MIMO-radar Waveform Covariance Matrices for High SINR and Low Side-lobe Levels

Sajid Ahmed *Member, IEEE* and Mohamed-Slim Alouini *Fellow, IEEE*

Abstract

MIMO-radar has better parametric identifiability but compared to phased-array radar it shows loss in signal-to-noise ratio due to non-coherent processing. To exploit the benefits of both MIMO-radar and phased-array two transmit covariance matrices are found. Both of the covariance matrices yield gain in signal-to-interference-plus-noise ratio (SINR) compared to MIMO-radar and have lower side-lobe levels (SLL)'s compared to phased-array and MIMO-radar. Moreover, in contrast to recently introduced phased-MIMO scheme, where each antenna transmit different power, our proposed schemes allows same power transmission from each antenna. The SLL's of the proposed first covariance matrix are higher than the phased-MIMO scheme while the SLL's of the second proposed covariance matrix are lower than the phased-MIMO scheme. The first covariance matrix is generated using an auto-regressive process, which allow us to change the SINR and side lobe levels by changing the auto-regressive parameter, while to generate the second covariance matrix the values of sine function between 0 and π with the step size of π/n_T are used to form a positive-semidefinite Toeplitiz matrix, where n_T is the number of transmit antennas. Simulation results validate our analytical results.

Index Terms

MIMO radar, phased MIMO, colocated antennas.

I. Introduction

Recently several researchers have considered the application of multiple-input multiple-output (MIMO) techniques developed for wireless communication systems to the radar systems [1]–[3]. In MIMO communication systems, n_T antennas are deployed at the transmitter and n_R antennas at the receiver to increase the data rate and provide

The authors are with the King Abdullah University of Science and Technology, Thuwal, Makkah Province, Saudi Arabia; Email: sajid.ahmed@kaust.edu.sa,slim.alouini@kaust.edu.sa, Phone: +966 (0)2 8080 286.

multiple paths to mitigate the fading in the channel. Like MIMO communications, which revolutionized the design, development and deployment of wireless networks over the last decade, MIMO-radar offers a new paradigm for signal processing research. MIMO-radars have many advantages over their phased-array counterparts: improved spatial resolution; better parametric identifiability, and greater flexibility to achieve the desired transmit beampattern. MIMO-radar is an emerging technology that has a number of non-defence applications as well. For example, in biomedical engineering MIMO-radar (using radio-frequency and acoustic waves) can be used for cancer detection and treatment [4], [5].

MIMO-radars can be classified into two categories: widely distributed [1] and colocated [2]. In the widely distributed case the transmitting antennas are separated so that each antenna may view a different aspect of the target. This topology can increase the spatial diversity of the system. In colocated systems the transmitting antennas are spaced so that all the transmit antennas view the same aspect of the target. The colocated antenna radar cannot provide spatial diversity but can increase the spatial resolution of the system. In contrast to phased-array, MIMO-radar allow each transmitting antenna to transmit independent waveforms, which provide extra degrees-of-freedom (DOF) that can be exploited to improve system performance [6], [7]. Therefore, in MIMO radar, waveform design is the focus of research from past few years. The waveform design methods to achieve specific goals for widely distributed radars are discussed in [8] (and the references therein) while the waveform design methods for colocated-radars to achieve a desired beampattern are discussed in [3], [9]–[12].

In phased-array radars the transmitted signals are coherent between different elements of the array that yields gain in signal-to-noise ratio (SNR) but it has poor parametric identifiability problem. MIMO-radar has better parametric identifiability but compared to phased-array radar it shows loss in SNR due to non-coherent processing. To exploit the benefits of both MIMO-radar and phased-array the available antennas are configured in number of ways. The configuration in [13] uses multiple independent phased-arrays at widely separated locations. The coherent processing between the individual phased-array at the fusion center is a practical issue. The configuration in both [14] and [15] divides the given transmit antennas into K overlapping sub-arrays, where $1 \leq K \leq n_T$. Each sub-array transmits the waveform, which is orthogonal to the waveforms transmitted by the other sub-arrays. The main advantage of this scheme is that it yields lower side-lobe levels (SLL)'s at the cost of different power transmission from different antennas.

Radio-frequency amplifiers (RFA)'s have non-linear relationships between their input and output and they cannot have maximum power efficiency at all power levels. If each antenna is required to transmit at a different power level then, for maximum power efficiency, multiple different RFA's with different bias voltage levels will be required. A better solution is to have identical RFA's all working at the same maximum power level.

In this work, two covariance matrices are proposed for the transmitted waveforms. To generate the first covariance matrix the values of an auto-regressive process are used to form a positive-definite Toeplitz matrix. While to generate the second covariance matrix the values of cosine function from 0 to π with the step size of π/n_T are used to form a positive-semidefinite Toeplitz matrix. The key benefits that can be obtained using the waveforms with the proposed covariance matrices are

- Lower SLL's compared to phased-array and MIMO-radar schemes.
- Higher SINR compared to MIMO-radar and close to the phased-array and phased-MIMO schemes.
- Same power transmission from each antenna, which is not possible with the phased-MIMO scheme.
- The SLL's using first covariance matrix are higher and the SLL's using second covariance matrix are significantly lower than the phased-MIMO scheme.

The remainder of this paper is organised as follows. In the following section the problem formulation and some background are given. The proposed algorithm is developed in section IV and the beamformer for the receiver is designed in section **??**. Simulation results are given in section **??**, followed by our conclusions in section **??**.

Notation: Bold upper case letters, $\mathbf{X}$, and lower case letters, $\mathbf{x}$, respectively denote matrices and vectors. The mth column vector of a matrix $\mathbf{X}$ is denoted by $\mathbf{x}_m$. The identity matrix of dimension $N \times N$ is denoted by $\mathbf{I}_N$ and the vector of N ones is denoted by $\mathbf{1}_N$. Conjugate transposition of a matrix is denoted by $(.)^H$ and statistical expectation is denoted by $\mathrm{E}\{.\}$.

II. Preliminaries

This section provides some results from trignometric identities that are essential for the proposed algorithms. For more details please see [16]

$$\sum_{n=0}^{N-1} u^n \sin(nx) = \frac{u\sin(x) - u^N \sin(Nx) + u^{N+1}\sin((N-1)x)}{1 - 2u\cos(x) + u^2} \tag{1}$$

and

$$\sum_{n=0}^{N-1} u^n \cos(nx) = \frac{1 - u\cos(x) - u^N \cos(Nx)}{1 - 2u\cos(x) + u^2} + \frac{u^{N+1}\cos((N-1)x)}{1 - 2u\cos(x) + u^2}. \quad (2)$$

If $u = 1$ then using (1) and (2) following result can be easily derived

$$\sum_{n=0}^{n_T-1} \sin\left(\frac{n\pi}{n_T}\right) = \frac{\sin\left(\frac{\pi}{n_T}\right)}{1 - \cos\left(\frac{\pi}{n_T}\right)}, \quad (3)$$

$$\sum_{n=0}^{n_T-1} \cos\left(\frac{n\pi}{n_T}\right) = 1, \quad (4)$$

$$\sum_{n=0}^{n_T-1} \cos\left(\frac{(n-p)\pi}{n_T}\right) = \frac{2\sin\left(\frac{p\pi}{n_T} + \frac{\pi}{2n_T}\right)\sin\left(\frac{\pi}{2n_T}\right)}{1 - \cos\left(\frac{\pi}{n_T}\right)}, \quad (5)$$

$$\sum_{n=0}^{n_T-1} \cos^2\left(\frac{n\pi}{n_T}\right) = \sum_{n=0}^{N-1} \sin^2\left(\frac{n\pi}{n_T}\right) = \frac{n_T}{2}. \quad (6)$$

Similarly, from (2) by assuming $x = 0$ and $u = e^{jn\pi\sin(\theta)}$ following result can be derived

$$\sum_{n=0}^{n_T-1} e^{jn\pi\sin(\theta)} = \frac{1 - e^{jn_T\pi\sin(\theta)}}{1 - e^{j\pi\sin(\theta)}}. \quad (7)$$

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are two matrices then from the matrix algebra following results can be easily obtained [17]

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}, \quad (8)$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD}). \quad (9)$$

III. Problem Formulation and Previous Work

Consider a uniform linear array of n_T transmit and n_R receive antennas, the inter-element-spacing between any two adjecent antennas is half of a wavelength of the transmitted waveform. In the given scenario, there is a target of interest located at an angle θ_t, and L interferers located at angles θ_1 to θ_L. The reflection coefficient of the target is β_t and interferer i is β_i. For the best detection performance, the receiver should be able to maximise the received power from the target direction and minimise it from all the other directions. In addition to this, it should be able to place deep nulls in the direction of interferers. To design such receiver, if $x_m(n)$ is the baseband signal transmitted from antenna m then the received signals after matched-filter at n_R antennas in vector form can be written as

$$\mathbf{y}(n) = \beta_t \mathbf{a}_R(\theta)\mathbf{a}_T^T(\theta)\mathbf{x}(n) + \sum_{i=1}^{L} \beta_i \mathbf{a}_R(\theta_i)\mathbf{a}_T^T(\theta_i)\mathbf{x}(n) + \mathbf{v}(n), \quad (10)$$

where

$$
\begin{aligned}
\mathbf{a}_T(\theta_p) &= \begin{bmatrix} 1 & e^{j\pi\sin(\theta_p)} & \ldots & e^{j(n_T-1)\pi\sin(\theta_p)} \end{bmatrix}^T \\
\mathbf{a}_R(\theta_p) &= \begin{bmatrix} 1 & e^{j\pi\sin(\theta_p)} & \ldots & e^{j(n_R-1)\pi\sin(\theta_p)} \end{bmatrix}^T, \\
\mathbf{x}(n) &= \begin{bmatrix} x_1(n) & x_2(n) & \cdots & x_{n_T}(n) \end{bmatrix}^T \\
\text{and} \qquad & \mathbf{v}(n) = \begin{bmatrix} v_1(n) & v_2(n) & \cdots & v_{n_R}(n) \end{bmatrix}^T
\end{aligned} \tag{11}
$$

are respectivly the, transmit and receive steering vectors corresponding to the target at location θ_p vector of symbols transmitted from n_T antennas at time index n, and the vector of circularly symmetric white Gaussian noise samples each of zero mean and σ_n^2 variance. Generally, in MIMO-radar, the transmitted waveforms from all antennas are fully uncorrelated i.e., $\mathrm{E}\{x_p(n)x_q^*(n)\} = 0$, for $p \neq q$. At each receive antenna the received signal is passed through a matched-filter and the output samples are correlated with n_T transmitted waveforms. The n_T outputs after correlation are collected from each of n_R receive antenna and cascaded into a vector after which the $n_T n_R \times 1$ received signal vector can be written as

$$
\mathbf{y}_m = \beta_t \mathbf{a}_T(\theta_t) \otimes \mathbf{a}_R(\theta_t) + \sum_{i=1}^{L} \beta_i \mathbf{a}_T(\theta_i) \otimes \mathbf{a}_R(\theta_i) + \mathbf{v}_m, \tag{12}
$$

where $\mathbf{a}_R(\theta_q) = \begin{bmatrix} 1 & e^{j\pi\sin(\theta_q)} & \ldots & e^{j(n_R-1)\pi\sin(\theta_q)} \end{bmatrix}^T$ is the receive-steering-vector of the target at location θ_q and $\mathbf{v} = \begin{bmatrix} v(0) & v(1) & \cdots & v(n_R n_T - 1) \end{bmatrix}^T$ is the vector of circularly symmetric white Gaussian noise samples each of zero mean and σ_n^2 variance.

To maximise the SINR in MIMO-radar concept a beamformer is designed at the receiver. To design a beamformer weight vector, $\mathbf{b}$, define $\mathbf{s}_m(\theta_q) = \mathbf{a}_T(\theta_q) \otimes \mathbf{a}_R(\theta_q)$ a virtual steering vector corresponding to the target/ interferer at location θ_q. By multiplying the received signal in (19) with the beamformer we can write

$$
\mathbf{b}^H \mathbf{y}_m = \beta_t \mathbf{b}^H \mathbf{s}_m(\theta_t) + \sum_{i=1}^{L} \beta_i \mathbf{b}^H \mathbf{s}_m(\theta_i) + \mathbf{b}^H \mathbf{v}. \tag{13}
$$

From (13) the SINR can be defined as

$$
\text{SINR} = \frac{|\mathbf{b}^H \mathbf{s}_m(\theta_t)|^2}{\mathbf{b}^H \mathbf{R}_{in} \mathbf{b}}, \tag{14}
$$

where $\mathbf{R}_{in} \in \mathcal{C}^{n_T n_R \times n_T n_R}$ is the covariance matrix of interference plus noise samples and is defined as

$$\mathbf{R}_{in} = \underbrace{\sum_{i=1}^{L} |\beta_i|^2 \mathbf{s}_m(\theta_i)\mathbf{s}_m^H(\theta_i)}_{\text{Interferer covariance matrix}} \quad + \quad \underbrace{\sigma_n^2 \mathbf{I}_{n_R n_T}}_{\text{Noise covariance matrix}} \quad .$$

To maximise SINR with respect to $\mathbf{b}$ using Schawarz's inequality (14) can be written as

$$\begin{aligned} \text{SINR} &= \frac{|\mathbf{b}^H \mathbf{R}_{\text{in}}^{1/2} \mathbf{R}_{in}^{-1/2} \mathbf{s}_m(\theta_t)|^2}{\mathbf{b}^H \mathbf{R}_{in} \mathbf{b}} \\ &\leq \frac{\mathbf{b}^H \mathbf{R}_{in} \mathbf{b} \ \ \mathbf{s}^H(\theta_t) \mathbf{R}_{in}^{-1} \mathbf{s}_m(\theta_t)}{\mathbf{b}^H \mathbf{R}_{in} \mathbf{b}}. \\ &\leq \mathbf{s}_m^H(\theta_t) \mathbf{R}_{in}^{-1} \mathbf{s}_m(\theta_t) \end{aligned} \tag{15}$$

Therefore, the optimal value of SINR is given by

$$\text{SINR}_\circ \quad = \quad \mathbf{s}_m^H(\theta_t) \mathbf{R}_{in}^{-1} \mathbf{s}_m(\theta_t). \tag{16}$$

It can be easily proved that the beamformer weight vector that brings the optimal value of SINR can be derived as

$$\mathbf{b} = \frac{\mathbf{R}_{in}^{-1} \mathbf{s}(\theta_t)}{\mathbf{s}^H(\theta_t) \mathbf{R}_{in}^{-1} \mathbf{s}(\theta_t)}, \tag{17}$$

Finding the beamformer vector $\mathbf{b}$ requires the inversion of covariance matrix $\mathbf{R}_{in}$, which can be computed in $\mathcal{O}(n_R n_T)^3$ computations. In the presence of only noise the matrix of eigen vectors, $\mathbf{U}$, can be replaced by an identity matrix, $\mathbf{I}_{n_R n_T}$, and the maximum value of SINR becomes

$$\text{SINR}_\circ \quad = \quad \frac{n_R n_T}{\sigma_n^2} \tag{18}$$

In the following different configuration of MIMO radar are discussed to see the leverage in improving the SINR of the system.

A. *Phased-Array radar*

In MIMO-radar if the transmitted waveform $x_m(n) = x_1(n) e^{-j\pi(m-1)\sin(\theta_t)}$ then all the transmitted waveforms will be fully correlated. Such configuration of MIMO-radar is called a phased-array radar, here at each receive antenna only one matched-filter corresponding to $x_1(n)$ is required. Therefore, the received samples collected after the matched-filtering in vector form can be written as

$$\mathbf{y}_p = \beta_t n_T \mathbf{a}_R(\theta_t) + \sum_{i=1}^{L} \beta_i \mathbf{a}_T^H(\theta_t) \mathbf{a}_T(\theta_i) \mathbf{a}_R(\theta_i) + \hat{\mathbf{v}}, \tag{19}$$

For this configuration of MIMO-radar the optimal value of SINR can be easily derived as

$$\text{SINR}_\circ = \mathbf{s}_p^H(\theta_t)\hat{\mathbf{R}}_{in}^{-1}\mathbf{s}_p(\theta_t),$$

where $\mathbf{s}_p(\theta_t) = n_T\mathbf{a}_R(\theta_t)$ and $\hat{\mathbf{R}}_{in} \in \mathcal{C}^{n_R \times n_R}$. The SINR for noise only case becomes

$$\text{SINR}_\circ = \frac{n_R n_T^2}{\sigma_n^2} \tag{20}$$

B. Phased-MIMO radar

To improve the SINR performance in [14] the concept of phased-MIMO is introduced, where in contrast to MIMO radar the beamforming is applied at the transmitter as well to transmit the maximum power in the direction of interest. In this work the n_T transmit antennas are divided into K uniform sub-arrays such that all the antennas in the sub-array k transmits the waveforms coherently. Here, an antenna in sub-array k can contribute to more than one other sub-arrays. The simulation results in this work shows improvement in the SINR using the proposed schemes when compared to MIMO-radar and phased-array. However, in [14] the comparison of the proposed phased-array with the MIMO-radar and phased-array is not a fair. In this work, the MIMO-radar and phased-array transmit total power of n_T while the proposed phased-array transmits the total power of $(n_T - K + 1)^2$. This problem is fixed in [18]. According to the phased-MIMO concept, if all the sub-arrays have equal number of antennas then the sub-array k will be consist of $n_T - K + 1$ antenna. Sub-array k transmits the waveform $x_k(n)$ at time index n. For transmit beamforming antenna m in the sub-array k multiply the waveform $x_k(n)$ by a weight w_{km}. Using this model the signal reflected by the target/interferer at location θ_p can be written as

The received signal at each receive antenna is passed through the matched-filter and correlated with the K transmitted waveforms. Similar to MIMO radar case after cascading all the samples from the n_R receive antennas the received signal in vector can be written as

$$\mathbf{y}_{pm} = \underbrace{\beta_t\tilde{\mathbf{s}}_{pm}(\theta_t)}_{\text{signal-term}} + \underbrace{\sum_{i=1}^{L}\beta_i\mathbf{s}_{pm}(\theta_i) + \tilde{\mathbf{v}}}_{\text{interferers-plus-noise term}} \tag{21}$$

where

$$\mathbf{s}_{pm}(\theta_p) = \sqrt{\frac{n_T}{K}} \begin{bmatrix} \mathbf{w}_1^H \mathbf{a}_{T_1}(\theta_p) \\ \mathbf{w}_2^H \mathbf{a}_{T_2}(\theta_p) \\ \vdots \\ \mathbf{w}_K^H \mathbf{a}_{T_K}(\theta_p) \end{bmatrix} \otimes \mathbf{a}_R(\theta_p)$$

while

$$\mathbf{w}_k = \begin{bmatrix} w_{k1} & w_{k2} & \cdots & w_{k(n_T-K+1)} \end{bmatrix}^T. \tag{22}$$

and $\mathbf{a}_{Tk}(\theta_t)$ contains k to $n_T - K + k$ elements of $\mathbf{a}_T(\theta_t)$. To maximise the SINR, in (21) $\mathbf{w}_k$'s can be optimised at the transmitter. To find the optimised values of $\mathbf{w}_k$'s Schawartz inequality tells us

$$|\mathbf{w}_k^H \mathbf{a}_{Tk}(\theta_t)|^2 \leq |\mathbf{w}_k|^2 |\mathbf{a}_{Tk}(\theta_t)|^2. \tag{23}$$

Therefore, in (21) the energy in the signal term will be maximum iff $\mathbf{w}_k = \mathbf{a}_{Tk}(\theta_t)$ for $k = 1, 2, \ldots, K$. With normalised $\mathbf{w}_k$'s to transmit total power of n_T from all antennas the optimum value of SINR, similar to MIMO-radar case, can be derived as

$$\text{SINR}_\circ = \mathbf{s}_{pm}^H(\theta_t) \tilde{\mathbf{R}}_{in}^{-1} \mathbf{s}_{pm}(\theta_t).$$

where

$$\tilde{\mathbf{R}}_{in} = \sum_{i=1}^{L} |\beta_i|^2 \mathbf{s}_{pm}(\theta_i) \mathbf{s}_{pm}^H(\theta_i) + \sigma_n^2 \mathbf{I}_{n_R K}.$$

In the absence of interferers the maximum SINR becomes

$$\text{SINR}_\circ = \frac{n_R n_T (n_T - K + 1)}{\sigma_n^2} \tag{24}$$

By comapring (18) with (24), we can say that in the absence of interferers the SINR of phased MIMO is $(n_T - K + 1)$ times higher than the MIMO radar.

However, phased-MIMO scheme performs better than the MIMO-radar, it has few drawbacks;

- Although the total power constraint is met but each antenna transmits different power, which require different power amplifiers for different antennas.
- It is difficult to find the optimal value of K to maximise the optimal SINR, and therefore hit and trial method is applied to find the optimal value of K.

- Introduction of phased-array decreases the parametric identifiability compared to MIMO-radar.

Due to the above drawbacks in the following, two covariance matrices are proposed.

IV. Proposed Correlated MIMO-radar formulation

In MIMO-radar, if the transmitted waveforms are correlated and the given waveform covariance matrix is $\mathbf{R}_x$ then by correlating the each element of $\mathbf{y}(n)$ in (10) with the n_T transmitted waveforms and cascading the outputs into a vector one can write

$$\mathbf{y}_c = \beta_t \mathbf{a}_R(\theta_t) \otimes \mathbf{R}_x \mathbf{a}_T(\theta_t) + \sum_{i=1}^{L} \beta_i \mathbf{a}_R(\theta_i) \otimes \mathbf{R}_x \mathbf{a}_T(\theta_i) + \mathbf{v}_c, \tag{25}$$

For the proposed model given in (25), similar to previous cases, the maximum SINR can be derived as

$$\text{SINR}_\circ = \mathbf{s}_c^H(\theta_t) \bar{\mathbf{R}}_{in}^{-1} \mathbf{s}_c(\theta_t), \tag{26}$$

where

$$\mathbf{s}_c(\theta_p) = \mathbf{a}_R(\theta_p) \otimes \mathbf{R}_x \mathbf{a}_T(\theta_p)$$

From (25), the covariance matrix of interference and noise can be found as

$$\bar{\mathbf{R}}_{in} = \sum_{i=1}^{K} |\beta_i|^2 \mathbf{s}_c(\theta_i) \mathbf{s}_c^H(\theta_i) + \sigma_n^2 (\mathbf{I}_{n_R} \otimes \mathbf{R}_x).$$

It can be noted here that the SINR depends on $\mathbf{R}_x$ and the noise is not white rather it is colored. In what follows, two waveform covariance matrices are proposed that yields lower SLL's compared to the phased-array, MIMO-radar, and phased-MIMO schemes.

A. Covariance matrix using auto-regressive process

The first covariance matrix of the waveform is generated using the auto-regressive model and is given by

$$\mathbf{R}_{x1} = \begin{bmatrix} 1 & \gamma^1 & \cdots & \gamma^{n_T-1} \\ \gamma^1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma^1 \\ \gamma^{n_T-1} & \cdots & \gamma^1 & 1 \end{bmatrix} \tag{27}$$

Here for $\gamma = 0$, the covariance matrix, $\mathbf{R}_{1x}$ will become diagonal and the correlated-MIMO system will become conventional MIMO, on the other hand for $\gamma = 1$ it will become a conventional phased-array. It can be easily proved that all the eigenvalues of $\mathbf{R}_{1x}$ are positive and hence it is a covariance matrix (Please see apppnedix for the proof). In Fig. 2, the transmit beampattern corresponding to different values of γ is shown. Now, we will derive the optimal SINR and the received power from the target direction using conventional receiver using this proposed covariance matrix.

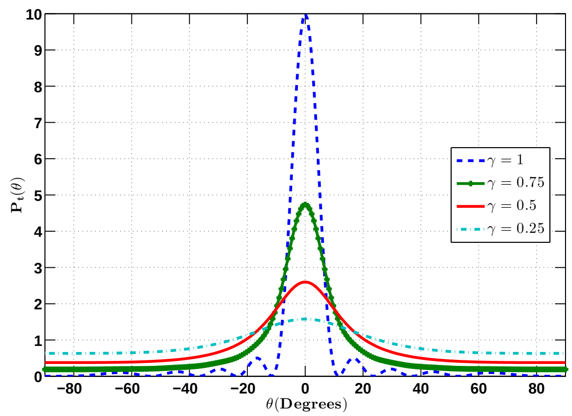

Fig. 1. Transmit beampatterns using the proposed covariance matrix $\mathbf{R}_{1x}$ for different values of γ.

1) Optimal SINR: The optimal SINR for noise only case can be derived as

$$\text{SINR}_o = \frac{\mathbf{a}_R^H(\theta_t) \otimes \mathbf{a}_T^H(\theta_t)\mathbf{R}_{1x}^H \cdot (\mathbf{I}_{n_R} \otimes \mathbf{R}_{1x})^{-1} \cdot \mathbf{a}_R(\theta_t) \otimes \mathbf{R}_{1x}\mathbf{a}_T(\theta_t)}{\sigma_n^2} \tag{28}$$

Using (8) and (9) the optimal SINR can be wriiten as

$$\begin{aligned} \text{SINR}_o &= \frac{\mathbf{a}_R^H(\theta_t) \otimes \mathbf{a}_T^H(\theta_t)\mathbf{R}_{1x}^H \cdot (\mathbf{I}_{n_R} \otimes \mathbf{R}_{1x}^{-1}) \cdot \mathbf{a}_R(\theta_t) \otimes \mathbf{R}_{1x}\mathbf{a}_T(\theta_t)}{\sigma_n^2} \\ &= \frac{\mathbf{a}_R^H(\theta_t) \otimes \mathbf{a}_T^H(\theta_t)\mathbf{R}_{1x}^H \cdot \mathbf{a}_R(\theta_t) \otimes \mathbf{a}_T(\theta_t)}{\sigma_n^2} \\ &= \frac{n_R\mathbf{a}_T^H(\theta_t)\mathbf{R}_{1x}^H\mathbf{a}_T(\theta_t)}{\sigma_n^2} \end{aligned} \tag{29}$$

Since $\mathbf{R}_{1x}$ is real and symmetric the main lobe for any value of γ will be symmetric about $\theta_t = 0$ as shown in Fig. 2 and the transmit steering vector corresponding to $\theta_t = 0$ will become

$$\mathbf{a}_T(\theta_t = 0) = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T \in \mathcal{C}^{n_T \times 1} \tag{30}$$

and using (2) for $x = 0$ yield

$$\mathbf{R}_{1x}\mathbf{a}_T(\theta_t = 0) = \begin{bmatrix} \sum_{n=0}^{0}\gamma^n + \sum_{n=0}^{n_T-1}\gamma^n - 1 \\ \sum_{n=0}^{1}\gamma^n + \sum_{n=0}^{n_T-1-1}\gamma^n - 1 \\ \vdots \\ \sum_{n=0}^{n_T-1}\gamma^n + \sum_{n=0}^{n_T-1-(n_T-1)}\gamma^n - 1 \end{bmatrix}. \tag{31}$$

Both (30) and (**??**) help us to write the optimal SINR in (29) as

$$\begin{aligned} \mathrm{SINR}_\circ &= \frac{n_R}{\sigma_n^2}\sum_{p=0}^{n_T-1}\left(\sum_{n=0}^{p}\gamma^n + \sum_{n=0}^{n_T-1-p}\gamma^n - 1\right) \\ &= \frac{n_R}{\sigma_n^2}\sum_{p=0}^{n_T-1}\left(\frac{1-\gamma^{p+1}}{1-\gamma} + \frac{1-\gamma^{n_T-p}}{1-\gamma} - 1\right) \\ &= \frac{n_R}{\sigma_n^2}\sum_{p=0}^{n_T-1}\left(\frac{1+\gamma}{1-\gamma} - \frac{\gamma^{p+1}+\gamma^{n_T-p}}{1-\gamma}\right) \\ &= \frac{n_R}{\sigma_n^2(1-\gamma)}\sum_{p=0}^{n_T-1}\left(1+\gamma-\left(\gamma^{p+1}+\gamma^{n_T-p}\right)\right) \\ &= \frac{n_R}{\sigma_n^2(1-\gamma)}\left(n_T(1+\gamma) - \gamma\left(\frac{1-\gamma^{n_T}}{1-\gamma}\right) - \gamma^{n_T}\left(\frac{1-\gamma^{-n_T}}{1-\gamma^{-1}}\right)\right), \\ &= \frac{n_R n_T}{\sigma_n^2} + \frac{2n_R\gamma}{\sigma^2(1-\gamma)}\left(n_T - \frac{1-\gamma^{n_T}}{1-\gamma}\right). \end{aligned} \tag{32}$$

Here, it can be noted that $\sum_{n=0}^{n_T-1}\gamma^n = \frac{1-\gamma^{n_T}}{1-\gamma} \leq n_T$ for $\gamma \in [0,\ 1]$ therefore SINR using $\mathbf{R}_{1x}$ is greater than MIMO-radar for any value of γ and as γ approaches 1 SINR using $\mathbf{R}_{1x}$ approaches the SINR of phased array.

2) Conventional Beamformer: For the proposed covariance matrix the conventional beamformer can be found as

$$\mathbf{b}_{c1} = \mathbf{a}_R(\theta_t = 0) \otimes \mathbf{R}_{1x}\mathbf{a}_T(\theta_t = 0). \tag{33}$$

The received power from the direction of θ at the output of a conventional receiver can be derived as

$$\begin{aligned} P_r(\theta) &= \left|\mathbf{b}_{c1}^H \cdot \mathbf{a}_R(\theta) \otimes \mathbf{R}_{1x}\mathbf{a}_T(\theta)\right|^2 \\ &= \left|\mathbf{a}_R^H(\theta_t = 0)\mathbf{a}_R(\theta) \otimes \mathbf{a}_T^H(\theta_t = 0)\mathbf{R}_{1x}^H\mathbf{R}_{1x}\mathbf{a}_T(\theta)\right|^2. \end{aligned} \tag{34}$$

To simplify (34), the first scalar term using (7) can be written as

$$\begin{aligned}
\mathbf{a}_R^H(\theta_t=0)\mathbf{a}_R(\theta) &= \sum_{n=0}^{n_R-1} e^{j\pi n\sin(\theta)}, \\
&= \frac{1-e^{j\pi n_R\sin(\theta)}}{1-e^{j\pi\sin(\theta)}}, \\
&= e^{j\frac{\pi}{2}(n_R-1)\sin(\theta)}\frac{\sin(\frac{\pi}{2}n_R\sin(\theta))}{\sin(\frac{\pi}{2}\sin(\theta))}.
\end{aligned} \tag{35}$$

While the second scalar term can be derived by first finding the individual vectors

$$\begin{aligned}
\mathbf{a}_T^H(\theta_t=0)\mathbf{R}_{1x} &= \begin{bmatrix} \sum_{n=0}^{n_T-1}\gamma^n & \sum_{n=0}^{1}\gamma^n+\sum_{n=0}^{n_T-1-1}\gamma^n-1 & \cdots & \sum_{n=0}^{n_T-1}\gamma^n+\sum_{n=0}^{n_T-1-(n_T-1)}\gamma^n-1 \end{bmatrix}, \\
\text{and}\quad \mathbf{R}_{1x}\mathbf{a}_T(\theta) &= \begin{bmatrix} \sum_{n=0}^{0}\gamma^n+\sum_{n=0}^{n_T-1}\gamma^n e^{j\pi n\sin(\theta)}-1 \\ \sum_{n=0}^{1}\gamma^{(1-n)}e^{j\pi n\sin(\theta)}+\sum_{n=0}^{n_T-2}\gamma^n e^{j\pi(n+1)\sin(\theta)}-e^{j\pi\sin(\theta)} \\ \vdots \\ \sum_{n=0}^{n_T-1}\gamma^{(n_T-1-n)}e^{j\pi n\sin(\theta)}+\sum_{n=0}^{n_T-n_T}\gamma^n e^{j\pi(n+n_T-1)\sin(\theta)}-e^{j\pi(n_T-1)\sin(\theta)} \end{bmatrix},
\end{aligned} \tag{36}$$

as

$$\mathbf{a}_T^H(\theta_t=0)\mathbf{R}_{1x}^H\mathbf{R}_{1x}\mathbf{a}_T(\theta)=\sum_{p=0}^{n_T-1}\left(\left(\sum_{n=0}^{p}\gamma^n+\sum_{n=0}^{n_T-1-p}\gamma^n-1\right)\left(\sum_{n=0}^{p}\gamma^{p-n}e^{j\pi n\sin(\theta)}+\sum_{n=0}^{n_T-1-p}\gamma^n e^{j\pi(n+p)\sin(\theta)}-e^{j\pi p\sin(\theta)}\right)\right) \tag{37}$$

Using (2), (37) can be written as

$$
\begin{aligned}
&= \sum_{p=0}^{n_T-1}\Bigg(\left(\frac{1-\gamma^{p+1}}{1-\gamma}+\frac{1-\gamma^{n_T-p}}{1-\gamma}-1\right) \\
&\quad \left(\gamma^{p}\left(\frac{1-\gamma^{-(p+1)}e^{j\pi(p+1)\sin(\theta)}}{1-\gamma^{-1}e^{j\pi\sin(\theta)}}\right)+e^{j\pi p\sin(\theta)}\left(\frac{1-\gamma^{(n_T-p)}e^{j\pi(n_T-p)\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}\right)-e^{j\pi p\sin(\theta)}\right) \\
&= \frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\Bigg((1+\gamma-\gamma^{p+1}-\gamma^{n_T-p}) \\
&\quad \left(\frac{\gamma^{p+1}-e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{e^{j\pi p\sin(\theta)}-\gamma^{n_T-p}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-e^{j\pi p\sin(\theta)}\right)\Bigg) \\
&= \frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left((1+\gamma)\left(\frac{\gamma^{p+1}-e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{e^{j\pi p\sin(\theta)}-\gamma^{n_T-p}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-e^{j\pi p\sin(\theta)}\right)\right) \\
&\quad -\frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left(\gamma^{p+1}\left(\frac{\gamma^{p+1}-e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{e^{j\pi p\sin(\theta)}-\gamma^{n_T-p}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-e^{j\pi p\sin(\theta)}\right)\right) \\
&\quad -\frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left(\gamma^{n_T-p}\left(\frac{\gamma^{p+1}-e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{e^{j\pi p\sin(\theta)}-\gamma^{n_T-p}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-e^{j\pi p\sin(\theta)}\right)\right) \\
&= \frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left((1+\gamma)\left(\frac{\gamma^{p+1}-e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{e^{j\pi p\sin(\theta)}-\gamma^{n_T-p}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-e^{j\pi p\sin(\theta)}\right)\right) \\
&\quad -\frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left(\frac{\gamma^{2(p+1)}-\gamma^{p+1}e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{\gamma^{p+1}e^{j\pi p\sin(\theta)}-\gamma^{n_T+1}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-\gamma^{p+1}e^{j\pi p\sin(\theta)}\right) \\
&\quad -\frac{1}{1-\gamma}\sum_{p=0}^{n_T-1}\left(\frac{\gamma^{n_T+1}-\gamma^{n_T-p}e^{j\pi(p+1)\sin(\theta)}}{\gamma-e^{j\pi\sin(\theta)}}+\frac{\gamma^{n_T-p}e^{j\pi p\sin(\theta)}-\gamma^{2(n_T-p)}e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}-\gamma^{n_T-p}e^{j\pi p\sin(\theta)}\right) \\
&= \frac{(\gamma^2-1)e^{j\pi\sin(\theta)}\left((1-\gamma^2e^{j\pi\sin(\theta)})+(1-e^{j\pi\sin(\theta)})\gamma^{n_T+1}e^{j\pi n_T\sin(\theta)}\right)}{(1-\gamma)(\gamma-e^{j\pi\sin(\theta)})(1-\gamma e^{j\pi\sin(\theta)})^2(1-e^{j\pi\sin(\theta)})} \\
&\quad +\frac{(\gamma^2-1)e^{j\pi\sin(\theta)}\left((e^{j\pi\sin(\theta)}-\gamma^2)e^{j\pi n_T\sin(\theta)}+(e^{j\pi\sin(\theta)}-1)\gamma^{n_T+1}\right)}{(1-\gamma)(\gamma-e^{j\pi\sin(\theta)})^2(1-\gamma e^{j\pi\sin(\theta)})(1-e^{j\pi\sin(\theta)})} \\
&\quad +\frac{\gamma}{\gamma-e^{j\pi\sin(\theta)}}\left[\frac{(1+\gamma)^2(1-\gamma^{n_T})-\gamma(1-\gamma^{2n_T})-n_T\gamma^{n_T}(1-\gamma^2)}{(1-\gamma)^2(1+\gamma)}\right] \\
&\quad -\frac{\gamma e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}\left[\frac{(1+\gamma)^2(1-\gamma^{n_T})-\gamma(1-\gamma^{2n_T})-n_T\gamma^{n_T}(1-\gamma^2)}{(1-\gamma)^2(1+\gamma)}\right]
\end{aligned}
$$

Further solution yields

$$
\begin{aligned}
&= \frac{(\gamma^2-1)e^{j\pi\sin(\theta)}\left((1-\gamma^2e^{j\pi\sin(\theta)})+(1-e^{j\pi\sin(\theta)})\gamma^{n_T+1}e^{j\pi n_T\sin(\theta)}\right)}{(1-\gamma)(\gamma-e^{j\pi\sin(\theta)})(1-\gamma e^{j\pi\sin(\theta)})^2(1-e^{j\pi\sin(\theta)})}\\
&\quad+\frac{(\gamma^2-1)e^{j\pi\sin(\theta)}\left((e^{j\pi\sin(\theta)}-\gamma^2)e^{j\pi n_T\sin(\theta)}+(e^{j\pi\sin(\theta)}-1)\gamma^{n_T+1}\right)}{(1-\gamma)(\gamma-e^{j\pi\sin(\theta)})^2(1-\gamma e^{j\pi\sin(\theta)})(1-e^{j\pi\sin(\theta)})}\\
&\quad+\left[\frac{(1+\gamma)^2(1-\gamma^{n_T})-\gamma(1-\gamma^{2n_T})-n_T\gamma^{n_T}(1-\gamma^2)}{(1-\gamma)^2(1+\gamma)}\right]\left(\frac{\gamma}{\gamma-e^{j\pi\sin(\theta)}}-\frac{\gamma e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}\right)\\
&= -(1+r)e^{j\pi\sin(\theta)}\left(\frac{(1-\gamma^2e^{j\pi\sin(\theta)})+(1-e^{j\pi\sin(\theta)})\gamma^{n_T+1}e^{j\pi n_T\sin(\theta)}}{(\gamma-e^{j\pi\sin(\theta)})(1-\gamma e^{j\pi\sin(\theta)})^2(1-e^{j\pi\sin(\theta)})}\right.\\
&\quad\left.+\frac{(e^{j\pi\sin(\theta)}-\gamma^2)e^{j\pi n_T\sin(\theta)}-(1-e^{j\pi\sin(\theta)})\gamma^{n_T+1}}{(\gamma-e^{j\pi\sin(\theta)})^2(1-\gamma e^{j\pi\sin(\theta)})(1-e^{j\pi\sin(\theta)})}\right)\\
&\quad+\left[\frac{(1+\gamma)^2(1-\gamma^{n_T})-\gamma(1-\gamma^{2n_T})-n_T\gamma^{n_T}(1-\gamma^2)}{(1-\gamma)^2(1+\gamma)}\right]\left(\frac{\gamma}{\gamma-e^{j\pi\sin(\theta)}}-\frac{\gamma e^{j\pi n_T\sin(\theta)}}{1-\gamma e^{j\pi\sin(\theta)}}\right)
\end{aligned}
$$

B. Covariance matrix using cosine function

The second covariance matrix is generated using the cosine function. To generate covariance matrix the values of cosine function from 0 to π with the step of π/n_T are used in the symmetric circulant matrix.

$$
\mathbf{R}_{x2}=\begin{bmatrix} 1 & \cos(\frac{\pi}{n_T}) & \cdots & \cos\left(\frac{(n_T-1)\pi}{n_T}\right)\\ \cos(\frac{\pi}{n_T}) & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \cos(\frac{\pi}{n_T})\\ \cos\left(\frac{(n_T-1)\pi}{n_T}\right) & \cdots & \cos(\frac{\pi}{n_T}) & 1 \end{bmatrix} \tag{38}
$$

It can be easily proved that this is a covariance matrix and has only two eigenvalues $n_T/2$ and $n_T/2$ (Please see apppnedix for the proof).

1) Optimal SINR: The optimal SINR for noise only case can be derived as

$$
\begin{aligned}
\mathrm{SINR}_o &= \frac{\mathbf{a}_R^H(\theta_t)\otimes\mathbf{a}_T^H(\theta_t)\mathbf{R}_{2x}^H\cdot(\mathbf{I}_{n_R}\otimes\mathbf{R}_{2x})^{-1}\cdot\mathbf{a}_R(\theta_t)\otimes\mathbf{R}_{2x}\mathbf{a}_T(\theta_t)}{\sigma_n^2}\\
&= \frac{n_R\otimes\mathbf{a}_T^H(\theta_t)\mathbf{R}_{2x}^H\mathbf{a}_T(\theta_t)}{\sigma_n^2}
\end{aligned} \tag{39}
$$

Since the matrix is real and symmetric the main lobe will be symmetric about $\theta_t=0$ and the transmit steering vector correspond to $\theta_t=0$ will become

$$
\mathbf{a}_T(\theta_t)=\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T\in\mathcal{C}^{n_T\times 1}.
$$

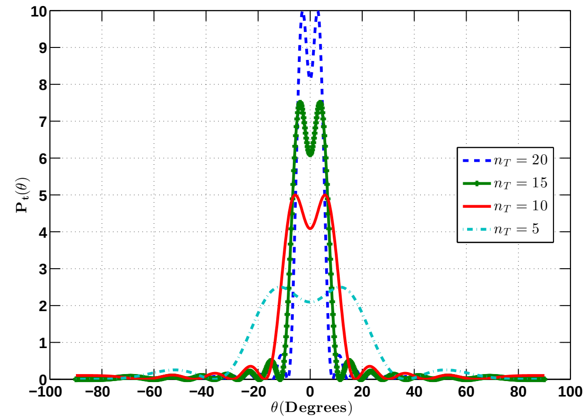

Fig. 2. Transmit beampatterns using the proposed covariance matrix $\mathbf{R}_{1x}$ for different values of γ.

To derive the optimal SINR, we can write

$$\mathbf{R}_{2x}\mathbf{a}_T(\theta_t = 0) = \begin{bmatrix} \sum_{n=0}^{n_T-1} \cos\left(\frac{n\pi}{n_T}\right) \\ \sum_{n=0}^{n_T-1} \cos\left(\frac{(n-1)\pi}{n_T}\right) \\ \vdots \\ \sum_{n=0}^{n_T-1} \cos\left(\frac{(n-(n_T-1))\pi}{n_T}\right) \end{bmatrix},$$

which help us to write (39) as

$$\mathrm{SINR}_\circ = \frac{n_R}{\sigma_n^2} \sum_{p=0}^{n_T-1} \left(\sum_{n=0}^{n_T-1} \cos\left(\frac{(n-(p-1))\pi}{n_T} \right) \right). \tag{40}$$

Using (5) in (40) we can write

$$\begin{aligned} \mathrm{SINR}_\circ &= \frac{n_R}{\sigma_n^2} \sum_{p=0}^{n_T-1} \left(\cos\left(\frac{(p-1)\pi}{n_T}\right) + \frac{\sin\left(\frac{\pi}{n_T}\right)}{1-\cos\left(\frac{\pi}{n_T}\right)} \sin\left(\frac{(p-1)\pi}{n_T}\right) \right), \\ &= \frac{n_R}{\sigma_n^2} \cos\left(\frac{\pi}{n_T}\right) \left(1 + \frac{1+\cos(\pi/n_T)}{1-\cos(\pi/n_T)}\right) \\ &= \frac{2n_R}{\sigma_n^2} \left(\frac{\cos(\pi/n_T)}{1-\cos(\pi/n_T)} \right) \end{aligned} \tag{41}$$

Interestingly, with the increase in the transmit antennas the gain in the SINR with the proposed covariance matrix increases exponentially compared to the MIMO-radar.

2) Optimal SINR: For the proposed covariance matrix the conventional beamformer can be found as

$$\mathbf{b}_{c2} = \mathbf{a}_R(\theta_t = 0) \otimes \mathbf{R}_{2x}\mathbf{a}_T(\theta_t = 0), \tag{42}$$

and the receive power from the direction of θ at the output of a conventional receiver can be found as

$$\begin{aligned} P_r(\theta) &= \left| \mathbf{b}_{c2}^H \cdot \mathbf{a}_R(\theta) \otimes \mathbf{R}_{2x} \mathbf{a}_T(\theta) \right|^2 \\ &= \left| \mathbf{a}_R^H(\theta_t = 0) \mathbf{a}_R(\theta) \otimes \mathbf{a}_T^H(\theta_t = 0) \mathbf{R}_{2x}^H \mathbf{R}_{2x} \mathbf{a}_T(\theta) \right|^2 . \end{aligned} \tag{43}$$

Using (7), we can write

$$\begin{aligned} \mathbf{a}_R^H(\theta_t = 0) \mathbf{a}_R(\theta) &= \sum_{n=0}^{n_R - 1} e^{j\pi n \sin(\theta)}, \\ &= \frac{1 - e^{j\pi n_R \sin(\theta)}}{1 - e^{j\pi \sin(\theta)}}, \\ &= e^{j\frac{\pi}{2}(n_R - 1)\sin(\theta)} \frac{\sin(\frac{\pi}{2} n_R \sin(\theta))}{\sin(\frac{\pi}{2}\sin(\theta))}. \end{aligned} \tag{44}$$

Similarly, using (1) and (2), the pth element of $\mathbf{R}_{2x}\mathbf{a}_T(\theta)$ can be derived as

$$\begin{aligned} \sum_{n=0}^{n_T-1} e^{j\pi n \sin(\theta)} \cos\left(\frac{\pi(n-p)}{n_T}\right) &= \cos\left(\frac{\pi p}{n_T}\right) \frac{\left(e^{-j\pi\sin(\theta)} - \cos(\frac{\pi}{n_T})\right)\left(1 + e^{j\pi n_T \sin(\theta)}\right)}{2\left(\cos(\pi \sin(\theta)) - \cos(\frac{\pi}{nT})\right)} \\ &\quad + \sin\left(\frac{\pi p}{n_T}\right) \frac{\sin(\frac{\pi}{n_T})\left(1 + e^{j\pi n_T \sin(\theta)}\right)}{2\left(\cos(\pi\sin(\theta)) - \cos(\frac{\pi}{n_T})\right)}, \\ &= \frac{\left(1 + e^{j\pi n_T \sin(\theta)}\right)}{2\left(\cos(\pi\sin(\theta)) - \cos(\frac{\pi}{nT})\right)} \\ &\quad \times \left[\cos\left(\frac{\pi p}{n_T}\right) e^{-j\pi \sin(\theta)} - \cos\left(\frac{\pi}{n_T}(p+1)\right)\right]. \end{aligned} \tag{45}$$

The result in (5) and (45) can be used to obtain

$$\begin{aligned}
\mathbf{a}_T^H(\theta_t=0)\mathbf{R}_{2x}^H\mathbf{R}_{2x}\mathbf{a}_T(\theta) &= \left[\begin{array}{ccc} \sum_{n=0}^{n_T-1}\cos\left(\frac{\pi n}{n_T}\right) & \cdots & \sum_{n=0}^{n_T-1}\cos\left(\frac{\pi(n-(n_T-1))}{n_T}\right)\end{array}\right]\left[\begin{array}{c} \sum_{n=0}^{n_T-1} e^{j\pi n\sin(\theta)}\cos\left(\frac{\pi n}{n_T}\right) \\ \vdots \\ \sum_{n=0}^{n_T-1} e^{j\pi n\sin(\theta)}\cos\left(\frac{\pi(n-(n_T-1))}{n_T}\right)\end{array}\right] \\
&= \sum_{p=0}^{n_T-1}\left(\frac{2\sin\left(\frac{(p+0.5)\pi}{n_T}\right)\sin\left(\frac{0.5\pi}{n_T}\right)}{1-\cos\left(\frac{\pi}{n_T}\right)}\right. \\
&\quad \left.\times\frac{\left(1+e^{j\pi n_T\sin(\theta)}\right)}{2\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\left[\cos\left(\frac{\pi p}{n_T}\right)e^{-j\pi\sin(\theta)}-\cos\left(\frac{\pi}{n_T}(p+1)\right)\right]\right) \qquad (46) \\
&= \left(\frac{2\sin\left(\frac{0.5\pi}{n_T}\right)}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)}\frac{e^{j\frac{n_T\pi\sin(\theta)}{2}}\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right. \\
&\quad \left.\times\sum_{p=0}^{n_T-1}\left[\sin\left(\frac{(p+0.5)\pi}{n_T}\right)\cos\left(\frac{\pi p}{n_T}\right)e^{-j\pi\sin(\theta)}-\sin\left(\frac{(p+0.5)\pi}{n_T}\right)\cos\left(\frac{\pi}{n_T}(p+1)\right)\right]\right) \qquad (47) \\
&= \left(\frac{2\sin\left(\frac{0.5\pi}{n_T}\right)}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)}\frac{e^{j\frac{n_T\pi\sin(\theta)}{2}}\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right. \\
&\quad \left.\times\frac{1}{2}\sum_{p=0}^{n_T-1}\left[\sin\left(\frac{(2p+0.5)\pi}{n_T}\right)+\sin\left(\frac{0.5\pi}{n_T}\right)\right]e^{-j\pi\sin(\theta)}-\left[\sin\left(\frac{(2p+1.5)\pi}{n_T}\right)-\sin\left(\frac{0.5\pi}{n_T}\right)\right]\right) \qquad (48) \\
&= \left(\frac{\sin\left(\frac{0.5\pi}{n_T}\right)}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)}\frac{e^{j\frac{n_T\pi\sin(\theta)}{2}}\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\sum_{p=0}^{n_T-1}\left(\sin\left(\frac{0.5\pi}{n_T}\right)e^{-j\pi\sin(\theta)}+\sin\left(\frac{0.5\pi}{n_T}\right)\right)\right) \qquad (49) \\
&= \left(\frac{\sin\left(\frac{0.5\pi}{n_T}\right)}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)}\frac{e^{j\frac{n_T\pi\sin(\theta)}{2}}\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}n_T\sin\left(\frac{0.5\pi}{n_T}\right)\left(e^{-j\pi\sin(\theta)}+1\right)\right) \qquad (50) \\
&= \left(2n_T\frac{e^{j\frac{(n_T-1)\pi\sin(\theta)}{2}}\sin^2\left(\frac{\pi}{2n_T}\right)\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)\cos\left(\frac{\pi\sin(\theta)}{2}\right)}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right) \\
&= \left(n_T\frac{e^{j\frac{(n_T-1)\pi\sin(\theta)}{2}}\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)\cos\left(\frac{\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right) \qquad (51)
\end{aligned}$$

Therefore, using (44) and (51), the received power in (43) can be written as

$$\begin{aligned}
P_r(\theta) &= \left(\frac{\sin\left(\frac{n_R\pi\sin(\theta)}{2}\right)}{\sin(\frac{\pi\sin(\theta)}{2})}\right)^2\left(n_T\frac{\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)\cos\left(\frac{\pi\sin(\theta)}{2}\right)}{\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right)^2 \\
&= n_T^2\left(\frac{\sin(\frac{n_R\pi\sin(\theta)}{2})\cos\left(\frac{n_T\pi\sin(\theta)}{2}\right)\cos\left(\frac{\pi\sin(\theta)}{2}\right)}{\sin(\frac{\pi\sin(\theta)}{2})\left(\cos(\pi\sin(\theta))-\cos(\frac{\pi}{nT})\right)}\right)^2 \qquad (52)
\end{aligned}$$

Here the maximum power using the conventional beamformer at $\theta = 0$ is $\frac{(n_T n_R)^2}{\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)^2}$, which in the case of

MIMO-radar and phased-array is $(n_T n_R)^2$. Therefore, the gain obtained using the proposed covariance matrix is $\left(1-\cos\left(\frac{\pi}{n_T}\right)\right)^{-2}$.

Above derivations are provided for the proposed real symmetric matrices to illuminate a taget located at $\theta_{=}0$. Both of the above proposed covariance matrices can be very easily modified to illuminate a target at location $\theta_t \neq 0$ as given by [11]

$$\tilde{\mathbf{R}}_x = \mathbf{R}_x \odot \mathbf{a}_T(\theta_t)\mathbf{a}_T^H(\theta_t), \tag{53}$$

where $\odot$ represesnts Hadamard product.

V. Simulation Results

In this section, to validate the performance of the proposed covariance matrices, some numerical examples are presented. In all of the following simulations half-wavelength inter-element spacing is used, target is located at $\theta_t = 10$ degrees, and two interferenrs are located at $\theta_1 = -10$ and $\theta_2 = +30$ degrees.

In the first simulation, to detect a target, Fig. 3 compares the normalised transmit beampattern of the proposed waveform covariance matrix with the normalised transmit beampatterns of phased-array, MIMO-radar, and phased-MIMO schemes. It can be seen in the figure that due to the coherent processing phased-array can focus the transmitted power in the direction of target and its main-lobe beam-width is minimum among all the other shown schemes. On the other hand MIMO-radar, with all independent waveforms, transmits power equally in all directions. Similarly, the first proposed covariance matrix with $\gamma = 0.5$, cannot efficiently focus the transmitted power in the direction of target, but it performs better than MIMO-radar. Proposed phased-MIMO scheme can focus the transmitted power in the direction of traget but compared to phased-array it has wider main-lobe beamwidth and poor resolution in the SLL's. Our proposed second covariance matrix, $\mathbf{R}_{2x}$, has similar main-lobe beamwidth to that of phased-MIMO but it has better resolution in the SLL's compared to phased-MIMO.

In the second simulation, the SINR of the proposed scheme is compared with the phased-array, MIMO-radar, and phased-MIMO schemes. For this simulation, the number of transmit and receive antennas is equal to 12 and for the covariance matrix of first schemes the value of $\gamma = 0.5$. Fig. 4 shows the corresponding simulation results. It can be seen in the figure that the SINR with both of the proposed schemes is much higher than the MIMO-radar as it was expected from the analytical results. The SINR with $\mathbf{R}_{2x}$ is close to phased-MIMO scheme. Similarly,

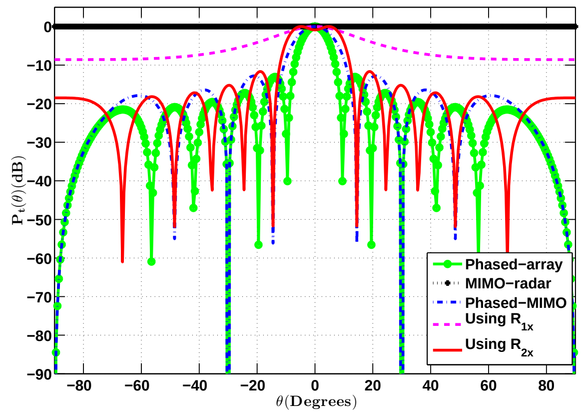

Fig. 3. Transmit beampattern of phased-array, MIMO-radar, phased-MIMO, and proposed schemes. Here, $n_T = n_R = 12$, for $\mathbf{R}_{1x}$ the value of $\gamma = 0.5$, and for phased-MIMO scheme $K = 6$.

Fig. 5 compares the SINR of proposed schemes with the phased-array, MIMO-radar and phased-MIMO schemes. Here, the number of transmit antennas are 20 and the number of receive antennas are 10. By comparing Fig. 4 and Fig. 5, it can be noted that as the number of transmit antennas is increasing, the difference between the SINR of first proposed scheme and phased-MIMO is increasing while the difference between the SINR of proposed second scheme and phased-MIMO is decreasing this is inline with our analytical results. On the other hand as the number of transmit antennas is increasing, the difference between the SINR of first and second proposed scheme compared to MIMO-radar is increasing.

The difference between the performance of phased array and with $\mathbf{R}_{2x}$ is 4.10 for $n_T = 10$ and for $n_T = 20$ it is only 3.17

In the third simulation, the received power with the conventional and MVDR receiver designed to detect the target at location $\theta_t = 10$ using phased-array, MIMO-radar, phased-MIMO, and proposed schemes is compared. For this simulation, the number of transmit and receive antennas is 12 and the value of $\gamma = 0.5$ (for the covariance matrix of first schemes). Fig. 6 shows the normalised received power from the direction of θ using all schemes with the conventional receiver. It can be seen in the figure that with the first proposed scheme the SLL's are lower compared to the MIMO-radar and phased-array but are higher compared to the phased-MIMO scheme. On the other hand with the proposed second scheme the SLL's are much lower even than the phased-MIMO scheme. The other thing

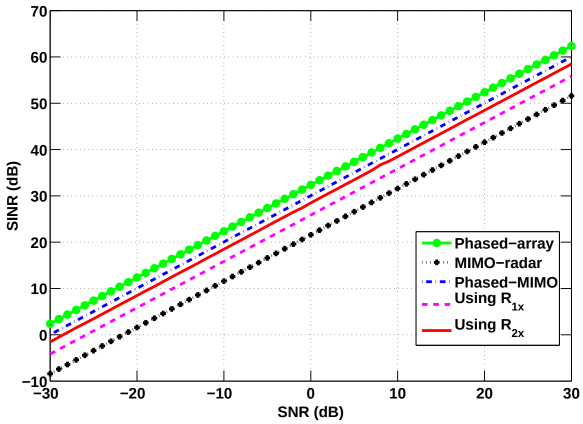

Fig. 4. The comparison of the SINR using the proposed schemes with the MIMO-radar, phased-array and phased-MIMO schemes. Here, for $\mathbf{R}_{1x}$ the value of $\gamma = 0.5$ and for phased-MIMO scheme $K = 6$, and $n_T = n_R = 12$.

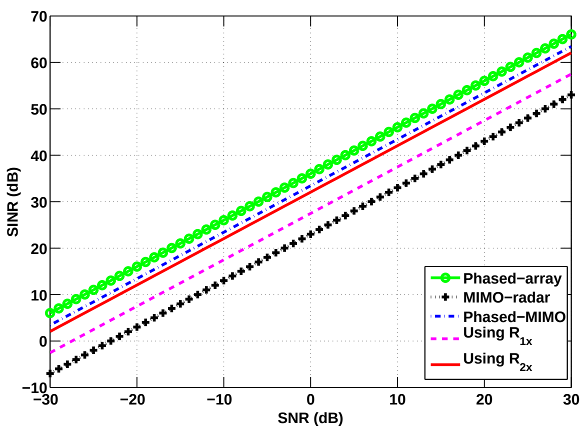

Fig. 5. The comparison of the SINR using the proposed schemes with the MIMO-radar, phased-array and phased-MIMO schemes. Here, for $\mathbf{R}_{1x}$ the value of $\gamma = 0.5$, for phased-MIMO scheme $K = 10$ and $n_T = 20$ and $n_R = 10$.

that can be noted with the second scheme is the better resolution in the SLL's compared to all the other shown schemes. Similarly, Fig. 6 shows the receive beampattern using the MVDR receiver. To design MVDR receiver, we assumed that there are two interferers located at angles $\theta = -10$ and $+30$ degrees. It can be seen in this figure that both of the proposed schemes have similar interferer supression capabilities to that of other schemes. Similar to conventional receiver, the first proposed scheme has lower SLL's compared to the MIMO-radar and phased-array but higher compared to the phased-MIMO scheme. While using the second proposed scheme SLL's are lowest and has better resolution compared to all the other shown schemes.

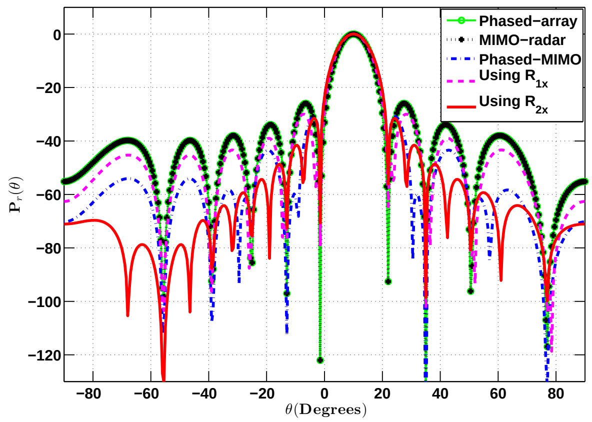

Fig. 6. Receive beampatterns using conventional beamformers of phased-array, MIMO-radar, phased-MIMO and proposed schemes. Here, for $\mathbf{R}_{1x}$, the value of $\gamma = 0.5$ and for phased-MIMO scheme $K = 5$.

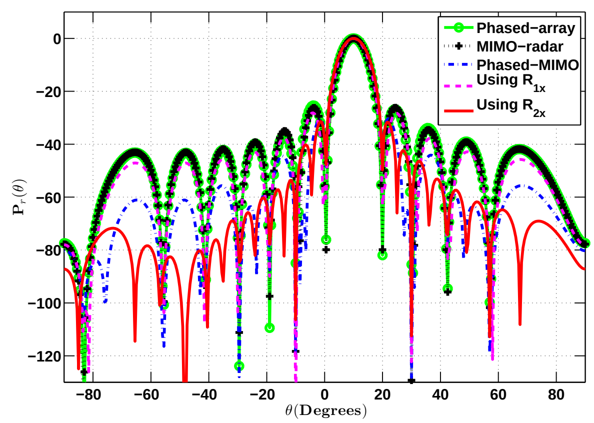

Fig. 7. Receive beampatterns using MVDR beamformers of phased-array, MIMO-radar, phased-MIMO and proposed schemes. Here, for $\mathbf{R}_{1x}$, the value of $\gamma = 0.5$ and for phased-MIMO scheme $K = 5$. The number of transmit and receive antennas is 12.

APPENDIX

A. *Eigenvalues of auto-regressive covariance matrix*

Eigenvalues of $\mathbf{R}_{x1}$ can be found by solving the following equation for λ

$$\det\left(\begin{bmatrix} 1 & \gamma^1 & \cdots & \gamma^{n_T-1} \\ \gamma^1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma^1 \\ \gamma^{n_T-1} & \cdots & \gamma^1 & 1 \end{bmatrix} - \lambda\mathbf{I}\right) = 0. \tag{54}$$

If $n_T = 2$ then (54) becomes

$$(1-\lambda)^2 = \gamma^2,$$

which yields the following two eigenvalues

$$\lambda_1 = 1 + \gamma,$$

$$\lambda_2 = 1 - \gamma.$$

$$\tag{55}$$

Since $\gamma \in [0\ ,\ 1]$ both the eigenvalues are positive, where $[a, b]$ represent a closed set.

If $n_T = 3$ then the following eigenvalues can be derived from (54) as

$$\begin{aligned} \lambda_1 &= 1 - \gamma^2, \\ \lambda_2 &= \frac{1}{2}\left(2 + \gamma^2 + \sqrt{(4+\gamma^2)^2 - 4^2}\right), \\ \lambda_3 &= \frac{1}{2}\left(2 + \gamma^2 - \sqrt{(4+\gamma^2)^2 - 4^2}\right). \end{aligned} \tag{56}$$

$$\tag{57}$$

It can be easily noted that λ_1 and λ_2 are positive. To prove λ_3 is also positive, following theorem is used.

Theorem 1: If A and B are two real positive numbers then if $A^2 > B^2$ then $A > B$.

It can be seen in (56) that $(2+\gamma^2)^2 - ((4+\gamma^2)^2 - 4^2) = 4 - 4\gamma^2 \geq 0$. Therefore, using Theorem 1, we can say that $(2+\gamma^2) \geq (\sqrt{(4+\gamma^2)^2 - 4^2})$ and $\lambda_3 \geq 0$. Therefore, all the eigenvalues are positive and $\mathbf{R}_{x1}$ for $n_T = 3$ is a covariance matrix.

Similarly, the eigenvalues for $n_T = 4$ can be derived from (54) as

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left(2-\gamma-\gamma^3-\gamma(\gamma-1)\sqrt{(\gamma+1)^2+4}\right), \\
\lambda_2 &= \frac{1}{2}\left(2-\gamma-\gamma^3+\gamma(\gamma-1)\sqrt{(\gamma+1)^2+4}\right), \\
\lambda_3 &= \frac{1}{2}\left(2+\gamma+\gamma^3-\gamma(\gamma+1)\sqrt{(\gamma-1)^2+4}\right), \\
\lambda_4 &= \frac{1}{2}\left(2+\gamma+\gamma^3+\gamma(\gamma+1)\sqrt{(\gamma-1)^2+4}\right),
\end{aligned}$$

Here, again

$$\begin{aligned}
(2-\gamma-\gamma^3)^2 - \left[\gamma^2(\gamma-1)^2((\gamma+1)^2+4)\right] &= 4(1-\gamma)^2(1+\gamma) \geq 0 \\
\text{and}\quad (2+\gamma+\gamma^3)^2 - \left[\gamma^2(\gamma+1)^2(\gamma^2-2\gamma+5)\right] &= 4(1-\gamma)(1+\gamma)^2 \geq 0,
\end{aligned}$$

which shows that $(2-\gamma-\gamma^3) \geq \gamma(\gamma-1)\sqrt{(\gamma+1)^2+4}$ and $(2+\gamma+\gamma^3) \geq \gamma(\gamma+1)\sqrt{(\gamma-1)^2+4}$. Therefore, all the eigenvalues are greater than or equal to zero. Similary, it can be easily proved that $\mathbf{R}_{1x}$ is positive semidefinite for all values of n_T.

B. Eigenvalues of cosine covariance matrix

Similar to previous case the eigenvalues of $\mathbf{R}_{x2}$ can be found by solving the following equation for λ

$$\det\left(\begin{bmatrix} 1 & \cos(\frac{\pi}{n_T}) & \cdots & \cos\left(\frac{(n_T-1)\pi}{n_T}\right) \\ \cos(\frac{\pi}{n_T}) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cos(\frac{\pi}{n_T}) \\ \cos\left(\frac{(n_T-1)\pi}{n_T}\right) & \cdots & \cos(\frac{\pi}{n_T}) & 1 \end{bmatrix} - \lambda\mathbf{I}\right) = 0.$$

Finding the determinant of above equation, we get

$$\lambda^{n_T-2}\left(\lambda^2 - n_T\lambda + \sum_{i=1}^{n_T-1}(n_T-i)\cos^2\left(\frac{(i-1)\pi}{n_T}\right)\right) = 0,$$

$$\lambda^{n_T-2}\left(\lambda^2 - n_T\lambda + \frac{n_T^2}{4}\right) = 0. \quad (58)$$

Equation (58) shows that $\mathbf{R}_{x2}$ has $(n_T - 2)$ zero eigenvalues. The remaining two eigenvalues can be found by finding the roots of inner quadratic equation, which yield

$$\lambda_1 = \frac{n_T}{2},$$
$$\text{and} \quad \lambda_2 = \frac{n_T}{2}. \tag{59}$$

By examining (58) and (59), it can be said that all the eigenvalues of $\mathbf{R}_{x2}$ are greater than or equal to zero, therefore it is a positive semidefinite matrix and hence can be a covariance matrix of the waveforms.

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