory is:



where $N_1 = N_2 = N_3 = N_4 = N$ and the superpotential is given by:

$$W = \epsilon_{\alpha\beta} \mathrm{Tr} \left[X_{34}^{(\alpha)} X_{41}^{(\beta)} X_{13} - X_{34}^{(\alpha)} X_{23}^{(\beta)} X_{42} + X_{12} X_{34}^{(3)} X_{41}^{(\alpha)} X_{23}^{(\beta)} \right] . \tag{3.10}$$

The charge assignment specified by the toric data is given by:

Field	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$
X_{12}	$1/2$	0	0	1
$X_{23}^{(1)}$	$1/2$	0	1	0
$X_{23}^{(2)}$	$1/2$	-1	-1	-1
$X_{34}^{(1)}$	1	0	1	1
$X_{34}^{(2)}$	1	-1	-1	0
$X_{34}^{(3)}$	$1/2$	1	0	0
$X_{41}^{(1)}$	$1/2$	0	1	0
$X_{41}^{(2)}$	$1/2$	-1	-1	-1
X_{13}	$1/2$	1	0	0
X_{41}	$1/2$	1	0	0

Let us perform the following transformation of the chemical potentials $\Delta_{1,2,3} \to \Delta_{1,2,3} - \frac{\tau}{2}$. Evaluating to leading order in N part of the superconformal index according to (2.38):

$$\begin{aligned} \log \mathcal{I} = & -\frac{i\pi N^2}{3\tau^2} [2K(\Delta_1,\tau) + K(\Delta_3,\tau) + 2K(\Delta_2,\tau) - 2K(-(-\Delta_1 - \Delta_2 - \Delta_3),\tau) \\ & + K(\Delta_2 + \Delta_3,\tau) - K(-(-\Delta_1 - \Delta_2),\tau)] \\ = & -\frac{i\pi N^2}{\tau^2} [-(2\Delta_1 + \Delta_2)\Delta_3^2 - 3\Delta_1\Delta_2(\Delta_1 + \Delta_2 + 1) \\ & - \left(\Delta_2^2 + 4\Delta_1\Delta_2 + \Delta_2 + 2\Delta_1(\Delta_1 + 1)\right)\Delta_3 + \tau(6\Delta_1\Delta_2 + 2(2\Delta_1 + \Delta_2)\Delta_3]. \end{aligned} \tag{3.11}$$

Introducing now Δ_4 via the constraint $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - 2\tau = -1$ we obtain:

$$\log \mathcal{I} = -\frac{i\pi N^2}{\tau^2} [2\Delta_1\Delta_2\Delta_3 + 3\Delta_1\Delta_2\Delta_4 + 2\Delta_1\Delta_3\Delta_4 + 2\Delta_2\Delta_3\Delta_4], \tag{3.12}$$

which coincides with the expectation (2.39).

3.4 $Y^{p,q}$ quiver gauge theories

The Y^{pq} model corresponds to quiver gauge theories with $2p$ gauge groups and a chiral field content of bifundamental fields. The charge assignment and the corresponding multiplicity of the fields are shown below:

Multiplicity	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$	$\mathrm{U}(1)_R$
$p+q$	1	0	0	1/2
p	0	1	0	1/2
$p-q$	0	0	1	1/2
p	-1	-1	-1	1/2
q	0	1	1	1
q	-1	-1	0	1

We proceed to perform the shifting of chemical potentials as follows: $\Delta_{1,2,3} \to \Delta_{1,2,3} - \frac{\tau}{2}$. Now we evaluate the leading, order $\mathcal{O}(N^2)$, part of the superconformal index (2.38):

$$\begin{aligned}\log \mathcal{I} = &-\frac{i\pi N^2}{3\tau^2}[(p+q)K(\Delta_1,\tau) + pK(\Delta_2,\tau) + (p-q)K(\Delta_3,\tau) \\ &- pK(-(-\Delta_1-\Delta_2-\Delta_3),\tau) + qK(\Delta_2+\Delta_3,\tau) - qK(-(-\Delta_1-\Delta_2),\tau)] \\ = &-\frac{i\pi N^2}{3\tau^2}[\Delta_2(\Delta_1-\Delta_3)(-q)(\Delta_1+\Delta_2+\Delta_3-2\tau+1) \\ &- p\left((\Delta_2+\Delta_3\Delta_1^2 + (\Delta_2+\Delta_3\Delta_1(\Delta_2+\Delta_3-2\tau+1) + \Delta_2\Delta_3(\Delta_2+\Delta_3-2\tau+1)\right)]\end{aligned} \tag{3.13}$$

Finally we eliminate τ from (3.13) using $\sum_{I=1}^4 \Delta_I - 2\tau = -1$, which successfully reproduce the structure of (2.39):

$$\log \mathcal{I} = -\frac{i\pi N^2}{3\tau^2}[p\Delta_1\Delta_2\Delta_3 + (p+q)\Delta_1\Delta_2\Delta_4 + p\Delta_1\Delta_3\Delta_4 + (p-q)\Delta_2\Delta_3\Delta_4]. \tag{3.14}$$

4 Corrections to the dual black hole entropy of the conifold theory

Thus far we have been focused on a specific region of chemical potentials that permited us to simplify all the computations associated with the function $[\Delta]_\tau$. Ultimately, it is necessary to identify the quantity $\log \mathcal{I}$ with the corresponding entropy function for the dual black hole gravity solution. The prototypical example is provided by $\mathcal{N}=4$ SYM, as discussed in [8], where the identification states:

$$\log \mathcal{I} = -\frac{i\pi N^2}{6\tau^2} C_{IJK}\Delta_I\Delta_J\Delta_K \Longleftrightarrow S_E = -\frac{i\pi N^2}{6\tau^2} C_{IJK}X_IX_JX_K, \tag{4.1}$$

provided $\Delta_I \Longleftrightarrow X_I$. More generically, $[\Delta_I]_\tau \Longleftrightarrow X_I$ where τ is the chemical potential associated to the two equal angular momenta $J_1 = J_2 = J$ and X_I are the chemical potentials associated to the cherges Q_I. As discussed in [8], the identification above is valid provided X_I is within the same analyticity domain that $[\Delta]_\tau$. This is the case, since both belong to the domain specified by (3.2), and they satisfy the constraint:

$$\sum_{I=1}^{d} X_I - 2\tau = -1. \tag{4.2}$$

The next step would be to obtain the black hole entropy by extremizing the Legendre transform of $\log \mathcal{I}$ with the appropriate constraint:

$$S\left(Q, \Lambda\right) = S_E + 2\pi i \left(\sum_{I=1}^{d} X_I Q_I - 2\tau J \right) + 2\pi i \Lambda \left(\sum_{I=1}^{d} X_I - 2\tau + 1 \right), \tag{4.3}$$

where Λ is a Lagrange multiplier imposing the constraint. The black hole entropy can be thus obtained exactly as done in [5, 16]. In particular the result obtained in [22], which appeared after the first version of our manuscript, confirms our results. Going away from the region of chemical potentials we have been restricting ourselves to so far would produce some modifications in the result.The technical reason being that, in different regions of the domain of complex chemical potentials, the functions $\left[\sum_I q_I \Delta_I\right]_\tau$ and $\sum_I q_I \left[\Delta_I\right]_\tau$ do not agree.

The structure (2.39) can be lost either by spoiling the homogeneity of $\log \mathcal{I}$ with respect to Δ_I and τ or by failing to completely cancel the term that only depends on τ and comes from the contribution of vector multiplets. Homogeneity in Δ_I and τ is crucial to perform the extremization procedure, whereas the appearance of an extra term exclusively dependent on τ could be easily incorporated in order to explore possible modifications to black the hole entropy. In the particular case of the conifold theory that we studied in section 3.1 we have that, for chemical potentials in the region:

$$\begin{gathered} \mathrm{Im}\left(-\frac{2}{\tau}\right) > \mathrm{Im}\left(\frac{[\Delta_1]_\tau + [\Delta_2]_\tau + [\Delta_3]_\tau}{\tau}\right) > \mathrm{Im}\left(-\frac{1}{\tau}\right) \\ \Downarrow \\ [\Delta_1 + \Delta_2 + \Delta_3]_\tau = [\Delta_1]_\tau + [\Delta_2]_\tau + [\Delta_3]_\tau + 1, \end{gathered} \tag{4.4}$$

therefore, the superconformal index takes the following form:

$$\begin{aligned} \log \mathcal{I} = -\frac{i\pi N^2}{\tau^2} \Big\{ & \left([\Delta_2]_\tau + \frac{1}{2}\right)\left([\Delta_3]_\tau + \frac{1}{2}\right)\left([\Delta_4]_\tau + \frac{1}{2}\right) \\ & + \left([\Delta_1]_\tau + \frac{1}{2}\right)\left([\Delta_3]_\tau + \frac{1}{2}\right)\left([\Delta_4]_\tau + \frac{1}{2}\right) \\ & + \left([\Delta_1]_\tau + \frac{1}{2}\right)\left([\Delta_2]_\tau + \frac{1}{2}\right)\left([\Delta_3]_\tau + \frac{1}{2}\right) \\ & + \left([\Delta_1]_\tau + \frac{1}{2}\right)\left([\Delta_2]_\tau + \frac{1}{2}\right)\left([\Delta_4]_\tau + \frac{1}{2}\right) \Big\} - i\pi N^2 \left(\frac{1}{2\tau} - 1\right) \end{aligned} \tag{4.5}$$

Notice that the appearance of a contribution that only depends on τ in this case is related to the specific details of how the function $[\cdots]_\tau$ behaves in the different domains of chemical potentials. The case of $\mathcal{N} = 4$ SYM is special because the function $\log \mathcal{I} \sim \Delta_1 \Delta_2 \Delta_3$ is quite simple and consists only of one term. For this reason one could hope to eliminate all contributions depending only on τ by modifying the constraint obeyed by the chemical potentials $\sum_{I=1}^{d} \Delta_I - 2\tau = -1 \rightarrow \sum_{I=1}^{d} \Delta_I - 2\tau = 1$. This is, indeed, the case verified in [8]. However, for more complicated theories where $\log \mathcal{I}$ has a more than one term dictated by the anomaly coefficients C_{IJK}, the extra pice persists as we see in (4.5).

Let us now investigate how this extra term modifies the entropy obtained by taking the Legendre transform of $\log \mathcal{I}$. Our starting point it to organize the computation as to maximally take advantage of the scaling properties of S_E, which implies that now we propose the identification $\left([\Delta_I]_\tau + \frac{1}{2}\right) \Longleftrightarrow X_I$ within the region (4.4). Notice that now the constraint is modified as:

$$\sum_{I=1}^{4} [\Delta_I]_\tau - 2\tau = -1 \tag{4.6}$$
$$\Downarrow$$
$$\sum_{I=1}^{4} \left(X_I - \frac{1}{2}\right) - 2\tau = -1$$
$$\Downarrow$$
$$\sum_{I=1}^{4} X_I - 2\tau = 1.$$

Even though the constraint (4.6) has been modified when identifying $\left([\Delta_I]_\tau + \frac{1}{2}\right) \Longleftrightarrow X_I$ we notice that the new constraint corresponds precisely to the other possible choice of relation among the chemical potentials as discussed in [12, 14]. If the chemical potentials are in the region (4.4), then:

$$\mathrm{Im}\left(-\frac{2}{\tau}\right) > \mathrm{Im}\left(\frac{X_1 + X_2 + X_3}{\tau} - \frac{3}{2\tau}\right) > \mathrm{Im}\left(-\frac{1}{\tau}\right) \tag{4.7}$$
$$\Downarrow$$
$$\mathrm{Im}\left(-\frac{1}{2\tau}\right) > \mathrm{Im}\left(\frac{X_1 + X_2 + X_3}{\tau}\right) > \mathrm{Im}\left(\frac{1}{2\tau}\right).$$

Note that this region does not coincides precisely with the fundamental domain over which the X_I are defined (3.2), however there is a non-empty intersection between the two as we illustrate in figure 2.

Thus we write:

$$\begin{aligned} \widehat{S} &= S_E + S_\tau \\ S_E &= -\frac{i\pi N^2}{\tau^2}\left(X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4\right) \\ S_\tau &= \frac{i\pi N^2}{\tau^2}\tau\left(\tau - \frac{1}{2}\right) \end{aligned} \tag{4.8}$$

Since S_τ is independent of X_I we have:

$$\begin{aligned} \frac{\partial \widehat{S}}{\partial X_I} &= \frac{\partial S_E}{\partial X_I} \\ \frac{\partial \widehat{S}}{\partial \tau} &= \frac{\partial S_E}{\partial \tau} + \frac{\partial S_\tau}{\partial \tau} \end{aligned} \tag{4.9}$$

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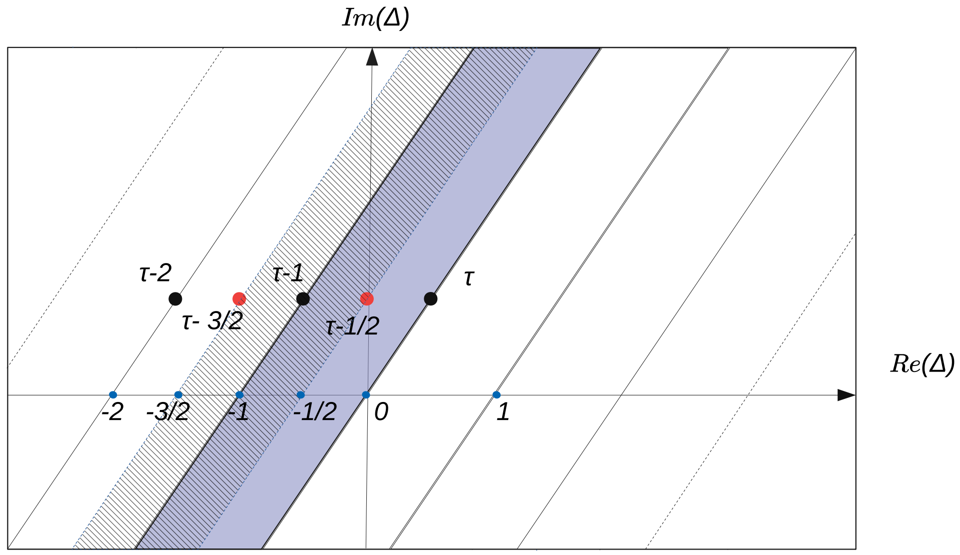

Figure 2. The figure shows the complex plane of chemical potentials for generic Δ including also the region for the corresponding X_I inside the dashed strip. notice that the grey and the dashed region overlap in a zone where the identification $\left([\Delta_I]_\tau + \frac{1}{2}\right) \Longleftrightarrow X_I$ is valid.

The function we need to extremize now is the following:

$$S\left(Q,\Lambda\right) = \widehat{S} + 2\pi i\left(\sum_{I=1}^{d} X_I Q_I - 2\tau J\right) + 2\pi i\Lambda\left(\sum_{I=1}^{d} X_I - 2\tau - 1\right). \tag{4.10}$$

The extremization condition implies:

$$\begin{aligned} \frac{\partial S}{\partial X_I} &= 0, \;\Rightarrow\; \frac{\partial \widehat{S}}{\partial X_I} = -2\pi i\left(Q_I + \Lambda\right) \\ \frac{\partial S}{\partial \tau} &= 0 \;\Rightarrow\; \frac{\partial S_E}{\partial \tau} = -4\pi i\left(\widetilde{J} - \Lambda\right) \\ \widetilde{J} &\equiv J + \frac{1}{4\pi i}\frac{\partial S_\tau}{\partial \tau}. \end{aligned} \tag{4.11}$$

The homogeneity of S_E leads to the important relation:

$$S_E = \sum_{I=1}^{d} X_I \frac{\partial S_E}{\partial X_I} + \tau\frac{\partial S_E}{\partial \tau}. \tag{4.12}$$

Following [16], we insert (4.12) in (4.9) and evaluating on the extremization solutions we find:

$$\begin{aligned} S\left(Q,J\right) &= 2\pi i\Lambda(Q,J) + S_\tau - 4\pi i\tau\left(\widetilde{J} - J\right) \\ &= 2\pi i\Lambda(Q,J) + S_\tau - \tau\frac{\partial S_\tau}{\partial \tau} \\ &= 2\pi i\Lambda(Q,J) + \frac{i\pi N^2}{\tau(J)}\left(\tau(J) - 1\right). \end{aligned} \tag{4.13}$$

For a particular set of values of X_I, the properties of S_E allow us to reconstruct $(S_E)^2$ from suitable combinations of products of its derivatives with respect to X_I which generically leads to a cubic equation to determine $\Lambda(Q,J)$. If we choose $X_1 = X_3$, then S_E for the conifold theory coincides with S_E for the $Y^{p,p}$ theory described in [16], namely:

$$\widetilde{S}_E \equiv S_E\big|_{X_1=X_3} = -\frac{i\pi N^2}{\tau^2}\left(X_1^2 X_2 + 2X_1X_2X_4 + X_1^2X_4\right). \tag{4.14}$$

Now we can follow the extremization procedure put forward in [16] keeping track of the correction S_τ when taking $Q_3 \to Q_1$ for the $Y^{1,1}$ quiver gauge theory. It can be shown that $\widetilde{S}_E$ satisfies:

$$\begin{aligned} 0 = \frac{\partial \widetilde{S}_E}{\partial X_1}&\left[2\left(2\frac{\partial \widetilde{S}_E}{\partial X_1}+\frac{\partial \widetilde{S}_E}{\partial X_4}\right)\frac{\partial \widetilde{S}_E}{\partial X_2}-\left(\frac{\partial \widetilde{S}_E}{\partial X_2}\right)^2-\left(2\frac{\partial \widetilde{S}_E}{\partial X_1}-\frac{\partial \widetilde{S}_E}{\partial X_4}\right)^2\right] \\ &+4pN^2\left(\frac{\partial \widetilde{S}_E}{\partial \tau}\right)^2. \end{aligned} \tag{4.15}$$

Using equation (4.9), we can obtain a cubic equation for Λ in the same spirit as [16] and also to keep track of the modification produced in the entropy by the presence of S_τ in (2.39), hence, we have:

$$\begin{aligned} 0 = &(Q_1+\Lambda)[2\left(2(Q_1+\Lambda)+(\Lambda+Q_4)\right)(\Lambda+Q_2)-(\Lambda+Q_2)^2 \\ &-(2(\Lambda+Q_1)-(\Lambda+Q_4))^2]+4pN^2(\Lambda-\widetilde{J})^2. \end{aligned} \tag{4.16}$$

Equation (4.16) can be written as:

$$\begin{aligned} 0 &= \Lambda^3+\widetilde{p}_2\Lambda^2+\widetilde{p}_1\Lambda+\widetilde{p}_0 \\ \widetilde{p}_0 &= N^2p\widetilde{J}^2-\frac{1}{4}Q_1Q_2^2+Q_1^2Q_2+\frac{1}{2}Q_1Q_2Q_4-Q_1^3-\frac{1}{4}Q_1Q_4^2+Q_1^2Q_4 \\ \widetilde{p}_0|_{\widetilde{J}=J} &\equiv p_0 \\ \widetilde{p}_1 &\equiv p_1-\frac{N^4}{4\tau^2}=2Q_1(Q_2+Q_4)+\frac{Q_4Q_2}{2}-\frac{Q_2^2}{4}-Q_1^2-\frac{Q_4^2}{4}-2N^2\widetilde{J} \\ \widetilde{p}_2 &\equiv p_2 = N^2p+2Q_1+Q_2. \end{aligned} \tag{4.17}$$

Demanding the condition

$$\widetilde{p}_0 = \widetilde{p}_1\widetilde{p}_2, \tag{4.18}$$

the assumption of real charges in [5, 16] led to purely imaginary values of Λ and therefore to a real entropy. We need to be more careful since (4.16) is a modified version of the one appearing in [16]. The modifications enter through $\widetilde{J}$ and $\frac{i\pi N^2}{\tau(J)}(\tau(J)-1)$ in (4.13). Let us still demand the condition (4.18), which a priory do not ensure real entropy but gives a simplified enough expression that we can work with. Reality of the entropy then would impose that, separately, the correction was a real number:

$$\mathrm{Re}\left(\frac{1}{\tau(J)}(\tau(J)-1)\right) = 0. \tag{4.19}$$

Imposing (4.19) would constraint the set of possible values τ could take. Eve though in some contexts [8, 22] the values of τ for which one can obtain a reasonable black hole entropy are constrained, we do not have any a priory reason for which τ should satisfy (4.19). Of course, a more rigorous approach is required, since the reality condition for the full entropy would imply a relation among the coefficients (4.17) far more complicated than (4.18). However, at least for the values of τ that ensure reality of (4.13), through (4.18) and (4.19) we can proceed as follows.

The solution of (4.16) when plugged into equation (4.13) leads to an entropy of the form:

$$\begin{aligned} S\left(Q,\Lambda\right) &= 2\pi\sqrt{2Q_1(Q_2+Q_4)+\frac{Q_4Q_2}{2}-\frac{Q_2^2}{4}-Q_1^2-\frac{Q_4^2}{4}-2N^2\widetilde{J}} \\ &\quad +\frac{i\pi N^2}{\tau(J)}\left(\tau(J)-1\right) \\ &= 2\pi\sqrt{2Q_1(Q_2+Q_4)+\frac{Q_4Q_2}{2}-\frac{Q_2^2}{4}-Q_1^2-\frac{Q_4^2}{4}-2N^2\left[J+\frac{N^2}{8\tau(J)^2}\right]} \\ &\quad +\frac{i\pi N^2}{\tau(J)}\left(\tau(J)-1\right) \end{aligned} \tag{4.20}$$

In the above expression we have left $\widetilde{J}=J+\frac{1}{4\pi i}\frac{\partial S_\tau}{\partial\tau}$ explicitly in the corrections to highlight its effect. The angular velocity $\tau(J)$ appears only formally, it should be substituted by the extremization procedure, we have indicated such operation as $\tau(J)$. The most dramatic effect is a shift in the angular momentum.

5 Conclusions

In this brief note we have explored the superconformal index following the Bethe Ansatz approach introduced by Benini and Milan [9]. We have shown that a class of solutions can be extended to solve the Bethe Ansatz equation for a large class of 4d $\mathcal{N}=1$ supersymmetric gauge theories. The Bethe Ansatz approach has the advantage that it does not require to take the Cardy limit and therefore provides a more complete large N expression. Indeed, for generic toric quiver gauge theories we determined that there is a region in the space of chemical potentials in which the $\mathcal{O}(N^2)$ result obtained in the cardy-like limit can be recovered buttessing previous results in the literature [14–16]. Furhtermore, at least for the simple case of the conifold theory we saw that one can obtain a similar structure of the superconformal index with extra corrections in τ, but sufficiently simple as to permit us to proceed with the extremization procedure and consequently a corrected black hole entropy. We hope that more work along this direction might eventually allow to understand the growth of states in the index in a more systematic fashion that covers all the possible regions in the space of chemical potentials. For example, by exploiting the Bethe Ansatz approach to the topologically twisted index a systematic study of $1/N$ corrections for the ABJM index was performed in [33]; a similar study for a Chern-Simons matter theory dual to massive IIA black holes was reported in [34]. Such understanding of $1/N$ corrections will naturally translate into interesting aspects in the dual quantum gravity side for AdS_5 black holes. For example, the statistical entropy of certain magnetically charged AdS_4 black holes has recently been given a microscopic explanation in terms of the topologically twisted index [1] (see [2, 3] for a reviews with comprehensive lists of references). The investigation of sub-leading (logarithmic in N) corrections such as those performed recently [35, 36] have helped clarify the nature of the degrees of freedom on the gravitational side of the duality. One would hope for similar developments in the context of AdS_5 black holes.

There are many other interesting open problems. At the technical level, it would be interesting to generalize the Bethe Ansatz approach to arbitrary fugacities such that a general expression depending on both angular momenta can be achieved. There is little doubt that such generalization will yield the expected results but it will clarify the inner workings of the evaluation of the superconformal index. In this manuscript we have completely avoided the subtle discussion concerning the space of solutions of the Bethe Ansatz equations, we limited ourselves to just one class and showed that it yields a contribution sufficient to extract the dual black hole entropy and its potential corrections in the appropriate domain of chemical potentials. It would be very illuminating to have a better understanding of all the solutions and how one should weight their contributions to the index.

Finally, it is an important open problem to construct explicitly the black holes dual to the field theories discussed in this manuscript. Our computation, as well as those in a number of recent publications [14–16], show that it is relatively easy to find the superconformal index in a large class of supersymmetric four-dimensional field theories some of which have known supergravity dual. Moreover, using the entropy formula one can evaluate the entropy and realize that it corresponds to that of large black holes in AdS_5. However, the explicit black hole construction on the gravity side is still in its infancy, not much is known beyond the AdS_5 black holes dual to $\mathcal{N} = 4$ SYM (and some of its orbifolds). It remains an outstanding challenge for the supergravity community to explicitly construct rotating electrically charged black holes which could be understood as dual of available field theory results. One particular example that comes to mind among the class discussed in this note would be the black holes in asymptotically $AdS_5 \times T^{1,1}$ and, more generally, $AdS_5 \times Y^{p,q}$.

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