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MILD SOLUTIONS FOR NONLOCAL FRACTIONAL SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS INVOLVING CAPUTO DERIVATIVE

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In this paper, we prove various existence results of a mild solution for a fractional nonlocal functional semilinear differential inclusion involving Caputo derivative in Banach spaces. We consider the case when the values of the orient field are convex as well as nonconvex. Moreover, we study the topological structure of solution sets. Our results extend or generalize results proved in recent papers.

1. Introduction

During the past two decades, fractional differential equations and fractional differential inclusions have gained considerable importance due to their applications in various fields, such as physics, mechanics and engineering. For some of these applications, one can see [18, 19, 27, 30] and the references therein.

El Sayed and Ibrahim [22] initiated the study of fractional multivalued differential inclusions. Recently, some basic theory for initial-value problems for fractional differential equations and inclusions was discussed by [1, 3, 6, 14, 15, 17, 23, 26, 35, 37, 38].

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Moreover, a strong motivation for investigating the nonlocal Cauchy problems, which is a generalization for the classical Cauchy problems with initial condition, comes from physical problems. For example, it used to determine the unknown physical parameters in some inverse heat condition problems. The nonlocal condition can be applied in physics with better effect than the classical initial condition $x(0) = x_0$. For example, $g(x)$ may be given by

$$g(x) = \sum_{J=1}^{n} c_J x(t_J),$$

where $c_J (J = 1,2,...n)$ are given constants and $0 \le t_1 < t_2 < .... < t_n \le b$. For the applications of nonlocal conditions problems, we refer the reader to [5, 24]. In the few past years, several papers have been devoted to study the existence of solutions for nonlocal differential equations or differential inclusions of positive integer order, for example [2, 7, 9, 10, 13, 28, 36].

Let $q \in]0,1[$, $J = [0,b], b > 0, r > 0$, E be a Banach space, $C(J,E)$ be the Banach space of continuous functions from J to E endowed with the uniform norm $\| u \|_\infty = Sup\{\| u(t) \| : t \in J\}$, $C_0 = C([-r,0],E), \Theta = C([-r,b],E)$ and A be the infinitesimal generator of a C_0-semigroup $\{T(t), t \ge 0\}$ in E [34]. Let $F : J \times C_0 \to 2^E$ be a multifunction, $g : \Theta \to E$ and $\tau(t) : \Theta \to C_0$ defined by $\tau(t)u(s) = u(s+t), \forall s \in [-r,0]$ and $u \in \Theta$.

In this paper we are interested in the nonlocal semilinear functional differential inclusion of order q of the type

$$(P_\Psi) \begin{cases} {}^cD^{\,q}x(t) \in Ax(t) + F(t,\tau(t)x), \ a.e. \text{ on } J, \\ x(t) = \Psi(t) - g(x), \forall t \in [-r,0], \end{cases}$$

where $\Psi \in C_0$ is given and ${}^cD^{\,q}x(t)$ is the Caputo derivative of order q to the function x at the point t.

To study the theory of abstract differential equations or inclusions with fractional order, the first step is how to define the mild solution. Zhou et al. [37, 38] introduced a suitable definition of mild solution based on Laplace transformation and probability density functions for a fractional evolution equation of the type

$$(Q) \begin{cases} {}^cD^q x(t) = A(t)x(t) + f(t,x(t)), \ t \in J, \\ x(0) = x_0 - g(x), x_0 \in E, \end{cases}$$

More recently, Wang et al. [35] proved the existence of a mild solution to the following semilinear fractional differential equation with local condition

$$(S) \begin{cases} {}^cD^{\,q}x(t) \in Ax(t) + G(t,x(t)), \ a.e. \text{ on } J, \\ x(0) = x_0 \in E, \end{cases}$$

where $G : J \times E \to 2^E$ is a multifunction. Among the previous research, little is concerned with nonlocal fractional functional differential inclusions.

In this paper, motivated by the researches mentioned above, we establish two criteria on existence of mild solutions of (P_Ψ). In the first result (Theorem 3.1) we assume that the operator $T(t), t > 0$, is compact and the values of F are convex. In the second result (Theorem 3.3), neither the compactness of operator $T(t), t > 0$, nor the convexity of the values of F is imposed. Moreover, we study the topological structure of solution sets (Theorem 3.2). In this regard, AL-Omair et al. [2] and Ibrahim et al. [21] studied (P_Ψ) when $q = 1$ and Henderson et al. [25] considered problem (P_Ψ) when $A = 0$, $\Psi(0) = 0$, $g = 0$. We mention, among others works [20] for the topological properties of solution sets. So, our obtained results improve or generalize some results obtained in [2, 7, 9, 10, 13, 23, 28, 35, 36, 37, 38].

The present paper is organized as follows: in section 2 we collect some background material and basic results of multivalued analysis and fractional calculus to be used later. In section 3 we prove the main results. Our basic tools are the methods and results for semilinear differential inclusions and C_0-semigroup theory and combined with suitable fixed point theorems techniques.

2. Preliminaries and Notations

We use the following notation:

E a real separable Banach space,

$C(J,E)$ the space of E-valued continuous functions on J with the uniform norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$,

$L^1(J,E)$ the space of E-valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\| dt$,

$P_b(E) = \{B \subseteq E : B$ is nonempty and bounded$\}$,

$P_{cl}(E) = \{B \subseteq E : B$ is nonempty and closed$\}$,

$P_k(E) = \{B \subseteq E : B$ is nonempty and compact$\}$,

$P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex$\}$,

$P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact$\}$,

$conv(B)$ (respectively, $\overline{conv}\,(B)$) be the convex hull (respectively, convex closed hull in E) of a subset B.

Definition 2.1. Let X and Y be two topological spaces. A multifunction $G : X \to P(Y)$ is said to be upper semicontinuous (*u. s. c.* for short) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. G is called closed if its graph $\Gamma_G = \{(x,y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset B of X. If the multifunction G is completely continuous with non empty compact values, then G is *u.s.c.* if and only if G is closed.

For more details about multifunctions we refer to [4, 12, 29, 31, 33].

Definition 2.2. A multivalued function $F : E \to P_{cl}(E)$ is called

(i) γ-Lipschitz if and only if there exists $\gamma > 0$, such that $H(F(x),F(y)) \leq \gamma d(x,y)$, where H is the Hausdorff distance,

(ii) contraction if and only if it is γ-Lipschitz with $\gamma < 1$,

(iii) F has a fixed point if there exists $x \in E$, such that $x \in F(x)$.

Definition 2.3 ([34])**.** A semigroup $T(t),\ 0 \leq t < \infty$, of bounded linear operators on a Banach space X is said to be

(i) uniformly continuous if
$\lim_{t \downarrow 0} \| T(t) - I \| = 0,$ where I is the identity operator,

(ii) strongly continuous if
$\lim_{t \downarrow 0} T(t)x = x,$ for every $x \in X$.

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0-semigroup. It is known that if $T(t),\ 0 \leq t < \infty$ is a C_0-semigroup, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\| T(t) \| \leq Me^{\omega t}, \text{ for } 0 \leq t < \infty.$$

A C_0-semigroup $T(t),\ 0 \leq t < \infty$ is called compact if for every $t > 0, T(t)$ is compact. It is known that ([34], Theorem 3.2) every compact C_0-semigroup is uniformly continuous.

Definition 2.4. Let $T(t),\ 0 \leq t < \infty$, be a semigroup of bounded linear operators on a Banach space X. The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\} \quad \text{and} \quad Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$$

is called the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

In all our paper we adopt the following definitions of fractional primitive and fractional derivative.

Definition 2.5 ([11])**.** The Riemann-Liouville fractional integral of order $q \in (0,1)$ for a function $f \in L^1(J,E)$ is defined as

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

where Γ is the Euler gamma function defined by

$$\Gamma(q) = \int_0^\infty t^{\,q-1} e^{-t}\, dt.$$

Definition 2.6 ([11])**.** The Riemann-Liouville fractional derivative of order $q \in (0,1)$ for a function $f \in L^1(J,E)$ is defined as

$$^{L}D^{\,q} f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^q} ds, t > 0.$$

The following definition is more restrict than the Riemann-Liouville fractional derivative, originally introduced by Caputo in the late sixties.

Definition 2.7 ([11])**.** The Caputo derivative of order $q \in (0,1)$ of a continuously differentiable function f is defined by

$$^{c}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f^{\,(1)}(s)\, ds = J^{\,(1-q)} f^{\,(1)}(t)$$

Note that $^{c}D^{\,q} J^{\,q} f(t) = f(t)$ for all $t \in J$. For more informations about the fractional calculus we refer to [30, 32].

Definition 2.8 ([35, 37, 38])**.** A function $x \in C(J,E)$ is said to be a mild solution for (S) if there exists an integrable selection f for $F(.,x(.))$ such that

$$x(t) = K_1(t)(x_0) + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, \quad t \in J, \tag{1}$$

where

$$K_1(t) = \int_0^\infty \xi_q(\theta) T(t^{\,q}\theta)\, d\,\theta, \quad K_2(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^{\,q}\theta)\, d\,\theta, \tag{2}$$

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \overline{w_q}(\theta^{-\frac{1}{q}}) \geq 0,$$

$$\overline{w_q}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(n\,q+1)}{n!} \sin(n\pi q), \theta \in (0,\infty),$$

and $\int_0^\infty \xi_q(\theta)\, d\theta = 1$.

For more informations about mild solutions of nonlocal semilinear fractional differential equation involving Caputo derivative we refer to [35, 37, 38].

According to the previous definition we can give the following definition for a mild solution of (P_Ψ).

Definition 2.9. One says that a continuous function $x:[-r,b]\to E$, is a mild solution of problem (P_Ψ), if there exists $f\in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$x(t)=\begin{cases}\Psi(t)-g(x),\ t\in[-r,0],\\ K_1(t)(\Psi(0)-g(x))+\int_0^t(t-s)^{q-1}K_2(t-s)f(s)ds,\ t\in J,\end{cases}\tag{3}$$

where

$$S_{F(\cdot,\tau(\cdot)x)}=\{f\in L^1(J,E):f(t)\in F(t,\tau(t)x),a.e.t\in J\}.$$

Lemma 2.10 ([38 lemma 3.2, Lemma 3.3 and Lemma 3.5]). *The operators K_1 and K_2 have the following properties:*

(i) *$K_1(t)$ and $K_2(t)$ are linear bounded operators and if there is $M>0$ such that $\operatorname*{Sup}_{t\in J}\parallel T(t)\parallel\leq M$, then for any $x\in E$ and any $t\in J$*

$$\parallel K_1(t)x\parallel\leq M\parallel x\parallel \text{ and } \parallel K_2(t)x\parallel\leq\frac{M}{\Gamma(q)}\parallel x\parallel.$$

(ii) *If $\{T(t),t\geq 0\}$ is strongly continuous, then $\{K_1(t),t\geq 0\}$and $\{K_2(t),t\geq 0\}$ are strongly continuous.*

(iii) *For every $t>0$, $K_1(t)$ and $K_2(t)$ are compact operators if $T(t)$, $t>0$ is compact.*

(iv) *For $\gamma\in[0,1]$, $\int_0^\infty\theta^\gamma\xi_q(\theta)\,d\,\theta=\frac{\Gamma(1+\gamma)}{\Gamma(1+q\,\gamma)}$.*

Lemma 2.11 ([35, Lemma 2.11]). *For $\delta\in(0,1]$ and $0<a\leq b$ we have $\mid a^\delta-b^\delta\mid\leq(b-a)^\delta$.*

Definition 2.12. A sequence $\{f_n{:}n\in\mathbb{N}\}\subset L^1(J,E)$ is said to be semicompact if

(i) It is integrably bounded, i.e. there is $\beta\in L^1(J,\mathbb{R}_+)$ such that $\parallel f_n(t)\parallel\leq\beta(t)$, for $a.e.\ t\in J$ and for every $n\in\mathbb{N}$.

(ii) The set $\{f_n(t):n\in\mathbb{N}\}$ is relatively compact in E for $a.e.\ t\in J$.

We recall one fundamental result which follows from Dunford-Petties theorem.

Lemma 2.13 ([29, Prop. 4.2.1]). *Every semicompact sequence in $L^1(J,E)$ is weakly compact in $L^1(J,E)$.*

Lemma 2.14 ([4, Theorem 8.2.8]). *Let (Ω, A, μ) be a complete σ-finite measure space, X a complete separable metric space and let $F : \Omega \to 2^X$ be a measurable multivalued function with non empty closed images. Consider a multivalued function G from : $\Omega \times X$ to $P(Y)$, Y is a complete separable metric space such that for every $x \in X$ the multivalued function $w \to G(w,x)$ is measurable and for every $w \in \Omega$ the multivalued function $x \to G(w,x)$ is continuous. Then the multivalued function $w \to \overline{G(w,F(w))}$ is measurable. In particular for every measurable single-valued function $z : \Omega \to X$, the multivalued function $w \to G(w,z(w))$ is measurable and for every Carathéodory single-valued function $\varphi : \Omega \times X \to Y$, the multivalued function $w \to \overline{\varphi(w,F(w))}$ is measurable.*

The following theorems are crucial in the proof of our main result .

Theorem 2.15 ([8]). *Let W be a nonempty subset of a Banach space E, which is bounded, closed and convex. Suppose $R : W \to 2^E$ is u.s.c. with closed, convex values, and such that $R(W) \subseteq W$ and $R(W)$ is compact. Then R has a fixed point.*

Theorem 2.16 ([29, Prop. 3.5.1]). *Let W be a closed subset of a Banach space E and $R : W \to P_k(E)$ be a closed multifunction which is χ-condensing on every bounded subset of W, where χ is a monotone measure of noncompactness. If the set of fixed points for R is a bounded subset of E then it is compact.*

The following auxiliary theorem due to Covitz and Nadler [16].

Theorem 2.17. *Let (X,d) be a complete metric space. If $R : X \to P_{cl}(X)$ is contraction, then R has a fixed point.*

3. Main Results

In this section, we prove two existence results for (P_Ψ) and describe the compactness of the solutions sets.

3.1. Convex Case

In the following we give our first main result for problem (P_Ψ) when the values F are convex.

Theorem 3.1. *Assume the following hypotheses:*

$[HA_1]$ *A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ and $T(t), t > 0$ is compact.*

$[HF_1]$ *$F : J \times C_0 \to P_{ck}(E)$ is a multifunction such that, for every $x \in C_0, t \to F(t,x)$ is measurable, for almost $t \in J, x \to F(t,x)$ is upper semicontinuous and for each $x \in \Theta$, the set $S_{F(\cdot,\tau(\cdot)x)} = \{f \in L^1(J,E) : f(t) \in F(t,\tau(t)x, a.e.\}$ is nonempty.*

$[HF_2]$ *There is a function $\gamma \in L^{\frac{1}{\sigma}}(J,\mathbb{R}_+), \sigma \in (0,q)$ such that for every $\varphi \in C_0$*

$$\sup\{\| z \| : z \in F(t,\varphi)\} \leq \gamma(t)(1+ \| \varphi(0) \|), \ for\ a.e. t \in J. \tag{4}$$

$[Hg]$ *$g : \Theta = C([-r,b],E) \to E$ is compact function and there exist positive constants c and d such that, $\| g(x) \|_E \leq c \| x \| + d, \forall x \in \Theta$.*

Then the problem (P_Ψ) has at least one mild solution on Θ provided that,

$$(M+1)c + \frac{M}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} < 1, \tag{5}$$

where $\eta = \frac{q-\sigma}{1-\sigma}$ and $M = \sup_{t \in J} \| T(t) \|$.

Proof. Transform the problem (P_Ψ) into a fixed point problem. Since for each $x \in \Theta$ the set $S_{F(\cdot,\tau(\cdot)x)}$ is nonempty, we can define a multivalued map $N_\Psi : \Theta \to 2^\Theta - \{\phi\}$ as: $y \in N_\Psi(x)$ if and only if

$$y(t) = \begin{cases} \Psi(t) - g(x), \ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds, \ t \in J. \end{cases} \tag{6}$$

where $f \in S_{F(\cdot,\tau(\cdot)x)}$. Clearly every fixed point for N_Ψ is a mild solution for (P_Ψ). So, our goal is to apply Theorem 2.15. We divide the proof into five steps.

Step (1). The values of N_Ψ are convex and closed subset in Θ.

(i) $N_\Psi(x)$ is convex for each $x \in \Theta$. Let $y_1, y_2 \in N_\Psi(x)$ and $\lambda \in (0,1)$. Then, for $t \in [-r,0]$

$$\begin{aligned} (1-\lambda)y_1(t) + \lambda y_2(t) &= (1-\lambda)[\Psi(t) - g(x)] + \lambda [\Psi(t) - g(x)] \\ &= \Psi(t) - g(x). \end{aligned}$$

Thus, $(1-\lambda)y_1(t) + \lambda y_2(t) \in N_\Psi(x)$. Let $t \in J$ and $f_1, f_2 \in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$y_i(t) = K_1(t)[\Psi(0) - g(x)] + \int_0^t (t-s)^{q-1} K_2(t-s) f_i(s) ds, i = 1,2$$

Then,

$$\begin{aligned} &(1-\lambda)y_1(t) + \lambda y_2(t) \\ &= K_1(t)[\Psi(0) - g(x)] + \int_0^t (t-s)^{q-1} K_2(t-s)[(1-\lambda) f_1(s) + \lambda f_2(s)] ds \end{aligned}$$

Since the values of F are convex then $(1-\lambda)f_1+\lambda f_2 \in S_{F(\cdot,\tau(\cdot)x)}$. Therefore,

$$(1-\lambda)y_1+\lambda y_2 \in N_\Psi(x).$$

(ii) $N_\Psi(x) \in P_f(\Theta)$ for each $x \in \Theta$. Let $(y_n)_{n\geq 1} \in N_\Psi(x)$, such that $y_n \to y$ in Θ. From (6), for any $n \geq 1$, there exists $f_n \in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$y_n(t) = \begin{cases} \Psi(t)-g(x),\ t\in[-r,0], \\ K_1(t)(\Psi(0)-g(x))+\int_0^t (t-s)^{q-1}K_2(t-s)f_n(s)ds,\ t\in J. \end{cases}$$

From (4), for every $n \geq 1$ and for $a.e.s \in J$

$$\begin{aligned} \| f_n(s) \| &\leq \gamma(s)(1+\| \tau(s)x\,(0) \|) \\ &\leq \gamma(s)(1+\| x(s) \|) \\ &\leq \gamma(s)(1+\| x \|), \text{for } a.e.s \in J. \end{aligned}$$

This shows that the set$\{f_n\ : n \geq 1\}$ is integrably bounded. Using the fact that F has compact values, the set $\{f_n\,(t) : n \geq 1\}$ is relativity compact in E and for a.e.$t \in J$. Therefore, the set $\{f_n\ : n \geq 1\}$ is semicompact and then, by Lemma 2.13, it is weakly compact in $L^1(J,E)$. So, we may pass to a subsequence if necessary to get that f_n converges weakly to a function $f \in L^1(J,E)$. From Mazur's theorem, there is a sequence $(g_n), n \geq 1$ such that $\{g_n(t) : n \geq 1\} \subseteq \overline{Conv}\{f_n\ (t) : n \geq 1\}; t \in J$ and g_n converges strongly to f. Since the values of F are convex, $g_n \in S_{F(.,x(.))}$ and hence, by the compactness of $F(.,x(.))$, $f \in S_{F(.,x(.))}$. Moreover, for every $t \in J$, $s \in (0,t]$ and for every $n \geq 1$,

$$\| (t-s)^{q-1}K_2(t-s)f_n(s) \| \leq |t-s|^{q-1}\frac{M}{\Gamma(q)}\gamma(t)\,(1+\| x \|_\Theta) \in L^1((0,t],R^+).$$

Therefore, by means of the Lebesgue dominated convergence theorem we get

$$y(t) = \begin{cases} \Psi(t)-g(x),\ t\in[-r,0], \\ K_1(t)(\Psi(0)-g(x))+\int_0^t (t-s)^{q-1}K_2(t-s)f(s)ds,\ t\in J. \end{cases}$$

This means that $y \in N_\Psi(x)$.

Step (2). There is a positive real number δ satisfying N_Ψ maps $B_\delta(x^*) = \{x \in \Theta\colon \| x-x^* \| \leq \delta\}$ into itself, where

$$x^*(t) = \begin{cases} \Psi(t), t\in[-r,0] \\ \Psi(0), t\in J. \end{cases}$$

If this is not true, then for any $\delta > 0$, there exists a function $x^{\delta} \in B_{\delta}(x^*)$ and $y \in N_{\Psi}(x^{\delta})$ such that $\| y - x^* \| > \delta$. According to (6) there is $f^{\delta} \in S_{F(\cdot,\tau(\cdot)x^{\delta})}$ such that

$$y(t) = \begin{cases} \Psi(t) - g(x^{\delta}),\ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x^{\delta})) + \int_0^t (t-s)^{q-1} K_2(t-s) f^{\delta}(s)ds,\ t \in J. \end{cases}$$

Thus,

$$\begin{aligned} \delta &< \| y - x^* \| = \sup_{t \in [-r,b]} \| y(t) - x^*(t) \| \\ &\leq \sup_{t \in [-r,0]} \| y(t) - x_*(t) \| + \sup_{t \in J} \| y(t) - x_*(t) \| \\ &\leq \| g(x^{\delta}) \| + \| \Psi(0) \| + \sup_{t \in J} \| K_1(t) \| [\| \Psi(0) \| + \| g(x) \|] \\ &+ \sup_{t \in J} \int_0^t (t-s)^{q-1} \| K_2(t-s) f^{\delta}(s) \| ds. \end{aligned}$$

Using Lemma 2.10 (i), [Hg] and Hölder's inequality we get

$$\begin{aligned} \delta &< (M+1) \left[\| \Psi(0) \| + c \| x^{\delta} \| + d \right] \\ &+ \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \gamma(s)(1 + \| \tau(s) x^{\delta}(0) \|) ds \\ &\leq (M+1) \left[\| \Psi(0) \| + c \| x^{\delta} - x_* \| + c \| x_* \| + d \right] \\ &+ \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \gamma(s)(1 + \| x^{\delta} \|) ds \\ &\leq (M+1) [\| \Psi(0) \| + c\, \delta + c \| \Psi \| + d] \\ &+ \frac{M(1 + \| x^{\delta} \|)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \gamma(s) ds. \\ &\leq (M+1) [\| \Psi(0) \| + c\, \delta + c \| \Psi \| + d] \\ &+ \frac{M(1 + \delta + \| \Psi \|)}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} [\int_0^t (t-s)^{\frac{q-1}{1-\sigma}} ds]^{1-\sigma} \\ &= (M+1) [\| \Psi(0) \| + c\, \delta + c \| \Psi \| + d] \\ &+ \frac{M(1 + \delta + \| \Psi \|)}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}}. \end{aligned}$$

Dividing both sides of the a above inequality by δ and taking the limit as $\delta \to \infty$, we get

$$1 < (M+1)c + \frac{M}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}}.$$

But this contradicts with condition (5). Thus, for some $\delta > 0$, $N_\Psi(B_\delta(x^*)) \subseteq B_\delta(x^*)$.

Step (3). $N_\Psi(B_\delta(x^*))$ is equicontinuous set of Θ. Let $x \in B_\delta(x^*)$, $y \in N_\Psi(x)$. Then there exists $f \in S_{F(\cdot,\tau(\cdot)x)}$ such that,

$$y(t) = \begin{cases} \Psi(t) - g(x),\ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s)ds,\ t \in J. \end{cases}$$

We consider the following cases:

Case(i). Let $t \in [-r,0)$, and $h > 0$ such that $-r \leq t < t+h \leq 0$. We have

$$\| y(t+h) - y(t) \| = \| \Psi(t+h) - \Psi(t) \| .$$

Since Ψ is continuous on $[-r,0]$, we get

$$\lim_{h\to 0} \| y(t+h) - y(t) \| = 0,$$

independently of x.

Case(ii). Let $t = 0$ and $h > 0$ such that $-r \leq -h < t = 0 < h \leq b$. By arguing as in (i) we obtain $\lim\limits_{h\to 0} \| y(-h) - y(0) \| = 0$, independently of x. Also

$$\begin{aligned}
&\| y(h) - y(0) \| \\
&\leq \| K_1(h) - K_1(0) \| \, \| \Psi(0) - g(x) \| + \int_0^h (h-s)^{q-1} \| K_2(h-s) f(s)ds \| \\
&\leq \| K_1(h) - K_1(0) \| \, \| \Psi(0) - g(x) \| \\
&\qquad + \int_0^h (h-s)^{q-1} \| K_2(h-s)\gamma(s)(1 + \| \tau(s)x(0) \|)ds \\
&\leq \| K_1(h) - K_1(0) \| \, \| \Psi(0) - g(x) \| + \frac{M(1+\delta + \| \Psi \|)}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{h^{q-\sigma}}{\eta^{1-\sigma}}.
\end{aligned}$$

Since $T(h)$ is compact, $K_1(h)$ is too (Lemma 2.10 (iii)), and hence, $K_1(h)$ is uniformly continuous on J (see [34]). Therefore, the last inequality tends to zero as $h \to 0$, independently of x.

Case (iii). Let $t \in (0,b]$ and $h > 0$ such that $0 < t < t+h \leq b$. We have

$$\begin{aligned}
&\| y(t+h) - y(t) \| \leq \| \big(K_1(t+h) - K_1(t)\big)\big(\Psi(0) - g(x)\big) \| \\
&+ \| \int_0^{t+h} (t+h-s)^{q-1} K_2(t+h-s) f(s) ds - \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds \| \\
&\leq \| \big(K_1(t+h) - K_1(t)\big)\big(\Psi(0) - g(x)\big) \| \\
&+ \| \int_0^t (t+h-s)^{q-1} K_2(t+h-s) f(s) ds - \int_0^t (t+h-s)^{q-1} K_2(t-s) f(s) ds \| \\
&+ \| \int_0^t (t+h-s)^{q-1} K_2(t-s) f(s) ds - \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds \| \\
&+ \| \int_t^{t+h} (t+h-s)^{q-1} K_2(t+h-s) f(s) ds \| \qquad (7)
\end{aligned}$$

We put

$$\begin{aligned}
I_1 &= [K_1(t+h) - K_1(t)]\,[\Psi(0) - g(x)]\,, \\
I_2 &= \int_0^t (t+h-s)^{q-1}\,[K_2(t+h-s) - K_2(t-s)]\, f(s) ds, \\
I_3 &= \int_0^t \left[(t+h-s)^{q-1} - (t-s)^{q-1}\right] K_2(t-s) f(s) ds \\
I_4 &= \int_t^{t+h} (t+h-s)^{q-1} K_2(t+h-s) f(s) ds. \qquad (8)
\end{aligned}$$

Now, we only need to check $\| I_i \| \to 0$, as $h \to 0, i = 1,2,3,4$. At first we note that, as we mention above the operators $K_1(t),\ K_2(t),\ t > 0$ are uniformly continuous on J. So, $\lim_{h \to 0} \| I_1 \| = 0$.

For I_2, using [HF$_2$] and Hölder's inequality, one obtain

$$\begin{aligned}
I_2 &\leq \sup_{s \in [0,t]} \| K_2(t+h-s) - K_2(t-s) \| \,(1 + \delta + \| \Psi \|)\, \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \\
&\quad \times \left[\int_0^t (t+h-s)^{\frac{q-1}{1-\sigma}} ds\right]^{1-\sigma} \\
&\leq \sup_{s \in [0,t]} \| K_2(t+h-s) - K_2(t-s) \| \| (1 + \delta + \| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \\
&\quad \times \frac{[(t+h)^{\eta} - \ h^{\eta}]^{1-\sigma}}{\eta^{1-\sigma}}
\end{aligned}$$

In view of Lemma 2.11, we get

$$\begin{aligned}
I_2 &\leq \sup_{s \in [0,t]} \| K_2(t+h-s) - K_2(t-s) \| \,(1 + \delta + \| \Psi \|)\, \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \\
&\quad \times \frac{[(t+h) - \ h]^{\eta(1-\sigma)}}{\eta^{1-\sigma}} =
\end{aligned}$$

$$= \sup_{s\in[0,t]} \| K_2(t+h-s) - K_2(t-s) \| \, (1+\delta+\| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{t^{\eta(1-\sigma)}}{\eta^{1-\sigma}}$$

$$\leq (1+\delta+\| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{\eta(1-\sigma)}}{\eta^{1-\sigma}} \sup_{s\in[0,t]} \| K_2(t+h-s) - K_2(t-s) \|$$

By the uniform continuity of $K_2(t), (t>0)$, we conclude that $\lim_{h\to 0} \| I_2 \| = 0$.

For I_3, we have

$$\begin{aligned}
\| I_3 \| &\leq \int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}| \, \| K_2(t-s) f(s) \| \, ds \\
&\leq \frac{M}{\Gamma(q)} \Big[\int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}|^{\frac{1}{1-\sigma}} ds\Big]^{1-\sigma} \\
&\times \| (1+\delta+\| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} .
\end{aligned}$$

We note that $\frac{q-1}{1-\sigma} \in (-1,0)$, then $(t-s)^{\frac{q-1}{1-\sigma}} \geq (t+h-s)^{\frac{q-1}{1-\sigma}}$. By applying lemma 2.11 and taking into account that $1-\sigma \in (0,1)$ we get

$$\left| \left[(t-s)^{\frac{q-1}{1-\sigma}}\right]^{1-\sigma} - \left[(t+h-s)^{\frac{q-1}{1-\sigma}}\right]^{1-\sigma} \right| \leq \left[(t-s)^{\frac{q-1}{1-\sigma}} - (t+h-s)^{\frac{q-1}{1-\sigma}}\right]^{1-\sigma}.$$

Thus,

$$|(t-s)^{q-1} - (t+h-s)^{q-1}| \leq \left[(t-s)^{\frac{q-1}{1-\sigma}} - (t+h-s)^{\frac{q-1}{1-\sigma}}\right]^{1-\sigma},$$

So,

$$|(t-s)^{q-1} - (t+h-s)^{q-1}|^{\frac{1}{1-\sigma}} \leq \left[(t-s)^{\frac{q-1}{1-\sigma}} - (t+h-s)^{\frac{q-1}{1-\sigma}}\right]. \tag{9}$$

Therefore by (4), Hölder's inequality, and (9) one obtain

$$\begin{aligned}
&\lim_{h\to 0} \| I_3 \| \leq \frac{M}{\Gamma(q)} (1+\delta+\| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \\
&\times \lim_{h\to 0} \Big[\int_0^t \left[(t-s)^{\frac{q-1}{1-\sigma}} - (t+h-s)^{\frac{q-1}{1-\sigma}}\right] ds\Big]^{1-\sigma} \\
&\quad = \frac{M}{\Gamma(q)} (1+\delta+\| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \lim_{h\to 0} \frac{[t^\eta - ((t+h)^\eta - h^\eta)]^{1-\sigma}}{\eta^{1-\sigma}} = 0,
\end{aligned} \tag{10}$$

independently of x.

For I_4, we have

$$\| I_4 \| \leq \frac{M\,(1+\delta+\| \Psi \|)\,\| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)}}{\Gamma(q)} [\int_t^{t+h} (t+h-s)^{\frac{q-1}{1-\sigma}} ds]^{1-\sigma} ds.$$

Arguing as above, we get

$$\lim_{h\to 0} \| I_4 \| \leq \lim_{h\to 0} \frac{M\,(1+\delta+\| \Psi \|)\,\| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)}\, h^{q-\sigma}}{\Gamma(q)\eta^{1-\sigma}} = 0 \qquad (11)$$

As a result from (7) $\to$ (11) we immediately obtain that

$$\lim_{h\to 0} \| y(t+h) - y(t) \| = 0,$$

independently of x, which means that $N_\Psi\ (B_\delta(x^*))$ is equicontinuous set of Θ.

Step (4). For any $t \in [-r,b]$, the set $\Omega\,(t) = \{y(t) : y \in N_\Psi(x), x \in B_\delta(x^*)\}$ is relatively compact in E.

At first, let $t \in [-r,0]$. Because $B_\delta(x^*)$ is a bounded subset in Θ and g is compact, then the set $\Omega\,(t) = \{\Psi(t) - g(x) : x \in B_\delta(x^*)\}$ is relatively compact in E. Now, let $t \in (0,b]$ be fixed and $x \in B_\delta(x^*)$. Let $y \in N_\Psi(x)$. Then there is $f_y \in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$y(t) = K_1(t)(\Psi(0) - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f_y(s) ds,\ t \in J.$$

For any $h \in (0,t)$ and any $\lambda \in (0,1)$ we define $\Omega_{h,\lambda}\,(t) = \left\{y_{h,\lambda}(t) : y \in N_\Psi(x)\right\}$, where

$$y_{h,\lambda}(t) = \int_\lambda^\infty \zeta_q(\theta) T(t^q\theta)\,[\Psi(0) - g(x)]\,d\,\theta +$$
$$q\int_0^{t-h} (t-s)^{q-1} \int_\lambda^\infty \theta\zeta_q(\theta) T((t-s)^q\theta) f_y(s)\ d\theta ds.$$

Since the operator $T(t), t > 0$ is compact and g is compact on Θ, the sets $\left\{y_{h,\lambda}(t) :\ y \in N_\Psi(x)\right\}$ are relatively compact in E. Moreover, by using [HF$_2$] and [Hg] we get

$$\| y(t) - y_{h,\lambda}(t) \| \leq \| \int_0^\lambda \zeta_q(\theta) T(t^q\theta)\,[\Psi(0) - g(x)]\,d\,\theta \|$$
$$+q \| \int_0^t \int_0^\infty \theta(t-s)^{q-1} \zeta_q(\theta) T((t-s)^q\theta) f_y(s) d\,\theta ds$$
$$- \int_0^{t-h} \int_\lambda^\infty \theta(t-s)^{q-1} \zeta_q(\theta) T((t-s)^q\theta) f_y(s)\ d\theta ds \|$$

$$\leq M(\| \Psi(0) \| + \| g(x) \|) \int_0^\lambda \zeta_q(\theta) d\ \theta$$
$$+ q \| \int_0^t \int_\lambda^\infty \theta (t-s)^{q-1} \zeta_q(\theta) T((t-s)^q \theta) f_y(s) d\ \theta ds$$
$$+ \int_0^t \int_0^\lambda \theta (t-s)^{q-1} \zeta_q(\theta) T((t-s)^q \theta) f_y(s) d\theta ds$$
$$- \int_0^{t-h} \int_\lambda^\infty \theta (t-s)^{q-1} \zeta_q(\theta) T((t-s)^q \theta) f_y(s) d\theta ds \|$$
$$\leq M(\| \Psi(0) \| + c \| x \| + d) \int_0^\lambda \zeta_q(\theta) d\theta$$
$$+ q \| \int_{t-h}^t \int_\lambda^\infty \theta (t-s)^{q-1} \zeta_q(\theta) T((t-s)^q \theta) f_y(s) d\theta ds \|$$
$$+ q \left\| \int_0^t \int_0^\lambda \theta (t-s)^{q-1} \zeta_q(\theta) T((t-s)^q \theta) f_y(s) d\theta ds \right\|$$
$$\leq M(\| \Psi(0) \| + c\delta + c \| x_* \| + d) \int_0^\lambda \zeta_q(\theta) d\theta$$
$$+ Mq \int_{t-h}^t \int_\lambda^\infty \theta (t-s)^{q-1} \zeta_q(\theta) \gamma(s)(1 + \| x \|) d\theta ds$$
$$+ Mq \int_0^t \int_0^\lambda \theta (t-s)^{q-1} \zeta_q(\theta) \gamma(s)(1 + \| x \|) d\ \theta ds$$
$$\leq M(\| \Psi(0) \| + c\gamma + c \| x_* \| + d) \int_0^\lambda \zeta_q(\theta) d\ \theta$$
$$+ Mq(1 + \delta + \| \Psi \|) \int_\lambda^\infty \theta \zeta_q(\theta) d\ \theta \int_{t-h}^t (t-s)^{q-1} \gamma(s) ds$$
$$+ Mq\ (1 + \delta + \| \Psi \|) \int_0^\lambda \theta \zeta_q(\theta) d\ \theta \int_0^t (t-s)^{q-1} \gamma(s) ds.$$

Using Hölder's inequality to get

$$\| y(t) - y_{h,\lambda}(t) \| \leq M(\| \Psi(0) \| + c\ \delta + c \| x_* \| + d) \int_0^\lambda \zeta_q(\theta) d\ \theta$$
$$+ Mq(1 + \delta + \| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{h^{q-\sigma}}{\eta^{1-\sigma}} \int_\lambda^\infty \theta \zeta_q(\theta) d\ \theta$$
$$+ Mq\ (1 + \delta + \| \Psi \|) \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{q-\sigma}}{\eta^{1-\sigma}} \int_0^\lambda \theta \zeta_q(\theta) d\ \theta.$$

Obviously, by Lemma 2.10 (iv), the right hand side of the previous inequality tend to zero as $\lambda, h \to 0$. Hence, there exists a relatively compact set that can be arbitrary close to the set $\Omega(t),\ t \in (0, b]$. Hence, this set is relatively compact in E.

A consequence of Steps (3) and (4) with Arzela-Ascoli theorem we conclude that $N\ (B_\delta(x^*))$ is relatively compact.

Step (5). N_Ψ has a closed graph on $B_\delta(x^*)$.

Let $x_n \in B_\delta(x^*), x_n \to x$ in Θ and $y_n \in N_\Psi(x_n), \forall n \geq 1$ with $y_n \to y$ in $B_\delta(x^*)$. We will show that $y \in N_\Psi(x)$. By recalling the definition of N_Ψ, for any $n \geq 1$ there exists $f_n \in S_{F(\cdot,\tau(\cdot)x_n)}$ such that

$$y_n(t) = \begin{cases} \Psi(t) - g(x_n),\ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x_n\ )) + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(s) ds,\ t \in J. \end{cases} \tag{12}$$

Since g is continuous function, so $g(x_n) \to g(x)$. We must prove that there exists $f \in S_{F(\cdot,\tau(\cdot)x)}$ such that

$$y(t) = K_1(t)(\Psi(0)\ - g(x))\ + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds, \forall t \in J.$$

Let us show that the sequence $(f_n)_{n\geq 1}$ is semicompact. From the uniform convergence of x_n towards x, for any $t \in J$

$$\lim_{n\to\infty} \|\ \tau(t)x_n - \tau(t)x\ \|_{C_0} = 0.$$

Moreover $F(t,.)$ is upper semicontinuous with compact values, then for every $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that for every $n \geq n_0$

$$f_n(t) \in F(t,\tau(t)x_n) \subseteq F(t,\tau(t)x) + \varepsilon B(0,1), a.e. t \in J, \tag{13}$$

where $B(0,1) = \{z \in E :\| z \| \leq 1\}$. Then, the compactness of $F(t,\tau(t)x)$ implies that the set $\{f_n(t) : n \geq 1\}$ is relatively compact for $a.e.$ In addition, assumption $[\mathrm{HF}_2]$ implies

$$\begin{aligned} \|\ f_n(t)\ \| &\leq \gamma(t) \parallel 1 + \tau(t)x_n(0) \parallel \\ &\leq \gamma(t)(1+ \parallel x_n \parallel) \\ &\leq \gamma(t)(1+\delta+ \parallel \Psi \parallel), a.e. t \in J. \end{aligned}$$

Then, by Lemma 2.13, $\{f_n, n \geq 1\}$ is semicompact, hence weakly compact. Arguing as in Step1 from Mazur's theorem, there is a sequence $(g_n), n \geq 1$ such that

$$\{g_n(t) : n \geq 1\} \subseteq \overline{Conv}\{f_n\ (t) : n \geq 1\}; t \in\ J$$

and g_n converges strongly to $f \in L^1(J,E)$. Since the values of F are convex, the relation (13) implies $g_n \in S_{F(.,\tau(.)x_n)}$ and hence, $f \in S_{F(.,\tau(.)x)}$. By passing to the limit in (12), we obtain

$$y(t) = \begin{cases} \Psi(t) - g(x),\ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x\ )) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds,\ t \in J. \end{cases}$$

This implies that $y \in N_\Psi(x)$.

Now, since the values of N_Ψ are compact, then by Step (5) N_Ψ is u.s.c. As a consequence of Steps 1 to 5, N_Ψ is compact multivalued map, u.s.c. with closed convex values. By applying Theorem 2.15, we can deduce that N_Ψ has a fixed point x which is a mild solution of problem (P_Ψ). □

Now, for each $\Psi \in C_0$ let $\check{S}_{[-r,b]}(\Psi)$ be the set of mild solutions of (P_Ψ) and consider the multivalued function $\check{S}_{[-r,b]} : C_0 \to 2^\Theta$. In the following theorem, we give some topological properties of $\check{S}_{[-r,b]}$.

Theorem 3.2. *If the hypotheses of Theorem 3.1 hold then we have the following:*

(i) for every $\Psi \in C_0$, the set $\check{S}_{[-r,b]}(\Psi)$ is nonempty and compact in Θ.

(ii) The multivalued function $\check{S}_{[-r,b]} : C_0 \to P_k(\Theta)$ is upper semicontinuous.

Proof. Let $\Psi \in C_0$. From Theorem 3.1, the set $\check{S}_{[-r,b]}(\Psi)$ is nonempty.In order to prove $\check{S}_{[-r,b]}(\Psi)$ is compact, we consider the multivalued function $N_\Psi : \Theta \to 2^\Theta - \{\phi\}$ defined in Theorem 3.1 as: for any $x \in \Theta$ we have $y \in N_\Psi(x)$ if and only if

$$y(t) = \begin{cases} \Psi(t) - g(x),\ t \in [-r,0], \\ K_1(t)(\Psi(0 - g(x)\ )) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds,\ t \in J. \end{cases}$$

Clearly $\check{S}_{[-r,b]}(\Psi) = Fix(N_\Psi)$. As a consequence of Theorem 3.1, N_Ψ is completely continuous with closed and compact values. Consequently, it is $\chi-$condensing where χ is the Hausdorff measure of noncompactness. From Theorem 2.16, we need only to show that the set of fixed points for N_Ψ is bounded. So, let $x \in Fix(N_\Psi)$.Then $x \in N_\Psi(x)$, therefore there is $f \in S_{F\ (\cdot,\tau(\cdot)x)}$ such that

$$x(t) = \begin{cases} \Psi(t) - g(x),\ t \in [-r,0], \\ K_1(t)(\Psi(0) - g(x\ )) + \int_0^t (t-s)^{q-1} K_2(t-s) f(s) ds,\ t \in J. \end{cases}$$

If $t \in [-r,0]$,then

$$\begin{aligned} \sup_{t\in[-r,0]} \| x(t) \| &\leq \| \Psi(t) \| + \| g(x) \| \\ &\leq \| \Psi \| + c \| x \| + d\ . \end{aligned} \tag{14}$$

If $t \in J$, then

$$\begin{aligned}\sup_{t\in J} \| x(t) \| &\leq M(\| \Psi(0) \| + \| g(x) \|) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1}\gamma(s)(1+ \| x \|)ds \\ &\leq M(\|\Psi(0)\| + c \| x \| + d) + \frac{M(1+ \| x \|)}{\Gamma(q)} \int_0^t (t-s)^{q-1}\gamma(s)(s)ds. \\ &\leq M\ (\| \Psi \| + c \| x \| + d) + \frac{M(1+ \| x \|)}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}.\end{aligned} \tag{15}$$

From (14) and (15) one obtains for every $t \in J$

$$\| x \|_\Theta \leq (M+1)(\| \Psi \| + c\ \| x \|_\Theta + d) + \frac{M(1+ \| x \|_\Theta)}{\Gamma(q))} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}.$$

Thus, by condition (5), the previous inequality can be written as

$$\| x \|_\Theta \leq \frac{(M+1)(\| \Psi \| + d) + \dfrac{M}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}}{1 - \left[c\ (M+1) + \dfrac{M}{\Gamma(q)} \| \gamma \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}\right]}.$$

This means that $Fix(N_\Psi)$ is bounded. By applying Theorem 2.16, we conclude that the set $\check{S}_{[-r,b]}(\Psi)$ is compact.

(ii) Arguing again as in Theorem 3.1, we can show that the multivalued function $\check{S}_{[-r,b]}$ is closed. Since its values are compact, then it is u. s. c. □

3.2. Nonconvex Case.

In this section, we are concerned with an existence result for (P_Ψ) with non-convex valued right-hand side. Let $E = \mathbb{R}^n$ and H be the Hausdorff distance on $P_{cl}(E)$. We make the following conditions:

[HA$_2$] $A : D(A) \to A$ is the infinitesimal generator of a C_0-semigroup of bounded linear operator $T(t)$ in E such that

$$\|T(t)\|_{\mathcal{L}(\mathbb{R}^n)} \leq M \text{ for some } M > 0$$

[HF$_3$] The multivalued function $F : J \times C_0 \to P_{cl}(E)$ has the property that for every $x \in C_0, t \to F(t,x)$ is measurable.

[HF$_4$] There exists $\omega \in L^{\frac{1}{\sigma}}(J,\mathbb{R}_+), \sigma \in (0,q)$ such that for every $\varphi, \psi \in C_0$

$$H(F(t,\varphi), F(t,\psi)) \leq \omega(t) \| \varphi - \psi \|_{C_0}\ a.e, for\ \ t \in J, \tag{16}$$

and

$$d\ (0, F\ (t, 0)) \leq \omega(t)\ for\ \ a.e, t \in J. \tag{17}$$

$\left[\mathrm{H}_g^*\right]$ $g : \Theta \to E$ is a function such that there is a positive constant ς with

$$\|\ g(z) - g(w)\ \| \leq \varsigma\ \ \| z - w\ \|_{\Theta}\ , \text{for all}\ \ z, w \in \Theta. \tag{18}$$

.

Theorem 3.3. *Let assumptions*$[HA_2]$, $[HF_3]$, $[HF_4]$ *and* $\left[H_g^*\right]$ *be satisfied. Then, the problem* (P_Ψ) *has at least one mild solution on* $[-r, b]$ *provided that,*

$$M\ (\varsigma + \frac{\|\ \omega\ \|_{L^{\frac{1}{\sigma}}(J, \mathbb{R}_+)}}{\Gamma(q)} \frac{b^{\ q-\sigma}}{\eta^{1-\sigma}}) < 1 \tag{19}$$

Proof. At first from Lemma 2.14, $[\mathrm{HF}_3]$ and $[\mathrm{HF}_4]$ we conclude that for every $x \in \Theta$ the multivalued function $t \to F(t, \tau(t)x)$ is measurable with closed values, hence from (17) the set $S_{F(\cdot, \tau(\cdot)x)}$ is nonempty. In order to transform the problem (P_Ψ) into a fixed point problem, we shall show that the mulivalued function N_Ψ, defined in (6), satisfies the assumptions of Theorem 2.17. We divide the proof into two steps.

First step. The values of N_Ψ are nonempty and closed.

The values of N_Ψ are nonempty since for each $x \in \Theta$, the set $S_{F\ (\cdot, \tau(\cdot)x)}$ is nonempty. Now let$\{y_n\}_{n \in \mathbb{N}} \in N_\Psi(x)$ such that $y_n \to y$ in Θ.Then for any $n \geq 1$ there exists $f_n \in S_{F(\cdot, \tau(\cdot)x)}$ such that

$$y_n(t) = \begin{cases} \Psi(t) - g(x),\ t \in [-r, 0], \\ K_1(t)(\Psi(0 - g(x)) + \int_0^t (t-s)^{q-1} K_2(t-s) f_n(s) ds,\ t \in J. \end{cases}$$

Obviously, $y(t) = \Psi(t) - g(x);\ t \in [-r, 0]$. Moreover, since $F(t, 0)$ is closed, for any $t \in J$, there is $v(t) \in F(t, 0)$ such that $\|\ v(t)\ \| =\ d\ (\ 0, F(t, 0))$. Then from (16) and (17) we conclude that for any $n \geq 1$ and for a.e.$t \in J$

$$\begin{aligned} \|\ f_n(t)\ \|\ | &\leq \|\ v(t)\ \| + \|\ v(t) - f_n(t)\ \| \\ &=\ d\ (\ 0, F(t, 0)) + \|\ v(t) - f_n(t)\ \| \\ &\leq\ \omega(t) + H(F(t, \tau(t)x), F(t, 0)) \\ &\leq\ \omega(t) + \omega(t)\ \|\ \tau(t)x\ \| \\ &\leq \omega(t)(1 + \|\ x\ \|). \end{aligned}$$

This show that the set $\{f_n\ \ : n \geq 1\}$ is integrably bounded, hence the set $\{f_n(t) : n \geq 1\}$ is relativity compact in $\mathbb{R}^n$ for a.e.$t \in\ J$. Then, there exists a subsequence still denoted $\{f_n\}$ which converges to a function f. Using the fact that the values of $F(t,\tau(t)x)$ are closed to obtain $f(t)\in F(t,\tau(t)x), a.e.$, hence $f\in S_{F\ (\cdot,\tau(\cdot)x)}$. Then the Lebesgue dominated convergence theorem implies that, as $n\to\infty$, for each $t\in J$

$$y(t)=\lim_{n\to\infty} y_n(t)=k_1(t)\left[\Psi(0)\ -\ g(x)\ \right]+\int_0^t (t-s)^{q-1}K_2(t-s)f(s)ds.$$

So, $y\in N_{\Psi}(x)$.

Second step. N_{Ψ} is contraction, that is there exists $1>\rho>0$, such that

$$H\ (N_{\Psi}(x_1),N_{\Psi}(x_2))\le\rho\left\|x_1-x_2\right\|,\forall x_1,x_2\in\Theta.$$

Let $x_1,x_2\in\Theta$ and $y_1\in N_{\Psi}(x_1)$.Then there exists $f_1\in S_{F(\cdot,\tau(\cdot)x_1)}$ such that,

$$y_1(t)=\begin{cases}\Psi(t)-g(x_1),\ t\in[-r,0],\\ K_1(t)(\Psi(0-g(x_1))+\int_0^t(t-s)^{q-1}K_2(t-s)f_1(s)ds,\ t\in J.\end{cases}$$

Condition $[\mathrm{HF}_4]$ tells us that

$$H(F(t,\tau(t)x_1),F(t,(\tau(t)x_2))\le\omega(t)\left\|\tau(t)x_1-\tau(t)x_2)\right\|_{C_0},a.e.t\in J.$$

Hence there is $u_t\in F(t,(\tau(t)x_2)$ such that

$$||f_1(t)-u_t||\ \le\omega(t)\left\|\tau(t)x_1-\tau(t)x_2\right\|_{C_0},t\in J.$$

Then consider the multifunction $U:J\to 2^E$ defined by

$$U(t)=\{\rho\in E:||f_1(t)-\rho||\ \le\omega(t)\left\|\tau(t)x_1-\tau(t)x_2\right\|_{C_0}\},t\in J.$$

Since $f_1,\omega,\tau(.)x_1,\tau(.)x_2$ are measurable, Theorem III.41 [12] tells us that the closed set $U(t)$ is measurable. Finally the set $\Lambda(t)=U(t)\cap F(t,\tau(t)x_2)$ is nonempty since it contains u_t. Therefore the multivalued map $t\to\Lambda(t)$ is measurable with nonempty closed values (see[12]). Hence,there exists a function f_2, which is measurable selection for Λ. Thus $f_2(t)\in F(t,\tau(t)x_2)$ and

$$||f_1(t)-f_2(t)||\ \le\omega(t)\left\|\tau(t)x_1-\tau(t)x_2\right\|_{C_0},t\in J. \tag{20}$$

Let us define

$$y_2(t)=\begin{cases}\Psi(t)-g(x_2),\ t\in[-r,0],\\ K_1(t)(\Psi(0)-g(x_2))+\int_0^t(t-s)^{q-1}K_2(t-s)f_2(s)ds,\ t\in J.\end{cases}$$

For $t\in[-r,0]$, by$\left[\mathrm{H}_g^*\right]$

$$||y_1(t)-y_2(t)||\ \le\parallel g(x_1)-g(x_2)\parallel\le\varsigma\parallel x_1-x_2\parallel_{\Theta} \tag{21}$$

For $t \in J$, we have from (20)

$$
\begin{aligned}
&\| y_1(t) - y_2(t) \| \\
&\quad \leq M \| g(x_1) - g(x_2) \| + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f_1(s) - f_2(s) \| \, ds \\
&\quad \leq M \, \varsigma \| x_1 - x_2 \|_\Theta + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \omega(s) \| \tau(s)x_1 - \tau(s)x_2 \|_{C_0} \, ds \\
&\quad \leq M \, \varsigma \| x_1 - x_2 \|_\Theta + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \omega(s) \| x_1 - x_2 \|_\Theta \, ds \\
&\quad \leq \| x_1 - x_2 \|_\Theta \, M(\varsigma + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \omega(s) ds \\
&\quad \leq \| x_1 - x_2 \|_\Theta \;\; M \, (\varsigma + \frac{\| \omega \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)}}{\Gamma(q)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}).
\end{aligned} \tag{22}
$$

By analogous relation, obtained by interchanging the roles of $y_1$and y_2, it follows from (21) and (22) that

$$
H(N_\Psi(x_1), N_\Psi(x_2)) \leq M \, (\varsigma + \frac{\| \omega \|_{L^{\frac{1}{\sigma}}(J,\mathbb{R}_+)}}{\Gamma(q)} \frac{b^{\,q-\sigma}}{\eta^{1-\sigma}}) \, \|x_1 - x_2\| \, .
$$

This proves that N_Ψ is contraction, and thus, by Theorem 2.17, N_Ψ has a fixed point which is a mild solution of problem (P_Ψ). □

Corollary 3.4. *In addition of assumptions of Theorem 3.3 if the values of F are nonempty and compact, then* $\check{S}_{[-r,b]}(\Psi)$ *is nonempty and compact.*

Proof. From Theorem 3.3 we conclude that the set of solutions for (P_Ψ) is nonempty. By arguing as in Theorem 3.3 (i), one can show that the set of fixed points for N_Ψ is bounded. Since the contraction multifunctions with compact values in normed vector space is condensing, hence by Theorem 2.16 we conclude that the set $\check{S}_{[-r,b]}(\Psi)$ is compact in Θ. □

Remark 3.5. Theorem 3.3 still holds for infinite dimensional separable Banach spaces if we suppose that the values of F are in $P_{ck}(E)$. Indeed, as in the proof of Theorem 3.3, one can show that N_Ψ is contraction with nonempty values. Moreover, by following the same arguments used in Step 1 (ii) in the proof of Theorem 3.1, we can show that the values of N_Ψ are closed. Thus, by Theorem 2.17, N_Ψ has a fixed point which is a mild solution of problem (P_Ψ).

Conclusion

In this paper, existence results of fractional-order semilinear functional differential inclusions with nonlocal conditions have been obtained in convex as well as nonconvex case. Conditions are strictly weaker than some of the existing ones. In addition, the topological structure of the solution set is studied.

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