

**Microstructural Randomness and Scaling
in Mechanics of Materials**

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¹Chapter written jointly with I. Jasiuk

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