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MILNOR'S $\bar{\mu}$-INVARIANTS AND MASSEY PRODUCTS

BY

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ABSTRACT. The main result of this paper gives an interpretation of Milnor's $\bar{\mu}$-invariants of a link in terms of Massey products in the complement of the link. The approach presented here can be used to give topological proofs of results about the $\bar{\mu}$-invariants obtained by Milnor using different methods.

The main result of this paper, Theorem 3, gives an interpretation of Milnor's $\bar{\mu}$-invariants of a link in terms of Massey products in the complement of the link. This settles the question raised by Stallings **[21]** of how Milnor's invariants are related to Massey products. The $\bar{\mu}$-invariants are isotopy invariants of a link defined in terms of certain presentations of the quotient of the fundamental group of the complement of a link by lower central series subgroups. The existence of a relationship between Massey products and Milnor's invariants is suggested by the result **[21]** that homological invariants of H_1 and H_2, such as Massey products of elements in H^1, are invariants of quotients of lower central series subgroups. Specific results relating the lower central series to products and coboundaries of one-dimensional cochains can be found in **[3]**, **[4]**, and **[6]**. Massey products are used to define linking invariants in **[13]** and **[19]**.

It was conjectured by Stallings **[22]** that Milnor's invariants can be described in terms of the spectral sequence of the fundamental group of the complement of a link. Since Massey products determine differentials in the cohomology spectral sequence of a group **[8]**, the main result of this paper implies that Milnor invariants determine some of the differentials in the spectral sequence of the fundamental group of the complement of a link (see **[9]**).

The $\bar{\mu}$-invariants of a link in the 3-sphere are defined by Milnor **[18]** as follows. Denote by F_1 the fundamental group of the complement of the link. For $q > 1$, set $F_{q+1} = [F_q, F_1]$. F_q is the qth lower central series subgroup of F_1. An ith parallel in F_1/F_q can be represented by a word w_i in the meridians $\alpha_1, \ldots, \alpha_n$ (one meridian for each component of the link). The group F_1/F_q then has the presentation $(\alpha_1, \ldots, \alpha_n: [\alpha_i, w_i] = 1, A_q = 1)$ where A_q is the qth lower central series subgroup of the free group generated by the α. Denote by $\mu(l_1, \ldots, l_p)$ the coefficient of $K_{l_1}, \ldots, K_{l_{p-1}}$ in the Magnus expansion of the word w_{l_p}.

For $p < q$, the residue class, $\bar{\mu}$, of $\mu(l_1, \ldots, l_p)$ modulo an integer, $\Delta(l_1, \ldots, l_p)$, determined by the μ, is an isotopy invariant of the link. If the indices $l_1, \ldots, l_p$ are all distinct, then the corresponding $\bar{\mu}$ is a homotopy invariant.

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Canonical elements u_i and $\gamma_{i,j}$ in the cohomology groups of the complement of a link are defined by taking the u_i's to be the Alexander duals to the components of the link and the $\gamma_{i,j}$'s to be Lefschetz duals to paths from one component of the link to another. The collection, u_i, forms a basis for $H^1(S^3 - L : R)$ and the $\gamma_{i,j}$'s generate $H^2(S^3 - L : R)$ subject to the relations $\gamma_{i,i} = 0$ and $\gamma_{i,j} + \gamma_{j,k} = \gamma_{i,k}$.

Each sequence $(l_1, \ldots, l_p)$ determines a subset $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ of $H^2(S^3 - L : R)$ called the generalized Massey product. The main result is that with R equal to the integers modulo $\Delta(l_1, \ldots, l_p)$ the product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ contains the single element $(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}$. The elements in a defining system for $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ are required to be cochains in the complement of certain sublinks. Examples 3 and 4 of §2 show that this restriction is generally necessary for Theorem 3 to be true.

§1 contains definitions and a precise statement of the main result. The indeterminacy of generalized Massey products is compared with the indeterminacy of ordinary Massey products in the complement of a link, and there is a discussion of how some of the results about $\bar{\mu}$-invariants can be reproved using Theorem 3 and the methods of **[13]** and **[19]**. Examples are given in §2.

The proofs are contained in §3. There are two key steps in the proof of the main result. First a theorem of Milnor's is used to construct a 2-dimensional CW complex whose Massey products determine the Massey products in the complement of the link. Massey products in the 2-dimensional CW complex are then evaluated using Theorem 2 which gives a formula for Massey products in a 2-dimensional CW complex in terms of the coefficients of the Magnus expansion of words corresponding to the attaching maps of the 2-cells. Theorem 2 is closely related to Proposition 4.1 of **[6]**. The proof of Theorem 2 is based on a geometric interpretation of cup products and coboundaries of cochains motivated by R. M. Goresky's geometric description of the algebraic topology of stratified objects, **[7]**. See also **[15]**, **[16]**, **[23]**, **[24]** and **[25]**. I am indebted to W. S. Massey for suggesting that Goresky's viewpoint be applied to the problem of calculating Massey products, and for several very helpful conversations. The referee's comments have resulted in a much improved exposition.

1. Definitions, statement of main result. Denote by $C(N)$ the space consisting of N disjoint oriented circles. An N-link in the three-sphere S^3 is an embedding L: $C(N) \to S^3$. Two links L and L' are called isotopic if there is a continuous 1-parameter family of links h_t with $h_0 = L$ and $h_1 = L'$. L and L' are called homotopic if there is a continuous 1-parameter family of maps h_t: $C(N) \to S^3$ such that for each t disjoint circles in $C(N)$ have disjoint images in S^3, $h_0 = L$ and $h_1 = L'$.

Given a link L in S^3 denote by F_1 the fundamental group of the complement of the link. Subgroups F_q of F_1 are defined by setting $F_{q+1} = [F_q, F_1]$ for $q > 1$ where $[F_q, F_1]$ denotes the subgroup of F_1 generated by elements of the form $aba^{-1}b^{-1}$ with $a \in F_q$ and $b \in F_1$. F_q is the qth lower central series subgroup of F_1. Meridians and parallels to a link are elements in F_1/F_q defined in **[18]** as follows. Choose $M_1^0, \ldots, M_N^0$ pairwise disjoint connected neighborhoods of the components $L_1, \ldots, L_N$ of the link. For each $i = 1, 2, \ldots, N$ choose a sequence $M_i^0 \supseteq$

$M_i^1 \supseteq \cdots \supseteq M_i^q$ of connected open neighborhoods of L_i such that M_i^j can be deformed into L_i within M_i^{j-1} for each $j = 1, 2, \ldots, q$. (That is, there is a homotopy r_t: $M_i^j \to M_i^{j-1}$ such that r_0 is the inclusion map and $r_1(M_i^j) \subseteq L_i$.) Such a sequence can be constructed inductively since L_i and M_i^{j-1} are both absolute neighborhood retracts. Choose the base point x_0 to be a point in $S^3 - (\bigcup_{i=1}^N M_i^0)$. For each i choose a path $p_i(t)$ $(0 \leq t \leq 1)$ from x_0 to L_i. An ith meridian α_i of L with respect to the path p_i is defined as follows: first traverse the path p_i to a point in $M_i^q - L_i$, then traverse a closed loop in $M_i^q - L_i$ which has linking number $+1$ with L_i and is homotopic to a constant in M_i^q, and finally return to x_0 along p_i. This procedure defines a unique element α_i of F_1/F_q.

An ith parallel β_i of L with respect to the path p_i is an element of F_1/F_q obtained by traversing p_i from x_0 to a point in $M_i^q - L_i$, then traversing a closed loop in $M_i^q - L_i$ which is homotopic to L_i within M_i^q and has linking number 0 with L_i, and finally returning to x_0 along p_i. This procedure defines a unique element β_i of F_1/F_q. If p_i is replaced by some other path then the pair (α_i, β_i) is replaced by some conjugate pair $(\lambda\alpha_i\lambda^{-1}, \lambda\beta_i\lambda^{-1})$.

The following result (Theorem 4 of **[18]**) is used to define the $\bar{\mu}$-invariants of a link L, and will be used in the proof of Theorem 3 to construct a 2-dimensional CW complex whose Massey products determine the Massey products in $S^3 - L$.

THEOREM 1 (MILNOR). *If L is an N-link in Euclidean space, then the group F_1/F_q has the presentation*

$$\{\alpha_1, \ldots, \alpha_N / [\alpha_i, w_i] = 1, i = 1, 2, \ldots, N; A_q = 1\}$$

where the α_i are meridians, and the w_i are certain words in $\alpha_1, \ldots, \alpha_N$ which represent parallels, and where A_q is the qth lower central series subgroup of the free group generated by the α.

The Magnus expansion of the word w_i is obtained by substituting

$$\alpha_i = 1 + K_i, \qquad \alpha_i^{-1} = 1 - K_i + K_i^2 - K_i^3 + \ldots$$

in w_i and multiplying out to get a formal power series in the noncommuting indeterminates K_i, $i = 1, \ldots, N$. Given a sequence $(l_1, \ldots, l_p)$ of integers with $1 \leq l_j \leq N$ and $p < q$ set $\mu(l_1, \ldots, l_p)$ equal to the coefficient of $K_{l_1}, \ldots, K_{l_{p-1}}$ in the Magnus expansion of w_{l_p}, and set $\Delta(l_1, \ldots, l_p)$ equal to the greatest common divisor of the numbers $\mu(j_1, \ldots, j_s)$ where $(j_1, \ldots, j_s)$, $s \geq 2$, ranges over all cyclic permutations of proper subsequences of $(l_1, \ldots, l_p)$. The Milnor invariant $\bar{\mu}(l_1, \ldots, l_p)$ of the link L is the residue class of $\mu(l_1, \ldots, l_p)$ modulo $\Delta(l_1, \ldots, l_p)$. In **[18]** it is shown that $\bar{\mu}(l_1, \ldots, l_p)$ is an isotopy invariant of L and a homotopy invariant of L if the l_i's are distinct. In addition, L is homotopic to the trivial link if and only if $\bar{\mu}(l_1, \ldots, l_p)$ is zero for all sequences with distinct l_i's **[17]**.

Massey products of elements in H^1 are defined, **[10]**, as follows. Let $\{X_i\}_{i=1}^p$ be a collection of subspaces of a space X. Given elements u_i in $H^1(X_i : R)$ for $i = 1, \ldots, p$; a defining system for the Massey product $\langle u_1, \ldots, u_p \rangle$ in the system $\{X_i\}_{i=1}^p$ with coefficients in the commutative ring with unit, R, is a collection of cochains, $m_{i,j}$; $1 \leq i \leq j \leq p$, $(i, j) \neq (1, p)$ satisfying:

1. $m_{i,j} \in C^1(X_i \cap X_{i+1} \cap \cdots \cap X_j : R)$.

2. $m_{i,i}$ is a cocycle representative of u_i, for $i = 1, 2, \ldots, p$.

3. $\delta(m_{i,j}) = \sum_{k=i}^{j-1} m_{i,k}m_{k+1,j}$ for $i < j$ where by abuse of notation $m_{i,k}m_{k+1,j}$ denotes the cup product in $C^*(X_i \cap \cdots \cap X_j : R)$ of the restrictions of $m_{i,k}$ and $m_{k+1,j}$ to $X_i \cap \cdots \cap X_j$.

$C^*(Y : R)$ denotes the singular cochains of Y with coefficients R. It follows that $\sum_{k=1}^{p-1} m_{1,k}m_{k+1,p}$ is a cocycle in $C^2(X_1 \cap \cdots \cap X_p : R)$. $\langle u_1, \ldots, u_p \rangle$ is defined if there is a defining system for it, in which case $\langle u_1, \ldots, u_p \rangle$ is the subset of $H^2(X_1 \cap \cdots \cap X_p : R)$ consisting of all elements representable by cocycles of the form $\sum_{k=1}^{p-1} m_{1,k}m_{k+1,p}$ with $\{m_{i,j}\}$ a defining system for $\langle u_1, \ldots, u_p \rangle$. Massey products in a system, $\{X_i\}_{i=1}^p$ are a special case of the products considered in **[14]**.

Given an N-link, L, in S^3 set u_i equal to the element in $H^1(S^3 - L_i)$ which corresponds by Alexander duality to the generator of $H_1(L_i)$ determined by the orientation of L_i. For i and j in $\{1, 2, \ldots, N\}$ set $\gamma_{i,j}$ equal to the element in $H^2(S^3 - (L_i \cup L_j))$ which corresponds by Lefschetz duality to the element in $H_1(S^3, L_i \cup L_j)$ determined by a path from L_i to L_j. The relationship between the $\bar{\mu}$-invariants of a link and Massey products in the complement of the link is given by the following result where $\mathbf{Z}_0$ denotes the ring of integers and $\mathbf{Z}_n$ the ring of integers modulo the positive integer n.

THEOREM 3. *Let L be an N-link in S^3. For any sequence, $(l_1, \ldots, l_p)$, of integers with $1 \leq l_j \leq N$, the Massey product $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ with coefficients $\mathbf{Z}_{\Delta(l_1, \ldots, l_p)}$ is defined and contains the single element $(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}$.*

Theorem 3, along with the techniques of **[13]**, **[19]**, and **[21]**, can be used to recover some of the properties of the $\bar{\mu}$-invariants. For example, the naturality of Massey products together with Alexander duality implies that Massey products in the complement of a link are isotopy invariants (see **[21]**). Hence the $\bar{\mu}$'s are isotopy invariants of a link. If the number l_1 occurs only once in the sequence $(l_1, \ldots, l_p)$, then the Massey product $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ with $R = \mathbf{Z}_{\Delta(l_1, \ldots, l_p)}$ can be identified with a functional Massey product (see **[13]** and **[19]**). It then follows that $\bar{\mu}(l_1, \ldots, l_p)$ is an invariant of the homotopy class of the inclusion of the l_1th component into $S^3 - (\bigcup_{i=2}^p L_{l_i})$. This together with the relation $\bar{\mu}(l_1, \ldots, l_p) = \bar{\mu}(l_2, \ldots, l_p, l_1)$ (obtained as part of the proof of Theorem 3), then implies that $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ and hence $\bar{\mu}(l_1, \ldots, l_p)$ are homotopy invariants of a link if the indices $(l_1, \ldots, l_p)$ are all distinct.

Massey products in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ are related to Massey products in $S^3 - L$ (the elements in a defining system for a product in $S^3 - L$ are only required to be cochains in $S^3 - L$) as follows. From the definition of Massey product it follows that $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ in $S^3 - (\bigcup_{i=1}^p L_{l_i})$. Hence Theorem 3 implies that the Massey product $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ is always a subset of the product $\langle u_{l_1}, \ldots, u_{l_p} \rangle$ in $S^3 - (\bigcup_{i=1}^p L_{l_i})$ with $R = \mathbf{Z}_{\Delta(l_1, \ldots, l_p)}$ is defined and contains the element $(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}$. For $p = 2, 3$ this is the only element in the product. Examples 4 and 5 of the next section indicate that for

$p > 4$, the product generally contains more than the one element $(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}$. In particular, Massey products in $S^3 - L$ do not, in general, determine the $\bar{\mu}$-invariants of a link.

If products in $\{S^3 - L_{l_i}\}_{i=1}^p$ are replaced by products in $S^3 - L$ in the proof of Theorem 3, then the following result is obtained: The Massey product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in $S^3 - L$ with coefficient ring $\mathbf{Z}_{D(l_1,\ldots,l_p)}$ is defined and contains the single element $(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}$ where $D(l_1, \ldots, l_p)$ is defined by $D(l_1, l_2) = 0$. For $p > 2$, $D(l_1, \ldots, l_p)$ is the greatest common divisor of the following numbers:

(i) $D(l_i, \ldots, l_j)$, $1 \leq j - 1 \leq p - 2$;
(ii) $\mu(l_i, \ldots, l_j)$, $1 \leq j - 1 \leq p - 2$;
(iii) $D(l_1, \ldots, l_{k-1}, *, l_{k+2}, \ldots, l_p)$ $* \in \{1, 2, 3, \ldots, N\}$;
(iv) $\mu(l_1, \ldots, l_{k-1}, *, l_{k+2}, \ldots, l_p)$ $* \in \{1, 2, 3, \ldots, N\}$.

Conditions (i) through (iv) can be explained as follows. (i) guarantees that each of the products $\langle u_{l_i}, \ldots, u_{l_j}\rangle$ in $S^3 - L$ is defined and has zero indeterminacy (that is the product contains only one element). (ii) then implies that each of the products $\langle u_{l_i}, \ldots, u_{l_j}\rangle$ contains only zero. (In the terminology of **[14]**, (i) and (ii) imply that $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ is strictly defined.) Conditions (iii) and (iv) now imply that $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ has zero indeterminacy. (See Propositions 2.4 and 2.7 of **[14]**.) To see that $D(l_1, \ldots, l_p)$ divides $\Delta(l_1, \ldots, l_p)$ note that, using the identity $\bar{\mu}(l_1, \ldots, l_p) = \bar{\mu}(l_2, \ldots, l_p, l_1)$, a definition of $\Delta(l_1, \ldots, l_p)$ is obtained by replacing the condition $* \in \{1, 2, \ldots, N\}$ by $* = l_k$ or l_{k+1} in the definition of D. The identity $\Delta(l_1, \ldots, l_p) = D(l_1, \ldots, l_p)$ for $p \leq 3$ and $L = \bigcup_{i=1}^p L_{l_i}$ follows from the relation $\bar{\mu}(l_1, l_1) = 0$.

2. Examples. There are a number of methods for calculating Massey products in the complement of a link. If the link is smooth, then Massey products with coefficients equal to the real numbers can be calculated using differential forms in the complement of a tubular neighborhood of the link. For a polygonal link, Massey products with rational coefficients can be calculated using the algebra of Q-polynomial forms on a simplicial subdivision of the complement of a neighborhood of the link **[4]**. For a polygonal link and arbitrary coefficient ring, Massey products can be calculated by generalizing Rules I and II of §3 to 3-manifolds **[7]** and drawing pictures of defining systems for Massey products. This is essentially the same as using duality theorems to translate the cup product on $C^*(S^3 - L: R)$ into an intersection theory on $C_*(S^3, L)$, **[5]**, **[13]**, and **[19]**. Another approach is to construct a 2-dimensional CW complex, Y, and a map f: $Y \to S^3 - L$ so that Massey products in $S^3 - L$ can be calculated by evaluating the corresponding product in Y (Lemma 3). This last approach is used to prove Theorem 3.

The purpose of Examples 1, 2, and 3 is to illustrate Theorem 3. Example 4 is an example of a fourth order product in $S^3 - L$ that contains more elements than the corresponding product in the system $\{S^3 - L_{l_i}\}_{i=1}^p$. Example 5 shows that Massey products, $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in $S^3 - L$ with $(l_1, \ldots, l_p)$ all distinct, cannot, in general, be used to define homotopy invariants of a link. (Recall that if $(l_1, \ldots, l_p)$ are all distinct then the Massey product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ can be viewed as a homotopy invariant of the link.) Additional calculations of Massey products in the complement of a link are in [5], [13], and [19]. Calculations related to the $\bar{\mu}$-invariants are in [1], [17], and [18].

EXAMPLE 1. Let α_1, α_2 be meridians to the link in Figure 1a with respect to the paths p_1 and p_2. The word $w_2 = [\alpha_1^{-1}, \alpha_2]^N[\alpha_1, \alpha_2]^N$ represents an element in $\pi_1(S^3 - L, x_0)$ which commutes with α_2 and is a parallel to L_2. The Magnus expansion of w_2 is $1 + NK_1^2K_2 - 2NK_1K_2K_1 + NK_2K_1^2 +$ (terms of order ≥ 4).

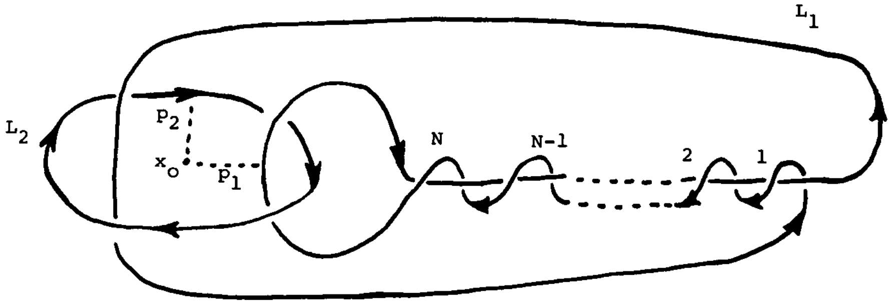


FIGURE 1a

Hence the nonzero Milnor invariants of order 4 are

$$\bar{\mu}(1, 1, 2, 2) = \bar{\mu}(1, 2, 2, 1) = \bar{\mu}(2, 2, 1, 1) = \bar{\mu}(2, 1, 1, 2) = N,$$
$$\bar{\mu}(1, 2, 1, 2) = \bar{\mu}(2, 1, 2, 1) = -2N.$$

The corresponding Massey products are

$$\langle u_1, u_1, u_2, u_2\rangle = N\gamma_{1,2}, \qquad \langle u_1, u_2, u_1, u_2\rangle = -2N\gamma_{1,2},$$
$$\langle u_2, u_2, u_1, u_1\rangle = N\gamma_{2,1}, \qquad \langle u_2, u_1, u_2, u_1\rangle = -2N\gamma_{2,1}.$$

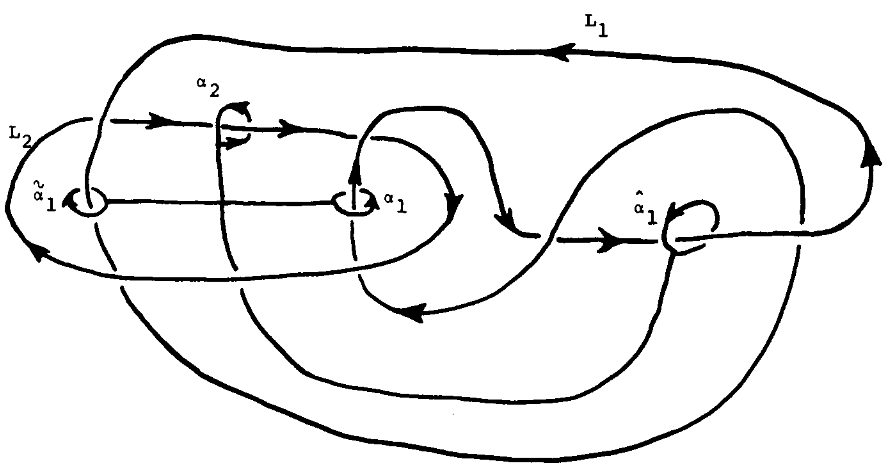


FIGURE 1b

Figure 1b can be used to show that w_2 is a parallel for $N = 1$ as follows. Clearly $\alpha_1^{-1}\tilde{\alpha}_1$ is a parallel to L_2 with respect to the path p_2. Using $\tilde{\alpha}_1 = \hat{\alpha}_1\alpha_1\hat{\alpha}_1^{-1}$, it follows that $\alpha_1^{-1}\hat{\alpha}_1\alpha_1\hat{\alpha}_1^{-1}$ is a parallel to L_2. But $\hat{\alpha}_1 = \alpha_2\alpha_1\alpha_2^{-1}$ so

$$\alpha_1^{-1}\alpha_2\alpha_1\alpha_2^{-1}\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1} = [\alpha_1^{-1}, \alpha_2][\alpha_1, \alpha_2] = w_2$$

is a parallel to L_2.

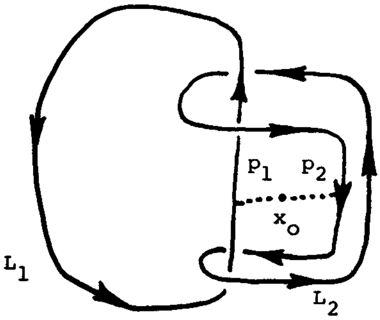


FIGURE 2

EXAMPLE 2. Let α_1, α_2 be meridians to the link in Figure 2 with respect to the paths p_1 and p_2. The word $w_2 = \alpha_1\alpha_2^{-1}\alpha_1\alpha_2$ represents an element in $\pi_1(S^3 - L, x_0)$ which commutes with α_2 and is a parallel to L_2. The coefficient of K_1 in the Magnus expansion of w_2 is 2 so $\bar{\mu}(1, 2) = \bar{\mu}(2, 1) = 2$ and each of the $\bar{\mu}$-invariants $\bar{\mu}(1, 1, 2)$, $\bar{\mu}(1, 2, 1)$, $\bar{\mu}(2, 1, 1)$, $\bar{\mu}(2, 2, 1)$, $\bar{\mu}(2, 1, 2)$, $\bar{\mu}(1, 2, 2)$ is an element of the integers mod 2. The coefficient of K_1^2 and the coefficient of K_1K_2 in the Magnus expansion of w_2 are both 1. Hence

$$\bar{\mu}(1, 1, 2) = \bar{\mu}(1, 2, 1) = \bar{\mu}(2, 1, 1) = 1 \quad \text{in } \mathbf{Z}_2$$

and

$$\bar{\mu}(1, 2, 2) = \bar{\mu}(2, 2, 1) = \bar{\mu}(2, 1, 2) = 1 \quad \text{in } \mathbf{Z}_2.$$

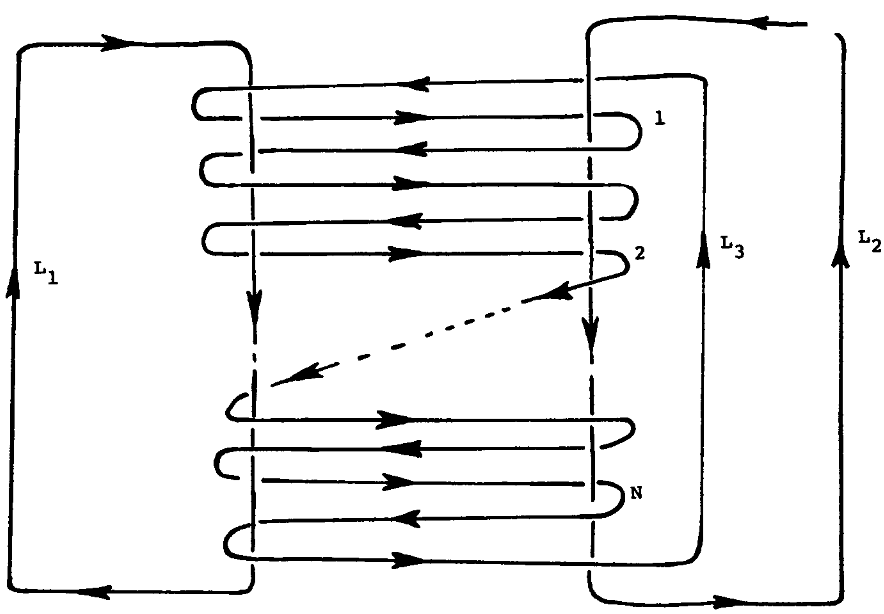


FIGURE 3

The corresponding products are

$$u_1u_2 = 2\gamma_{1,2},$$
$$\langle u_1, u_1, u_2\rangle = \langle u_2, u_1, u_1\rangle = \langle u_1, u_2, u_2\rangle = \langle u_2, u_2, u_1\rangle = \gamma_{1,2},$$
$$(\mathbf{Z}_2 \text{ coefficients}).$$

EXAMPLE 3 (SEE FIGURE 3). There is a word, w_3, in the α_i's representing an element in $\pi_1(S^3 - L, x_0)$ which commutes with α_3 and is a parallel to L_3. If α_3 is set equal to 1 in w_3, the resulting word is $[\alpha_1, \alpha_2]^N$. The Magnus expansion of $[\alpha_1, \alpha_2]^N$ is $1 + NK_1K_2 - NK_2K_1 +$ (terms of order ≥ 3). Hence

$$\bar{\mu}(1, 2, 3) = \bar{\mu}(2, 3, 1) = \bar{\mu}(3, 1, 2) = N,$$
$$\bar{\mu}(2, 1, 3) = \bar{\mu}(1, 3, 2) = \bar{\mu}(3, 2, 1) = -N.$$

All other Milnor invariants of length ≤ 3 are zero.

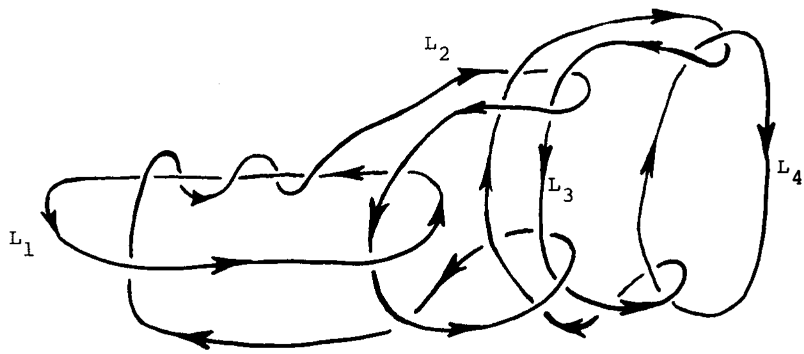


FIGURE 4

EXAMPLE 4. If either of the components L_1 or L_2 is removed from the link in Figure 4, then the resulting link is trivial. If L_3 and L_4 are removed, the remaining link is isotopic to the link in Example 2. Hence the only nonzero $\bar{\mu}$-invariants of length ≤ 3 are those in Example 2. By drawing a picture of a defining system for $\langle u_1, u_2, u_3, u_4\rangle$ in $\{S^3 - L_i\}_{i=1}^4$ with $\mathbf{Z}_2$ coefficients (see **[5]**, **[13]** or **[19]**), it follows that $\langle u_1, u_2, u_3, u_4\rangle$ contains the element $\gamma_{1,4}$. From Theorem 3, it follows that this is the only element in the product. The product $\langle u_1, u_2, u_3, u_4\rangle$ in $S^3 - L$ with $\mathbf{Z}_2$ coefficients contains $\langle u_1, u_2, u_2\rangle$ in its indeterminacy. From Example 2, $\langle u_1, u_2, u_2\rangle = \gamma_{1,2}$ ($\mathbf{Z}_2$ coefficients). Hence the product $\langle u_1, u_2, u_3, u_4\rangle$ in $S^3 - L$ with $\mathbf{Z}_2$ coefficients contains both $\gamma_{1,4}$ and $\gamma_{1,4} + \gamma_{1,2}$. This gives an example of a fourth order product in $S^3 - L$ which contains more elements than the corresponding product in the system $\{S^3 - L_i\}_{i=1}^4$.

EXAMPLE 5. For the link in Figure 5a, the Massey product $\langle u_1, u_2, u_3, u_4, u_5, u_6\rangle$ in $S^3 - L$ is defined and consists of all integer multiples of $\gamma_{1,6}$. For the link in Figure 5b, the Massey product $\langle u_1, u_2, u_3, u_4, u_5, u_6\rangle$ in $S^3 - L$ contains the single element $\gamma_{1,6}$. Since the links in Figures 5a and 5b are homotopic, the example indicates that Massey products in $S^3 - L$ with distinct u_j's do not, in general, determine homotopy invariants of the link. For the link in Figure 5a and the link in

Figure 5b, the Massey product $\langle u_1, u_2, \ldots, u_6 \rangle$ in $\{S^3 - L_i\}_{i=1}^6$ contains the single element $\gamma_{1,6}$. The link in Figure 5b is one of the examples considered in **[17]**.

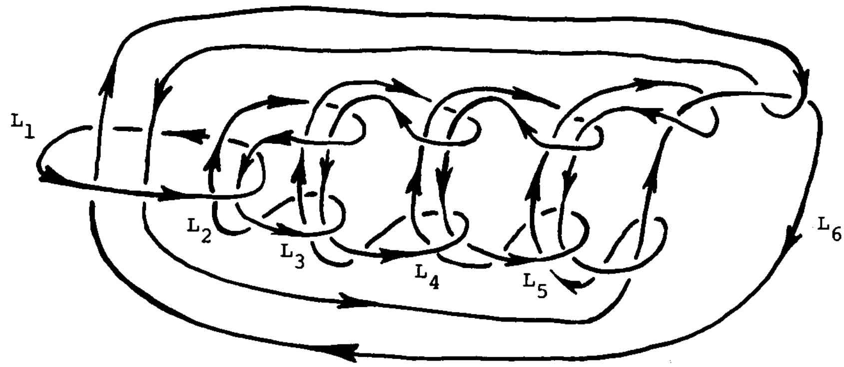


FIGURE 5a

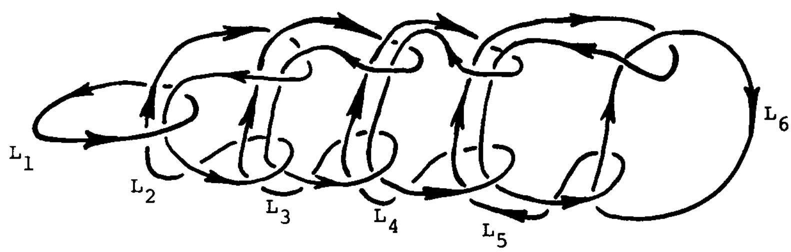


FIGURE 5b

3. Proofs. This section is organized as follows. First the notion of a special 2-dimensional cell structure is defined. A special cell structure is a regular 2-dimensional CW complex each of whose 2-cells is either a simplex or a cube, together with an ordering of the vertices. The ordering of the vertices is used to give the cellular cochains the structure of an associative differential graded algebra whose Massey products can be identified with those given by the algebra of singular cochains (Lemma 1). The cellular cochains of a CW complex do not, in general, admit such a product, **[24]**. The next step is to give a geometric description of the coboundary of one-dimensional cochains and the cup product of certain pairs of one-dimensional cochains (Rules I and II). The rule for cup products, Rule II, depends on the existence of a suitable ordering of the vertices but is independent of the ordering chosen. Using Rules I and II it is possible to draw pictures of defining systems for Massey products and calculate the corresponding element of the product (Lemma 2, Theorem 2). Theorem 2 gives a formula for Massey products in a 2-dimensional CW complex in terms of coefficients in the Magnus expansion of words corresponding to the attaching maps of the 2-cells.

The main result, Theorem 3, is proved as follows. The first step is to construct a 2-dimensional CW complex, X, a map $f\colon X \to S^3$, and a collection, $\{X_i\}_{i=1}^N$, of subcomplexes of X, one for each component of a link L, with $f(X_i) \subseteq S^3 - L_i$ for

$i = 1, 2, \ldots, N$. (The subcomplex $X_1 \cap \cdots \cap X_N$ is the complex Y of Lemma 3.) The naturality of Massey products implies that information about the products in a system $\{S^3 - L_{l_i}\}_{i=1}^p$ can be obtained by calculating the corresponding product in the system $\{X_{l_i}\}_{i=1}^p$. Massey products in $\{X_{l_i}\}_{i=1}^p$ are calculated using Theorem 2. A technical result of May, Lemma 4, is then used to show that the Massey product $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$ in $\{X_{l_i}\}_{i=1}^p$ completely determines the product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in $\{S^3 - L_{l_i}\}_{i=1}^p$.

Definition. A special cell structure is a regular 2-dimensional cell complex together with a partial ordering of the vertices such that:

(i) Each 2-cell is either a simplex or a cube;

(ii) The vertices of any cell are totally ordered and for each 2-cube the smallest and largest vertices are the endpoints of a diagonal of the cube.

Condition (ii) is used to define a boundary operator and diagonal approximation on the cellular chain complex. Note: There are regular 2-dimensional cell complexes (on the real projective plane for example) satisfying (i) for which there is no ordering of the vertices satisfying (ii).

The cells of a special cell structure will be indicated by listing the vertices of the cell in increasing order. A triple of vertices thus denotes a 2-simplex, a four-tuple indicates a 2-cube. If a space, X, has been given a special cell structure, set $C_*(X)$ equal to the cellular chains of X and $C^i(X) = \mathrm{Hom}(C_i(X); \mathbf{Z})$.

A boundary operator on $C_*(X)$ is defined by

$$\begin{aligned} \partial(a, b) &= (b) - (a), \\ \partial(a, b, c) &= (a, b) + (b, c) - (a, c), \\ \partial(a, b, c, d) &= (a, b) + (b, d) - (a, c) - (c, d). \end{aligned}$$

∂ is the usual boundary operator for singular theory combined with the boundary operator for cubical singular theory where the vertices of the 2-cube, (a, b, c, d), correspond to the vertices of the standard 2-cube by: $a \to (0, 0)$; $b \to (1, 0)$; $c \to (0, 1)$; $d \to (1, 1)$.

A diagonal approximation ϕ: $C_*(X) \to C_*(X) \otimes C_*(X)$ is defined by

$$\begin{aligned} \phi(a) &= (a) \otimes (a), \\ \phi(a, b) &= (a) \otimes (a, b) + (a, b) \otimes (b), \\ \phi(a, b, c) &= (a) \otimes (a, b, c) + (a, b) \otimes (b, c) + (a, b, c) \otimes (c), \\ \phi(a, b, c, d) &= (a) \otimes (a, b, c, d) + (a, b) \otimes (b, d) \\ &\quad - (a, c) \otimes (c, d) + (a, b, c, d) \otimes (d). \end{aligned}$$

ϕ is the usual Whitney diagonal approximation combined with the diagonal approximation for cubical theory given in [20].

Lemma 1. *Suppose the space X has a special cell structure with cellular cochains $C^*(X)$. Then the map ϕ^*: $C^*(X) \otimes C^*(X) \to C^*(X)$ induced by the map ϕ defined above gives $C^*(X)$ the structure of an associative differential graded algebra. Massey products calculated in $C^*(X)$ can be identified with those given by the algebra of singular cochains on X.*

PROOF. A simplicial subdivision of the special cell structure is obtained by subdividing each cube (a, b, c, d) into the two 2-simplices (a, b, d), (a, c, d). Denote by $C_*(\Delta)$, $(C^*(\Delta))$, the corresponding cellular chains (cochains). The ordering of the vertices in the special cell structure is an ordering of the vertices in the simplicial subdivision. Use the ordering to define ∂ and ϕ on $C_*(\Delta)$. An inclusion $C_*(X) \to C_*(\Delta)$ is defined by

$$\begin{aligned} i(a) &= (a), \\ i(a, b) &= (a, b), \\ i(a, b, c) &= (a, b, c), \\ i(a, b, c, d) &= (a, b, c) - (a, c, d). \end{aligned}$$

i is a chain map and the map ϕ on $C_*(X)$ is the Whitney diagonal approximation on $C_*(\Delta)$ restricted to $C_*(X)$. Since the Whitney diagonal approximation on $C_*(\Delta)$ gives $C^*(\Delta)$, the structure of an associative differential graded algebra and i^*: $C^*(\Delta) \to C^*(X)$ is an epimorphism, it follows that ϕ^* gives $C^*(X)$ the structure of an associative differential graded algebra. i^*: $C^*(\Delta) \to C^*(X)$ is a map of algebras inducing an isomorphism of cohomology groups so Massey products in $C^*(\Delta)$ can be identified with those in $C^*(X)$. Similarly, Massey products in $C^*(\Delta)$ can be identified with Massey products given by the algebra of singular cochains on X.

The following geometric interpretation of one- and two-dimensional cochains together with Rules I and II below make it possible to draw pictures of defining systems for Massey products and calculate the corresponding element of the Massey product.

FIGURE 6a

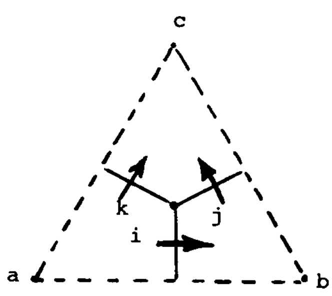


FIGURE 6b

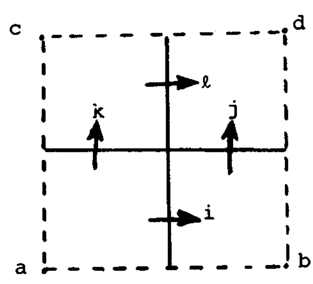


FIGURE 6c

Suppose a space, X, has been given a special cell structure. Each 1-cochain and each 2-cochain determine a picture in X. For $h \in C^1(X)$, the picture of h intersected with a 1-cell, (a, b), a 2-simplex, (a, b, c), and a 2-cube, (a, b, c, d), are given in Figures 6a, 6b, and 6c, where the cells of X are indicated by dotted lines and the picture of a cochain is drawn with solid lines. The integers i, j, k, l are defined by $i = h$ evaluated on the 1-cell $(a, b) = h(a, b)$, $k = h(a, c)$ and $l = h(c, d)$. In Figure 6b, $j = h(b, c)$. In Figure 6c, $j = h(b, d)$. For $h \in C^2(X)$, the picture of h intersected with a 2-simplex, (a, b, c), and a 2-cube, (a, b, c, d), are given in Figures 7a and 7b, where $i = h(a, b, c)$ and $j = h(a, b, c, d)$. The picture of an i-cochain intersected with any cell of dimension $< i$ is empty. The pictures of cochains are motivated by the proof of Poincaré duality which involves associating to each cell in a manifold a dual cell of complementary dimension. Recall from Chapter III of **[11]** that if X is a regular cell complex, then there is a simplicial subdivision of X whose vertices are in a 1-1 correspondence with the cells of X and whose simplices are denoted by sequences of cells of X, $(\sigma_0, \sigma_1, \ldots, \sigma_p)$, with σ_{i-1} contained in the boundary of σ_i for $i = 1, 2, \ldots, p$. For each cell σ of X, denote by D_σ the subcomplex of the simplicial subdivision consisting of all simplices $(\sigma_0, \ldots, \sigma_p)$ with σ contained in the closure of σ_0. In the terminology of **[11]**, D_σ is the closure of the transverse complex of σ. Note that if X is an N-manifold and σ is an i-cell of X, then D_σ is a homological $(N - i)$ disc. The D_σ's play an important role in the proof of Poincaré duality and in the intersection theory of **[11]**.

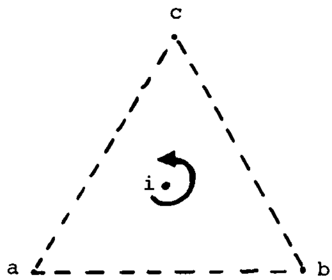


FIGURE 7a

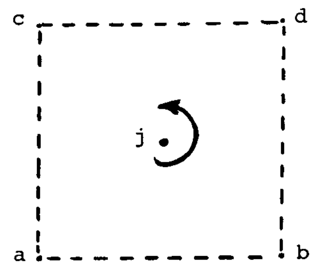
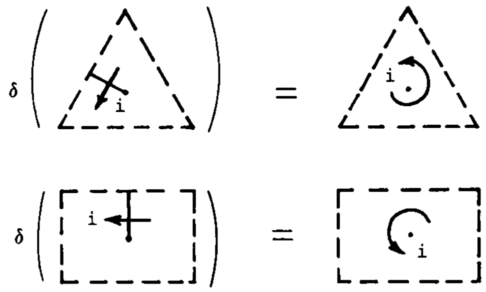


FIGURE 7b

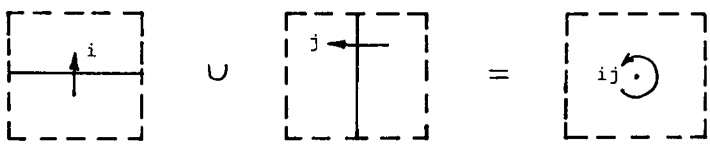
Assume now that X has been given a special cell structure. For a cell $(\sigma_0, \ldots, \sigma_p)$ in the simplicial subdivision of X, define the codimension of $(\sigma_0, \ldots, \sigma_p)$ to be $\dim(\sigma_p) - p$. For cells in X of positive dimension, i, a co-orientation of D_σ is by definition an orientation of the normal bundle to the codimension i cells in D_σ. Note that co-orientations of D_σ are in a 1-1 correspondence with orientations of σ. Suppose h is a positive dimensional cochain which is nonzero on only one cell, σ. Then the picture of h is the triple (the complex D_σ, a co-orientation, θ, of D_σ; an integer k) where the cochain h evaluated on the cell σ, oriented by θ, is the integer k. In general, write a positive dimensional cochain as a sum of cochains which are nonzero on only one cell. The picture of the cochain is then the union of the pictures of the summands. Note that the picture of a cochain of positive dimension, i, restricted to any i-cell, σ, consists of an orientation, θ, of σ and an integer k. The cochain is recovered from its picture by the rule: The cochain evaluated on the cell σ, with orientation θ, is the integer k. The pictures of cochains are a special case of the geometric cochains in stratified objects defined in **[7]**. See also **[15]**, **[16]**, **[23]** and **[25]**.

Suppose a space X has been given a special cell structure. The coboundary of one-dimensional cochains and the cup product of certain pairs of one-dimensional cochains can then be calculated in terms of pictures by the following rules which follow from the definition of ∂ and ϕ.

I



II



Rule II says that if the pictures of two 1-cochains are transverse, then the intersection of their pictures is a picture of their cup product. Rule II is the motivation for allowing some of the 2-cells to be cubes.

Using geometric pictures of cochains and Rules I and II, the technique for calculating Massey products in a 2-dimensional CW complex is as follows. View the space X as a set of disjoint 2-discs whose boundaries have been attached to a wedge of circles. Calculate cup products by drawing pictures of cocycle representatives so that the cup product can be evaluated by Rule II. If cup products are cohomologous to zero use Rules I and II to draw a picture of cochains in a defining system for a triple product. Choose the pictures so that the corresponding element of Massey product can be evaluated by Rule II. A special cell structure on X such that the pictures are pictures of cochains can be constructed as follows. First draw a small cube around each point where Rule II was used to calculate a cup product. Order the vertices of the cubes so condition (ii) in the definition of special cell structure is satisfied. Complete the cell structure by taking a simplicial subdivision of the complement of the interiors of the cubes. Extend the ordering of the vertices. Taking the cells in the subdivision transverse to the pictures guarantees that the pictures determine cochains in the special cell structure and that calculations based on Rules I and II are valid. The following example illustrates these ideas. Denote by X the quotient of the rectangle in Figure 8 obtained by attaching the boundary of the rectangle to a wedge of two oriented circles c_1, c_2, so that the attaching map is given by $[\alpha_1^2, \alpha_2] = \alpha_1^2\alpha_2\alpha_1^{-2}\alpha_2^{-1}$. Orient the rectangle by the ordered basis $\{(1, 0), (0, 1)\}$ for $\mathbf{R}^2$. The image of the oriented rectangle in X is a cycle whose boundary class generates $H_2(X : \mathbf{Z}) \simeq \mathbf{Z}$. A basis for H_1 is given by $\{[c_1], [c_2]\}$ where $[c_i]$ denotes the homology class determined by the oriented circle c_i. Let u_1 and u_2 be the elements in $H^1(X : \mathbf{Z})$ dual to $[c_1]$, $[c_2]$ and let e denote the element in $H^2(X : \mathbf{Z})$ dual to the homology class determined by the oriented rectangle. By drawing pictures of cochains and using Rules I and II to calculate coboundaries and cup products it will be shown that $u_1^2 = u_2^2 = 0$, $u_1u_2 = 2e$, and with $\mathbf{Z}_2$ coefficients the triple product $\langle u_1, u_1, u_2\rangle$ is the mod 2 reduction of e.

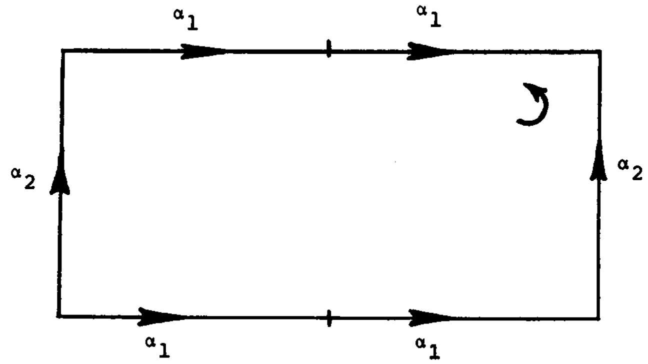


FIGURE 8

b_1 in Figure 9 is a picture of a cocycle representative for u_1. b_2 in Figure 10 is a picture of a cocycle representative for u_2. By moving the picture of b_1 parallel to itself, it is possible to get a picture of a cocycle, b_1', which represents u_1 and does not intersect the picture of b_1. $u_1^2 = 0$ by Rule II. Similarly $u_2^2 = 0$. From the picture of $b_1 \cup b_2$ (Figure 11) it follows that $u_1u_2 = 2e$.

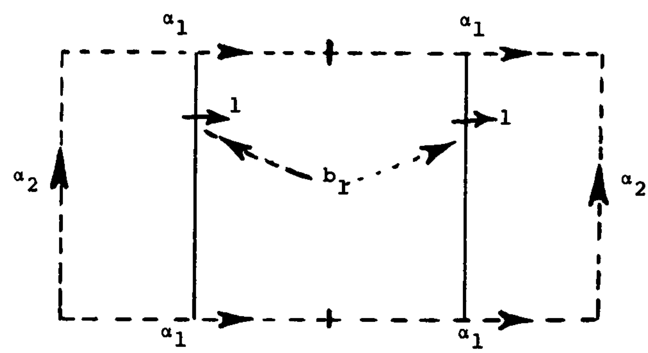


FIGURE 9

With $\mathbf{Z}_2$ coefficients all cup products of elements in H^1 are zero. Hence all triple products of elements in H^1, with $\mathbf{Z}_2$ coefficients, are defined and contain only one element. Let b_1', b_1, b_2 and $b_{1,2}$ be the cochains pictured in Figures 12 and 13. With $\mathbf{Z}_2$ coefficients, $b_1' \cup b_1 = 0$ and $\delta b_{1,2} = b_1 \cup b_2$. Hence $b_1' \cup b_{1,2}$ is a cocycle representative of $\langle u_1, u_1, u_2\rangle$. Rule II implies that $\langle u_1, u_1, u_2\rangle$ is the mod 2 reduction of e.

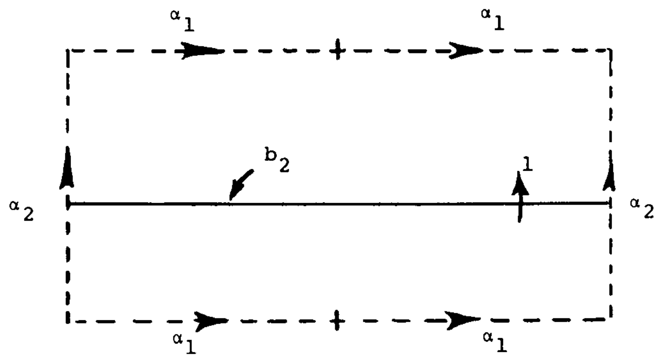


FIGURE 10

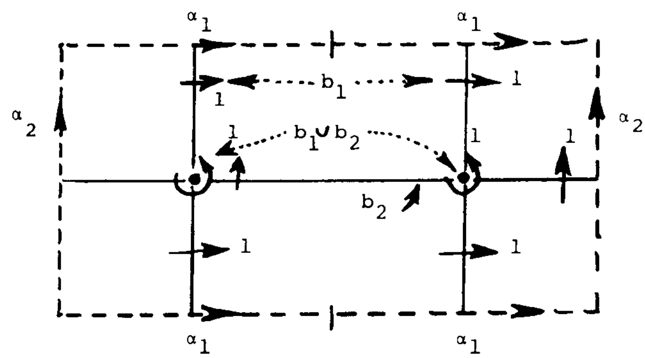


FIGURE 11

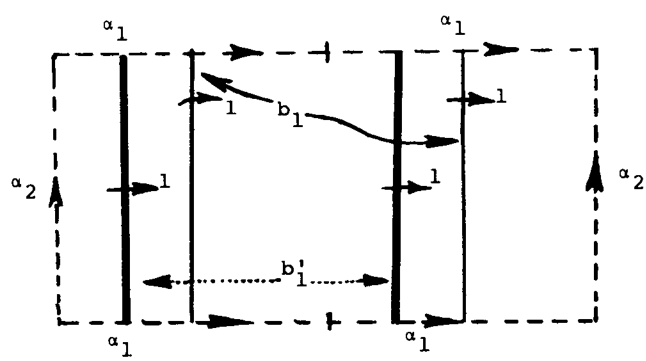


FIGURE 12

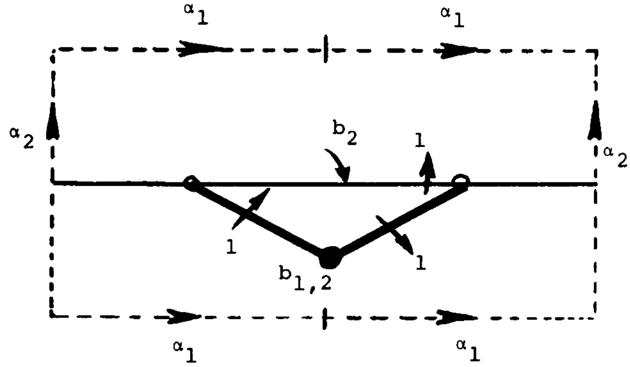


FIGURE 13

The purpose of Lemma 2, below, is to construct 1-cochains, v, and 2-cochains, a, in a 2-dimensional cell complex so that coboundaries and cup products of the v are given by formulas involving coefficients in the Magnus expansions of words corresponding to the attaching maps of the 2-cells. These cochains are used in the proof of Theorem 2 to construct defining systems for Massey products and derive a formula for the corresponding product which involves coefficients in the Magnus expansions of words corresponding to the attaching maps of the 2-cells. Theorem 2 applied to the example above implies that with $\mathbf{Z}_2$ coefficients each triple product of the form $\langle u_i, u_j, u_k\rangle$ is the mod 2 reduction of $\mu(i, j, k : [\alpha_1^2, \alpha_2])e$ where $\mu(i, j, k : [\alpha_1^2, \alpha_2])$ denotes the coefficient of $K_iK_jK_k$ in the Magnus expansion of $[\alpha_1^2, \alpha_2]$. Since $\mu(1, 1, 2 : [\alpha_1^2, \alpha_2]) = 1$, the formula checks with the explicit calculation carried out above. Theorem 2 applied to the example above also yields the more general formula that for any three elements h_1, h_2, h_3 in $H^1(X : \mathbf{Z}_2)$, the triple product $\langle h_1, h_2, h_3\rangle$ with $\mathbf{Z}_2$ coefficients is $\Sigma h_1(i_1)h_2(i_2)h_3(i_3)\mu(i_1, i_2, i_3\colon [\alpha_1^2, \alpha_2])e$ where the sum is over all sequences (i_1, i_2, i_3) with $i_t = 1$ or 2 and $h_s(i)$ denotes h_s evaluated on the homology class $[c_i]$. The cochains constructed by Lemma 2 to evaluate $\langle u_1, u_1, u_2\rangle$ are essentially different (and less obvious) than those pictured in Figures 8–13. This less obvious approach seems necessary in order to derive the general formula of Theorem 2. Pictures of the cochains constructed in Lemma 2 which can be used to evaluate $\langle u_1, u_1, u_2\rangle$ in the above example are described after the statement of Lemma 2.

Denote by $X(\alpha_1, \ldots, \alpha_J\colon \{W_\lambda\}_{\lambda\in\Lambda})$ the 2-dimensional CW complex determined by the group presentation $(\alpha_1, \ldots, \alpha_J\colon \{W_\lambda\}_{\lambda\in\Lambda})$. There is a single 0-cell, one edge for each generator α and a 2-cell for each relator W. The attaching map of the boundary of a 2-cell is determined by the recipe that the relator gives as a word in the α. $\mu(l_1, \ldots, l_k\colon W_\lambda)$ denotes the coefficient of $K_{l_1}, \ldots, K_{l_k}$ in the Magnus expansion of the word W_λ.

LEMMA 2. *Given the 2-dimensional CW complex, $X = X(\alpha_1, \ldots, \alpha_J\colon \{W_\lambda\}_{\lambda\in\Lambda})$ and a positive integer p; there is a subdivision of the cell structure on X which is a special cell structure having the following cochains. For $\lambda \in \Lambda$ and $j = 1, 2, \ldots, p$*

there is a 2-cochain, $a_\lambda(j)$. For each sequence $(l_1, \ldots, l_k)$ of integers with $1 < l_i < J$ and each $j = 1, 2, \ldots, p$ there is a 1-cochain $v(j : l_1, \ldots, l_k)$. The cochains a and v have the following properties.

1. *The 2-cochain, $a_\lambda(j)$, evaluated on the oriented 2-chain determined by the relator, $W_{\lambda'}$, is 1 if $\lambda = \lambda'$ and 0 otherwise.*
2. *The 1-cochain, $v(j : l_1)$, evaluated on the oriented 1-chain determined by the generator α_i is 1 if $i = l_1$ and 0 otherwise.*
3. $\delta v(j : l_1) = \Sigma_{\lambda \in \Lambda}\, \mu(l_1 : W_\lambda) a_\lambda(j)$.
4. $v(j : l_1, \ldots, l_k) v(j' : i_1, \ldots, i_{k'}) = 0$ *if* $j < j' - 1$ *and* $k' > 2$.
5. $v(j - 1 : l_1, \ldots, l_k) v(j : l_{k+1}) + \delta v(j : l_1, \ldots, l_{k+1}) = \Sigma_{\lambda \in \Lambda}\, \mu(l_1, \ldots, l_{k+1} : W_\lambda) a_\lambda(j)$.

The following example illustrates the construction of the cochains a and v and shows how these cochains can be used to evaluate Massey products. A 2-dimensional cell complex whose fundamental group has presentation $(\alpha_1, \alpha_2 : [\alpha_1^2, \alpha_2])$ is obtained as a quotient of the rectangle $[0, 6] \times [-1, 5]$ by making the identifications $(0, y) = (6, y)$ for all y in $[1 - , 5]$; $(x, 5) = (x', 5)$ for x and x' in $[0, 6]$; and by identifying the interval $[0, 6] \times \{-1\}$ to a wedge of two circles according to the word $[\alpha_1^2, \alpha_2] = \alpha_1\alpha_1\alpha_2\alpha_1^{-1}\alpha_1^{-1}\alpha_2^{-1}$. (See the proof of Lemma 2 for more details.) The quotient space, X, has a cell structure with one 0-cell, two oriented edges, and one oriented 2-cell. The ordered basis $\{(1, 0), (0, 1)\}$ for $\mathbf{R}^2$ orients the 2-cell of X. The oriented 2-cell is a cycle whose homology class generates $H_2(X : \mathbf{Z}) \simeq \mathbf{Z}$. $\{a_1, a_2\}$ is a basis for $H_1(X : \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}$ where a_i denotes the homology class determined by the oriented edge corresponding to the generator α_i. For $i = 1, 2$, denote by u_i the element in $H^1(X : \mathbf{Z})$ dual to a_i and denote by e the element in $H^2(X : \mathbf{Z})$ dual to the generator of $H_2(X : \mathbf{Z})$ determined by the oriented 2-cell. It will be shown that:

1. $u_2^2 = u_1^2 = 0$;
2. $u_1u_2 = 2e$;
3. with $\mathbf{Z}_2$ coefficients, the triple product $\langle u_1, u_1, u_2\rangle$ is defined and contains the mod 2 reduction of e.

Each of these calculations corresponds to a certain coefficient in the Magnus expansion of $[\alpha_1^2, \alpha_2]$ as follows:

$u_1^2 = 0$, coefficient of K_1^2 is 0;

$u_2^2 = 0$, coefficient of K_2^2 is 0;

$u_1u_2 = 2e$, coefficient of K_1K_2 is 2;

$\langle u_1, u_1, u_2\rangle = e$ ($\mathbf{Z}_2$ coefficients),

coefficient of $K_1^2K_2$ reduced mod 2 is nonzero.

A cocycle representative, $v(1 : 1)$, for u_1 is constructed in steps as follows. First choose a point, p, in the interior of the edge corresponding to the generator α_1. From each point of the rectangle which is in the inverse image of p, under the attaching map, draw a vertical line from the point to the horizontal line $y = 1$. On each vertical line draw an arrow which points in the direction of increasing values of x. Label each of the arrows with either $+1$ or -1 so that the result is a picture of a cochain which is $+1$ when evaluated on the oriented edge corresponding to α_1 (see Figure 14). Draw in the line $y = 1$ together with arrows pointing down. The next step is to label the vertical arrows with integers. The arrow furthest to the left is labeled 0. The other arrows are labeled so that the coboundary of the resulting cochain is 0 except possibly in a small neighborhood of the point (6, 1) (see Figure 15). This completes the construction of $v(1:1)$ a cocycle representative of u_1. It is important to note that the integers which label the vertical arrows are the coefficients of K_1 in the Magnus expansion of certain words as follows (reading Figure 15 from left to right).

$$
\begin{aligned}
0 &= \text{coefficient of } K_1 \text{ in the expansion of } 1,\\
1 &= \text{coefficient of } K_1 \text{ in the expansion of } \alpha_1,\\
2 &= \text{coefficient of } K_1 \text{ in the expansion of } \alpha_1^2 \text{ and } \alpha_1^2\alpha_2,\\
1 &= \text{coefficient of } K_1 \text{ in the expansion of } \alpha_1^2\alpha_2\alpha_1^{-1},\\
0 &= \text{coefficient of } K_1 \text{ in the expansion of } \alpha_1^2\alpha_2\alpha_1^{-2} \text{ and } \left[\alpha_1^2, \alpha_2\right].
\end{aligned}
$$

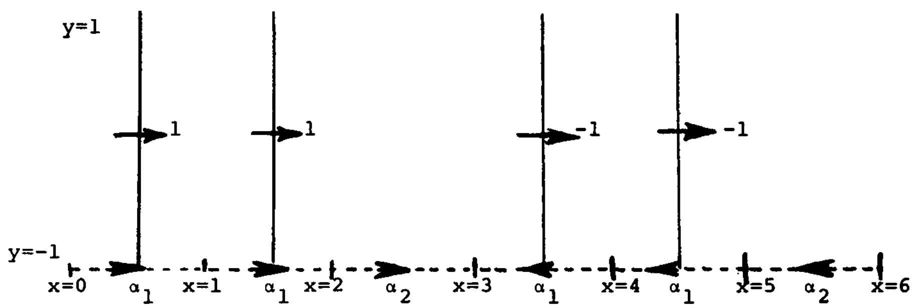


FIGURE 14

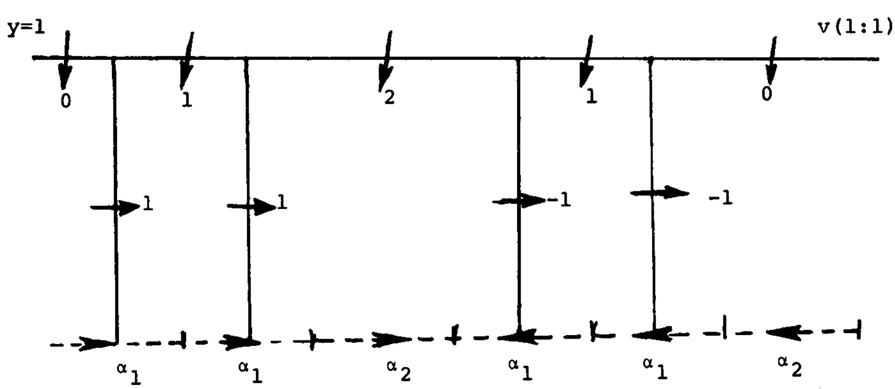


FIGURE 15

The product u_1^2 can be calculated by constructing another cocycle representative, $v(2:1)$, for u_1 and then using Rule II to evaluate $v(1:1) \cup v(2:1)$. $v(2:1)$ is constructed in the same manner as $v(1:1)$ with the following changes. First choose a point, p', in the interior of the edge corresponding to α_1 so that if the edge is traversed in the direction indicated by the orientation, then the point p' occurs before p. Secondly the vertical lines go from $y=-1$ to $y=2$. Figure 16 contains $v(1:1)$ and $v(2:1)$ with $v(1:1)$ dotted. A cochain $v(2:1,1)$ with $v(1:1)v(2:1) + \delta v(2:1,1) = 0$ is constructed as follows. The product $v(1:1)v(2:1)$ consists of two oppositely oriented points on the line $y=1$. From each of these points draw a vertical line to the horizontal line $y=2$. Put arrows going from left to right on each of the lines. Label the arrows with integers so that when restricted to a small horizontal strip about the line $y=1$ we have that the coboundary of the cochain $+ v(1:1)v(2:1) = 0$. The result is Figure 17. The construction of $v(2:1,1)$ is completed by putting vertical arrows along the line $y=2$ and labeling the arrows with integers so that the integer farthest to the left is 0 and the coboundary of $v(2:1,1)$ is 0 when restricted to the line $y=2$ except possibly in a small neighborhood of the point (6, 2) (see Figure 18).

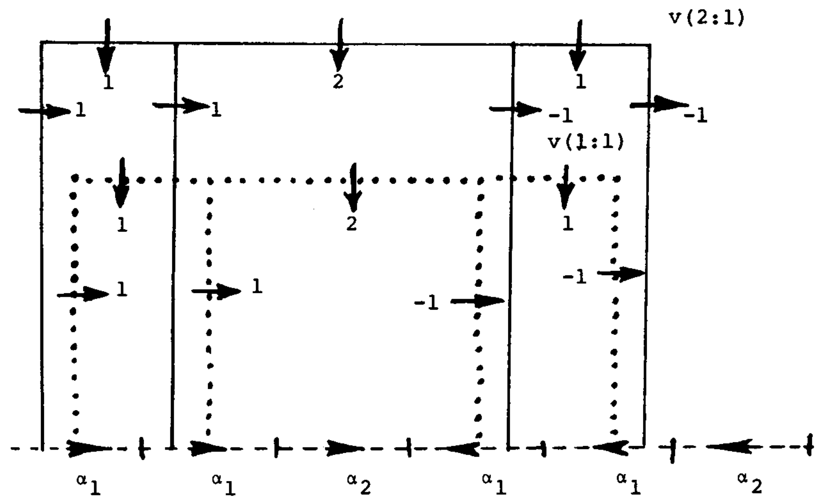


FIGURE 16

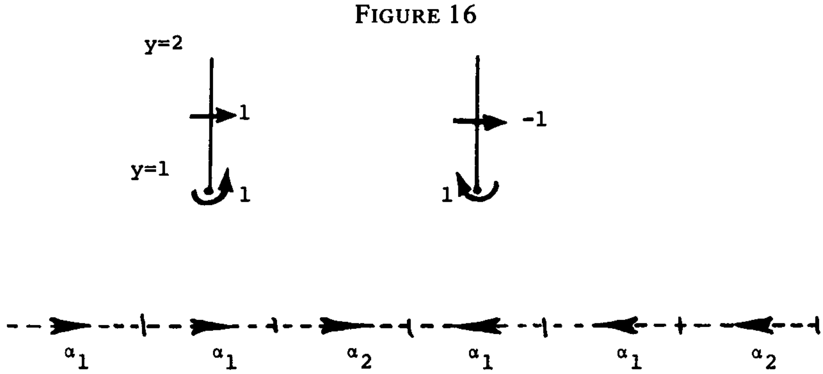


FIGURE 17

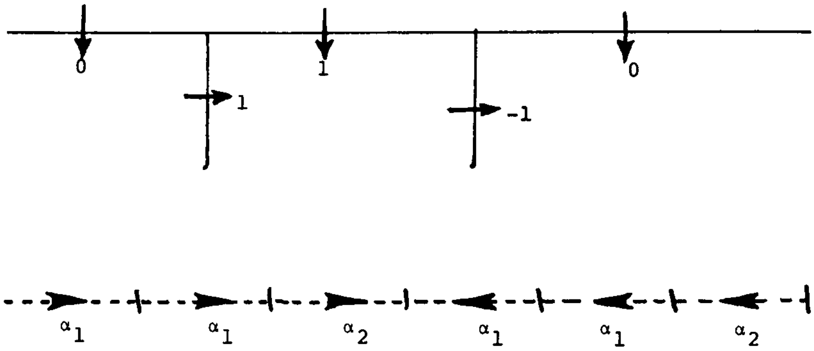


FIGURE 18

The numbers which label the vertical arrows are the coefficients of K_1^2 in the Magnus expansion of certain words as follows (reading Figure 18 from left to right).

$0 =$ coefficient of K_1^2 in the expansion of 1 and α_1,

$1 =$ coefficient of K_1^2 in the expansion of α_1^2 and $\alpha_1^2\alpha_2$,

$0 =$ coefficient of K_1^2 in the expansion of $\alpha_1^2\alpha_2\alpha_1^{-1}$; $\alpha_1^2\alpha_2\alpha_1^{-2}$ and $\left[\alpha_1^2, \alpha_2\right]$.

Since $v(1:1)v(2:1) + \delta v(2:1,1)$, it follows that $u_1^2 = 0$. A similar argument shows that u_2^2 is 0. The product u_1u_2 is calculated by constructing the cocycle representative, $v(3:2)$, for u_2 (Figure 19). The picture of $v(2:1) \cup v(3:2)$ is $\circlearrowleft 2$. Hence $u_1u_2 = 2e$ in $H^2(X:\mathbf{Z})$.

Since all cup products of elements in $H^1(X:\mathbf{Z}_2)$ are zero, the triple product $\langle u_1, u_1, u_2\rangle$ (with $\mathbf{Z}_2$ coefficients) is defined and contains a single element. Set $v_2(1:1)$; $v_2(2:1)$; $v_2(2:1,1)$ and $v_2(3:2)$ equal to the mod 2 reductions of the corresponding cochains v. With $\mathbf{Z}_2$ coefficients, $\delta v_2(2:1,1) = v_2(1:1)v_2(2:1)$ and $v_2(2:1)v_2(3:2) = 0$. Hence $v_2(2:1,1)v_2(3:2)$ is a cocycle representative of the unique element in $\langle u_1, u_1, u_2\rangle$. The product $v(2:1,1)v(3:2)$ is indicated in Figure 19. A cochain, $v(3:1,1,2)$ is constructed as follows. From the point where $v(2:1,1)$ intersects $v(3:2)$ draw a vertical line up to the line $y = 3$. Put arrows on this line and on the line $y = 3$. Label the arrows so that $v(2:1,1)v(3:2) + \delta v(3:1,1,2) = 0$ in the complement of a small neighborhood of the point (6, 3) (see Figure 20). Set $a(3)$ equal to the 2-cocycle whose picture as $\circlearrowleft^1$ located at the point (6, 3). $a(3)$ is a cocycle representative of e and $v(2:1,1)v(3:2) + \delta v(3:1,1,2) = a(3)$. Hence with $\mathbf{Z}_2$ coefficients, the product $\langle u_1, u_1, u_2\rangle$ contains the mod 2 reduction of e. (This also follows directly from Figure 19.) The numbers 0 and 1 on the vertical arrows in Figure 20 are the coefficients of $K_1^2K_2$ in the Magnus expansion of certain words as follows.

$0 =$ coefficient of $K_1^2K_2$ in the expansion of 1, α_1, and α_1^2,

$1 =$ coefficient of $K_1^2K_2$ in the expansion of $\alpha_1^2\alpha_2$, $\alpha_1^2\alpha_2\alpha_1^{-1}$,
$\alpha_1^2\alpha_2\alpha_1^{-2}$ and $\left[\alpha_1^2, \alpha_2\right]$.

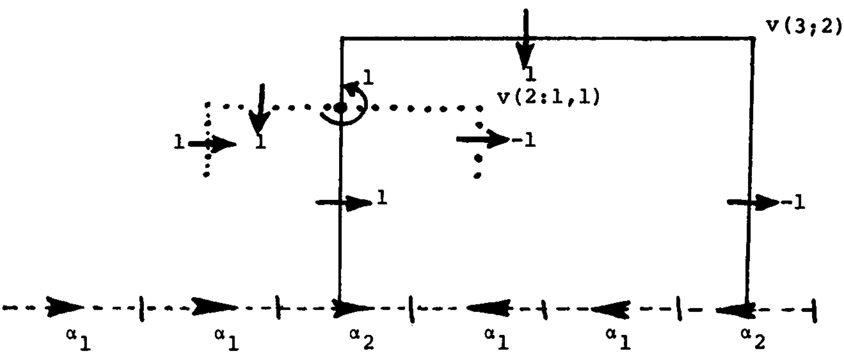


FIGURE 19

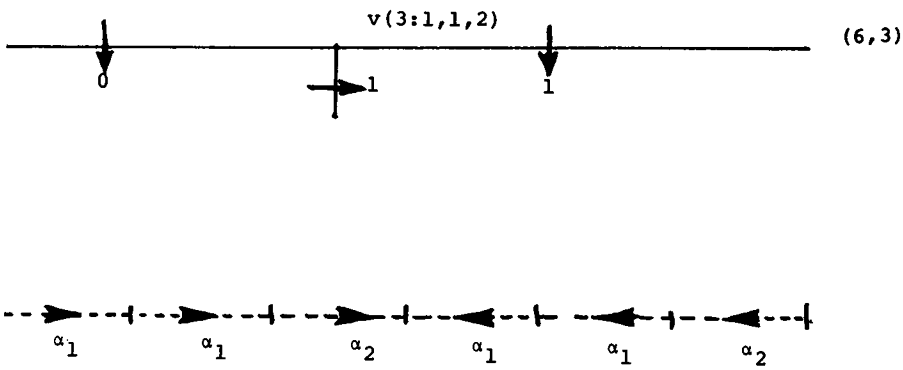


FIGURE 20

PROOF OF LEMMA 2. The cochains a and v are constructed by first drawing their pictures and then describing a procedure for giving X a special cell structure with cochains whose pictures are the given ones. Properties 1–5 are then proved by applying Rules I and II to the pictures.

The first step is to give an explicit description of the cell complex $X(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda\in\Lambda})$. The one-skeleton is a wedge of J-oriented circles, $c_1 \vee c_2 \vee \cdots \vee c_J$, one circle for each generator α. The attaching maps of the 2-cells are described as follows. For each $\lambda \in \Lambda$ write

$$W_\lambda = \alpha_{q_1}^{\varepsilon_1}, \ldots, \alpha_{q_s}^{\varepsilon_s} \quad \text{with } \varepsilon_i = \pm 1,$$

$$W_{\lambda,0} = 1,$$

$$W_{\lambda,r} = \alpha_{q_1}^{\varepsilon_1}, \ldots, \alpha_{q_r}^{\varepsilon_r} \quad \text{for } 1 < r < s.$$

Consider the 2-cell, E_λ, corresponding to the relator, W_λ, as the quotient of the rectangle $[0, s] \times [-1, p + 2]$ by the relations $(0, y) = (s, y)$ for all y in $[-1, p + 2]$ and $(x, p + 2) = (x', p + 2)$ for x and x' in $[0, s]$. For each $j, j = 1, 2, \ldots, J$, choose an orientation-preserving homeomorphism, f_j, from the unit circle in the complex plane to the circle c_j. The attaching map, g_λ, of the boundary of E_λ to

$c_1 \vee c_2 \vee \cdots \vee c_J$ is given by the formula

$$g_\lambda(x, -1) = f_{q_r}(\cos[2\pi(x - r + 1)], \varepsilon_r \sin[2\pi(x - r + 1)])$$

for x in the closed interval $[r - 1, r]$.

The next step is to specify the pictures of the cochains a and v. The picture of $a_\lambda(j)$ intersected with E_λ is the triple (the point with coordinates (s, j); the orientation of E_λ given by the ordered basis $\{(1, 0), (0, 1)\}$; the integer 1). The picture of $a_\lambda(j)$ intersected with any cell other than E_λ is empty.

To describe pictures of the cochains $v(j : l_1, \ldots, l_k)$ choose a number h with $0 < h < 1/2p$. The picture of $v(j : l_1)$ intersected with the strip $[r - 1, r] \times [-1, p + 2]$ in the cell E_λ is given in Figure 21a where $a_r = \mu(l_1 : W_{\lambda,r-1})$, $c_r = \mu(l_1 : W_{\lambda,r})$ and $a_r + b_r = c_r$. For $k > 2$ the picture of $v(j : l_1, \ldots, l_k)$ intersected with the same strip is given in Figure 21b where $a_r = \mu(l_1, \ldots, l_k : W_{\lambda,r-1})$, $c_r = \mu(l_1, \ldots, l_k : W_{\lambda,r})$ and $a_r + b_r = c_r$.

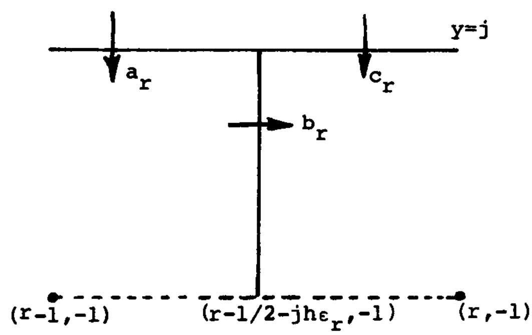


FIGURE 21a

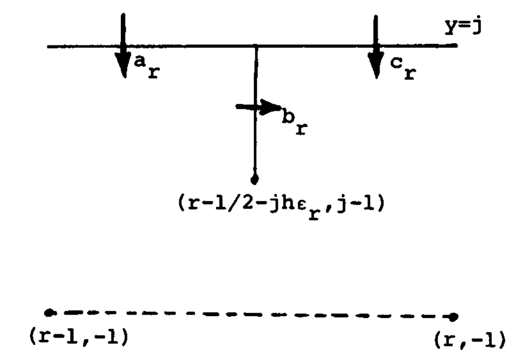


FIGURE 21b

In order to describe the restriction of Figure 21a to the wedge of circles $c_1 \vee c_2 \vee \cdots \vee c_J$, note that for any word W in the α,

(i) $\mu(l : W\alpha_i^\varepsilon) = \mu(l : W)$ if $i \neq l$, $\varepsilon = \pm 1$ and

(ii) $\mu(l : W\alpha_l^\varepsilon) = \mu(l : W) + \varepsilon$; $\varepsilon = \pm 1$.

It follows that the numbers b_r in Figure 21a for the picture of $v(j : l_1)$ satisfy (iii)

$$b_r = \begin{Bmatrix} 0 & \text{if } q_r \neq l_1 \\ \varepsilon_r & \text{if } q_r = l_1 \end{Bmatrix}.$$

Hence the picture of $v(j : l_1)$ restricted to the circle c_i is empty if $i \neq l_1$. The picture of $v(j : l_1)$ restricted to the circle c_{l_1} consists of the triple (the point $f_{l_1}(\cos[2\pi(\frac{1}{2} - jh\varepsilon_r)]$, $\varepsilon_r \sin[2\pi(\frac{1}{2} - jh\varepsilon_r)])$; the orientation of c_{l_1}; the integer 1). The picture of $v(j : l_1, \ldots, l_k)$ restricted to the wedge of circles $c_1 \vee c_2 \vee \cdots \vee c_J$ is empty for $k > 2$.

The next step is to show that there is a subdivision of $X(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda \in \Lambda})$ which is a special cell structure with cochains a and v whose pictures are the ones described above. First take a regular subdivision of $c_1 \vee c_2 \vee \cdots \vee c_J$ so that the points

$$f_i\left(\cos\left[2\pi\left(\tfrac{1}{2} - jh\right)\right], \sin\left[2\pi\left(\tfrac{1}{2} - jh\right)\right]\right), \qquad 1 < i < J, 1 < j < p,$$

are in the interiors of different one-cells. Next, take a small cube around each of the points $(r - \frac{1}{2} - jh\varepsilon_r, j - 1)$ and $(r - \frac{1}{2} - jh\varepsilon_r, j)$ that occurs in Figure 21b, and the points (s, j) used in defining the picture of $a_\lambda(j)$. Draw the cubes with edges parallel to the x and y axes. Take the cubes small enough so that the cubes are disjoint and do not intersect $c_1 \vee c_2 \vee \cdots \vee c_J$. Now order the vertices so condition (ii) in the definition of special cell structure is satisfied. Complete the cell structure by adding 2-simplices without subdividing any of the 1-cells already described. Extend the ordering of the vertices to get a special cell structrue. Add the 2-simplices so that each cell in the special cell structure is transverse to each of the pictures in Figures 21a and 21b. Then each picture of an i-cochain restricted to an i-cell of the special cell structure consists of an orientation of the cell and an integer. Each of the pictures above thus determines a cochain in the special cell structure and calculations based on the pictures using Rules I and II are valid.

Properties 1–5 in the statement of Lemma 2 are proved as follows. Property 1 follows directly from the definition of $a_\lambda(j)$ and the rule for recovering a cochain from its picture. Property 2 follows since the picture of $v(j : l_1)$ restricted to c_{l_1} consists of the triple (a point on c_{l_1}; the orientation of c_{l_1}; the integer 1) and the picture of $v(j : l_1)$ restricted to c_i is empty for $i \neq l_1$. Property 3 follows from Rule I (recall that $a_r + b_r = c_r$ in Figure 21a). Property 4 holds since the picture of $v(j : l_1, \ldots, l_k)$ does not intersect the picture of $v(j' : i_1, \ldots, i_{k'})$ if $j < j' - 1$ and $k' > 2$. This leaves Property 5. From Rule II and the definition of the cochains v, it follows that the picture of $v(j - 1 : l_1, \ldots, l_k)v(j : l_{k+1})$ restricted to the 2-cell, E_λ, is the collection of triples (the point $(r - \frac{1}{2} - jh\varepsilon_r, j - 1)$; the orientation of E_λ; the number d_r) where $1 < r < s$ and the integers d_r satisfy

$$d_r = \begin{Bmatrix} 0 \quad \text{if } q_r \neq l_{k+1} & \\ \mu(l_1, \ldots, l_k : W_{\lambda, r-1}) & \text{if } q_r = l_{k+1} \text{ and } \varepsilon_r = 1 \\ -\mu(l_1, \ldots, l_k : W_{\lambda, r}) & \text{if } q_r = l_{k+1} \text{ and } \varepsilon_r = -1 \end{Bmatrix}.$$

$d_r = 0$ when $q_r \neq l_{k+1}$ since the number b_r in Figure 21a for $v(j : l_{k+1})$ is 0. If $q_r = l_{k+1}$, then the formula for d_r follows from Figures 22a and 22b.

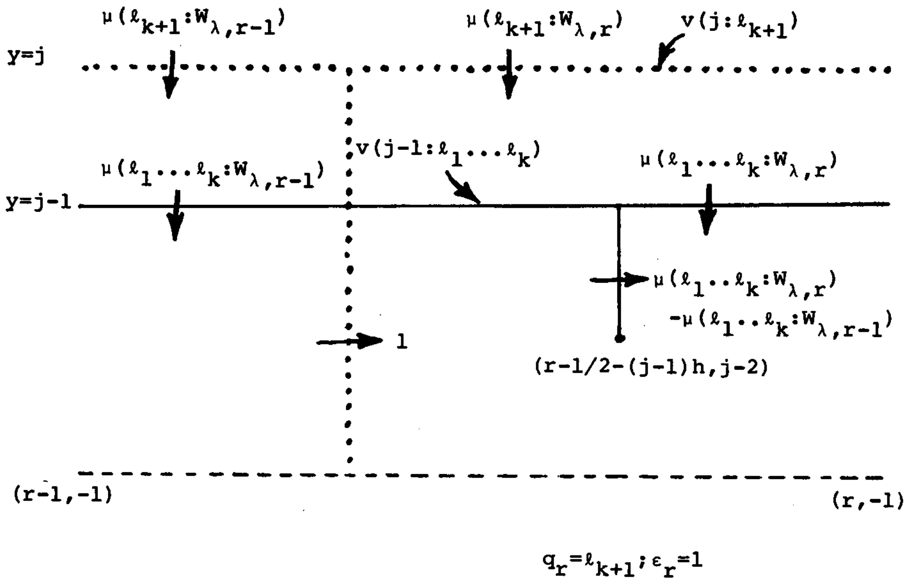


FIGURE 22a

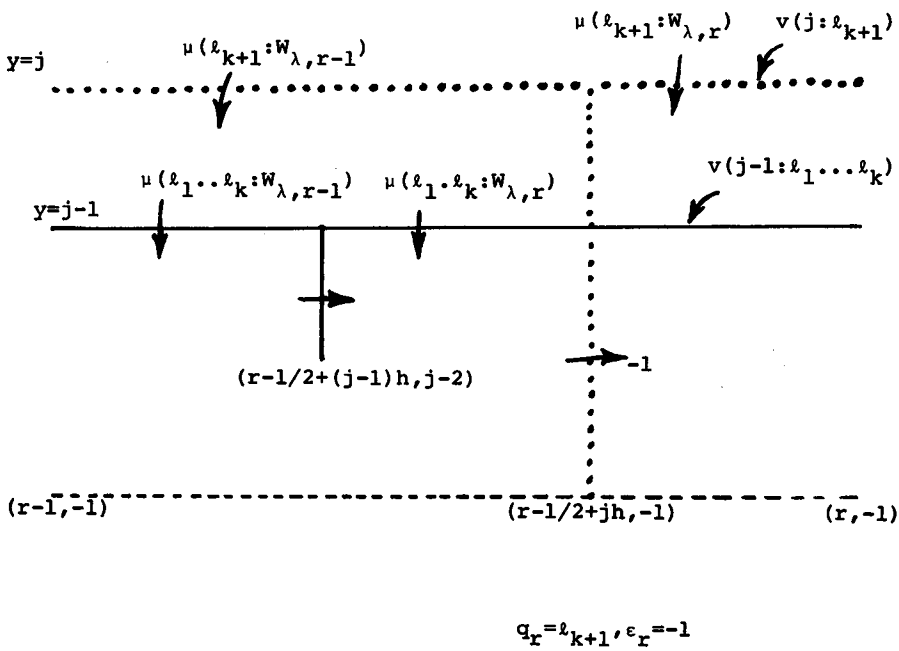


FIGURE 22b

On the other hand, $\delta v(j : l_1, \ldots, l_{k+1}) - \Sigma_{\lambda \in \Lambda} \mu(l_1, \ldots, l_{k+1} : W_\lambda) a_\lambda(j)$ restricted to the 2-cell E_λ is the collection of triples (the point $(r - \frac{1}{2} - jh\varepsilon_r, j - 1)$; the opposite orientation of E_λ; the integer b_r), where $1 < r < s$ and b_r is the number given in Figure 21b. Thus it suffices to show that $d_r = b_r$ for $1 < r < s$. This is done using the following properties of coefficients in the Magnus expansion. For any two words W_0, W_1, in the generators α:

$$\begin{aligned}\mu(l, \ldots, l_{k+1} : W_0 W_1) = \mu(l_1, \ldots, l_{k+1} : W_0) \\ + \sum_{i=1}^{k} \mu(l_1, \ldots, l_i : W_0)\mu(l_{i+1}, \ldots, l_{k+1} : W_1) \\ + \mu(l_1, \ldots, l_{k+1} : W_1).\end{aligned}$$

Hence:

(a) $\mu(l_1, \ldots, l_{k+1} : W\alpha_i^\varepsilon) = \mu(l_1, \ldots, l_{k+1} : W)$ if $i \neq l_{k+1}$, $\varepsilon = \pm 1$;
(b) $\mu(l_1, \ldots, l_{k+1} : W\alpha_{l_{k+1}}) = \mu(l_1, \ldots, l_{k+1} : W) + \mu(l_1, \ldots, l_k : W)$; and
(c) $\mu(l_1, \ldots, l_{k+1} : W\alpha_{l_{k+1}}^{-1}) = \mu(l_1, \ldots, l_{k+1} : W) - \mu(l_1, \ldots, l_k : W\alpha_{l_{k+1}}^{-1})$.

To prove (c) set $l_{k+1} = l$ and write $(l_1, \ldots, l_{k+1})$ as

$$l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i}$$

with $i > 1$ and $l_j \neq l$. The expansion of α_l^{-1} is $1 - K_l + K_l^2 + \cdots + (-1)^i K_l^i + \ldots$ so

$$\begin{aligned}\mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i} : W\alpha_l^{-1}\Big) = \mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i} : W\Big) \\ - \mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i-1} : W\Big) \\ + \cdots + (-1)^i \mu(l_1, \ldots, l_j : W).\end{aligned}$$

Similarly

$$\begin{aligned}\mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i-1} : W\alpha_l^{-1}\Big) = \mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i-1} : W\Big) \\ - \mu\Big(l_1, \ldots, l_j, \underbrace{l, \ldots, l}_{i-2} : W\Big) \\ + \cdots + (-1)^{i-1} \mu(l_1, \ldots, l_j : W).\end{aligned}$$

Adding these two equalities yields property (c) in the form

$$\mu\big(l_1, \ldots, l_{k+1} : W\alpha_{l_{k+1}}^{-1}\big) + \mu\big(l_1, \ldots, l_k : W\alpha_{l_{k+1}}^{-1}\big) = \mu(l_1, \ldots, l_{k+1} : W).$$

From the definition of the cochains $v(j : l_1, \ldots, l_{k+1})$ it follows that

$$b_r = c_r - a_r = \mu(l_1, \ldots, l_{k+1} : W_{\lambda,r}) - \mu(l_1, \ldots, l_{k+1} : W_{\lambda,r-1}).$$

Since

$W_{\lambda,r} = W_{\lambda,r-1}\,\alpha_{q_r}^{\varepsilon_r}$, $\quad b_r = \mu(l_1, \ldots, l_{k+1} : W_{\lambda,r-1}\alpha_{q_r}^{\varepsilon_r}) - \mu(l_1, \ldots, l_{k+1} : W_{\lambda,r-1})$, formulas (a), (b) and (c) imply

$$b_r = \begin{cases} 0 & \text{if } q_r \neq l_{k+1} \\ \mu(l_1, \ldots, l_k : W_{\lambda,r-1}) & \text{if } q_r = l_{k+1} \text{ and } \varepsilon_r = 1 \\ -\mu(l_1, \ldots, l_k : W_{\lambda,r}) & \text{if } q_r = l_{k+1} \text{ and } \varepsilon_r = -1 \end{cases}.$$

Hence $b_r = d_r$ for $1 < r < s$ and this completes the proof of Lemma 2.

Lemma 2 can be used to calculate cup products and Massey products as follows. Suppose for simplicity that each relator, W_λ, is in the commutator subgroup of the free group generated by the α. The Magnus expansion of each relator then has the form $1 +$ (terms of order > 2), [**12**]. The generators, α, determine a basis for H_1 and the relators determine a basis for H_2. From Properties 2 and 3 it follows that for fixed j the cochains $v(j : i)$, $i = 1, 2, \ldots, J$ are cocycles which give the basis for H^1 which is dual to the basis for H_1 determined by the α and, from Property 1, it follows that for fixed j the collection, $a_\lambda(j)$, $\lambda \in \Lambda$, of 2-cocycles determines the basis for H^2 which is dual to the basis for H_2 determined by the relators. Cup products can now be evaluated using Property 5. For $i = 1, 2, \ldots, J$, set u_i equal to the element in H^1 represented by the cocycle $v(j : i)$. The cup product $u_{l_1} \cup u_{l_2}$ is represented by $v(j - 1 : l_1)v(j : l_2)$ and from Property 5 it follows that $u_{l_1}u_{l_2}$ evaluated on the homology class determined by the relator W_λ is the coefficient of $K_{l_1}K_{l_2}$ in the Magnus expansion of W_λ.

Set N equal to the greatest common divisor of the numbers $\mu(l_1, l_2 : W_\lambda)$. Then all cup products of elements in $H^1(X : \mathbf{Z}_N)$ are zero. Hence, with coefficients $\mathbf{Z}_N$, all triple products $\langle u_{l_1}, u_{l_2}, u_{l_3}\rangle$ are defined and contain a single element. A defining system for the triple product $\langle u_{l_1}, u_{l_2}, u_{l_3}\rangle$ is given by setting $m_{i,i}$ equal to the mod N reduction of $v(i : l_i)$ for $i = 1, 2, 3$ and by setting $m_{i,i+1}$ equal to the mod N reduction of $-v(i + 1 : l_i, l_{i+1})$ for $i = 1, 2$. Then the mod N reduction of $-v(2 : l_1, l_2)v(3 : l_3)$ is a cocycle representative of $\langle u_{l_1}, u_{l_2}, u_{l_3}\rangle$. Property 5 implies that $\langle u_{l_1}, u_{l_2}, u_{l_3}\rangle$ evaluated on the homology class determined by the relator, W_λ, is the reduction mod N of the negative of the coefficient of $K_{l_1}K_{l_2}K_{l_3}$ in the Magnus expansion of W_λ.

Suppose the following are given: a group presentation $(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda\in\Lambda})$, a collection $\{X_i\}_{i=1}^p$ of subcomplexes of the 2-dimensional complex $X(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda\in\Lambda})$ determined by the presentation, and for each $i = 1, 2, \ldots, p$ an element h_i in $H^1(X_i : R)$, R a commutative ring with unit. In general, the Massey product $\langle h_1, \ldots, h_p\rangle$ in the system $\{X_i\}_{i=1}^p$ with coefficients R will not be defined. However, after replacing R with the quotient of R by an ideal $(h_1, \ldots, h_p)$, it is possible to use the cochains of Lemma 2 to construct a defining system for $\langle h_1, \ldots, h_p\rangle$ and describe the corresponding element of the product in terms of the coefficients in the Magnus expansions of the relators W_λ.

THEOREM 2. *Suppose the following are given*:

$$X = X(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda\in\Lambda}),$$

the 2-dimensional CW complex determined by the group presentation $(\alpha_1, \ldots, \alpha_J : \{W_\lambda\}_{\lambda\in\Lambda})$, *a collection* $\{X_i\}_{i=1}^p$, *of subcomplexes of* X *and for each*

$i = 1, 2, \ldots, p$ an element h_i in $H^1(X_i : R)$. Define the elements $I(i, j : \lambda)$ and the ideal $(h_1, \ldots, h_p)$ in R as follows: For $1 \leq i \leq j \leq p$ and λ the index of a 2-cell in $X_i \cap X_{i+1} \cap \cdots \cap X_j$ set

$$I(i, j : \lambda) = \sum h_i(l_1), \ldots, h_j(l_{j-i+1})\mu(l_1, \ldots, l_{j-i+1} : W_\lambda),$$

where the sum is over all sequences $(l_1, \ldots, l_{j-i+1})$ with $1 \leq l_t \leq J$ and $h_i(l)$ denotes h_i evaluated on the homology class determined by the generator α_l. ($h_i(l)$ is zero if the circle in X corresponding to α_l is not in the subcomplex X_i.) Set $(h_1, \ldots, h_p)$ equal to the ideal in R generated by the $I(i, j : \lambda)$ with $(i, j) \neq (1, p)$. Then the Massey product $\langle h_1, \ldots, h_p \rangle$ in the system $\{X_i\}_{i=1}^p$ with coefficients $R/(h_1, \ldots, h_p)$ is defined and contains the cohomology class given by the homomorphism which is $(-1)^p I(1, p : \lambda)$ when evaluated on the 2-cell of $X_1 \cap \cdots \cap X_p$ indexed by λ.

PROOF. For $1 \leq i \leq j \leq p$ set

$$m_{i,j} = (-1)^{j-i} \sum h_i(l_1), \ldots, h_j(l_{j-i+1})\bar{v}(j : l_1, \ldots, l_{j-i+1}),$$

where the sum is taken over all sequences $(l_1, \ldots, l_{j-i+1})$ with $1 \leq l_t \leq J$ and $\bar{v}(j : l_1, \ldots, l_{j-i+1})$ denotes the cochain $v(j : l_1, \ldots, l_{j-i+1})$ of Lemma 2 restricted to the subcomplex $X_i \cap \cdots \cap X_j$. It will be shown that $\{m_{i,j}\}$ with $(i, j) \neq (1, p)$ is a defining system for $\langle h_1, \ldots, h_p \rangle$ with coefficients $R/(h_1, \ldots, h_p)$. The cochains $m_{i,j}$ satisfy:

1. $\delta m_{i,i} = \sum_\lambda I(i, i : \lambda)a_\lambda(i)$ for $i = 1, 2, \ldots, p$ in $C^2(X_i : R)$.
2. With coefficients $R/(h_1, \ldots, h_p)$, $m_{i,i}$ is a cocycle representative of h_i.
3. $m(i, j)m(k, k') = 0$ if $j < k < k'$.
4. $\delta m(i, j) = m(i, j - 1)m(j, j) + (-1)^{j-i}\sum_\lambda I(i, j : \lambda)a_\lambda(j)$ in $C^2(X_i \cap \cdots \cap X_j : R)$. The sum is over all λ which index a 2-cell of $X_i \cap \cdots \cap X_j$.

From the definition of the $m_{i,j}$ it follows that $m_{i,i} = \sum_{l=1}^J h_i(l)\bar{v}(i : l)$. Property 3 of Lemma 2 implies $\delta m_{i,i} = \sum_{l=1}^J \sum_\lambda h_i(l)\mu(l : W_\lambda)a_\lambda(i)$. The sum is over all λ which indexed a 2-cell of X_i. Applying the definition of $I(i, j : \lambda)$ yields 1 above.

From 1 and the definition of the ideal $(h_1, \ldots, h_p)$ it follows that with coefficients $R/(h_1, \ldots, h_p)$ each of the $m_{i,i}$ is a cocycle. Property 2 of Lemma 2 and the definition of $m_{i,i}$ imply that $m_{i,i}$ and h_i are the same when evaluated on any of the oriented 1-cells in X_i. Hence $m_{i,i}$ is a cocycle representative of h_i.

Property 3 above follows directly from Property 4 of Lemma 2. From the definition of $m(i, j)$ it follows that

$$\delta m(i, j) = (-1)^{j-i} \sum h_i(l_1), \ldots, h_j(l_{j-i+1})\delta\bar{v}(j : l_1, \ldots, l_{j-i+1}).$$

Property 5 of Lemma 2 implies

$$\delta m(i, j) = (-1)^{j-i} \sum \sum_\lambda h_i(l_1), \ldots, h_j(l_{j-i+1})\mu(l_1, \ldots, l_{j-i+1} : W_\lambda)a_\lambda(j)$$
$$+ (-1)^{j-i+1} \sum h_i(l_1), \ldots, h_j(l_{j-i+1})\bar{v}(j - 1 : l_1, \ldots, l_{j-i})\bar{v}(j : l_{j-i+1}),$$

where the sum over λ is the sum over all λ which index a 2-cell of $X_i \cap \cdots \cap X_j$, and the other sum is over all sequences $(l_1, \ldots, l_{j-i+1})$ with $1 \leq l_t \leq J$. From the definition of $m(i,j)$ it follows that

$$m(i,j-1)m(j,j) \\ = (-1)^{j-i}\sum h_i(l_1),\dots,h_j(l_{j-i+1})\bar{v}(j-1:l_1,\dots,l_{j-1})\bar{v}(j:l_{j-i+1}).$$

Hence

$$\delta m(i,j) = m(i,j-1)m(j,j) \\ + (-1)^{j-i}\sum \sum_{\lambda} h_i(l_1),\dots,h_j(l_{j-i+1})\mu(l_1,\dots,l_{j-i+1}:W_\lambda)a_\lambda(j).$$

The definition of $I(i,j:\lambda)$ implies

$$\delta m(i,j) = m(i,j-1)m(j,j) + (-1)^{j-i}\sum_{\lambda} I(i,j:\lambda)a_\lambda(j).$$

Properties 1–4 above imply that with coefficients $R/(h_1,\dots,h_p)$ the collection $\{m_{i,j}\}$, $1 < i < j < p$, $(i,j) \neq (1,p)$, is a defining system for the Massey product, $\langle h_1,\dots,h_p\rangle$, in the system $\{X_i\}_{i=1}^p$. Property 4 above with $(i,j) = (1,p)$ implies that the corresponding element of $\langle h_1,\dots,h_p\rangle$ is represented by the cocycle

$$(-1)^p\sum_{\lambda} I(1,p:\lambda)a_\lambda(p) \quad \text{in } C^2(X_1\cap\cdots\cap X_p : R/(h_1,\dots,h_p)).$$

For polygonal (or smooth) links, Lemma 3, below, follows directly from Theorem 1. For arbitrary (i.e., wild) links, the proof is more complicated and will be omitted. This is the only point in the proof of Theorem 3 where it is assumed that the link is polygonal.

Lemma 3. *Given L, a polygonal N-link in S^3, q, an integer > 3, and $\alpha_1,\dots,\alpha_N$, meridians in F_1/F_q, then there are words $w_1,\dots,w_N$ in the α which represent parallels to L, a presentation*

$$P = (\alpha_1,\dots,\alpha_N,\alpha_{N+1},\dots,\alpha_J : [w_i,\alpha_i]a_i = 1 \text{ for } i = 1,2,\dots,N)$$

and a map f: $Y(P)\to S^3 - L$ (where $Y(P)$ denotes the complex determined by the presentation) so that the presentation, P, and the map f satisfy:

1. *Each of the words $a_1,\dots,a_N$ represents an element in the qth lower central series subgroup of the free group on $\alpha_1,\dots,\alpha_J$;*

2. *f restricted to the circle in $Y(P)$ corresponding to the generator α_i, $i = 1,\dots,N$ is the closed loop α_i;*

3. *$f^*(\gamma_{i,j})$ evaluated on the homology class determined by the relator $[w_i,\alpha_i]a_i$ is*

$$\begin{Bmatrix} -1 & \text{if } j = i \neq k \\ +1 & \text{if } j \neq i = k \\ 0 & \text{otherwise} \end{Bmatrix},$$

where $\gamma_{j,k}$ denotes the element of $H^2(S^3 - L)$ Lefschetz dual to a path from L_j to L_k.

Properties 1 and 2 are a direct consequence of the statement of Theorem 1 (which is Theorem 4 in [**18**]). From the proof of Theorem 1 for a polygonal link, it follows that $\alpha_{N+1},\dots,\alpha_J$ can be taken to be meridians and the image of the 2-cell corresponding to the relator $[w_i,\alpha_i]a_i$ can be assumed to be homologous to a torus that separates L_i from the other components of the link which implies Property 3.

Lemma 4 is a special case of Proposition 2.4 in [**14**].

LEMMA 4 (MAY). *Assume a collection of subspaces, $\{Y_i\}_{i=1}^p$, of a space Y, and a collection of cohomology classes $h_i \in H^1(Y_i : R)$, $i = 1, \ldots, p$, have been given. Assume further that with coefficient ring R, each of the following products in the specified system consists only of the zero element.*

1. *$\langle h_i, \ldots, h_j\rangle$ in $\{Y_i, Y_{i+1}, \ldots, Y_j\}$ for $1 < j - i < p - 2$.*

2. *$\langle h_1, \ldots, h_{k-1}, h, h_{k+2}, \ldots, h_p\rangle$ in $\{Y_1, \ldots, Y_{k-1}, Y_k \cap Y_{k+1}, Y_{k+2}, \ldots, Y_p\}$ for $k = 1, 2, \ldots, (p-1)$, and h any element of $H^1(Y_k \cap Y_{k+1} : R)$.*

Then the product $\langle h_1, \ldots, h_p\rangle$ in the system $\{Y_i\}_{i=1}^p$ with coefficients R is defined and contains only one element.

PROOF OF THEOREM 3. Let L be an N-link in S^3. Let α_i, $i = 1, \ldots, N$, be meridians in F_1/F_q. Choose words $w_1, \ldots, w_N$ in $\alpha_1, \ldots, \alpha_N$ representing parallels to L in F_1/F_q, choose a presentation

$$P = \left(\alpha_1, \ldots, \alpha_N, \ldots, \alpha_J : \left[w_i, \alpha_i\right]a_i, i = 1, 2, \ldots, N\right)$$

and a map f of the complex Y, determined by the presentation P into $S^3 - L$ satisfying Properties 1, 2, and 3 of Lemma 3. For any coefficient ring R, the map f^*: $H^2(S^3 - L) \to H^2(Y)$ is a monomorphism by Property 3 of Lemma 3. So information about Massey products in $S^3 - L$ can be obtained by calculating Massey products in Y. By adding 2-cells to Y it is possible to construct a collection of subcomplexes X_i, $i = 1, \ldots, N$ of a complex X, so that Massey products in a system $\{S^3 - L_{l_i}\}_{i=1}^p$ can be calculated by evaluating Massey products in the system $\{X_{l_i}\}_{i=1}^p$. Specifically: set X equal to the complex determined by the presentation

$$\left(\alpha_1, \ldots, \alpha_N, \ldots, \alpha_J : \left[w_i, \alpha_i\right]a_i = 1, \alpha_i = 1, i = 1, 2, \ldots, N\right),$$

and for $i = 1, \ldots, N$, set X_i equal to the subcomplex of X obtained by deleting the relator α_i from the presentation. Note that $X_1 \cap \cdots \cap X_N$ is the complex Y of Lemma 3. Clearly the map f: $Y \to S^3 - L$ extends to a map f: $X \to S^3$ with $f(X_i) \subseteq S^3 - L_i$ for $i = 1, \ldots, N$. From the naturality of Massey products it follows that if the product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ is defined, then the product $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$ in $\{X_{l_i}\}_{i=1}^p$ is defined and contains $f^*(\langle u_{l_1}, \ldots, u_{l_p}\rangle)$.

The relators $[w_i, \alpha_i]a_i$ give a basis for $H_2(X)$. Set r_i, $i = 1, \ldots, N$ equal to the dual basis for $H^2(X)$. If $\bar{X}$ is a subcomplex of X which contains $X_1 \cap \cdots \cap X_N$, then the inclusion of $\bar{X}$ into X induces an isomorphism on H^2. By abuse of notation, the restriction of r_i to such a complex will also be denoted by r_i.

The proof of Theorem 3 is completed by verifying the following statements.

1. $f^*(\gamma_{i,j}) = r_j - r_i$ in $H^2(X_i \cap X_j)$ where f is viewed as a map of $X_i \cap X_j$ to $S^3 - (L_i \cup L_j)$. Hence for any nonempty subset D of $\{1, 2, \ldots, N\}$ and any coefficient ring R, the map

$$f^*H^2\Big(\bigcap_{i\in D}(S^3 - L_i) : R\Big) \to H^2\Big(\bigcap_{i\in D} X_i : R\Big)$$

is a monomorphism.

Let $(l_1, \ldots, l_p)$ be a sequence of integers with $1 \leq l_i \leq N$ and $p < q$ where q is the integer occurring in the statement of Lemma 3. Then,

2. The Massey product $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$ in the system $\{X_{l_i}\}_{i=1}^p$ with coefficients $\mathbf{Z}_{\Delta(l_1,\ldots,l_p)}$ is defined and contains the single element

$$(-1)^p\big[\bar{\mu}(l_1, \ldots, l_p)r_{l_p} - \bar{\mu}(l_2, \ldots, l_p, l_1)r_{l_1}\big].$$

3. The Massey product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ with coefficients $\mathbf{Z}_{\Delta(l_1,\ldots,l_p)}$ is defined and contains only one element.

Statements 1, 2, and 3 above, together with the naturality of Massey products, imply that $\bar{\mu}(l_1, \ldots, l_p) = \bar{\mu}(l_2, \ldots, l_p, l_1)$ and that the product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$ with coefficients $Z_{\Delta(l_1,\ldots,l_p)}$ is defined and contains the single element

$$(-1)^p\bar{\mu}(l_1, \ldots, l_p)\gamma_{l_1,l_p}.$$

Statement 1 follows from Lemma 3, so the proof is completed by proving Statements 2 and 3 above. The first step in the proof of Statement 2 is to use Theorem 2 to show that $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$ in $\{X_{l_i}\}_{i=1}^p$ with coefficients $\mathbf{Z}_{\Delta(l_1,\ldots,l_p)}$ is defined and contains the element

$$(-1)^p\big[\bar{\mu}(l_1, \ldots, l_p)r_{l_p} - \bar{\mu}(l_2, \ldots, l_p, l_1)r_{l_1}\big].$$

Set $(f^*(u_{l_1}), \ldots, f^*(u_{l_p}))$ equal to the greatest common divisor of the numbers $I(i,j : \lambda)$ with $1 \leq i \leq j \leq p$, $(i,j) \neq (1,p)$ and W_λ a relator corresponding to a 2-cell in $X_{l_i} \cap \cdots \cap X_{l_j}$, where

$$I(i,j : \lambda) = \sum f^*(u_{l_i})(d_1), \ldots, f^*(u_{l_j})(d_{j-i+1})\mu(d_1, \ldots, d_{j-i+1} : W_\lambda).$$

The sum is over all sequences $(d_1, \ldots, d_{j-i+1})$ with $1 \leq d_t \leq J$. It will be shown that $\Delta(l_1, \ldots, l_p)$ divides $(f^*(u_{l_1}), \ldots, f^*(u_{l_p}))$.

First note that the summand

$$f^*(u_{l_i})(d_1), \ldots, f^*(u_{l_j})(d_{j-i+1})\mu(d_1, \ldots, d_{j-i+1} : W_\lambda)$$

is zero if W_λ is one of the relators α_i. Thus $(f^*(u_{l_1}), \ldots, f^*(u_{l_p}))$ is the greatest common divisor of the numbers $I(i,j : [w_k, \alpha_k]a_k)$. a_k is in the qth lower central series subgroup of the free group generated by $\alpha_1, \ldots, \alpha_N, \ldots, \alpha_J$. So the Magnus expansion of a_k has the form $1 +$ (terms of order $\geq q$) (see **[12]**). Thus

$$\mu\big(d_1, \ldots, d_{j-i+1} : [w_k, \alpha_k]a_k\big) = \mu\big(d_1, \ldots, d_{j-i+1} : [w_k, \alpha_k]\big)$$

since $p < q$.

So

$$\begin{aligned} I\big(i,j : [w_k, \alpha_k]a_k\big) \\ &= \sum f^*(u_{l_i})(d_1), \ldots, f^*(u_{l_p})(d_{j-i+1})\mu\big(d_1, \ldots, d_{j-i+1} : [w_k, \alpha_k]\big), \end{aligned}$$

where the sum is over all sequences $(d_1, \ldots, d_{j-i+1})$ with $1 \leq d_t \leq N$. For $1 \leq d_t \leq N$, $f^*(u_{l_t})(d)$ equals u_{l_t} evaluated on the homology class, $f_*(\alpha_d)$, which is the same as the linking number of L_{l_t} and α_d. α_d is a meridian to the dth component of L. Hence $f^*(u_{l_t})(d)$ = linking # of L_{l_t} and

$$\alpha_d = \begin{Bmatrix} 1 & \text{if } d = l_i \\ 0 & \text{otherwise} \end{Bmatrix}.$$

So $I(i, j : [w_k, \alpha_k]a_k) = \mu(l_i, \ldots, l_j : [w_k, \alpha_k])$ and $(f^*(u_{l_1}), \ldots, f^*(u_{l_p}))$ is the greatest common divisor of the numbers $\mu(l_i, \ldots, l_j : [w_k, \alpha_k])$ where $1 \leq i \leq j \leq p$, $(i,j) \neq (1,p)$ and $1 \leq k \leq N$. Recall that $\Delta(l_1, \ldots, l_p)$ is the greatest common divisor of the numbers $\mu(j_1, \ldots, j_s : w_{j_{s+1}})$ with $(j_1, \ldots, j_{s+1})$ some cyclic permutation of a proper subsequence of $(l_1, \ldots, l_p)$. To show that $\Delta(l_1, \ldots, l_p)$ divides $(f^*(u_{l_1}), \ldots, f^*(u_{l_p}))$ it suffices to show that $\mu(l_i, \ldots, l_j : [w_k, \alpha_k]) = 0 \mod \Delta(l_1, \ldots, l_p)$.

First note that if $l_i = l_{i+1} = \cdots = l_j$, then $\mu(l_i, \ldots, l_j : [w_k, \alpha_k]) = 0$ since $[w_k, \alpha_k]$ is a commutator. It will be assumed that not all of the l_t's are the same, $i \leq t \leq j$. $\mu(l_i, \ldots, l_j : [w_k, \alpha_k])$ is the sum

$$\sum \mu(l_i, \ldots, l_{i+p_1-1} : W_k)\mu(l_{i+p_1}, \ldots, l_{i+p_2-1} : \alpha_k) \\ \times \mu\left(l_{i+p_2}, \ldots, l_{i+p_3-1} : w_k^{-1}\right)\mu\left(l_{i+p_3}, \ldots, l_{i+p_4-1} : \alpha_k^{-1}\right)$$

where the sum is over all (p_1, p_2, p_3, p_4) with $0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 = j - i + 1$. If for some $s, p_s = p_{s+1}$, then the sequence $l_{i+p_s}, \ldots, l_{i+p_{s+1}-1}$ is empty and $\mu(l_{i+p_s}, \ldots, l_{i+p_{s+1}-1} : W)$ is to be replaced by 1 for all words W. If $p_2 - p_1 \geq 2$, then $\mu(l_{i+p_1}, \ldots, l_{i+p_2-1} : \alpha_k) = 0$ since the expansion of α_k is $1 + K_k$. The sum over (p_1, p_2, p_3, p_4) with $p_2 = p_1$ is the coefficient of $K_{l_i}, \ldots, K_{l_j}$ in the expansion of $w_k w_k^{-1}\alpha_k^{-1} = \alpha_k^{-1}$ but $\mu(l_i, \ldots, l_j : \alpha_k^{-1}) = 0$ from the assumption that the l_t's are not all the same. Hence $\mu(l_i, \ldots, l_j : [w_k, \alpha_k])$ is the sum

$$\sum \mu(l_i, \ldots, l_{i+t_1-1} : w_k)\mu(l_{i+t_1} : \alpha_k) \\ \times \mu\left(l_{i+t_1+1}, \ldots, l_{i+t_2-1} : w_k^{-1}\right)\mu\left(l_{i+t_2}, \ldots, l_{i+t_3-1} : \alpha_k^{-1}\right).$$

The sum over all (t_1, t_2, t_3) with $0 \leq t_1 < t_2 \leq t_3, = j - i + 1$. If $l_{i+t_1} \neq k$, then $\mu(l_{i+t_1} : \alpha_k) = 0$. So it can be assumed that $l_{i+t_1} = k$.

If the sequence $(l_i, \ldots, l_{i+t_1-1})$ is nonempty, then the corresponding summand contains the factor $\mu(l_i, \ldots, l_{i+t_1-1} : w_{l_{i+t_1}})$. In this case $(l_i, \ldots, l_{i+t_1})$ is a proper subsequence of $(l_1, \ldots, l_p)$ so $\mu(l_i, \ldots, l_{i+t_1-1} : w_{l_{i+t_1}})$ equals zero mod $\Delta(l_1, \ldots, l_p)$. Hence we can assume $t_1 = 0$. If the sequence $(l_{i+1}, \ldots, l_{i+t_2-1})$ is nonempty, the corresponding summand contains the factor $\mu(l_{i+1}, \ldots, l_{i+t_2-1} : w_{l_i}^{-1})$. The definition of $\Delta(l_1, \ldots, l_p)$ implies that $\mu(j_1, \ldots, j_s : w_{l_i}) = 0 \mod \Delta(l_1, \ldots, l_p)$ for every subsequence $(j_1, \ldots, j_s)$ of $(l_{i+1}, \ldots, l_{i+t_2-1})$. Hence $\mu(j_1, \ldots, j_s : w_{l_i}^{-1}) = 0 \mod \Delta(l_1, \ldots, l_p)$ for every subsequence $(j_1, \ldots, j_s)$ of $(l_{i+1}, \ldots, l_{i+t_2-1})$. In particular, $\mu(l_{i+1}, \ldots, l_{i+t_2-1} : w_{l_i}^{-1}) = 0 \mod \Delta(l_1, \ldots, l_p)$. Hence

$$\mu(l_i, \dots, l_j : [w_k, \alpha_k]) = \mu(l_i : \alpha_k)\mu(l_{i+1}, \dots, l_j : \alpha_k^{-1}) \mod \Delta(l_1, \dots, l_p) = 0,$$

since not all of the l_t's, $i < t < j$, are the same. This completes the argument that $\Delta(l_1, \dots, l_p)$ divides $(f^*(u_{l_1}), \dots, f^*(u_{l_p}))$. Since $\Delta(l_1, \dots, l_p)$ divides $(f^*(u_{l_1}), \dots, f^*(u_{l_p}))$ it follows from Theorem 2 that the Massey product $\langle f^*(u_{l_1}), \dots, f^*(u_{l_p})\rangle$ in the system $\{X_{l_i}\}_{i=1}^p$ with coefficients $\mathbf{Z}_{\Delta(l_1, \dots, l_p)}$ is defined and contains the element

$$\sum_{k=1}^{N} (-1)^p I(1, p : [w_k, \alpha_k]a_k) r_k.$$

The arguments used to show that $\Delta(l_1, \dots, l_p)$ divides $(f^*(u_{l_1}), \dots, f^*(u_{l_p}))$ imply that

$$\begin{aligned} I(1, p : [w_k, \alpha_k]a_k) &= \mu(l_1, \dots, l_p : [w_k, \alpha_k]) \\ &= \mu(l_1, \dots, l_{p-1} : w_k)\mu(l_p : \alpha_k) \\ &\quad + \mu(l_1 : \alpha_k)\mu(l_2, \dots, l_p : w_k^{-1}) \mod \Delta(l_1, \dots, l_p). \end{aligned}$$

Equivalently

$$I(1, p : [w_k, \alpha_k]) = \left\{ \begin{array}{l} \mu(l_1, \dots, l_{p-1} : w_{l_p}) \quad \text{if } l_1 \neq k = l_p \\ -\mu(l_2, \dots, l_p : w_{l_1}) \quad \text{if } l_1 = k \neq l_p \\ \mu(l_1, \dots, l_{p-1} : w_{l_p}) - \mu(l_2, \dots, l_p\, l_1 : w_{l_1}) \quad \text{if } l_1 = k = l_p \\ 0 \quad \text{otherwise} \end{array} \right\}.$$

So

$$\begin{aligned} &\sum_{k=1}^{N} (-1)^p I(1, p : [w_k, \alpha_k]a_k) r_k \\ &\qquad = (-1)^p[\mu(l_1, \dots, l_p) r_{l_p} - \mu(l_2, \dots, l_p, l_1) r_{l_1}]. \end{aligned}$$

This completes the argument that $\langle f^*(u_{l_1}), \dots, f^*(u_{l_p})\rangle$ is defined with coefficients $\mathbf{Z}_{\Delta(l_1, \dots, l_p)}$ and contains $(-1)^p[\bar{\mu}(l_1, \dots, l_p) r_{l_p} - \bar{\mu}(l_1, \dots, l_p, l_1) r_{l_1}]$.

This same reasoning shows that 0 is an element of each of the products $\langle f^*(u_{l_i}), \dots, f^*(u_{l_j})\rangle$ in the systems $\{X_{l_i}, \dots, X_{l_j}\}$ with coefficients $\mathbf{Z}_{\Delta(l_1, \dots, l_p)}$ and $j - i < p - 2$. Similarly 0 is an element of each of the products of the form

$$\langle f^*(u_{l_i}), \dots, f^*(u_{l_{k-1}}), h, f^*(u_{l_{k+2}}), \dots, f^*(u_{l_j})\rangle$$

in the systems

$$\{X_{l_i}, \dots, X_{l_{k-1}}, X_{l_k} \cap X_{l_{k+1}}, X_{l_{k+2}}, \dots, X_{l_j}\}$$

with coefficients $\mathbf{Z}_{\Delta(l_1, \dots, l_p)}$ and $1 < i < j < p$ (including $i = 1, j = p$), any element of $H^1(X_{l_k} \cap X_{l_{k+1}})$.

Lemma 4 together with an inductive argument on the order of the product implies that 0 is the only element in each of the products above. This means that the hypotheses of Lemma 4 are satisfied for the product $\langle f^*(u_{l_1}), \dots, f^*(u_{l_p})\rangle$.

Hence $(-1)^p[\bar{\mu}(l_1, \ldots, l_p)r_{l_p} - \bar{\mu}(l_2, \ldots, l_p, l_1)r_{l_1}]$ is the only element of $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$ and the proof of Statement 2 is complete.

Statement 1, naturality of Massey products and the fact that the hypotheses of Lemma 4 are satisfied for $\langle f^*(u_{l_1}), \ldots, f^*(u_{l_p})\rangle$, imply that the hypotheses of Lemma 4 are also satisfied for the product $\langle u_{l_1}, \ldots, u_{l_p}\rangle$ in the system $\{S^3 - L_{l_i}\}_{i=1}^p$. This proves Statement 3, and completes the proof of Theorem 3.

References

1. K. T. Chen, *Commutator calculus and link invariants*, Proc. Amer. Math. Soc. **3** (1952), 44–55.

2. ______, *Isotopy invariants of links*, Ann. of Math. (2) **56** (1952), 343–353.

3. ______, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. (2) **97** (1973), 217–246.

4. P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.

5. R. Fenn and P. Taylor, *Generalized Borromean rings and higher order linking numbers* (to appear).

6. W. G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), 177–190.

7. R. M. Goresky, *Geometric cohomology and homology of stratified objects*, Ph. D. Thesis, Brown University, 1976.

8. V. K. A. M. Gugenheim and J. Peter May, *On the theory and applications of differential torsion products*, Mem. Amer. Math. Soc. No. 142 (1974).

9. M. A. Gutierrez, *The $\bar{\mu}$-invariants for groups*, Proc. Amer. Math. Soc. **55** (1976), 293–298.

10. D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

11. S. Lefschetz, *Topology*, Amer. Math. Soc. Colloq. Publ., no. 12, Amer. Math. Soc., Providence, R. I., 1930.

12. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Applied Mathematics, Vol. XIII, Interscience, New York, 1966.

13. W. S. Massey, *Higher order linking numbers*, Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, 1968), University of Chicago at Chicago Circle, Chicago, Ill., pp. 174–205.

14. J. P. May, *Matric Massey products*, J. Algebra **12** (1969), 533–568.

15. C. G. McCrory, *Cone complexes and P. L. transversality*, Trans. Amer. Math. Soc. **207** (1975), 269–291.

16. ______, *Poincaré duality in spaces with singularities* (to appear).

17. J. Milnor, *Link groups*, Ann. of Math. (2) **59** (1954), 177–195.

18. ______, *Isotopy of links. Algebraic geometry and topology*. A symposium in honor of S. Lefschetz, pp. 280–306. Princeton Univ. Press, Princeton, N. J., 1957.

19. E. J. O'Neill, *Higher order cohomology products and links*, Ph. D. Thesis, Yale University, 1976.

20. J. P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505.

21. J. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.

22. ______, *Quotients of the powers of the augmentation ideal in a group ring. Knots, groups, and 3-manifolds*, Ann. of Math. Studies, No. 84, Princeton Univ. Press, Princeton, N. J., 1975, pp. 101–128.

23. A. W. Tucker, *An abstract approach to manifolds*, Ann. of Math. (2) **34** (1933), 191–243.

24. H. Whitney, *On products in a complex*, Ann. of Math. (2) **39** (1938), 397–432.

25. S. Wylie, *Duality and intersection in general complexes*, Proc. London Math. Soc. **46** (1940), 174–198.

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