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"Mid-Auction Information Acquisition"

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EPGE, 10º andar, Sala 3

"Free lunch" para alunos e professores: sanduíches, refrigerantes e café. ☺

Mid-Auction Information Acquisition

Leonardo Rezende*

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Abstract

This paper studies a model of a sequential auction where bidders are allowed to acquire further information about their valuations of the object in the middle of the auction. It is shown that, in any equilibrium where the distribution of the final price is atomless, a bidder's best response has a simple characterization. In particular, the optimal information acquisition point is the same, regardless of the other bidders' actions. This makes it natural to focus on symmetric, undominated equilibria, as in the Vickrey auction. An existence theorem for such a class of equilibria is presented. The paper also presents some results and numerical simulations that compare this sequential auction with the one-shot auction. Sequential auctions typically yield more expected revenue for the seller than their one-shot counterparts. So the possibility of mid-auction information acquisition can provide an explanation for why sequential procedures are more often adopted.

1 Introduction

This paper studies a continuously ascending price independent private values auction, with the added richness that bidders are allowed to acquire further information about the value of the good in the middle of the auction. The information structure allows bidders to have different initial signals of their valuation and different privately known costs of acquiring information. The

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*Department of Economics, Stanford University. Still preliminary. I thank Prof. Paul Milgrom for suggesting me this research topic. I acknowledge the support of a Melvin and Joan Lane Stanford Graduate Fellowship. All errors are mine.

framework can also accommodate for the possibility of some bidders already knowing their valuation at the outset of the auction or, conversely, not being able to acquire information with some probability.

It turns out that in some aspects the optimal strategy of a bidder in such an auction is quite simple. For example, the price at which information is acquired does not depend on the expected behavior of the other players. This greatly simplifies the characterization, numerical computation, and proving the existence of the equilibrium.

Why is it interesting to study models with mid-auction information acquisition? A direct reason is its potential application in complex environments, where a bidder participates in many different auctions, such as in simultaneous ascending auctions or online auctions. In complex environments, gathering information and computing valuations for all goods and combinations of goods can be an overwhelming task. It is not unreasonable to imagine that bidders approach the problem with just crude estimates of the valuations, and as the auctions proceed, they elect to concentrate their computational resources in evaluating the most promising alternatives. This paper is a first step towards modeling this behavior.

The current one good model also offers an explanation on why sequential auctions seem to be so much more popular than their one-shot counterparts. Several explanations have been forwarded for this puzzle. Milgrom and Weber (1982) have shown that under affiliation a sequential English auction generates more revenue than one-shot, sealed-bid rules. An English auction may also in practice be superior to a Vickrey auction because it is more immune to manipulation by the auctioneer.

However, affiliated models are hard to generalize to more complex settings, such as auctions of multiple goods. In such complex situations one issue that becomes important to the bidders is the cost of collecting information and processing it into bidding strategies. This suggests an alternative explanation for why sequential auctions might be useful: they allow bidders to revise their decisions in information acquisition in the middle of the auctions, and this option might be valuable not only to bidders but to the seller as well.

Engelbrecht-Wiggans (1988) has attempted to formally model this insight working with two-stage auctions. However, a two-stage auction is a formally difficult object and the existing results have been limited to very restrictive functional assumptions. The current model presents a more flexible and tractable way to capture this economic intuition.

This model may also be helpful in econometric applications. Data in the "serious" tail of the bid distribution usually reflect much more accurately the valuations of the bidders than the rest of the distribution. Since the equilibrium of this model also has this feature, the model may potentially be helpful in structural estimation of auctions.

Given the fundamental role that asymmetric information takes in Contract Theory and Mechanism Design, there has been surprisingly little work that treats the information acquisition process as endogenous. Some authors have studied information acquisition in the context of Baron-Myerson-style agency models (Crémer and Khalil 1992, Lewis and Sappington 1997, Crémer, Khalil, and Rochet 1998b, Crémer, Khalil, and Rochet 1998a).

Several authors have studied information acquisition in the context of auctions. In most cases, the analysis is restricted to ex-ante information acquisition, and to particular functional forms of the valuation and signal distributions.[1]

Matthews (1984) and Persico (2000) study models where bidders can purchase information out of a continuum of alternative degrees of informativeness. To do so, they resolve in different ways the non-trivial problem of ranking distributions in terms of informativeness.[2] Due to the simple structure of the information acquisition problem that is imposed in this paper, this ranking is immediate here.

All papers cited in the last two paragraphs study situations where bidders are allowed to acquire information before the auction begins. Besides the already mentioned work Engelbrecht-Wiggans (1988), I am not aware of any literature on information acquisition in sequential auction procedures.

The paper is structured as follows: Next section presents the setup that is used throughout the paper. The problem of characterizing the best response of a given bidder is then investigated. Section 4 contains a proof that an equilibrium of this game exists. Section 5 compares this equilibrium to what would arise in a one-shot game. In section 6 results of some numerical

[1]Examples are Milgrom (1981b), Schweizer and von Ungern-Sternberg (1983), Lee (1985), Hausch and Li (1993). Guzman and Kolstad (1997) study a setting similar to the one assumed here for one-shot procedures; however, these authors elect to characterize a rational expectations equilibrium in the spirit of Grossman and Stiglitz (1980), rather than use game-theoretical concepts.

[2]To do so, Persico (2000) uses the statistical notion of efficacy, due to Lehmann (1988). Using efficacy, Athey and Levin (1998) develop a theory that provides comparative static results in a variety of models.

simulations are presented.[3]

2 The Setup

I seek to investigate an auction model that is conventional in all aspects, except for the mid-auction information acquisition decision.

All bidders are symmetric and have independent private valuations for the good to be auctioned. I represent these valuations by i.i.d. random variables $v_1, \ldots, v_n$, where n is the commonly known number of bidders. I assume that the distribution function of v_i, F_v, is absolutely continuous with support $[0, \bar{v}]$.

The auction rules are also conventional: I model a Japanese ascending auction, where the price p begins at a low level (that I assume for simplicity to be 0) and increases continuously. Bidders should decide at which price to drop out. The auction ends when only one bidder is left, and he or she pays the price at which the last of the other bidders dropped out. If all remaining bidders drop out at the same time, the winner is selected at random, with equal probability.

At any point in the auction each bidder can have two possible levels of information about her own valuation. Each bidder initially has a signal w_i about her valuation v_i, but can learn it perfectly and instantaneously at any point in the auction at a cost c_i. Both c_i and w_i are known by player i in the beginning of the game, but are not observed by anyone else. It is common knowledge however that all w_i are i.i.d. F_w, and likewise the c_i are i.i.d. F_c. I represent the distribution of v_i conditional on w_i by $F_{v|w}$, and adopt a similar notation for other conditional distribution functions. Except for w_i and v_i, all other variables are assumed to be independent.[4]

It is convenient to impose some further assumptions on the distributions of c_i, w_i and v_i. I assume that they have bounded intervals as supports; the support of v_i is $[0, \bar{v}]$ and the one for c_i is $[0, \bar{c}]$. The support of w_i can be anywhere, but I assume that the supremum of the support of $E[v_i|w_i]$ is strictly below $\bar{v}$. I assume that $\bar{c}$ is high enough, so that there is always the

[3]The paper also contains an appendix that sketches a non-existence result for some distributional assumptions.

[4]Independence across different bidders is an important simplifying assumption. Independence between c_i and w_i or v_i is not important, and the analysis would not change much without it.

possibility that some bidder elects not to acquire information. Also, I assume that the distribution of c_i has an atom at 0. Besides analytical convenience, these assumptions allow me to accommodate in the same framework bidders that cannot acquire information and bidders that already have all information at the outset of the auction.

Apart from the atom at 0, all distributions are assumed to have densities, and those are bounded above and away from zero everywhere in the support.

I assume that a higher w_i is good news about v_i, in the sense of Milgrom (1981a); that is, if $w > w'$, then $F_{v|w}$ first-order stochastically dominates $F_{v|w'}$.[5]

During the auction, the only information about the behavior of other players that a bidder observes is whether all have dropped out or not, i.e., if the auction has ended or not. So I am ruling out both the possibility of observing early drop-out points or information acquisition points.

I conjecture that the assumption on the unobservability of drop-out points does not qualitatively affect the analysis (even though it greatly simplifies part of it). Notice that due to the IPV assumption, knowledge of other player's drop-out point/valuation does not change i's estimate of her own v_i, so a linkage effect as in Milgrom and Weber (1982) is not expected to exist.

The second assumption is possibly not innocuous; direct observation of the information acquisition point can in principle generate additional strategic effects that have not been accounted for in the present model.

3 The individual bidder's problem

I begin by studying the individual bidder's problem, taking as given the behavior of the other bidders. For notational simplicity I drop the subscript i in this section and call $v_i = v$, etc.

It is convenient to summarize the other bidders' behavior in a reduced form fashion. Let the random variable y represent the price at which the last of the other bidders drops out.

Since y is a function of (c_{-i}, w_{-i}, v_{-i}), it is initially independent of the bidder's private information on c_i and w_i (and v_i, if she elects to acquire information right from start). I will denote this distributions of y by F_y. Under the unobservability assumptions made in the end of last section, no

[5]I also assume that the slope of $E[v|w]$ is bounded away from zero.

information about y is obtained during the auction, except that y is greater than the current price. This allows me to express the update of i's information about y in a simple way. If bidder i were allowed to observe the other bidders' drop-out points, then the update formula would be more complex.

The information acquisition process that has been imposed allows me to separate the possible strategies of the bidder in two groups: either she decides never to acquire information and drop out at a price $\check{p}$, or she decides to wait until a price $\hat{p}$ and acquire information at this point. $\hat{p} = 0$ represents immediate information acquisition, and $\hat{p} = \infty$ represents never acquiring information (or dropping out of the auction), so all pure strategies are represented by this parameterization. I consider in turn the optimal strategy within each group, and later compare the two to find the overall optimal strategy.

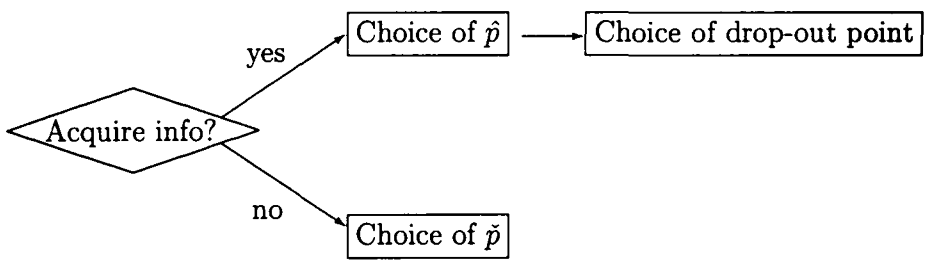

Figure 1: Schematic representation of a bidder's strategy.

As will be seen in the equilibrium analysis in the next section, ones needs to consider only the optimal response to the cases where F_y is absolutely continuous. The appendix deals with the case where F_y contains jumps.

3.1 If the bidder decides to acquire information

The task in this section is to find the function $\hat{p}(w, c, F_y)$, defined as the optimal information acquisition point for a bidder that observes a signal w of her valuation, faces a cost c to acquire information, and expects that the other bidders behave in such a way that the highest price at which any of them stays in the auction is distributed according to y.

Define the function $\hat{U}(p, q)$ as the expected utility for a bidder that decides to acquire information when the price reaches q, conditional on the price of the auction having reached p, that is, the expected (over v and y) profits conditional on w, c and $y \geq p$. In a subgame perfect equilibrium $\hat{p}$ should be chosen as the q that maximizes this function at any given p.

After acquiring the information v, the bidder's dropping decision is simple. It is dominant for her to drop out of the auction when $p \geq v$ and stay otherwise. Throughout the paper I shall assume that this dominant strategy is always followed in this subgame.

Given that $\hat{U}$ for the region where $p > q$ can be evaluated. In this region the bidder would have dropped and incurred profit 0 if $v \leq p$, and otherwise would expect to earn $E[v - y | v > y > p]$.[6] So defining

$$\begin{aligned} u(p, v) &= E[v - y | v, v > y > p] \Pr[v > y | v, y > p] \\ &= \begin{cases} \int_p^v (v - y)\, dF_{y|y \geq p}(y) & \text{if } v > p, \\ 0 & \text{if } v \leq p, \end{cases} \end{aligned}$$

then $\hat{U}(p, q) = E[u(p, v) | w] - c$, for all $p > q$. It is convenient to rewrite this double integral as

$$\hat{U}(p, q) = \int_p^\infty \int_y^\infty (v - y)\, dF_{v|w}(v)\, dF_{y|y \geq p}(y) - c,$$

for all $p > q$, since from this expression it is easy to verify that $\hat{U}$ is differentiable with respect to p in this region, and its derivative obeys

$$\frac{\partial}{\partial p} \hat{U}(p, q) = f_{y|y \geq p}(p) \left[\hat{U}(p, q) + c - \int_p^\infty (v - p)\, dF_{v|w}(v) \right].$$

In the region where $p < q$, the bidder is staying in the auction without knowing her v. Let Δp be an interval of time sufficiently small, so that still $p + \Delta p < q$. Staying in the auction through this period, the bidders is subject to one out of two outcomes: she may win the auction alone, if $y \in (p, p + \Delta p]$, or the auction may continue, in which case she obtains an expected profit of $\hat{U}(p + \Delta p, q)$. So we can write

$$\hat{U}(p, q) = \int_p^{p + \Delta p} (E[v|w] - y)\, dF_{y|y \geq p} + (1 - F_{y|y \geq p}(p + \Delta p))\, \hat{U}(p + \Delta p, q),$$

[6]It is also possible that there is a tie, if $y = v$, and the bidder must share the prize with others. In this case, however, the prize is worth 0 for her, and we do not need to incorporate a term for that.

for all $p < p + \Delta p < q$. Notice that independence was required to write the expectation over v inside the integration sign. Taking $\Delta p \to 0$, we conclude that $\hat{U}$ is continuous with respect to p; rearranging and taking limits again we obtain that $\hat{U}$ is differentiable with respect to p in this region as well, and the derivative now obeys

$$\frac{\partial}{\partial p}\hat{U}(p,q) = f_{y|y\geq p}(p)\left[\hat{U}(p,q) - E[v-p|w]\right].$$

At $p = q$ the function $\hat{U}$ is not necessarily differentiable, since its left- and right-hand derivatives exist but are generally different. It turns out that by an application of the Envelope Theorem, the optimal $\hat{p}$ is the point where $\hat{U}$ is differentiable with respect to p:

Proposition 1 *Suppose c is not too large,*[7] *i.e., $c < \bar{v} - E[v|w]$. The optimal $\hat{p}(w, c, F_y)$ is the point where the derivative of $\hat{U}$ with respect to p evaluated at $(\hat{p}, \hat{p})$ exists, i.e., at the point $\hat{p}$ uniquely determined by*

$$c = \int_0^{\hat{p}} (\hat{p} - v)\, dF_{v|w}.$$

Proof: We first notice that $\hat{U}$ is continuous with respect to q; it is constant with respect to q in the region where $p > q$ and depends on q only through the boundary condition at $p = q$ in the region where $p < q$. Also, any $q > \bar{v}$ leads to negative profits, that may be easily avoided by choosing $q = 0$. So the selection of the optimal information acquisition point may be circumscribed to the compact interval $[0, \bar{v}]$ and by Weierstrass Theorem an optimum must exist.

We proceed by showing that at any interior point if the left- and right-derivatives of $\hat{U}$ do not coincide, then this point cannot be an optimum. Suppose that at q the right-derivative is smaller than the left-derivative. Then there exists some $\epsilon > 0$ such that at $p < q - \epsilon$, $\hat{U}(p, q - \epsilon) > \hat{U}(p, q)$. So it is suboptimal to wait until q.

Conversely, suppose now that the right-derivative is larger than the left-derivative. Consider an $\epsilon > 0$ small enough so that, at all points between q and $q + \epsilon$, this relation still holds. Then $\hat{U}(p, q + \epsilon) > \hat{U}(p, q)$, for all p between q and $q + \epsilon$, and it would be suboptimal to acquire information at q.

[7] As will be seen in the next section, if c is very large the bidder will elect not to acquire information in the first place, and the determination of $\hat{p}$ is irrelevant.

Finally, notice that, at 0, $c > \int_0^0 (0 - v)\, dF_{v|w}$, so the argument of the last paragraph holds. Likewise, at $\bar{v}$ the right-derivative is smaller, so this cannot be an optimum either. So the optimum must be interior.

Uniqueness of $\hat{p}$ comes from the fact that $\int_0^p (p - v)\, dF_{v|w}$ is a strictly increasing function of p inside the support of $F_{v|w}$, since its derivative is $0 + \int_0^p dF_{v|w} = F_{v|w}(p) > 0$. $\square$

The optimality condition has a sensible economic interpretation: the cost of acquiring information, c, must be balanced against the benefit of doing so, namely, avoiding the potential loss from buying the good at a price higher than the bidder's valuation.

One remarkable feature about the condition that determines $\hat{p}$ is that it depends solely on the distribution of $v|w$; it does not depend on the specified distributions for c or w, or on the behavior of the other players.

An immediate corollary is the following:

Corollary 1 *If $c > 0$, then it is never optimal to acquire information at the start of the auction.*

So positive information acquisition costs always have an impact on bidder behavior, if this bidder plans to acquire information.

Let's investigate the properties of $\hat{p}$ as a function of c and w. This is the function obtained once one solves $c = \int_0^{\hat{p}} (\hat{p} - v)\, dF_{v|w}$ for $\hat{p}$. $\hat{p}$ is an increasing function of c (since its derivative with respect to c, by the implicit function theorem, is $1/F_{v|w}(\hat{p}(c, w)) \geq 0$).

As for w, notice that from the assumption that a higher w is good news about v, for a fixed p the term $\int_0^p (v - p) dF_{v|w} = \int \min\{v - p, 0\} dF_{v|w}$ is increasing in w, since the integrand is a weakly increasing function. So with a higher w one needs a higher $\hat{p}$ to reduce that term.

We conclude that $\hat{p}$ is an increasing function of both c and w, and does not depend on the behavior of the other players.

3.2 If the bidder decides not to acquire information

This case can be handled in an analogous fashion, defining $\check{U}(p, r)$ as the expected utility at p of a bidder that will drop out at r. Let $\check{p}$ be the optimal drop-out point, given that the bidder does not acquire information.

$\check{U}(p, r) = 0$ at the region where $p > r$, since then the bidder would have dropped the auction. Before that, $\check{U}$ should obey the same differential equation that $\hat{U}$ obeys before the information is acquired, since the bidder's situation is the same in both instances:

$$\frac{\partial}{\partial p}\check{U}(p,r) = f_{y|y\geq p}(p)\left[\check{U}(p,r) - E[v-p|w]\right].$$

Also, as before, the chosen $\check{p}$ will be such that the $\check{U}$ is differentiable at $(\check{p},\check{p})$. So we have $0 = \frac{\partial}{\partial p}\hat{U}(\check{p},\check{p}) = f_{y|y\geq\check{p}}(\check{p})\left[0 - E[v-\check{p}|w]\right]$, or simply $\check{p} = E[v|w]$. This is expected: this is simply the dominant strategy for a player that could not acquire information in the first place, staying in the auction until her expected valuation is reached.

3.3 The overall optimal response

In order to obtain the optimal response, one compares the payoff of the bidder under each of the two alternatives discussed above. Since before both $\hat{p}$ and $\check{p}$ the functions $\hat{U}$ and $\check{U}$ follow the same differential equation, this comparison can be made at any point $p \leq \min\{\hat{p},\check{p}\}$.

A convenient comparison point is $p = 0$, since at this point it is easy to directly obtain expressions for $\check{U}$ and $\hat{U}$: they are simply the (unconditional) expected payoffs for a bidder at the outset of the auction under each alternative.

A bidder that elects not to acquire information and drops at $\check{p} = E[v|w]$ wins if $y < \check{p}$, and gets $v - y$, and looses (and gains 0) otherwise. So she has an expected payoff of

$$\check{U}(0,\check{p}) = \int_0^{\check{p}}\int_0^{\infty}(v-y)\,dF_{v|w}\,dF_y;$$

while a bidder that plans to acquire information at $\hat{p}$ wins the auction if $y < \hat{p}$ or if $\hat{p} < y < v$, and incurs cost c if $y > \hat{p}$. Her payoff is then

$$\begin{aligned}\hat{U}(0,\hat{p}) &= \int_0^{\hat{p}}\int_0^{\infty}(v-y)\,dF_{v|w}\,dF_y \\ &\quad + \int_{\hat{p}}^{\infty}\int_y^{\infty}(v-y)\,dF_{v|w}\,dF_y - c(1-F_y(\hat{p})).\end{aligned}$$

The information acquisition decision will depend on comparing these two quantities for each (c,w)-type.

3.3.1 Information acquisition as an option trade

It is convenient to rewrite the information acquisition decision in the following fashion:

$$\begin{aligned}
\hat{U}(0,\hat{p}) - \check{U}(0,\check{p}) &= \int_{\check{p}}^{\infty} (\check{p} - y) dF_y + \int_{\hat{p}}^{\infty} \left[\int_0^y (y - v)\, dF_{v|w} - c \right] dF_y \\
&= -\int \max\{y - \check{p}, 0\} dF_y \\
&\quad + \int \max\left\{ \int_0^y (y - v)\, dF_{v|w} - c, 0 \right\} dF_y \\
&= \int r(t, c, w) dF_y(t),
\end{aligned}$$

where $r(t,c,w) = \max\{\int_0^t F_{v|w}(s)ds - c, 0\} - \max\{t - \check{p}, 0\}$.

So the decision of acquiring information looks like the decision of trading options on the underlying asset y; $\int r dF_y$ is the expected profit from selling a call option on y at strike price $\check{p}$ and buying a call option on $\int_0^y F_{v|w}(s)ds - c$ at strike price $\hat{p}$. Figure 2 shows the shape of this r function. It is an asymmetric spread, that pays if y is close to $\check{p}$, has negative value if y is too high, and 0 if y is too low. This suggests that information acquisition depends negatively on the variance of y with respect to $v|w$.

3.3.2 Comparative statics on the information acquisition decision

In the (c,w)-space, there will be a region A where condition $\hat{U}(0,\hat{p}) > \check{U}(0,\check{p})$ is satisfied. In order to investigate the properties of the region A, we need comparative statics results about the effect of c and w on $\check{U}$ and $\hat{U}$, and, to do so, again we invoke the Envelope Theorem.

Observe that $\hat{U}(0,\hat{p})$ is the value function of a problem with c and w as parameters, and where $\hat{p}$ is chosen to maximize $\hat{U}(0,q)$, with respect to q. The Envelope Theorem[8] then yields that $\frac{d}{dc}\hat{U}(0,\hat{p}(c,w)) = -(1 - F_y(\hat{p}(c,w)) < 0$, and of course $\frac{d}{dc}\check{U}(0,\check{p}) = 0$. We conclude that A shrinks as c increases.[9]

[8]Notice that the conditions for the Envelope Theorem are satisfied: $\hat{U}_0$ is differentiable with respect to c and this derivative is continuous; and the solution $\phi(c,w)$ is unique (Milgrom 1999, Corollary 2).

[9]Formally, if $c < c'$, $(c',w) \in A \Rightarrow (c,w) \in A$.

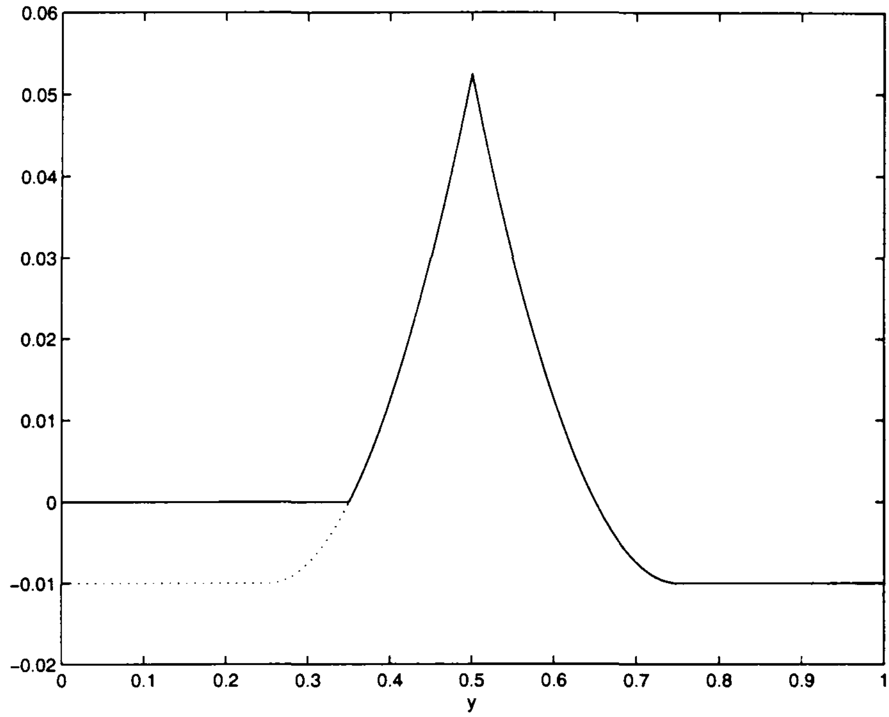

Figure 2: Graph of the r function (solid line) and the r_0 function (dotted line), for the case where $v|w \sim U[0.25, 0.75]$ and $c = 0.01$.

To assess the effect of w we again invoke the Envelope Theorem to disregard the effect through $\hat{p}$ or $\check{p}$, and use the assumption that higher w's are good news.

Both $v-y$ and $\max\{v-y,0\}$ are increasing functions of v; so the integrals with respect to $v|w$ that appear in both expressions increase with w. We conclude that $\hat{U}(0,\hat{p})$ and $\check{U}(0,\check{p})$ both increase with w.

In the case where $\hat{p} < \check{p}$, the sign of effect will depend on the comparison of the effect of w on two integrals of $v-y$: $\int_{\check{p}}^{\infty}\int_{y}^{\infty}(v-y)dF_{v|w}dF_y$ versus $\int_{\hat{p}}^{\check{p}}\int_{0}^{\infty}(v-y)dF_{v|w}dF_y$. The impact of a larger w on both terms is positive. If the effect on the latter expression is larger than on the former, A shrinks with w, and vice-versa.

One does not need to investigate the other case, where $\hat{p} > \check{p}$, because of the following result:

Proposition 2 *For any type that strictly prefers to acquire information, $\hat{p} < \check{p}$.*

Proof: The strategy of the proof is to establish that if $\hat{p} \leq \check{p}$, then $\hat{U}(0,\hat{p}) - \check{U}(0,\check{p}) \leq 0$. We start by noticing that, using equation 1, $\hat{U}(0,\hat{p}) - \check{U}(0,\check{p}) = \int_{\check{p}}^{\hat{p}}\int_{0}^{\infty}(v-y)\,dF_{v|w}\,dF_y + \int_{\hat{p}}^{\infty}\int_{y}^{\infty}(v-y)\,dF_{v|w}\,dF_y + \int_{\hat{p}}^{\infty}\int_{0}^{\hat{p}}(v-\hat{p})\,dF_{v|w}\,dF_y$. If $\hat{p} > \check{p} = E[v|w]$, the first term is non-positive. So it remains to show that the third negative term dominates the second. For any $y > \hat{p}$, of course $\int_{y}^{\infty}(y-\hat{p})\,dF_{v|w} \geq 0$ and $\int_{0}^{y}(v-\hat{p})\,dF_{v|w} \geq 0$. So $\int_{y}^{\infty}(v-y)\,dF_{v|w} + \int_{0}^{\hat{p}}(v-\hat{p})\,dF_{v|w} \leq \int_{y}^{\infty}(v-\hat{p})\,dF_{v|w} + \int_{0}^{\hat{p}}(v-\hat{p})\,dF_{v|w} \leq \int_{0}^{\infty}(v-\hat{p})\,dF_{v|w} = \check{p}-\hat{p} \leq 0$. So the second and third terms are the integral of a non-positive integrand. □

We collect the conclusions about a bidder's best response in the following

Proposition 3 *Suppose that the other bidders act in such a way that the highest drop-out point among them is an absolutely continuous random variable with distribution F_y. Then the best response to it by a bidder with type (c,w) is as follows: If $\int r(t,c,w)dF_y > 0$, stay in the auction until the price reaches $\hat{p}$; then acquire information, drop out if $v \leq \hat{p}$ and otherwise drop out when the price reaches v. If $\int r(t,c,w)dF_y < 0$, do not acquire information and drop out at $\check{p}$. Here $\check{p} = E[v|w]$, $\hat{p}$ the unique solution to $c = \int_{0}^{\hat{p}}(\hat{p}-v)dF_{v|w}$, and $r(t,c,w) = \max\{\int_{0}^{t}F_{v|w}(s)ds - c, 0\} - \max\{t - E[v|w], 0\}$.*

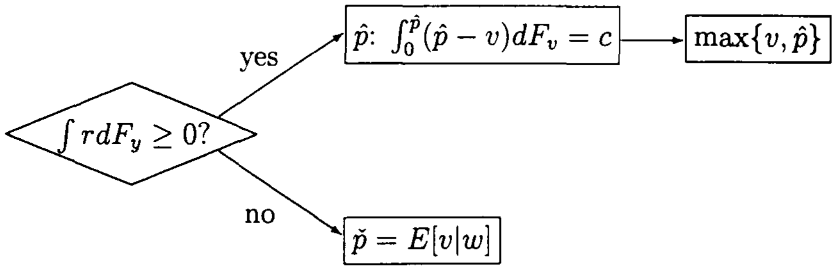

Figure 3: Schematic representation of an optimal strategy.

4 Equilibrium

From Proposition 3, we know that the distribution of the drop-out point of a bidder that follows the strategy described there is a mixture of $\max\{\hat{p}, v\}$ and $\check{p}$. So the distribution of y_i and, likewise, y, inherits the smoothness properties imposed on the distributions of c, v and w.

To obtain an existence result, it remains to verify that there exists a distribution F_y such that a best response to it generates itself. This is done in the next subsection.

4.1 Existence

Let $\mathcal{F}$ be the set of all absolutely continuous distributions over $[0, \bar{v}]$ and $\mathcal{A}$ the collection of all measurable subsets of types (c, w).

Define two applications between these spaces. $T : \mathcal{A} \to \mathcal{F}$ gives the distribution of y_i that would arise if a bidder was acquiring information if her type was in A; i.e.,

$$\begin{aligned} T(A)(x) &= \Pr[A]\Pr[\max\{\hat{p}, v\} \leq x|A] + (1 - \Pr[A])\Pr[\check{p} \leq x|A^c] \\ &= \int_A \Pr[\max\{\hat{p}, v\} \leq x|c, w]dF_{c,w} + \int_{A^c} \Pr[\check{p} \leq x|c, w]dF_{c,w} \end{aligned}$$

Notice that $T(A) \in \mathcal{F}$, since it is a mixture of absolutely continuous distributions. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ under the sup norm.[10]

Define $R : \overline{\mathcal{F}} \to \mathcal{A}$ as follows:

$$R(F) = \left\{ (c, w) \mid \int r(t, c, w)dF^{n-1}(t) \geq 0 \right\},$$

[10] $\overline{\mathcal{F}} \setminus \mathcal{F}$ contains continuous distribution functions with a singular part.

where $r(t,c,w) = \max\{\int_0^t F_{v|w}(s)ds - c, 0\} - \max\{t - E[v|w], 0\}$. For an absolutely continuous distribution, this application selects the best response A to it. Notice that, since any distribution function is of finite variation and r is continuous with respect to y, the integral $\int r dF^{n-1}$ is well defined (Natanson 1961, Ch. 8, §6, Thm. 1), and for a sequence $F_k \to F$, $\lim_{k\to\infty} \int r dF_k^{n-1} = \int r dF^{n-1}$, by Helly's Second Theorem (Natanson 1961, Ch. 8, §7, Thm. 3).

Using this notation, the last object that we need to find to obtain a symmetric equilibrium is a distribution $F^* \in \mathcal{F}$ such that the information acquisition decisions consistent with it generate it; that is, we need to find a fixed point

$$F^* = T(R(F^*)).$$

I will prove existence of an equilibrium applying the Schauder Fixed Point Theorem. To do so, I begin with some definitions from Topology.

A set is *relatively compact* if it is a subset of a compact set. A continuous application is a *compact map* if its image is relatively compact. I recall the following important result:

Theorem 1 (Ascoli-Arzelà) *Let X be a compact metric space. If $S \subset C(X)$ is equicontinuous and bounded, then S is relatively compact.*

It is convenient at this point to impose bounds in the densities of w, c and v:

Assumption 1 (Density Bounds) *Assume the distributions of w and v are absolutely continuous; the distribution of c is absolutely continuous, except possibly for an atom π at 0; and there are positive constants M_c, M_w, M_v, and m such that $f_w \leq M_w$, $f_v \leq M_v$, $\frac{\partial}{\partial w}(E[v|w]) \geq 1/m$, and $f_c(t) \leq M_c, \forall t > 0$.*

From the Ascoli-Arzelà Theorem, we obtain that

Proposition 4 *$T(\mathcal{A})$ is relatively compact.*

Proof: Take $X = [0, \bar{v}]$ and $S = T(\mathcal{A})$ in the Ascoli-Arzelà Theorem statement. Since all distributions are bounded in the sup norm, it only remains to verify equicontinuity.

Take $\epsilon > 0$. Let $\delta < \epsilon/[(n-1)(M_c+M_v+M_w/m)]$. Let F be a distribution in $T(\mathcal{A})$.

So, for any $x \in [0, \bar{v}]$, one can write $0 \leq F(x+\delta) - F(x) = \Pr[y_1 \in [x, x+\delta]] \leq \Pr[v \in [x, x+\delta]] + \Pr[\hat{p} \in [x, x+\delta]] + \Pr[\breve{p} \in [x, x+\delta]]$.

We next observe that, using the Jacobian rule and the definition of $\hat{p}$, we have that $f_{\hat{p}}(s) = \int F_{v|w}(s) f_c \left(\int_0^s F_{v|w}(t)dt\right) f_w(w)dw$. Since $f_c \leq M_c$ and $F_{v|w} \leq 1$, we obtain $f_{\hat{p}} \leq M_c$. Also, $f_{\breve{p}} = \left(\frac{\partial}{\partial w}(E[v|w])\right)^{-1} f_w \leq M_w/m$. It then follows that, for sufficiently small δ,

$$0 \leq F(x+\delta) - F(x) \leq M_c\delta + M_v\delta + M_w/m\delta < \epsilon.$$

As for the case where $x = 0$, notice that $v \geq \hat{p}(0, w) = 0$. So the atom in the $\hat{p}$ distribution is irrelevant, since we can write $0 \leq F(\delta) - F(0) \leq \Pr[v \in [x, x+\delta]] + \Pr[\breve{p} \in [x, x+\delta]] < (M_v + M_w/m)\delta < \epsilon$. So $T(\mathcal{A})$ is equicontinuous. □

We now need to investigate continuity of the $T \circ R$ operator.

Proposition 5 *If $F \in \mathcal{F}$ is such that $\Pr\{\int r dF^{n-1}\}$ is zero, then $T \circ R$ is continuous at F.*

Proof:

Take $F_t \to F$ uniformly. Then, by Helly's Second Theorem, $\int r dF_t^{n-1} \to \int r dF^{n-1}$. We can write

$$T \circ R(F)(x) =$$
$$\int \mathbb{I}_{\{\breve{p} \leq x\}} + \mathbb{I}_{\{\int r dF^{n-1} \geq 0\}}(\Pr[\max\{\hat{p}, v\} \leq x | c, w] - \mathbb{I}_{\{\breve{p} \leq x\}}) dF_{(c,w)},$$

where $\mathbb{I}_{\{\}}$ denotes the indicator function. Applying Cauchy-Schwarz and the fact that indicator functions are bounded by 1, we obtain that

$$(T \circ R(F_k)(x) - T \circ R(F)(x))^2 =$$

$$\left(\int (\mathbb{I}_{\{\int r dF_k^{n-1} \geq 0\}} - \mathbb{I}_{\{\int r dF^{n-1} \geq 0\}})(\Pr[\max\{\hat{p}, v\} \leq x | c, w] - \mathbb{I}_{\{\hat{p} \leq x\}}) dF_{(c,w)} \right)^2$$

$$\leq \int (\mathbb{I}_{\{\int r dF_k^{n-1} \geq 0\}} - \mathbb{I}_{\{\int r dF^{n-1} \geq 0\}})^2 dF_{(c,w)}$$

$$\times \int (\Pr[\max\{\hat{p}, v\} \leq x | c, w] - \mathbb{I}_{\{\hat{p} \leq x\}})^2 dF_{(c,w)}$$

$$\leq \int (\mathbb{I}_{\{\int r dF_k^{n-1} \geq 0\}} - \mathbb{I}_{\{\int r dF^{n-1} \geq 0\}})^2.$$

For any point outside $\{ (c, w) \mid \int r dF^{n-1} = 0 \}$, $\mathbb{I}_{\{\int r dF_k^{n-1} \geq 0\}}$ converges pointwise to $\mathbb{I}_{\{\int r dF_k^{n-1} \geq 0\}}$. So the limit of the integral of the last expression is of a function that is zero almost everywhere. Also, since the last expression does not depend on x, convergence is uniform and continuity is established. □

So existence would be guaranteed, if one could only restrict the analysis to distributions where $\Pr[\{c, w \mid \int r dF^{n-1}\}] = 0$. Unfortunately, this is not necessarily true for distributions that concentrate mass in low values: Since $r = 0$ for sufficiently low values of y (see Figure 2), against these distributions a positive mass of types will be indifferent about acquiring or not information and proposition 5 cannot be applied.

It is not hard to impose assumptions that avoid this technical problem. For example, suppose that with some positive probability π bidders start the game already knowing v. This is the same as assuming that there is an atom π in the distribution of c at 0, since bidders that start knowing v behave in exactly the same way as bidders with zero cost.

This assumption is sufficient for existence; to see that, define, for $\pi > 0$ $\mathcal{F}_\pi = \{ (1-\pi)F + \pi F_v \mid F \in \overline{\mathcal{F}} \}$, where F_v is the (unconditional) distribution of v. The next propositions show that we can restrict our attention to this set.

Proposition 6 *Suppose* $\Pr[c = 0] = \pi > 0$. *Then* $T \circ R(\overline{\mathcal{F}}) \subset \mathcal{F}_\pi$.

Proof: Notice that $r(y, 0, w) \geq 0$. So, since we are resolving any indifference in favor of acquiring information, for any $A \subset R(\overline{\mathcal{F}})$, $\{c = 0\} \subset A$ (meaning that all types with $c = 0$ are in A). Also, $\hat{p}(0, w) = 0$. So for such A, separating the types where $c = 0$, we obtain

$$
\begin{aligned}
T(A)(x) = & \int_{A^c \cap \{c>0\}} \mathbb{I}_{\{\hat{p} \leq x\}} dF_{c,w} + \int_{A \cap \{c>0\}} \Pr[\max\{\hat{p}, v\} \leq x | c, w] dF_{c,w} \\
& + \int_{\{c=0\}} \Pr[\max \hat{p}, v \leq x | c, w] dF_{c,w} \\
= & (1-\pi) \left[\int_{A^c \cap \{c>0\}} \mathbb{I}_{\{\hat{p} \leq x\}} dF_{c|c>0,w} + \int_{A \cap \{c>0\}} \Pr[\max\{\hat{p}, v\} \leq x | c, w] dF_{c|c>0,w} \right] \\
& + \pi \int_{\{c=0\}} \Pr[v \leq x | w] dF_{c,w} \\
= & (1-\pi) F(x) + \pi \int F_{v|w}(x) dF_w \\
= & (1-\pi) F(x) + \pi F_v(x),
\end{aligned}
$$

by the law of iterated expectations. Here F is the distribution defined as the term between square brackets. □

Proposition 7 *For $\pi > 0$, $T \circ R$ is continuous in $\mathcal{F}_\pi$.*

Proof: From proposition 5, it is enough to verify that the measure of $\{\int r dF^{n-1} = 0\}$ is zero. From the Envelope Theorem, $\frac{\partial}{\partial c} \int r dF^{n-1} = 1 - F^{n-1}(\hat{p})$. But for all distributions in $\mathcal{F}_\pi$, and any $x < \bar{v}$, $F(x) < 1 - \pi F_v(x) < 1$. So $\int r dF^{n-1}$ is strictly increasing in c everywhere, and for each w, there is at most one $c > 0$ such that $(c, w) \in \{\int r dF^{n-1} = 0\}$. Furthermore, this c can never be zero, because if F is in $\mathcal{F}_\pi$, there is a positive probability of y_i occurring in any interval in the support $[0, \bar{v}]$. So the integral $\int r(t, 0, w) dF^{n-1}$ is positive for these types. □

Now the fixed point result comes from applying the following

Theorem 2 (Schauder Fixed Point Theorem) *Let C be a closed, convex subset of a normed linear space and let $h : C \to C$ be a compact map. Then h has a fixed point.*

Then we can finally state that

Proposition 8 *If $\Pr[c = 0] = \pi > 0$, a symmetric equilibrium exists.*

Proof: After proposition 3, it only remains to show that there exists a fixed point to the $T \circ R$ map in $\mathcal{F}$.

The set $\mathcal{F}_\pi$ is convex and closed. By propositions 4 and 7, the restriction of $T \circ R$ on $\mathcal{F}_\pi$ is a compact map, and by Proposition 6 its image is in $\mathcal{F}_\pi$. So Schauder's Theorem applies, and a fixed point F^* exists in $\mathcal{F}_\pi$. But F^* is in the image of $T \circ R$, so it also belongs to $\mathcal{F}$. □

5 Comparison with the One-shot Auction

One virtue of the present analysis of the sequential auction is that it easily accommodates the case of a one-shot, Vickrey auction.

In a Vickrey auction, the bidders can act exactly as they would in the sequential auction, except that is not feasible anymore to acquire information in the middle of the auction. So a model of this auction is the same as the one studied so far, with the added restriction that $\hat{p} = 0$.

Suppose this restriction is added to the individual bidder's best response problem. Define $\check{U}$ and $\hat{U}$ as before. The optimal $\check{p}$ is still $E[v|w]$, but now the "maximization" of $\hat{U}$ forcefully leads to $\hat{p} = 0$. The decision to acquire information will still depend on the comparison of two quantities, $\hat{U}(0,0)$ versus $\check{U}(0,\check{p})$. Again, we can write $\hat{U}(0,0) - \check{U}(0,\check{p}) = \int r_0(t,c,w)dF_y(t)$, where $r_0(t,c,w) = \int_0^t F_{v|w}(s)ds - c - \max\{0, y - \check{p}\}$. This payoff difference is also a combination of two options. Figure 2 shows how r_0 differs from r.

Notice that the derivative of that integral with respect to c is 1, no matter what F_y is expected to be.

Define, as before, $R_0 : \overline{\mathcal{F}} \to \mathcal{A}$ as follows:

$$R_0(F) = \left\{ (c,w) \mid \int r_0(t,c,w)dF^{n-1}(t) \geq 0 \right\},$$

and $T_0 : \mathcal{A} \to \mathcal{F}$ as

$$T_0(A)(x) = \int_A \Pr[v \leq x|c,w]dF_{c,w} + \int_{A^c} \Pr[\check{p} \leq x|c,w]dF_{c,w}.$$

An equilibrium of the one-shot auction corresponds to a fixed point of $T_0 \circ R_0$. Notice that the properties of T and R described in propositions 4 and 5 still hold for T_0 and R_0; but now the set $\{\int r_0 dF^{n-1} = 0\}$ is of measure zero for any distribution $F \in \overline{\mathcal{F}}$, because $\int r_0 dF^{n-1}$ is always strictly increasing in c.[11]

We conclude that

Proposition 9 *An equilibrium of the one shot-auction exists (even if* $\pi = 0$*).*

How do the equilibria of the two auctions compare? I have found out that, through numerical simulations, the distribution of the bids in the sequential auction frequently dominates the one in the one-shot auction. Because of that, one typically finds a higher expected revenue for the seller in the sequential procedure.

In this section I formally show that, at least when n is large, the expected revenue in the sequential auction is indeed higher. The numerical computations presented in the next section show that this ranking can also be true for $n = 2$, so the conclusion of this result can be valid under more general conditions.

As with the case of existence, I obtain the revenue comparison result through a series of propositions. I begin with a convenient definition:

Definition 1 *A distribution* F *dominates* G *at the upper tail if there exists some* $\bar{x} < \bar{v}$ *such that* $F(x) < G(x)$ *for all* $x \in (\bar{x}, \bar{v})$*, where* $\bar{v}$ *is the supremum of the support of* F*.*

The usefulness of this definition comes from the following proposition:

Proposition 10 *If* F *dominates* G *at the upper tail*[12] *then, for sufficiently high* n*, the expected value of the* r^{th}*-order statistic of an i.i.d. sample of size* n *from* F *is higher than the same expectation for* G*.*

Proof: We first note that, after a change of variables, we can write this expectation as $B(n+1-r,r)^{-1}\int_0^1 F^{-1}(u)u^{n-r}(1-u)^{r-1}du$, and likewise for G, and where B is the Beta function (Arnold and Balakrishnan 1989, expression 2.1).

Therefore the difference in expected second-order statistics is proportional to $\int_0^1[F^{-1}(u) - G^{-1}(u)][u^{n-r}(1-u)^{r-1}]du$. Let's look at the shape of each of the factors in square brackets to assess the sign of this expression.

[11] See the proof of proposition 7 for the precise argument of why these ideas are related.

[12] ... and F has a compact support $[0, \bar{v}]$.

From the upper-tail dominance, there is a point $\bar{x}$ such that $F(x) < G(x)$, for all $x \in (\bar{x}, \bar{v})$, and $F(x) \geq G(x)$, for $x \in (0, \bar{x})$. Let $\bar{u} = F(\bar{x})$; then there is some set $(\bar{u}, 1)$ such that $F^{-1} - G^{-1} > 0$ there.

Now let's look at the behavior of $u^{n-r}(1-u)^{r-1}$. Observe that, for n large enough, it is increasing in $(0, \bar{u})$, and $\bar{u}^{n-r}(1-\bar{u})^{r-1} \to 0$ as $n \to \infty$. So we can bound the negative part of the integral, writing $\int_0^{\bar{u}}[F^{-1}(u) - G^{-1}(u)]u^{n-r}(1-u)^{r-1}du \geq \bar{u}^{n-r}(1-\bar{u})^{r-1} \int_0^{\bar{u}}[F^{-1}(u) - G^{-1}(u)]du \geq -\bar{v}\bar{u}\bar{u}^{n-r}(1-\bar{u})^{r-1} \to 0$. So for large enough n, the difference becomes positive. □

This is an useful result for revenue comparisons, because expected revenues in auction models are expectations over second-order statistics, and also to efficiency comparisons, being those related to comparisons of moments of first-order statistics.

How the distributions of drop-out points in the sequential and the one-shot auction compare?

Proposition 11 *For any $F \in \overline{\mathcal{F}}$, $R_0(F) \subset R(F)$.*

Proof: By inspection, $r \geq r_0$. □

Proposition 12 *For any $A \in \mathcal{A}$, $T(A)$ first-order stochastically dominates $T_0(A)$.*

Proof: Since $\Pr[\max\{\hat{p}, v\} \leq x|c, w] - \Pr[v \leq x|c, w] \leq 0$, we obtain $T(A)(x) - T_0(A)(x) = \int_A (\Pr[\max\{\hat{p}, v\} \leq x|c, w] - \Pr[v \leq x|c, w])\, dF_{c,w} \leq 0$. □

Proposition 13 *Suppose that the supremum of the support of $E[v|w]$ (call it $\bar{w}$) is strictly below $\bar{v}$. Then for any F, $T(R(F))$ dominates $T_0(R_0(F))$ at the upper tail.*

Proof: $T(R(F)) - T_0(R_0(F)) = T(R(F))(x) - T_0(R(F))(x) + T_0(R(F)) - T_0(R_0(F)) \leq T_0(R(F)) - T_0(R_0(F))$, by proposition 12.

Substituting formulas we can write that $T_0(R(F))(x) - T_0(R_0(F))(x) = \int_{R \setminus R_0}(\Pr[v \leq x|c, w] - \Pr[\check{p} \leq x|c, w])dF_{c,w}$; for any $x \in (\bar{w}, \bar{v})$, $\Pr[\check{p} \leq x|c, w] = 1$, so this term is non-positive.

To see that the term is strictly negative, look at types with w close to $\bar{w}$ and low (but not zero) c. In the move from the one-shot to the sequential auction, a positive mass of those types switched their information acquisition decision, since varies continuously with (at least) c. So $R \setminus R_0$ has a positive mass for high w-types. This means that $\Pr[v \leq x|c, w]$ in a region with positive mass, and the inequality is indeed strict. □

So we conclude that, holding the behavior of the opponents fixed, the effect of the change in the rules is an upper-tail dominance for drop-out points of an individual bidder. This in turn implies a ranking in expected revenue. All that remains is to obtain the result in equilibrium comparisons, as well. This is done through the following result from Milgrom and Roberts (1994):

Theorem 3 *Let $\phi(x,t) = [\phi_L(x,t), \phi_H(x,t)] : [0,1] \times \mathbf{T} \to [0,1]$, where $\mathbf{T}$ is any partially ordered set. Suppose that, for all $t \in \mathbf{T}$, ϕ is continuous but for upward jumps*[13] *in x and that, for all $x \in [0,1]$, ϕ_L and ϕ_H are monotone nondecreasing in t. Then the set of fixed points of ϕ is nondecreasing in t.*

An application of this result to the problem at hand yields:

Proposition 14 *For large enough n, the set expected revenues of all symmetric equilibria of the sequential auction is higher than the set of expected revenues of the one-shot auction equilibria.*

Proof: Take an appropriate closed, convex restriction of the domain of $T \circ R$ such that it is continuous, like $\mathcal{F}_\pi$, and consider its image (that belongs in it, by proposition 6). According to proposition 4, the closure of its image, K, is compact, is connected and lies inside $\mathcal{F}_\pi$. Any fixed points will be in K as well, and we can safely restrict attention to this set.

Call the second-order expectation functional $\mu : K \to [0, \bar{v}]$, $\mu(F) = B(n-1,2)^{-1} \int_0^1 F^{-1}(u) u^{n-2}(1-u) du$. For every F in K, by the previous propositions we know there is a $n^*(F)$ so that, for $n > n^*(F)$, $\mu(T(R(F))) > \mu(T_0(R_0(F)))$. Take an open ball around F so that this property still holds inside it. Doing that for every F, we obtain an open covering of K. But K is compact; so there is a finite subcovering, and a maximal n^*, such that $n > n^*$ makes $\mu(T(R(F))) > \mu(T_0(R_0(F)))$ for any $F \in K$.

[13]Continuity but for upward jumps means that, for any x, $\limsup_{x_k \nearrow x} \phi_H(x_k, t) \leq \phi_H(x,t)$ and $\liminf_{x_k \searrow x} \phi_L(x_k,t) \geq \phi_L(x,t)$.

Being K compact and connected, $\mu(K)$ is a compact, connected set as well. It has to be a compact interval; call it $[a, b]$. Define the ϕ of Theorem 3 to be $\phi : [a, b] \times \{0, 1\} \to [a, b]$, with $\phi(m, 0) = \mu \circ T_0 \circ R_0 \circ \mu^{-1}$, and $\phi(m, 1) = \mu \circ T \circ R \circ \mu^{-1}$.

Let's first verify that $\phi(m, t)$ is a compact interval, as required by Theorem 3. Since $\{m\}$ is closed and μ is continuous, $\mu^{-1}(m)$ is a closed set within a compact, and therefore is compact. It is also path-connected.[14] Its image through the continuous application $\mu \circ T \circ R$ is also connected and compact. So $\phi(m, t)$ is a compact interval in $[0, \bar{v}]$, since these are the only connected sets on the line.

To verify continuity but for upward jumps, start with a sequence $m_k \nearrow m$, and the corresponding sequence $\phi_H(m_k, t)$. Take a subsequence that converges to $\limsup \phi_H(m_k, t)$. To each element of it there is a function F_k in $\mu^{-1}(m_k)$. This sequence of functions belong to a compact set, so it has a subsequence that converges to a function F. Being μ continuous, $\mu(F) = \lim \mu(F_k) = m$. So $\phi_H(m, t) \geq \limsup \phi_H(m_k, t)$. The argument for the lim inf part is analogous. Applying Theorem 3 leads to the conclusion that there are two sets, I_0 and I_1, such that (i) the expectations of equilibria in the one-shot game are in I_0 and those in the sequential auction are in I_1, (ii) $\inf I_0 \leq \inf I_1$ and (iii) $\sup I_0 \leq \sup I_1$.

This conclusion is not quite sufficient for our purposes because I_0 and I_1 may potentially be very large. There are many fixed points of ϕ that are not equilibria: any distribution with the property that $\mu(F) = \mu(T(R(F)))$ would "look like" a fixed point from the point of view of ϕ.

To fix this important flaw, consider the family $\{\phi^k\}_{k \in \{0,1,2,\ldots\}}$ of correspondences defined as $\phi^k(m, t) = \{\, \min\{b, \mu(T_t(R_t(F)) + k\|F - T_0(R_0(F))\| \times \|F - T(R(F))\|\} \mid \mu(F) = m \,\}$. In this way $\phi^0 = \phi$, and for $k > 0$ we add to ϕ k times the distance (in the sup norm) between F and each of the $T_t(R_t(F))$'s (If the result falls outside $[a, b]$, we just truncate).

Since all involved operations are continuous, all the arguments done before for ϕ apply again for each ϕ^k, and we obtain sets I_0^k and I_1^k with properties (i), (ii) and (iii) as before. Consider $I_0^* = \bigcap_k I_0^k$ and $I_1^* = \bigcap_k I_1^k$. These sets also have the same properties, and furthermore cannot contain any point that does not correspond to a true equilibrium expected revenue.

[14] This is because μ is "linear" with respect to F^{-1}; to see that, fix F and G in $\mu^{-1}(m)$ and for $\lambda \in [0, 1]$, let H_λ be the function such that $H_\lambda^{-1} = \lambda F^{-1} + (1 - \lambda) G^{-1}$. Then $\mu(H_\lambda) = m$.

To see that, fix $m \in I_t^*, m < b$. For each $k \in \{0, 1, 2, \dots\}$, there is a F^k with $m = \mu(F^k) = \mu(T_t(R_t(F^k))) + k\|F^k - T_0(R_0(F^k))\| \, \|F^k - T(R(F^k))\|$. This is a sequence in a compact; it has a subsequence converging to a distribution F and, by continuity, $m = \mu(F) = \mu(T_t(R_t(F))) + \|F - T_0(R_0(F))\| \|F - T(R(F))\| \lim k$. The corresponding set of indices is exploding; so this equation can only be satisfied if $\|F - T_0(R_0(F))\| = 0$ or $\|F - T(R(F))\| = 0$. If $\|F - T_t(R_t(F))\| = 0$, then F is indeed a fixed point. If $\|F - T_r(R_r(F))\| = 0$, for $r \neq t$ only, then we have $\mu(T_t(R_t(F))) = \mu(F) = \mu(T_r(R_t(F)))$, and this contradicts proposition 13. □

6 Examples

This section discusses the computation for some choices of distributions for c, w and v. The motivation for this exercise is twofold: first, it shows how an equilibrium can be computed.[15] Second, it establishes some quantitative meaning to the comparative statics finding that the sequential auction revenue-dominates the one-shot procedure.

As mentioned before, for all examples simulated this revenue ranking holds for any number of bidders between 2 and 10. Numerical results also suggest that it might be true that in fact the distribution of drop-out points in the sequential auction in fact first-order stochastically dominates the one in the one-shot auction.

I begin by discussing the computational method.

6.1 Computational Method

To compute the equilibrium, I iterate until a fixed point is found, but instead of working with the $\mathcal{F}$ space, I work on the $\mathcal{A}$ space; that is, I seek to find a set A^* of (c, w)-types such that $A^* = R \circ T(A^*)$.

All expectations are calculated through a quasi-Monte Carlo method. More specifically, 3 Weyl sequences with K elements have been drawn.[16]

[15] In particular, equilibria exist and are easy to compute also in conditions not covered by the existence theorem.

[16] The number of draws utilized so far has ranged from 3000 to 9000. This is admittedly quite small, and I plan to report in a later version results from a much larger sample.

Two of these sequences have been used to construct samples of (c, w)-types, and the corresponding $\check{p}$ and $\hat{p}$ have been computed.[17]

Given a candidate set A_t, the algorithm computes the distribution of y_i (the individual bidder drop-out point) that would arise if only types in A_t acquired information. For each type, it is then computed what is the best information acquisition decision against the highest order-statistic of y, and this leads to a new A_{t+1}. The method then iterates until convergence.[18]

An advantage of this rather crude procedure is that the representation of A is left free; I have tried before parameterizations for the border of A,[19] but polynomials or splines did not seem to fit this function well.

A disadvantage is that of course the method need not necessarily converge. My experience so far is that the distance $\Pr(A_t \setminus A_{t+1} \cup A_{t+1} \setminus A_t)$ goes down quite fast in the first couple of iterations, so the initial guess does not seem to be much important. So "almost" convergence is easy to achieve in most cases. Literal convergence, that is, to drive the distance of A_t and A_{t+1} to literally zero, sometimes is somewhat harder. So some loops may exist, but the sets that loop seem to be close to each other for the cases that have been studied so far.

6.2 Numerical Results

Here results for the case with $w \sim U[0,1]$ and $c \sim U[0, 0.05]$ are considered. I analyze three alternatives for the distribution of $v|w$, for n between 2 and 10.

The three alternatives for the distribution of $v|w$ were $U[0, 2w]$, $U[w, w+1]$ and $U[w, 1]$. The reason for these choices was to look at distributions where the variance increases, stays constant, and decreases with w. This is of interest because according to the discussion of section 3.3.1, the impact of w through variance is a potentially important determinant of information acquisition.[20]

[17]The last sequence is used to obtain a sample of $v|w$ where needed.

[18]Notice that by focusing on symmetric equilibria and y_i rather than directly computing a sample $y = \max\{y_1, \dots, y_{n-1}\}$, one can avoid the curse of dimensionality: In the present algorithm the number of q-MC draws does not depend on n.

[19]Recall that, as long as $1 - F_y(\hat{p}) > 0$, this border is the graph of a function $\bar{c}(w)$ in the (w, c)-plane.

[20]Notice that these distributional assumptions violate several of the conditions imposed in the theoretical part. This illustrates the fact that those assumptions were made for

Besides computing equilibria for these 24 cases, I have also computed the equilibria of the corresponding one-shot auction in each case. This allowed me to calculate the expected revenue for the seller in each case.

Table 1 presents the computed expected revenue of the seller under each circumstance. In order to provide a benchmark, the first column shows what would be the revenue if information was costless to all bidders (i.e., if every bidder would drop out at v).[21] The second and third columns show the expected revenue in the one shot and the sequential auctions. Finally, the last column shows the percentage difference of revenue (in terms of the one-shot auction).

In percentage terms, the increased revenue of a sequential procedure ranges from 0 to 6% — arguably, an economically significant figure. In almost all cases the gain is positive. A negative gain has been computed in the last specification for large values of n. It is not clear whether this is in fact true or it is due to the imprecision of the computation for high values of n.

It is interesting to note that as n grows large, the gain becomes small, both in absolute and percentage terms. This observation, coupled with the asymptotic comparison result, suggests that the expected revenue is generally higher with the sequential procedure.

Table 2 shows the *ex-ante* expected payoff of an individual bidder under each rule for all settings, i.e., the expected profit average over all (c, w)-types. In most cases the expected payoff under the sequential procedure is lower than in the one-shot auction. So sequential auctions seem to benefit the seller partially at the expense of the bidders.

Figures 4, 5 and 6 exhibit how the sets of types that acquire information (top panels) and the distributions of the individual drop-out points (bottom panels) are under each alternative. For convenience, only equilibria with $n = 2$ are depicted. Equilibria with more bidders have smaller information acquisition regions, but the the shape of these regions and of the drop-out

convenience, and are not necessary for existence or revenue rankings.

[21] A counterintuitive finding is that sometimes the sequential auction is more profitable than if information was for free. This can only occur however for $n = 2$. The logic is the following: suppose c is extremely high, so that nobody effectively buys information. In this case the revenue is the expected value of the second order statistic of a sample of $E[v|w_i]$, rather than of v_i. With many bidders, the latter is larger than the former, but not when the number of bidders is 2: in this case, $E[\min\{E[v|w_1], E[v|w_2]\}] > E[\min\{v_1, v_2\}]$. (I thank Paul Milgrom for pointing me that.)

$v \mid w \sim$:	n	if info was free	one-shot	sequential	% gain
$U[0, 2w]$	2	0.2600	0.2697	0.2872	6.48%
	3	0.4419	0.4376	0.4520	3.29%
	4	0.5758	0.5614	0.5703	1.57%
	5	0.6795	0.6554	0.6607	0.82%
	6	0.7628	0.7310	0.7332	0.30%
	7	0.8317	0.7925	0.7939	0.17%
	8	0.8899	0.8450	0.8456	0.07%
	9	0.9399	0.8882	0.8888	0.07%
	10	0.9836	0.9261	0.9262	0.01%
$U[w, w+1]$	2	0.7659	0.7877	0.8012	1.71%
	3	1.0002	0.9856	0.9951	0.96%
	4	1.1278	1.0928	1.1037	1.00%
	5	1.2128	1.1673	1.1790	1.01%
	6	1.2756	1.2241	1.2324	0.68%
	7	1.3247	1.2669	1.2752	0.66%
	8	1.3648	1.3028	1.3105	0.59%
	9	1.3983	1.3337	1.3389	0.39%
	10	1.4270	1.3594	1.3635	0.30%
$U[w, 1]$	2	0.6287	0.6454	0.6537	1.28%
	3	0.7793	0.7639	0.7653	0.17%
	4	0.8463	0.8131	0.8136	0.06%
	5	0.8837	0.8438	0.8441	0.03%
	6	0.9074	0.8658	0.8661	0.03%
	7	0.9235	0.8821	0.8821	-0.00%
	8	0.9352	0.8951	0.8952	0.01%
	9	0.9439	0.9052	0.9053	0.01%
	10	0.9508	0.9135	0.9134	-0.00%

Table 1: Expected revenue for the seller.

$v\|w \sim$:	n	one-shot	sequential	percentage gain
$U[0, 2w]$	2	0.2180	0.2129	-2.35%
	3	0.1340	0.1304	-2.72%
	4	0.0927	0.0909	-1.96%
	5	0.0691	0.0682	-1.36%
	6	0.0539	0.0536	-0.59%
	7	0.0435	0.0434	-0.40%
	8	0.0360	0.0359	-0.19%
	9	0.0305	0.0305	-0.21%
	10	0.0263	0.0263	-0.04%
$U[w, w+1]$	2	0.2014	0.1986	-1.37%
	3	0.1014	0.1006	-0.82%
	4	0.0652	0.0639	-2.05%
	5	0.0464	0.0449	-3.21%
	6	0.0349	0.0341	-2.39%
	7	0.0277	0.0269	-3.00%
	8	0.0225	0.0218	-3.22%
	9	0.0186	0.0182	-2.25%
	10	0.0157	0.0154	-1.93%
$U[w, 1]$	2	0.0998	0.0960	-3.74%
	3	0.0400	0.0396	-0.89%
	4	0.0233	0.0232	-0.38%
	5	0.0155	0.0155	-0.25%
	6	0.0111	0.0110	-0.35%
	7	0.0083	0.0083	0.12%
	8	0.0064	0.0064	-0.01%
	9	0.0052	0.0052	-0.13%
	10	0.0043	0.0043	0.13%

Table 2: *Ex-ante* expected payoff for each bidder.

distributions are qualitatively similar.

The lower panels show that typically the distribution of drop-out points in the sequential auction almost dominates the one for the one-shot auction.[22] The impact of the sequential rule can occur either at lower, intermediate or upper quantiles.

As top panels show, the information acquisition regions are indeed monotonic in c, but not necessarily so in w. A more optimistic signal about the good's valuation can make the bidder more (as in the first specification) or less (as in the second one) eager to acquire information, depending on how this news affect the dispersion of her valuation *vis-à-vis* the auction price.

A Appendix: Non-existence in the degenerate case

This appendix shows that, if the set of types is degenerate, an equilibrium may not exist.

Consider the best response to a distribution of y that is mixed, i.e., has an absolutely continuous component and a finite set of atoms. Define $\hat{U}$ and $\hat{p}$ as before. I contend that, as long as an atom of y does not occur at $\hat{p}$, this is still the optimal information acquisition point.

The reason for that is that atoms at $p \neq \hat{p}$ do not fundamentally affect the derivation of the differential equation characterization done before, once derivatives are appropriately replaced by discrete jumps.

Take an interval $[p, p + dp)$. If no atom of y falls in this interval for small enough dp, the derivation done before is unchanged. There may however be an atom at p. We can always take dp small enough so that there are no atoms in $(p, p + dp)$. In this case, it is still possible to write, say,

$$\hat{U}(p, q) = \int_p^{p+dp} (E[v] - t) dF_{y|y \geq p}(t) \\ + [1 - (F_{y|y\geq p}(p + dp) - F_{y|y\geq p}(p))]\hat{U}(p + dp, q)$$

for $p + dp < q$. The only problem is that $F_{y|y\geq p}(p + dp) \to F_{y|y\geq p}(p^+) > F_{y|y\geq p}(p)$. So $\hat{U}$ is discontinuous at this point; but the discontinuity point

[22] The "almost" is due to the fact that there is usually a region where the comparison is slightly reverse. It is not clear at this point whether this is a feature of the problem or just due to numerical errors.

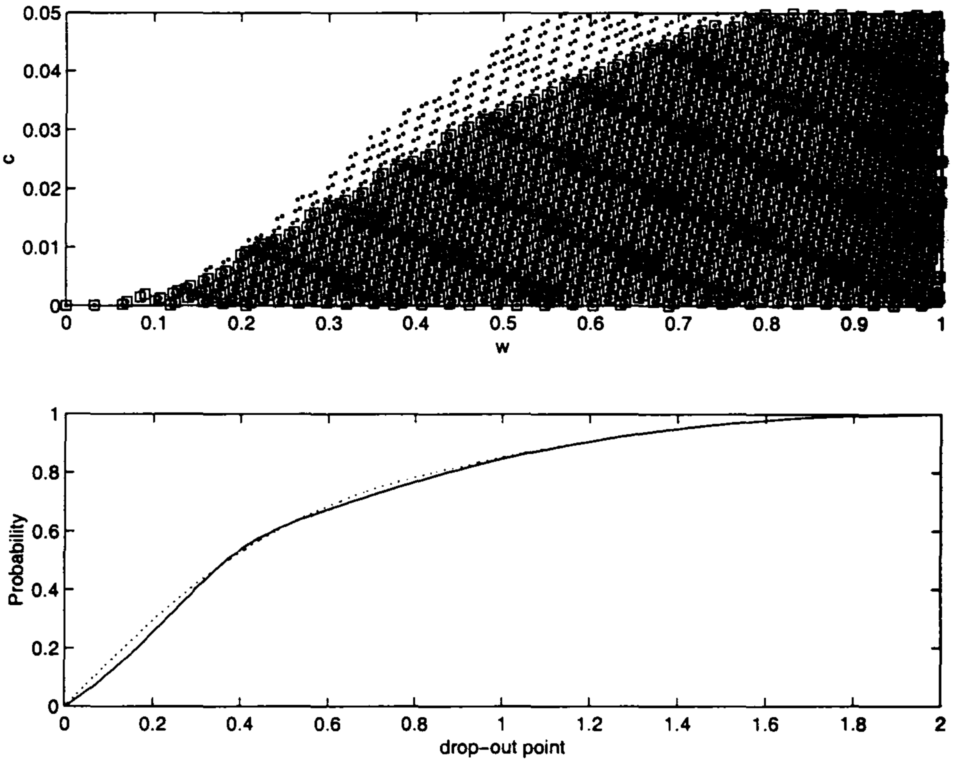

Figure 4: Information acquisition sets and drop-out point distributions when $v|w \sim U[0, 2w]$. Top panel: Shape of the equilibrium R_0 (squares) and R (dots) sets. Bottom panel: Distribution function of the bidder's drop-out point at the sequential (solid line) and one-shot (dotted line) auctions. All graphs assume $n = 2$, $w \sim U[0, 1]$, and $c \sim U[0, 0.05]$.

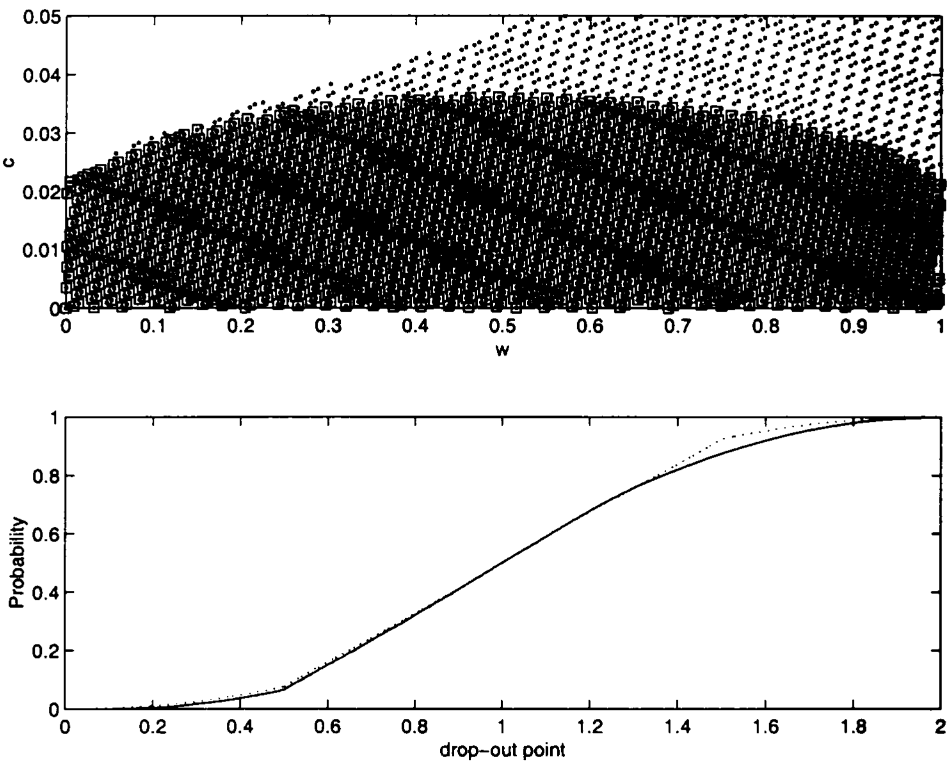

Figure 5: Information acquisition sets and drop-out point distributions when $v|w \sim U[w, w+1]$. Top panel: Shape of the equilibrium R_0 (squares) and R (dots) sets. Bottom panel: Distribution function of the bidder's drop-out point at the sequential (solid line) and one-shot (dotted line) auctions. All graphs assume $n = 2$, $w \sim U[0, 1]$, and $c \sim U[0, 0.05]$.

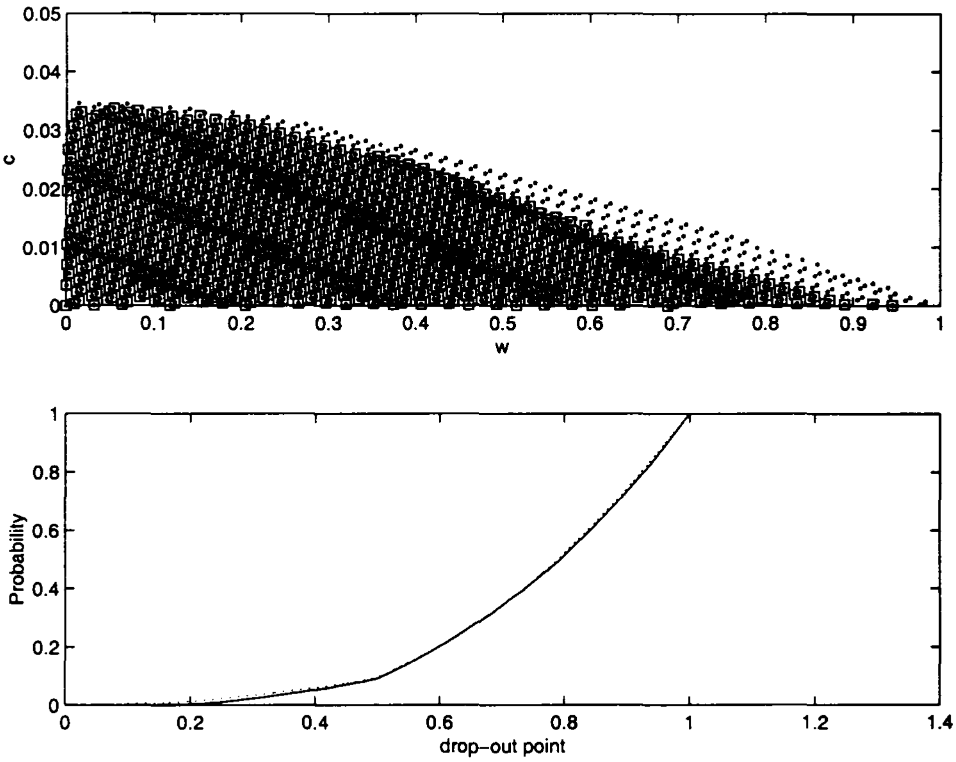

Figure 6: Information acquisition sets and drop-out point distributions when $v|w \sim U[w,1]$. Top panel: Shape of the equilibrium R_0 (squares) and R (dots) sets. Bottom panel: Distribution function of the bidder's drop-out point at the sequential (solid line) and one-shot (dotted line) auctions. All graphs assume $n = 2$, $w \sim U[0,1]$, and $c \sim U[0, 0.05]$.

has the same "size" as its derivative would have otherwise, in the following sense: the jump $\Delta\hat{U}(p,q) = \lim_{dp\to 0}\hat{U}(p+dp,q) - \hat{U}(p,q)$ obeys

$$\Delta\hat{U}(p,q) = \frac{\Pr[y=p]}{\Pr[y\geq p]}[\hat{U}(p,q) - (E[v]-p)],$$

while if we didn't have an atom we would write

$$\frac{\partial}{\partial p}\hat{U}(p,q) = \frac{f_y(p)}{1-F_y(p)}[\hat{U}(p,q) - (E[v]-p)].$$

The same observation applies to the case where $p > q$.

So the discontinuities in the distribution of y are immaterial in the choice of $\hat{p}$ (and, likewise $\check{p}$) *as long as the probability of a tie between this bidder and others is still 0.* That will be the case if atoms occur before $\hat{p}$, since the bidder will not exit at this point, or after $\hat{p}$, since the probability of a tie is the probability of v falling in a measure 0 set.

What happens however if there is an atom at exactly $\hat{p}$? If the bidder follows the strategy described in Proposition 3 (which is the only sensible candidate for a best response by what we have seen so far) she plans to drop out with strictly positive probability at $\hat{p}$, in the event she finds out that $v < \hat{p}$. If there is a positive probability of the last of the others dropping out at the same time, a tie occurs with positive probability. Furthermore, the valuation for the good will be smaller than its price if she wins the auction under these circumstances. So the expected profit of adopting this strategy includes a negative term that could be avoided if she acquired the information slightly before $\hat{p}$. The information acquisition decision problem becomes discontinuous and does not have a solution.

We conclude that, if the distribution of types is degenerate in such a way that $\hat{p}$ has an atom, then an equilibrium may not exist.

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