Min-max Nonlinear Model Predictive Control with Guaranteed Input-to-State Stability

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***Abstract*— In this paper we consider discrete-time nonlinear systems that are affected, possibly simultaneously, by parametric uncertainties and disturbance inputs. The min-max model predictive control (MPC) methodology is employed to obtain a controller that robustly steers the state of the system towards a desired equilibrium. The aim is to provide a priori sufficient conditions for robust stability of the resulting closed-loop system via the input-to-state stability framework. First, we show that only input-to-state practical stability can be ensured in general for perturbed nonlinear systems in closed-loop with min-max MPC schemes and we provide explicit bounds on the evolution of the closed-loop system state. Then, we derive new sufficient conditions that guarantee input-to-state stability of the min-max MPC closed-loop system, via a dual-mode approach.**

***Keywords*— Min-max, Nonlinear model predictive control, Input-to-state stability, Input-to-state practical stability.**

I. INTRODUCTION

The problem of robustly regulating discrete-time linear and nonlinear systems affected, possibly simultaneously, by parametric uncertainties and disturbance inputs towards a desired equilibrium has attracted the interest of many researchers. The reason is that this situation is often encountered in practical control applications. In case hard constraints are imposed on state and input variables, the robust model predictive control (MPC) methodology provides a reliable solution for addressing this control problem. The research related to robust MPC is focused on solving efficiently the corresponding optimization problems on one hand, and guaranteeing robust stability of the controlled system, on the other. In this paper we are interested in the latter problem. For a complete overview, also regarding computational aspects, see the survey [1] and the references therein.

There are several ways for designing robust MPC controllers for perturbed nonlinear systems. One way is to rely on the inherent robustness properties of nominally stabilizing nonlinear MPC algorithms, e.g. as it was done in [2]–[5]. Another approach is to incorporate knowledge about the disturbances in the MPC problem formulation via open-loop worst case scenarios. This includes MPC algorithms based on tightened constraints, e.g. as the one of [6], and MPC algorithms based on open-loop min-max optimization problems, e.g. see the survey [1]. To incorporate feedback to the disturbance inputs, the closed-loop or feedback min-max MPC problem set-up was introduced in [7] and further developed in [8]–[10]. The open-loop approach is computationally somewhat easier than the feedback approach, but the set of feasible states corresponding to the feedback min-max MPC optimization problem is usually much larger. Sufficient conditions for asymptotic stability of nonlinear systems in closed-loop with feedback min-max MPC controllers were presented in [8] under the assumption that the (additive) disturbance input converges to zero as time tends to infinity. These results were extended in [10] to the case when persistent (additive) disturbance inputs affect the system.

In this paper we employ the input-to-state stability (ISS) framework [11]–[13] to derive new sufficient conditions for robust asymptotic stability of nonlinear min-max MPC. Firstly, we show that in general, only input-to-state practical stability (ISpS) [14]–[16] can be a priori ensured for min-max nonlinear MPC. This is because the min-max MPC controller takes into account the effect of a non-zero disturbance input, even if the disturbance input vanishes in reality. In other words, ISpS does not imply asymptotic stability for zero disturbance inputs, as it is the case for ISS. However, we derive explicit bounds on the evolution of the min-max MPC closed-loop system state.

Still, in the case when the disturbance input converges to zero, it is desirable that *asymptotic stability* is recovered for the controlled system, i.e. that the state also converges to zero, which is guaranteed by the ISS property, but not by the ISpS property, as explained above. One of the main results of this paper is to provide novel a priori sufficient conditions for ISS of min-max nonlinear MPC. ISS is achieved via a dual-mode approach and using a new technique based on $\mathcal{KL}$-estimates of stability, e.g. see [17]. This result is important because it unifies the properties of [10] and [8], i.e. it guarantees both ISpS in the presence of persistent disturbances and robust asymptotic stability in the presence of asymptotically decaying disturbances (without assuming a priori that this property holds for the disturbances).

The paper is organized as follows. After introducing the notation in Section II, the ISS framework is presented in Section III. The min-max MPC problem set-up is briefly described in Section IV. The ISpS results for min-max nonlinear MPC are presented in Section V and the sufficient

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conditions for ISS of dual-mode min-max nonlinear MPC are given in Section VI. Conclusions are summarized in Section VII.

II. Notation and basic notions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ to denote the set $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ for some $c_1 \in \mathbb{Z}_+$. For any set $\mathbb{S} \subseteq \mathbb{R}^n$, $\mathbb{S}^N$ denotes the N-times Cartesian product $\mathbb{S}\times\mathbb{S}\times\ldots\times\mathbb{S}$, for some $N \in \mathbb{Z}_{\geq 1}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\operatorname{int}(\mathcal{S})$ its interior. We use $\|\cdot\|$ to denote an arbitrary p-norm. For a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ with $z_j \in \mathbb{R}^q$, let $\|\{z_j\}_{j\in\mathbb{Z}_+}\| \triangleq \sup\{\|z_j\| \mid j \in \mathbb{Z}_+\}$. Let $z_{[k]} \in \{\mathbb{R}^q\}^{k+1}$ denote the truncation of a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k],j} = z_j$, $j \leq k$. A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set.

III. Input-to-state stability

In this section we present the ISS framework [11]–[13] for discrete-time autonomous nonlinear systems, which will be employed in this paper to study the behavior of perturbed nonlinear systems in closed-loop with min-max MPC controllers.

Consider the discrete-time autonomous perturbed nonlinear system described by

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, unknown *parametric uncertainties* and other *disturbance inputs* (possibly additive) at discrete-time k and, $G : \mathbb{R}^n \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function. We assume that $\mathbb{W}$ is a known compact set and $\mathbb{V}$ is a known C-set.

Definition III.1 A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a *robust positive invariant (RPI) set* for system (1) if for all $x \in \mathcal{P}$ it holds that $G(x, w, v) \in \mathcal{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$.

Definition III.2 A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class* $\mathcal{K}_\infty$ if $\varphi(\cdot) \in \mathcal{K}$ and it is radially unbounded (i.e. $\varphi(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Next, we define the notions of input-to-state practical stability (ISpS) [14]–[16] and input-to-state stability (ISS) [11]–[13] for the discrete-time nonlinear system (1).

Definition III.3 The system (1) is said to be *ISpS* for initial conditions in $\mathbb{X} \subseteq \mathbb{R}^n$ if there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$, a $\mathcal{K}$-function $\gamma(\cdot)$ and a non-negative number d such that, for each $x_0 \in \mathbb{X}$, all $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2}$$

If the origin is an equilibrium in (1) for zero disturbance input v (i.e $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$), $\mathbb{X}$ contains the origin in its interior and inequality (2) is satisfied for $d = 0$, the system (1) is said to be *ISS* for initial conditions in $\mathbb{X}$.

In what follows we state a *discrete-time* version of the *continuous-time* ISpS sufficient conditions of Proposition 2.1 of [16]. This result will be used throughout the paper to prove ISpS and ISS for the particular case of min-max nonlinear MPC.

Theorem III.4 *Let d_1, d_2 be non-negative numbers, let a, b, c, λ be positive numbers with $c \leq b$ and let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ and $\sigma(\cdot) \in \mathcal{K}$. Furthermore, let $\mathbb{X}$ be a RPI set for system (1) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1, \tag{3a}$$

$$V(G(x, w, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2, \tag{3b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Then it holds that:

(i) *The system (1) is ISpS for initial conditions in $\mathbb{X}$ and the ISpS property of Definition III.2 holds for*

$$\beta(s, k) \triangleq \alpha_1^{-1}(4\rho^k\alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \quad d \triangleq \alpha_1^{-1}(4\xi), \tag{4}$$

where $\xi := d_1 + \frac{d_2}{1-\rho}$ and $\rho := \frac{c}{b} \in [0, 1)$.

(ii) *If $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$, $0 \in \operatorname{int}(\mathbb{X})$ and the inequalities (3) hold for $d_1 = d_2 = 0$, the system (1) is ISS for initial conditions in $\mathbb{X}$ and the ISS property of Definition III.2 (i.e. when $d = 0$) holds for*

$$\beta(s, k) \triangleq \alpha_1^{-1}(2\rho^k\alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{5}$$

where $\rho := \frac{c}{b} \in [0, 1)$.

Proof: (i) From $V(x) \leq \alpha_2(\|x\|) + d_1$ for all $x \in \mathbb{X}$, we have that for any $x \in \mathbb{X} \setminus \{0\}$ it holds:

$$V(x) - \alpha_3(\|x\|) \leq V(x) - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}(V(x) - d_1) = \rho V(x) + (1-\rho)d_1,$$

where $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$. In fact, the above inequality holds for all $x \in \mathbb{X}$, since $V(0) - \alpha_3(0) = V(0) = \rho V(0) + (1-\rho)V(0) \leq \rho V(0) + (1-\rho)d_1$. Then, inequality (3b) becomes

$$V(G(x, w, v)) \leq \rho V(x) + \sigma(\|v\|) + (1-\rho)d_1 + d_2, \tag{6}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Due to robust positive invariance of $\mathbb{X}$, inequality (6) yields repetitively:

$$V(x_{k+1}) \leq \rho^{k+1}V(x_0) + \sum_{i=0}^{k} \rho^i(\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2),$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Then, taking (3a) into account and using the property $\sigma(\|v_i\|) \leq \sigma(\|v_{[k]}\|)$ for all $i \leq k$ and the identity $\sum_{i=0}^{k} \rho^i = \frac{1-\rho^{k+1}}{1-\rho}$, the following inequalities hold:

$$\begin{aligned}
&V(x_{k+1}) \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1}d_1 \\
&\quad + \sum_{i=0}^{k} \rho^i(\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2) \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1}d_1 \\
&\quad + (\sigma(\|v_{[k]}\|) + (1-\rho)d_1 + d_2)\sum_{i=0}^{k} \rho^i \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1-\rho^{k+1}}{1-\rho}\sigma(\|v_{[k]}\|) \\
&\quad + d_1 + \frac{1-\rho^{k+1}}{1-\rho}d_2 \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1}{1-\rho}\sigma(\|v_{[k]}\|) + d_1 + \frac{1}{1-\rho}d_2,
\end{aligned}$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Let $\xi \triangleq d_1 + \frac{d_2}{1-\rho}$. Taking (3a) into account and letting $\alpha_1^{-1}(\cdot)$ denote the inverse of $\alpha_1(\cdot)$, we obtain:

$$\begin{aligned}
\|x_{k+1}\| &\leq \alpha_1^{-1}(V(x_{k+1})) && (7)\\
&\leq \alpha_1^{-1}\left(\rho^{k+1}\alpha_2(\|x_0\|) + \xi + \frac{\sigma(\|v_{[k]}\|)}{1-\rho}\right). && (8)
\end{aligned}$$

Applying the following inequality,

$$\alpha_1^{-1}(z+y) \leq \alpha_1^{-1}(2\max(z,y)) \leq \alpha_1^{-1}(2z) + \alpha_1^{-1}(2y), \quad (9)$$

it is easy to see that

$$\begin{aligned}
\|x_{k+1}\| &\leq \alpha_1^{-1}(4\rho^{k+1}\alpha_2(\|x_0\|)) \\
&\quad + \alpha_1^{-1}(2\frac{\sigma(\|v_{[k]}\|)}{1-\rho}) + \alpha_1^{-1}(4\xi),
\end{aligned}$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$.

We distinguish between two cases: $\rho \neq 0$ and $\rho = 0$. First, suppose $\rho \in (0,1)$ and let $\beta(s,k) \triangleq \alpha_1^{-1}(4\rho^k\alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(\cdot,k) \in \mathcal{K}$ due to $\alpha_2(\cdot) \in \mathcal{K}_\infty$, $\alpha_1^{-1}(\cdot) \in \mathcal{K}_\infty$ and $\rho \in (0,1)$. For a fixed s, it follows that $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$, due to $\rho \in (0,1)$ and $\alpha_1^{-1}(\cdot) \in \mathcal{K}_\infty$. Thus, it follows that $\beta(\cdot,\cdot) \in \mathcal{KL}$.

Now let $\gamma(s) \triangleq \alpha_1^{-1}(\frac{2\sigma(s)}{1-\rho})$. Since $\frac{1}{1-\rho} > 0$, it follows that $\gamma(\cdot) \in \mathcal{K}$ due to $\alpha_1^{-1}(\cdot) \in \mathcal{K}_\infty$ and $\sigma(\cdot) \in \mathcal{K}$.

Finally, let $d \triangleq \alpha_1^{-1}(4\xi)$. Since $\rho \in (0,1)$ and $d_1, d_2 \geq 0$, we have that $d \geq 0$.

Otherwise, if $\rho = 0$ we have that

$$\begin{aligned}
\|x_k\| &\leq \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(2\xi) \\
&\leq \beta(\|x_0\|,k) + \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(2\xi),
\end{aligned}$$

for any $\beta(\cdot,\cdot) \in \mathcal{KL}$, $\xi \triangleq d_1 + d_2$ and $k \geq 1$.

Hence, the perturbed system (1) is ISpS in the sense of Definition III.3 for initial conditions in $\mathbb{X}$ and property (2) is satisfied with the functions given in (4).

(ii) Note that, following the proof of statement (i) it is trivial to observe that, when the sufficient conditions (3) are satisfied for $d_1 = d_2 = 0$, then ISS is achieved, since $d = \alpha_1^{-1}(4\xi) = \alpha_1^{-1}(0) = 0$. From (7) and (9) and following the reasoning as above, it is easily shown that the ISS property of Definition III.3 actually holds with the functions given in (5). ■

Definition III.5 A function $V(\cdot)$ that satisfies the hypothesis of Theorem III.4 is called an *ISpS (ISS) Lyapunov function*.

Note that the hypothesis of Theorem III.4 including part (i) does not require continuity of $G(\cdot,\cdot,\cdot)$ or $V(\cdot)$, nor that $G(0,w,0) = 0$ or $V(0) = 0$. This makes the ISpS framework suitable for analyzing stability of nonlinear systems in closed-loop with min-max MPC controllers, since the min-max MPC value function is not zero at zero in general (see Section V for details). The hypothesis of Theorem III.4 including part (ii), which deals with ISS, also does not require continuity of $G(\cdot,\cdot,\cdot)$ or $V(\cdot)$. However, it requires that $G(0,w,0) = 0$ for all $w \in \mathbb{W}$ and it implies that $V(0) = 0$. Also, it only implies continuity of both $G(\cdot,\cdot,\cdot)$ and $V(\cdot)$ at the point $x = 0$, and not on a neighborhood of $x = 0$.

Due to the use of a special type of $\mathcal{K}$-functions (which is not restrictive for min-max MPC, as shown in Section V) Theorem III.4 provides an a priori explicit bound on the evolution of the state. This is relevant because previous ISpS (ISS) results, e.g. as the ones of [10], [13], [16], establish the existence of bounds on the evolution of the state without deriving an explicit relation.

IV. MIN-MAX NONLINEAR MPC: PROBLEM SET-UP

The results presented in this paper can be applied to both open-loop and feedback min-max MPC strategies. Although the stability results are proven only for feedback min-max nonlinear MPC set-ups, it is possible to prove, via a similar reasoning and using the *same* hypotheses, that all the results developed in this paper also hold for open-loop min-max MPC schemes.

Consider the discrete-time non-autonomous perturbed nonlinear system:

$$x_{k+1} = g(x_k, u_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \quad (10)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, the control action, unknown *parametric uncertainties* and *disturbance inputs* at discrete-time k and, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function with $g(0,0,w,0)=0$ for all $w \in \mathbb{W}$. Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be C-sets that represent state and input constraints for system (10).

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}_+$ with $F(0)=0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ with $L(0,0)=0$ be continuous, convex and bounded functions on bounded sets.

Feedback min-max MPC obtains a sequence of feedback control laws that minimizes a worst case cost function, while assuring robust constraint handling. In this paper we employ the *dynamic programming approach* to feedback min-max nonlinear MPC proposed in [7] for linear systems and in [8] for nonlinear systems. In this approach, the feedback min-max optimal control input is obtained as follows:

$$V_i(x) \triangleq \min_{u\in\mathbb{U}}\{\max_{w\in\mathbb{W},v\in\mathbb{V}}[L(x,u)+V_{i-1}(g(x,u,w,v))] \\ \text{s.t. } g(x,u,w,v)\in\mathbb{X}_f(i-1), \forall w\in\mathbb{W}, \forall v\in\mathbb{V}\}, \quad (11)$$

where the set $\mathbb{X}_f(i)$ contains all the states $x_i \in \mathbb{X}$ which are such that (11) is feasible, $i=1,\ldots,N$, and "s.t." is a short term for "such that". The optimization problem is defined for $i=1,\ldots,N$ where N is the prediction horizon.

The boundary conditions are:

$$V_0(x) \triangleq F(x), \\ \mathbb{X}_f(0) \triangleq \mathbb{X}_T, \quad (12)$$

where $\mathbb{X}_T \subseteq \mathbb{X}$ is a desired target set that contains the origin in its interior. Taking into account the definition of the min-max problem, $\mathbb{X}_f(i)$ is now the set of all states that can be robustly controlled into the target set $\mathbb{X}_T$ in $i \in \mathbb{Z}_{\geq 1}$ steps.

The control law is applied to system (10) in a receding horizon manner. At each sampling time the problem is solved for the current state x and the value function $V_N(x)$ is obtained. The *feedback* min-max MPC control law is defined as

$$\bar{u}(x) \triangleq u_N^*, \quad (13)$$

where u_N^* is the optimizer that yields the min-max MPC value function $V(x)=V_N(x)$, which will be used in the next section as the candidate ISpS Lyapunov function to establish ISpS of the nonlinear system (10) in closed-loop with the feedback min-max MPC control (13).

V. ISpS results for min-max nonlinear MPC

In this section we present sufficient conditions for ISpS of system (10) in closed-loop with the feedback min-max MPC control (13) and we derive explicit bounds on the evolution of the closed-loop system state. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote an arbitrary nonlinear function with $h(0)=0$. Let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$ denote the safe set with respect to both state and input constraints for $h(\cdot)$.

Assumption V.1 There exist $a, b, a_1, \lambda > 0$ with $a \leq b$, non-negative numbers e_1, e_2, a function $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $h(0)=0$ and a $\mathcal{K}$-function $\sigma_1(\cdot)$ such that:

1) $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$ and $0 \in \text{int}(\mathbb{X}_T)$;
2) $\mathbb{X}_T$ is a RPI set for system (10) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$;
3) $L(x,u) \geq a\|x\|^\lambda$ for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$;
4) $a_1\|x\|^\lambda \leq F(x) \leq b\|x\|^\lambda + e_1$ for all $x \in \mathbb{X}_T$;
5) $F(g(x,h(x),w,v)) - F(x) \leq -L(x,h(x)) + \sigma_1(\|v\|) + e_2$ for all $x \in \mathbb{X}_T$, $w \in \mathbb{W}$ and $v \in \mathbb{V}$.

Note that Assumption V.1, which implies that $F(\cdot)$ is a local ISpS (ISS) Lyapunov function, can be regarded as a generalization of the usual sufficient conditions for nominal stability of MPC, which imply that $F(\cdot)$ is a local Lyapunov function, see, for example, the survey [1]. Techniques for computing a terminal cost and a function $h(\cdot)$ such that Assumption V.1 is satisfied have been recently developed in [18] for relevant subclasses of system (10).

The next result is directly obtained via Theorem III.4 by showing that the terminal cost $F(\cdot)$ is a local (i.e. for all $x \in \mathbb{X}_T$) ISpS (ISS) Lyapunov function for system (10) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$.

Proposition V.2 *Suppose that Assumption V.1 holds. Then, system* (10) *in closed-loop with* $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, *is ISpS for initial conditions in* $\mathbb{X}_T$. *Moreover, if Assumption V.1 holds with* $e_1 = e_2 = 0$, *system* (10) *in closed-loop with* $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, *is ISS for initial conditions in* $\mathbb{X}_T$.

Now we state the main result of this section.

Theorem V.3 *Suppose that* $F(\cdot)$, $L(\cdot,\cdot)$, $\mathbb{X}_T$ *and* $h(\cdot)$ *are such that Assumption V.1 holds for system* (10). *Then, the perturbed nonlinear system* (10) *in closed-loop with the feedback min-max MPC control* (13) *is ISpS for initial conditions in* $\mathbb{X}_f(N)$. *Moreover, the property* (2) *holds with the following functions:*

$$\beta(s,k) \triangleq \left(\frac{4\theta}{a}\right)^{\frac{1}{\lambda}} \tilde{\rho}^k s, \quad \gamma(s) \triangleq \left(\frac{2\delta}{a(1-\rho)}\right)^{\frac{1}{\lambda}} s, \\ d \triangleq \left(\frac{4\xi}{a}\right)^{\frac{1}{\lambda}}, \quad (14)$$

where $\theta \triangleq \max(b, \frac{\Gamma}{r^\lambda})$ *for some constants* $\Gamma, r > 0$, $\tilde{\rho} \triangleq \rho^{\frac{1}{\lambda}} \in (0,1)$, $\rho \triangleq \frac{a}{\theta} \in (0,1)$, $\delta > 0$ *can be taken arbitrarily small, and* $\xi \triangleq d_1 + \frac{d_2}{1-\rho}$, *with* $d_1 \triangleq e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2)$ *and* $d_2 \triangleq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2$.

Proof: The proof consists in showing that the min-max MPC value function $V(\cdot)$ is an ISpS Lyapunov function, i.e. it satisfies the hypothesis of Theorem III.4. First, it is known (see [8], [19]) that under Assumption V.1-1,2), the set $\mathbb{X}_f(N)$ is a RPI set for system (10) in closed-loop with the feedback min-max MPC control (13).

Now we obtain lower and upper bounding functions on the min-max MPC value function that satisfy (3a). From Assumption V.1-3) it follows that $V(x) = V_N(x) \geq L(x, \bar{u}(x)) \geq a\|x\|^\lambda$, for all $x \in \mathbb{X}_f(N)$, where $\bar{u}(x)$ is the feedback min-max MPC control law defined in (13).

Next, letting $x_0 \triangleq x \in \mathbb{X}_T$, from Assumption V.1-2),5) it follows that for any $w_{[N-1]} \in \mathbb{W}^N$ and any $v_{[N-1]} \in \mathbb{V}^N$

$$F(x_N)+\sum_{i=0}^{N-1} L(x_i, h(x_i)) \leq F(x_0)+\sum_{i=0}^{N-1} \sigma_1(\|v_i\|)+Ne_2,$$

where $x_i \triangleq g(x_{i-1}, h(x_{i-1}), w_{i-1}, v_{i-1})$ for $i = 1, \ldots, N$. Then, by optimality and Assumption V.1-4) we have that for all $x \in \mathbb{X}_T$,

$$\begin{aligned} V(x) = V_N(x) &\leq F(x) + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) \\ &\leq b\|x\|^\lambda + d_1, \end{aligned}$$

where $d_1 \triangleq e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) > 0$.

To establish a global upper bound on $V(\cdot)$ in $\mathbb{X}_f(N)$, let $r > 0$ be such that $\mathcal{B}_r \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq \mathbb{X}_T$. Due to compactness of $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$, $\mathbb{V}$ and convexity and boundedness of $F(\cdot)$, $L(\cdot, \cdot)$ there exists a number $\Gamma > 0$ such that $V(x) \leq \Gamma$ for all $x \in \mathbb{X}_f(N)$.

Letting $\theta \triangleq \max(b, \frac{\Gamma}{r^\lambda})$ we obtain

$$V(x) \leq \theta\|x\|^\lambda \leq \theta\|x\|^\lambda + d_1 \quad \text{for all} \quad x \in \mathbb{X}_f(N) \setminus \mathbb{X}_T.$$

Then, due to $\theta \geq b$ it also follows that

$$V(x) = V_N(x) \leq b\|x\|^\lambda + d_1 \leq \theta\|x\|^\lambda + d_1$$

for all $x \in \mathbb{X}_T$. Hence, $V(\cdot)$ satisfies condition (3a) for all $x \in \mathbb{X}_f(N)$ with $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq \theta s^\lambda$ and $d_1 = e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) > 0$.

Next, we show that $V(\cdot)$ satisfies condition (3b). By Assumption V.1-5) and optimality, for all $x \in \mathbb{X}_T = \mathbb{X}_f(0)$ we have that:

$$\begin{aligned} &V_1(x) - V_0(x) \\ &\leq \max_{w\in\mathbb{W}, v\in\mathbb{V}} [L(x, h(x)) + F(g(x, h(x), w, v))] - F(x) \\ &\leq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2. \end{aligned}$$

Then, we obtain via induction that:

$$\begin{aligned} V_{i+1}(x) - V_i(x) &\leq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 \\ &\forall x \in \mathbb{X}_f(i),\ \forall i \in 0, \ldots, N-1. \end{aligned} \tag{15}$$

At time $k \in \mathbb{Z}_+$, for a given state $x_k \in \mathbb{X}$ and a fixed prediction horizon N the min-max MPC control law $\bar{u}(x_k)$ is calculated and then applied to system (10). The state evolves to $x_{k+1} = g(x_k, \bar{u}(x_k), w_k, v_k) \in \mathbb{X}_f(N)$. Then, by Assumption V.1-5) and applying recursively (15) it follows that

$$\begin{aligned} &V_N(x_{k+1}) - V_N(x_k) \\ &\leq V_N(x_{k+1}) \\ &\quad - \max_{w\in\mathbb{W}, v\in\mathbb{V}} [L(x_k, \bar{u}(x_k)) + V_{N-1}(g(x_k, \bar{u}(x_k), w, v))] \\ &\leq V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) \\ &\quad - V_{N-1}(g(x_k, \bar{u}(x_k), w_k, v_k)) \\ &= V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) - V_{N-1}(x_{k+1}) \\ &\leq -L(x_k, \bar{u}(x_k)) + \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 \\ &\leq -a\|x_k\|^\lambda + \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 \\ &\leq -a\|x_k\|^\lambda + d_2, \end{aligned} \tag{16}$$

for all $x_k \in \mathbb{X}_f(N)$, $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$ and all $k \in \mathbb{Z}_+$, where $d_2 \triangleq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. Hence, the feedback min-max nonlinear MPC value function $V(\cdot)$ satisfies (3b) with $\alpha_3(s) \triangleq as^\lambda$, any $\sigma(\cdot) \in \mathcal{K}$ and $d_2 = \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. The statements then follow from Theorem III.4.

The functions $\beta(\cdot, \cdot), \gamma(\cdot)$ and the constant d defined in (14) are obtained by letting $\sigma(s) \triangleq \delta s^\lambda$ for some (any) $\delta > 0$ and substituting the functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot), \sigma(\cdot)$ and the constants d_1, d_2 obtained above in relation (4). ■

VI. Main result: ISS dual-mode min-max nonlinear MPC

As shown in the proof of Theorem V.3, ISS cannot be proven for system (10) in closed-loop with $\bar{u}(\cdot)$, *if the min-max MPC value function $V(\cdot)$ is used as the candidate ISS Lyapunov function*, which is the most common method of proving stability in MPC, e.g. see the survey [1]. This is due to the fact that by construction, $V(\cdot)$ satisfies the conditions (3) with $d_1, d_2 > 0$, even if Assumption V.1 holds for $e_1 = e_2 = 0$. In the case of persistent disturbances this is not necessarily a drawback, since ultimate boundedness in a RPI subset of $\mathbb{X}_f(N)$ can be established. However, in the case when the disturbance input vanishes after a certain time it is desirable to have an ISS closed-loop system, since then ISS implies (robust) asymptotic stability.

In this section we present sufficient conditions for ISS of system (10) in closed-loop with a dual-mode min-max MPC strategy. Therefore, a standing assumption throughout this section is that the origin is an equilibrium in (10) for zero inputs u and v. By this we mean that $g(0, 0, w, 0) = 0$ for all $w \in \mathbb{W}$. The following technical result will be employed to prove the main result for dual-mode min-max nonlinear MPC.

For any τ with $0 < \tau < a$ define

$$\begin{aligned} &\mathbb{M}_\tau \triangleq \left\{ x \in \mathbb{X}_f(N) \mid \|x\|^\lambda \leq \frac{d_2}{a - \tau} \right\} \quad \text{and} \\ &\overline{\mathbb{M}}_\tau \triangleq \mathbb{X}_f(N) \setminus \mathbb{M}_\tau, \end{aligned} \tag{17}$$

where a is the constant of Assumption V.1-3) and $d_2 = \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. Note that $0 \in \text{int}(\mathbb{M}_\tau)$, as for d_2 and a defined above it holds that $\frac{d_2}{a-\tau} > 0$ and $0 \in \mathrm{int}(\mathbb{X}_T) \subseteq \mathrm{int}(\mathbb{X}_f(N))$.

Theorem VI.1 *Suppose that $F(\cdot)$, $L(\cdot,\cdot)$, $\mathbb{X}_T$ and $h(\cdot)$ are such that Assumption V.1 holds for system* (10) *and there exists a $\tau \in (0,a)$ such that $\overline{\mathbb{M}}_\tau \neq \emptyset$. Then, for each $x_0 \in \overline{\mathbb{M}}_\tau$ and any disturbances realizations $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Let $i(x_0)$ denote the minimal one, i.e. $i(x_0) \triangleq \arg\min\{i \in \mathbb{Z}_{\geq 1} \mid x_0 \in \overline{\mathbb{M}}_\tau,\ x_i \in \mathbb{M}_\tau\}$.*

Moreover, there exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that for all $x_0 \in \overline{\mathbb{M}}_\tau$ and any disturbances realizations $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, the trajectory of the closed-loop system (10)-(13) *satisfies $\|x_k\| \leq \beta(\|x_0\|,k)$ for all $k \in \mathbb{Z}_{\leq i(x_0)}$, where $x_k \in \overline{\mathbb{M}}_\tau$ for all $k \in \mathbb{Z}_{< i(x_0)}$.*

Proof: We prove the second statement of the theorem first. As shown in the proof of Theorem V.3, the hypothesis implies that

$$a\|x\|^\lambda \leq V(x) \leq \theta\|x\|^\lambda + d_1, \quad \forall x \in \mathbb{X}_f(N).$$

Let $\tilde{r} > 0$ be such that $\mathcal{B}_{\tilde{r}} \subseteq \mathbb{M}_\tau$. For all the state trajectories that satisfy $x_k \in \overline{\mathbb{M}}_\tau$ (and thus $x_k \notin \mathbb{M}_\tau$) for all $k \in \mathbb{Z}_{<i(x_0)}$ we have that $\|x_k\| \geq \tilde{r}$ for all $k \in \mathbb{Z}_{<i(x_0)}$.

This yields:

$$V(x_k) \leq \theta\|x_k\|^\lambda + d_1\left(\frac{\|x_k\|}{\tilde{r}}\right)^\lambda \leq \left(\theta + \frac{d_1}{\tilde{r}^\lambda}\right)\|x_k\|^\lambda,$$

for all $x_k \in \overline{\mathbb{M}}_\tau$, and all $k \in \mathbb{Z}_{<i(x_0)}$. The hypothesis also implies (see (16)) that

$$V(x_{k+1}) - V(x_k) \leq -a\|x_k\|^\lambda + d_2,$$

for all $x_k \in \mathbb{X}_f(N)$, $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$ and all $k \in \mathbb{Z}_+$, and by the definition (17) it follows that

$$V(x_{k+1}) - V(x_k) \leq -\tau\|x_k\|^\lambda, \tag{18}$$

for all $x_k \in \overline{\mathbb{M}}_\tau$, $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$ and all $k \in \mathbb{Z}_{<i(x_0)}$. Then, following the steps of the proof of Theorem III.4, it is straightforward to show that the state trajectory satisfies for all $k \in \mathbb{Z}_{\leq i(x_0)}$,

$$\|x_k\| \leq \beta(\|x_0\|,k);$$
$$\beta(s,k) \triangleq \alpha_1^{-1}(\rho^k\alpha_2(s)) = \left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}}\left(\rho^{\frac{1}{\lambda}}\right)^k s, \tag{19}$$

where $\alpha_2(s) \triangleq \bar{b}s^\lambda$, $\bar{b} \triangleq \theta b + \frac{d_1}{\tilde{r}^\lambda}$, $\alpha_1(s) \triangleq as^\lambda$ and $\rho \triangleq \frac{\tau}{\bar{b}}$. Note that $\rho \in (0,1)$ due to $0 < \tau < a \leq b$.

Next, we prove that there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Let $\bar{r} > \tilde{r} > 0$ be such that $\mathbb{X}_f(N) \subseteq \mathcal{B}_{\bar{r}}$. Such an $\bar{r}$ exists due to the fact that the compactness of $\mathbb{X}$ implies that $\mathbb{X}_f(N)$ is bounded. Assume that there does not exist an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Then, for all $i \in \mathbb{Z}_+$ we have that

$$\|x_i\| \leq \beta(\|x_0\|,i) = \left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}}\|x_0\|\left(\rho^{\frac{1}{\lambda}}\right)^i \leq \left(\rho^{\frac{1}{\lambda}}\right)^i\left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}}\bar{r}.$$

Since $\rho^{\frac{1}{\lambda}} \in (0,1)$, we have that $\lim_{i\to\infty}\left(\rho^{\frac{1}{\lambda}}\right)^i = 0$. Hence, there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathcal{B}_{\tilde{r}} \subseteq \mathbb{M}_\tau$ and we reached a contradiction. ■

To state the main result, let the dual-mode feedback min-max MPC control law be defined as:

$$\bar{u}^{\mathrm{DM}}(x) \triangleq \begin{cases} \bar{u}(x) \text{ if } x \in \mathbb{X}_f(N) \setminus \mathbb{X}_T \\ h(x) \text{ if } x \in \mathbb{X}_T. \end{cases} \tag{20}$$

Theorem VI.2 *Suppose that Assumption V.1 holds with $e_1 = e_2 = 0$ for system* (10). *Furthermore, suppose there exists $\tau \in (0,a)$ such that $\mathbb{M}_\tau \subseteq \mathbb{X}_T$. Then, the perturbed nonlinear system* (10) *in closed-loop with the dual-mode feedback min-max MPC control $\bar{u}^{DM}(\cdot)$ is ISS for initial conditions in $\mathbb{X}_f(N)$.*

Proof: In order to prove ISS, we consider two situations: in Case 1 we assume that $x_0 \in \mathbb{X}_T$ and in Case 2 we assume that $x_0 \in \mathbb{X}_f(N)\setminus\mathbb{X}_T$. In Case 1, $F(\cdot)$ satisfies the hypothesis of Proposition V.2 with $e_1 = e_2 = 0$ and hence, the closed-loop system (10)-(VI) is ISS. Then, using the reasoning employed in the proof of Theorem III.4, it can be shown that there exist a $\mathcal{KL}$-function $\beta_1(s,k) \triangleq \alpha_1^{-1}(2\rho_1^k\alpha_2(s))$, with $\alpha_1(s) \triangleq a_1s^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\rho_1 \triangleq \frac{a}{b}$, and a $\mathcal{K}$-function $\gamma(\cdot)$ such that for all $x_0 \in \mathbb{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_1(\|x_0\|,k) + \gamma(\|v_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{21}$$

In Case 2, since $\mathbb{M}_\tau \subseteq \mathbb{X}_T$, by Theorem VI.1 there exists a $p \in \mathbb{Z}_{\geq 1}$ such that $x_p \in \mathbb{X}_T$. From Theorem VI.1 we also have that there exists a $\mathcal{KL}$-function $\beta_2(s,k) \triangleq \bar{\alpha}_1^{-1}(\rho_2^k\bar{\alpha}_2(s))$, with $\bar{\alpha}_1(s) \triangleq as^\lambda$, $\bar{\alpha}_2(s) \triangleq \bar{b}s^\lambda$, $\rho_2 \triangleq \frac{\tau}{\bar{b}}$ such that for all $x_0 \in \mathbb{X}_f(N) \setminus \mathbb{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_2(\|x_0\|,k), \quad \forall k \in \mathbb{Z}_{\leq p} \quad \text{and} \quad x_p \in \mathbb{X}_T.$$

Then, for all $p \in \mathbb{Z}_{\geq 1}$ and all $k \in \mathbb{Z}_{\geq p+1}$ it holds that

$$\begin{aligned}\|x_k\| &\leq \beta_1(\|x_p\|,k-p) + \gamma(\|v_{[k-p,k-1]}\|)\\ &\leq \beta_1(\beta_2(\|x_0\|,p),k-p) + \gamma(\|v_{[k-p,k-1]}\|)\\ &\leq \beta_3(\|x_0\|,k) + \gamma(\|v_{[k-1]}\|),\end{aligned}$$

where $v_{[k-p,k-1]}$ denotes the truncation between time $k-p$ and $k-1$. In the above inequalities we used

$$\begin{aligned}\beta_1(\beta_2(s,p),k-p) &= \alpha_1^{-1}\left(2\rho_1^{k-p}\alpha_2\left(\left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}}s\left(\rho_2^{\frac{1}{\lambda}}\right)^p\right)\right)\\ &\leq \left(\frac{2b\bar{b}}{a^2}\right)^{\frac{1}{\lambda}}s\left(\rho_3^{\frac{1}{\lambda}}\right)^k \triangleq \beta_3(s,k),\end{aligned}$$

and $\rho_3 \triangleq \max(\rho_1, \rho_2) \in (0,1)$. Hence, $\beta_3(\cdot,\cdot) \in \mathcal{KL}$.

Then, we have that

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geq 1},$$

for all $x_0 \in \mathbb{X}_f(N)$, $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, where $\beta(s,k) \triangleq \max(\beta_1(s,k), \beta_2(s,k), \beta_3(s,k))$.

Since $\beta_{1,2,3}(\cdot,\cdot) \in \mathcal{KL}$ implies that $\beta(\cdot,\cdot) \in \mathcal{KL}$, and we have $\gamma(\cdot) \in \mathcal{K}$, the statement then follows from Definition III.3. ■

The interpretation of the condition $\mathbb{M}_\tau \subseteq \mathbb{X}_T$ is that the min-max MPC controller steers the state of the system inside the terminal set $\mathbb{X}_T$ for all disturbances w in $\mathbb{W}$ and v in $\mathbb{V}$. Then, ISS can be achieved by switching to the local feedback control law when the state enters the terminal set.

Note that in principle it is sufficient that there exist a nonlinear function $\tilde{h}(\cdot)$ that satisfies Assumption V.1, a RPI set $\widetilde{\mathbb{X}} \subseteq \widetilde{\mathbb{X}}_{\mathbb{U}}$ for (10) in closed-loop with $u_k = \tilde{h}(x_k)$, $k \in \mathbb{Z}_+$, and a $\tau \in (0,a)$ such that $\mathbb{M}_\tau \subseteq \widetilde{\mathbb{X}}$ for the result of Theorem VI.2 to hold, where $\widetilde{\mathbb{X}}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid \tilde{h}(x) \in \mathbb{U}\}$. By this we mean that if one takes the modified dual-mode control law

$$\bar{u}^{\text{DM}}_{\text{mod}}(x) \triangleq \begin{cases} \bar{u}(x) \text{ if } x \in \mathbb{X}_f(N) \setminus \widetilde{\mathbb{X}} \\ \tilde{h}(x) \text{ if } x \in \widetilde{\mathbb{X}}, \end{cases}$$

the result of Theorem VI.2 still holds. Hence, it is not necessary to use the terminal set $\mathbb{X}_T$ employed in the min-max MPC optimization problem as the "switch set" (i.e., for example, $\widetilde{\mathbb{X}}$ defined above).

VII. Conclusions

In this paper we have revisited the robust stability problem of min-max nonlinear model predictive control. The input-to-state practical stability framework has been employed to study stability of perturbed nonlinear systems in closed-loop with min-max MPC controllers. A priori sufficient conditions for ISpS were presented and explicit bounds on the evolution of the closed-loop system state were derived. Then, new sufficient conditions under which input-to-state stability can be achieved in min-max nonlinear MPC were derived via a dual-mode approach. This result is important because it unifies the properties of [10] and [8], i.e. it guarantees both ISpS in the presence of persistent disturbances and robust asymptotic stability in the presence of decaying uncertainties, without assuming a priori that this property holds for the disturbances.

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