Minimal Critical Sets For Some Small Latin Squares

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Abstract

A general algorithm for finding a minimal critical set for any latin square is presented. By implementing this algorithm, minimal critical sets for all the latin squares of order six have been found. In addition, this algorithm is used to prove that the size of the minimal critical set for a back circulant latin square of order seven is twelve, and for order nine is twenty. These results provide further support for the conjecture that the back circulant latin square of odd order n has minimal critical set of size $(n^2-1)/4$.

Preliminaries

A *latin square* of order n is an $n \times n$ array with entries chosen from a set N, of size n such that each element of N occurs precisely once in each row and column. In this paper the set N is taken to be $\{0, 1, \ldots, n-1\}$. A latin square may be represented with a set of ordered triples $\{(i, j; k) \mid$ element k occurs in position $(i, j)\}$. Let $L_1 = \{(i_1, j_1; k_1) \mid i_1, j_1, k_1 \in N\}$ and $L_2 = \{(i_2, j_2; k_2) \mid i_2, j_2, k_2 \in N\}$ be two latin squares of order n. Then L_1 is said to be isotopic to L_2 if there exist permutations α, β and γ such that $L_2 = \{(i_1\alpha, j_1\beta; k_1\gamma) \mid (i_1, j_1; k_1) \in L_1\}$. In this case L_2 is said to be an *isotope* of L_1.

Each latin square $L = \{(i, j; k) \mid i, j, k \in N\}$ has five *conjugates* associated with it. These conjugates result from interchanging rows with columns and/or elements of L, and are:

- $L^* = \{(j, i; k) \mid (i, j; k) \in L\}$;
- ${}^{-1}L = \{(k, j; i) \mid (i, j; k) \in L\}$;
- $L^{-1} = \{(i, k; j) \mid (i, j; k) \in L\}$;
- ${}^{-1}(L^{-1}) = \{(j, k; i) \mid (i, j; k) \in L\}$; and

- $({}^{-1}L)^{-1} = \{(k, i; j) \mid (i, j; k) \in L\}$.

For every latin square L of order n, the *main class* of L consists of the set of latin squares $\{M \mid M$ is a conjugate or an isotope of $L\}$. For more details on latin squares, isotopisms and conjugates, see [6].

If a latin square L contains an $s \times s$ subarray S and if S is a latin square of order s then S is said to be a latin *subsquare* of L. A *partial latin square* P, of order n, is an $n \times n$ array where the entries in non-empty positions are chosen from a set N, in such a way that each element of N occurs at most once in each row and at most once in each column of the array. Let P be a partial latin square of order n. Then $|P|$ is said to be the *size* of the partial latin square and the set of positions $\{(i, j) \mid (i, j; k) \in P, \exists k \in N\}$ is said to determine the *shape* of P. Let P and P' be two partial latin squares of the same order, with the same size and shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position of P. A *latin interchange* I is a partial latin square for which there exists another partial latin square I', of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. The partial latin square I' is said to be a *disjoint mate* of I. See Table 1 for an example. An *intercalate* is a latin interchange of size four, and this is the smallest possible size for a latin interchange.

0	1	2	*	*		1	2	0	*	*
1	*	0	3	*		0	*	3	1	*
*	2	3	1	*		*	1	2	3	*
*	*	*	*	*		*	*	*	*	*
*	*	*	*	*		*	*	*	*	*

Table 1: A latin interchange of order 5 and size 9 with its disjoint mate.

A partial latin square $C = \{(i,\ j;\ k) \mid i,\ j,\ k \in N\}$, of order n is said to be *uniquely completable* (UC) (or to have *unique completion*) if there is precisely one latin square L of order n that has element k in position $(i,\ j)$ for each $(i,\ j;\ k) \in C$. If $C \subset L$ is a UC set of a latin square L, of order n, with the property that any proper subset of C is contained in at least two latin squares of order n, then C is said to be a *critical set*. A *minimal critical set* is a critical set for L of smallest possible size. An example is presented in Table 2.

The *back circulant* latin square of order n, denoted by BC_n, has the integer $i + j \pmod{n}$ in position $(i,\ j)$. The critical set in Table 2 completes to BC_7.

The following lemmas form an integral part of the algorithm presented in this paper.

0	1	2	*	*	*	*
1	2	*	*	*	*	*
2	*	*	*	*	*	*
*	*	*	*	*	*	*
*	*	*	*	*	*	3
*	*	*	*	*	3	4
*	*	*	*	3	4	5

Table 2: A critical set of size 12 for a latin square of order 7.

Lemma 1: *Let C denote a critical set for $L = \{(i,\, j;\, k) \mid i, j, k \in N\}$. If α, β, γ are permutations of N and $L' = \{(i\alpha,\, j\beta;\, k\gamma) \mid (i,\, j;\, k) \in L\}$ is an isotope of L, then $C' = \{(i\alpha,\, j\beta;\, k\gamma) \mid (i,\, j;\, k) \in C\}$ is a critical set for L'. Similarly, if L^c is a conjugate of L then set C^c, the relevant conjugate of C, will be a critical set for L^c.*

Note from Lemma 1 that in order to produce a critical set for every latin square of order n, it is sufficient to find a critical set for some representative from each main class of latin squares of order n. This idea is utilized for the latin squares of order six.

Lemma 2: *A partial latin square $C \subset L$, of size s and order n, is a UC set for a latin square L if and only if C contains an element of every latin interchange that occurs in L.*

Lemma 3: *A uniquely completeable set $C \subset L$, of size s and order n, is a critical set for a latin square L if and only if for each element $(i,\, j;\, k)$ of C, there exists a latin interchange I_{ijk} such that the intersection of I_{ijk} with C is precisely $\{(i,\, j;\, k)\}$.*

Any critical set must satisfy the following property:

Lemma 4: *If $C = \{(i,\, j;\, k) \mid i,\, j,\, k \in N\}$ is a critical set for a latin square L of order n, then each of the n values of i, j, k, does not occur more than $n-1$ times in C.*

Proof. If C contains n triples with the same row i_r, then every column in row i_r contains an entry. By removing any one of these triples, say $(i_r,\, j;\, k)$, then all columns in row i_r are filled except column j which must contain element k. A similar argument applies for columns and elements.

Lemma 5: *Let L be a latin square containing the partial latin square C. If L has a latin subsquare S, such that $S \cap C$ is not UC in S, then C cannot be UC in L, and hence cannot be a critical set.*

Lemma 6: *(Donovan and Cooper [3].) The partial latin square*

$$\begin{aligned} S &= \{(i,\, j;\, i+j (\operatorname{mod} n) \mid 0 \le i \le a,\, 0 \le j \le a-i\} \cup \\ &\quad \{(i,\, j;\, i+j (\operatorname{mod} n) \mid a+2 \le i \le n-1,\, n+1+a-i \le j \le n-1\}, \end{aligned}$$

where $\frac{n-3}{2} \leq a \leq n-2$, *is a critical set for* BC_n.

An example of S, for $n = 7$ and $a = 2$, is displayed in Table 2. In the even case, it is easily shown that a minimal critical set for BC_n has size $\frac{n^2}{4}$, as $\frac{n^2}{4}$ independent intercalates exist in BC_n and each must have non-zero intersection with a critical set. Note that when $a = \frac{n-4}{2}$, S provides an example of a minimal critical set. It is still undetermined for odd n, whether the corresponding critical set of size $\frac{n^2-1}{4}$ is minimal.

Let $lcs(n)$ denote the size of the largest critical set in any latin square of order n and $scs(n)$ denote the size of the smallest critical set in any latin square of order n.

In 1978 Curran and van Rees [5] established an upper bound of $n^2 - n$ for $lcs(n)$. If we let $a = n-2$ in the set S from Lemma 5, then a critical set is produced with size $\frac{n^2-n}{2}$. Hence, $lcs(n) \geq \frac{n^2-n}{2}$. This gives the following:

$$\frac{n^2-n}{2} \leq lcs(n) \leq n^2 - n.$$

Cooper, McDonough and Mavron [4] show that $scs(n) \geq n+1$, and a more recent result, by Fu, Fu and Rodger [10], establishes that $scs(n) \geq \lfloor \frac{7n-3}{6} \rfloor$ for $n \geq 20$. A conjecture by Nelder is that $scs(n) = \lfloor \frac{n^2}{4} \rfloor$ and no evidence has been presented to date to suggest otherwise.

For particular values of n, Curran and van Rees [5] verify the following exact results:

n	1	2	3	4	5
$scs(n)$	0	1	2	4	6

For all other values of n, only the bounds for $scs(n)$ are known. As mentioned, $\frac{n^2}{4}$ is the minimal size of a critical set for BC_n for even n, however the minimal size for odd n is undetermined. It is hence desirable to investigate BC_n for odd values of n and to extend results for the smaller sized latin squares. Recently, Donovan in [7] investigates the existence of critical sets in latin squares of order less than 11. Further, in [12], Adams and Khodkar produce minimal critical sets for all groups of order 8.

In this paper, we shall utilize the relationship that exists between latin interchanges and critical sets to produce an algorithm which is then used to determine $scs(6)$ and the minimal size of a critical set for BC_7 and BC_9.

2. Main Results

In order to produce a critical set C, for a latin square L of order n, it is evident from Lemmas 2 and 3 that the latin interchanges that occur within L must be identified. Latin interchanges are relatively easy to locate; for suppose any partial latin square

P has two completions, then the differing cells of these two completions form a latin interchange. In the event that P does complete uniquely (some subset of P is thus a critical set) then remove one or more elements from P and test again for UC. Note that when verifying Lemma 2, it is unnecessary to consider latin interchanges that contain smaller latin interchanges. That is, if latin interchanges I and J exist where $I \subset J$, then disregard J.

Suppose that a latin square L contains v distinct latin interchanges I_t, where $|I_t| = r_t$ for $1 \leq t \leq v$. Let $\mathcal{V}$ denote the set of all latin interchanges occurring in L.

For each position (i, j) in L, where $0 \leq i, j \leq n-1$, let $b_{(i,j)}$ denote the set of all latin interchanges in L containing position (i, j), and let $\mathcal{X}$ denote the set of latin interchange sets $b_{(i,j)}$, for every position (i, j) in L. We shall use the notation $b_{(i,j)} \in C$ to denote that the partial latin square C contains a triple corresponding to position (i, j). In analogy with Lemmas 2 and 3, a critical set C must then satisfy:

$\mathcal{L}2)$ For each $I_t \in \mathcal{V}$, $|I_t \cap C| \geq 1$; and

$\mathcal{L}3)$ for each $b_{(i,j)} \in C, \exists I_t \in \mathcal{V}$ so that $b_{(i',j')} \in (C \cap I_t)$, if and only if $i' = i$ and $j' = j$.

Note that finding a minimal critical set C is equivalent to finding a subset $\mathcal{W} \subset \mathcal{X}$, of smallest size, such that $\cup\mathcal{W} = \mathcal{V}$ and $\forall b_{(i,j)} \in \mathcal{W}$, the set $b_{(i,j)} \setminus \cup\{\mathcal{W} \setminus b_{(i,j)}\}$ is non-empty.

A general description of the algorithm will now be presented. In what follows, our goal is to produce a minimal critical set for some latin square L. When searching for a minimal critical set, we normally have some lower bound for the smallest possible size. As smaller sized partial latin squares require less computational time, it is generally more effective to test whether critical sets exist of smallest expected values. Then, if a critical set is found of size m, we search for a critical set of size $m-1$ and if none exists, then m must be the size of a minimal critical set. Note that if theoretical lower bounds do not exist for a particular latin square, we can find an upper bound for the size as follows.

If L has order n, then the set $C = \{(i, j; k) \,|\, 0 \leq i, j \leq n-2, (i, j; k) \in L\}$ is a uniquely completable set, as each row and column contains $n-1$ entries and so the last entry is forced in each case. The size of C is $(n-1)^2$. Note that we can remove any element from C and check for UC. While we still have UC, continue to remove elements from C. When C is reduced to the point where every element is necessary for UC, then we have a critical set of size m, where m is the number of elements remaining in set C. This method is well-suited for computer implementation. We now describe the steps for determining whether a critical set of size m exists.

Step 1. Find all sets C_w corresponding to a subset $\mathcal{W}$ of $\mathcal{X}$ of size m, where $C_w = \{(i, j; k) \mid b_{(i,j)} \in \mathcal{W}\}$, and $|C_w| = m$. Keep all sets that satisfy Lemma 4: that is, no row or column is completely filled and no element occurs n times.

Further, keep only the sets that contain at least $n-1$ different rows, columns and elements. Note that as any pair of rows, columns or elements form a latin interchange, this last requirement covers any latin interchanges of this type.

Step 2. For each m-set C_w check that Lemma 2 is satisfied. That is, each C_w must intersect with every latin interchange $I_t \subset L$.

Step 3. Check that each m-set C_w satisfies Lemma 3, (equivalent to requirement $\mathcal{L}3$ above).

Sets C_w that satisfy Steps 1, 2 and 3, are critical sets of size m. Thus, once a critical set of size m is found, repeat Steps 1, 2, 3, replacing m with $m-1$ to find a critical set of size $m-1$. Note that if all m-sets satisfy Steps 1 and 2 but fail Step 3, then they are UC but contain superfluous elements, so also repeat Steps 1, 2 and 3 to find a critical set of size $m-1$ or less. If *no* sets of size m have UC, that is they fail Steps 1 and 2, then there cannot be any critical set of size less than $m+1$.

For a latin square of order n there are $\binom{n^2}{m}$ sets of positions of size m that require checking at Step 1 of the algorithm. For $n = 6$ it is computationally feasible to check all m-sets. For example, with $m = 9$, approximately 94×10^6 sets must be checked. In some cases, this search space may be reduced if elements can be fixed without loss of generality (for example in symmetric latin squares). This concept is used in the search for BC_7, and for BC_9. Generally however, for larger values of n, the size of the search space limits the feasibility of conducting complete searches for critical sets. In these cases, the algorithm may be adapted for use in a probabilistic manner. Alternatively, Steps 1 and 2 may be used to check whether a UC set exists of size m, with the algorithm being halted as soon as the first one is found.

In the implementation of Step 2, it is not always necessary to identify every single latin interchange, I_t, in L. Quite often a minimal critical set can be found by utilizing just a selection of I_t's. It is desirable to start with small sized latin interchanges first, as these are the most 'easily missed' and are also computationally faster to check for. Perform the above steps with some subset of $\mathcal{V}$ containing all known smaller sized latin interchanges. If some m-set covers these I_t's, then additional latin interchanges need to be found. However, if no m-set is found that covers these latin interchanges, then clearly there cannot be a critical set of size m, or less.

We shall now proceed to find $scs(6)$.

In [9], Fisher and Yates produced a classification of all latin squares of order six. There are 9408 latin squares of order six when put into standard form. These latin squares are partitioned into twelve main classes. A relatively simple computer program will produce all of these, and isomorphism programs such as *Nauty* by B McKay [13], may be used to produce the necessary classification into main classes. It is noteworthy that this classification was accurately done by hand by Fisher et al in 1934.

In what follows, the notation L_i refers to a latin square from main class i, and C_i denotes a minimal critical set for L_i. Table 3 displays a representative of each main

Main Class	1	2	3	4
latin square	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5
	1 2 3 4 5 0	1 2 0 5 3 4	1 2 0 5 3 4	1 0 3 2 5 4
	2 3 4 5 0 1	2 0 1 4 5 3	2 0 1 4 5 3	2 3 4 5 0 1
	3 4 5 0 1 2	3 4 5 0 1 2	3 4 5 1 2 0	3 2 5 4 1 0
	4 5 0 1 2 3	4 5 3 2 0 1	4 5 3 0 1 2	4 5 0 1 3 2
	5 0 1 2 3 4	5 3 4 1 2 0	5 3 4 2 0 1	5 4 1 0 2 3
critical set	0 1 - - - -	- - - - 4 5	- 1 - - 4 -	- 1 - - - -
	1 - - - - -	1 - - - - 4	- - - - - -	1 - 3 - 5 -
	- - - - - -	2 - 1 - - -	2 - - 4 5 -	- - 4 - - -
	- - - - - 2	- - - 0 - -	- - - - 2 0	3 - - 4 - 0
	- - - - 2 3	- - 3 2 0 -	- - 3 - - 2	- 5 - - 3 -
	- - - 2 3 4	5 3 - - - -	- 3 4 - - -	- - - 0 - -

Main Class	5	6	7	8
latin square	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5
	1 0 3 2 5 4	1 0 3 4 5 2	1 0 3 2 5 4	1 0 3 2 5 4
	2 4 0 5 1 3	2 3 1 5 0 4	2 4 0 5 3 1	2 4 0 5 1 3
	3 5 1 4 0 2	3 5 4 1 2 0	3 5 4 0 1 2	3 5 4 0 2 1
	4 2 5 1 3 0	4 2 5 0 1 3	4 2 5 1 0 3	4 3 5 1 0 2
	5 3 4 0 2 1	5 4 0 2 3 1	5 3 1 4 2 0	5 2 1 4 3 0
critical set	- 1 - - 4 -	0 1 - - - 5	0 - 2 3 - -	- - - 3 4 -
	- - 3 - - 4	1 0 - - - -	- - - - - 4	1 - - - - -
	2 - - 5 - -	- - 1 - - -	- 4 - - - 1	- - 0 - - -
	- - - - 0 -	- - - - 2 -	3 - - 0 - -	3 5 - - - -
	- 2 5 1 - -	4 - - - - -	- - - - 0 -	- - - - 0 2
	5 - - - - -	- - - 2 3 -	5 - - - - -	5 2 1 - - -

Main Class	9	10	11	12
latin square	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5
	1 0 3 2 5 4	1 0 3 2 5 4	1 0 3 2 5 4	1 0 3 2 5 4
	2 3 4 5 0 1	2 4 0 5 1 3	2 4 0 5 3 1	2 3 4 5 0 1
	3 4 5 0 1 2	3 5 1 4 2 0	3 5 4 0 1 2	3 4 5 1 2 0
	4 5 0 1 2 3	4 3 5 1 0 2	4 3 5 1 2 0	4 5 1 0 3 2
	5 2 1 4 3 0	5 2 4 0 3 1	5 2 1 4 0 3	5 2 0 4 1 3
critical set	- - - - 4 5	0 - - - - -	0 1 2 - - -	0 - 2 - - -
	1 - 3 - - -	- - 3 - - 4	- - - - 5 -	1 - - - 5 4
	2 - - - 0 -	- 4 - - 1 3	- - - 5 3 -	- - - 5 - 1
	3 - - - - 2	- - - - - -	- - 4 - - 2	- - - 1 - -
	- - - 1 - -	- 3 - - - 2	- - - - - -	- - - - - -
	- - 1 4 - -	5 - - 0 - -	- 2 1 - - -	- 2 0 - - -

Table 3: The twelve main classes of latin squares of order six and their minimal critical sets.

class and its corresponding minimal critical set. It is relatively easy to prove that each C_i is UC and that each element is essential. Proof that each critical set produced is minimal is provided by following the general algorithm. In each case, no set of $|C_i| - 1$ positions in L_i passes both Steps 1 and 2 of the algorithm, thus confirming that C_i is a minimal critical set. For each main class, a selection of latin interchanges occurring in L_i is used to determine a critical set, and the number of various sizes required is displayed in Table 4.
The first latin square, L_1, is the back circulant latin square of order six, and from Lemma 6, the size of a minimal critical set is known to be nine. The second latin square, L_2, corresponds to the Dihedral group of order three. A critical set of size twelve was produced by Keedwell in [16] but was not known to be minimal. Using all 27 intercalates and all 36 latin interchanges of size 6, it is found that no set of 11 positions in this latin square satisfies Steps 1 and 2 of the algorithm. Thus, a minimal critical set for L_2 has size 12. For classes 6 and 12, Mortimer [14] produced critical sets of size 10 which are now known to be minimal. All remaining results are new.

$\lvert I_t \rvert$	4	6	8	9	10	11	12	13	14	15	16	18
L_1	9											
L_2	27	36										
L_3		36			162	691	463					108
L_4	9	36			96		252		443			
L_5	19	12	12		24		212		69	68	108	
L_6	4	12	61		24		252					
L_7	11	12	60		132							
L_8	15	20	90		60							
L_9	9	36	18		123	35	405	80	32	138		
L_{10}	15	8	30		60		60		36			
L_{11}	7	12	60	32	133	39	40					
L_{12}	5	16	52	53	139	46	12					

Table 4: Number of latin interchanges required to identify minimal critical sets for each latin square of order 6.

Theorem 1: $scs(6) = 9$.

Proof. Implementing the general algorithm outlined previously, produces minimal critical sets for every latin square of order six. The size of the minimal critical set for each latin square L_i is summarized below and out of these, the smallest size is nine, occurring for BC_6.

Main Class	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert C_i \rvert$	9	12	11	11	11	10	10	11	11	10	10	10

We now consider BC_7.

An example of a critical set of size 12 for BC_7 was given earlier in Table 2, and hence $scs(7) \leq 12$. We are interested in finding an exact value for the size of the minimal critical set for BC_7. As a critical set of size 12 exists, it suffices to produce all sets of 11 positions in BC_7 and test firstly whether they are UC.

Our goal is to produce a list of all sets of 11 positions in BC_7 which are possible candidates for critical sets. We know that at most six elements can occur in any row or column and that at least six rows and six columns must be represented. The following result tells us that we must have all seven distinct elements:

Lemma 6. *(Donovan, Cooper, Nott and Seberry [2].) Let L be a back circulant latin square of odd order $n \geq 7$. Then a critical set based on $n-1$ distinct elements has size at least $2(n-1)$.*

Implementing the algorithm described earlier confirms that no 11-set in BC_7 can be found that satisfies Steps 1 and 2. That is, for each 11-set that passes Step 1, a latin interchange can be found that does not intersect this 11-set. In all cases, a latin interchange I of size 9 (and type 3 of Keedwell's classification [11]) as in Table 1, can be found at Step 2. In BC_7, this is the smallest sized latin interchange occurring and there are 196 distinct latin interchanges of this size and type. These are distributed evenly throughout BC_7 and each position in BC_7 occurs in 36 of these latin interchanges. For each position (i, j) two such latin interchanges $I^1_{i,j}$ and $I^{-1}_{i,j}$, are produced with the following nine triplets, where $k = 1$ for $I^1_{i,j}$ and $k = -1$ for $I^{-1}_{i,j}$. The notation $\oplus$ and $\ominus$ denote addition and subtraction modulo 7.

$$\begin{aligned} I^k_{i,j} = \{ & (i,\, j;\, i\oplus j),\ (i,\, j\oplus k;\, i\oplus j\oplus k),\ (i,\, j\ominus k;\, i\oplus j\ominus k), \\ & (i\oplus k,\, j;\, i\oplus j\oplus k),\ (i\oplus k,\, j\ominus k;\, i\oplus j),\ (i\oplus k,\, j\oplus 3k;\, i\oplus j\oplus 4k), \\ & (i\ominus 2k,\, j\oplus k;\, i\oplus j\ominus k),\ (i\ominus 2k,\, j\ominus k;\, i\oplus j\oplus 4k),\ (i\ominus 2k,\, j\oplus 3k;\, i\oplus j\oplus k)\}. \end{aligned}$$

There are 49 values for (i, j) and the sets $I^k_{i,j}$ produce 98 distinct latin interchanges. These are not symmetric, so taking the transpose of each, gives an additional 98 latin interchanges, a total of 196.

As there are $\binom{49}{11} \approx 29 \times 10^9$ possible sets of 11 positions, it is preferential to split the search for all 11-sets in BC_7 into smaller groups. Five groups are formed by fixing q elements in the last row of BC_7 for $2 \leq q \leq 6$. Additionally, an upper limit, q, is imposed on the number of elements that can occur in any one row. For each group, a search is then conducted to find all sets of $(11-q)$ positions from the first six rows of BC_7. Each $(11-q)$-set, along with the set of q fixed positions is then required to cover at least six rows and six columns and seven distinct elements. Within each group, no single row is permitted to be represented more than q times. The following relates to the steps of the algorithm, using $m = 11$. The rows, columns and elements of BC_7 are indexed in the following steps with $0, 1, \ldots, 6$.

Step 1. For $q = 2$ to 6, fix q elements in the last row of BC_7, and search for all $\binom{42}{11-q}$ positions in the first six rows. Only sets satisfying the following are kept:

(a) A maximum of q elements is included from any one row;

(b) Each 11-set intersects at least six rows;

(c) Each 11-set intersects at least six columns; and

(d) Each 11-set represents seven elements.

Step 2. For each 11-set found above, find a latin interchange $I_w = \{(i_t, j_t; k_t) \mid 1 \leq t \leq 9\}$ of size 9 and type 3 or it's transpose, that does not intersect with this 11-set.

A flowchart for the search algorithm is displayed in Figure 1.

Search Results

Table 5 contains a summary of the searches performed for each group. Recall that q elements were fixed in the last row of BC_7. The program searched for all $\binom{42}{11-q}$ sets of positions in the first six rows of BC_7 and kept sets that satisfied requirements $1(a), (b), (c)$ and (d). For $2 \leq q \leq 5$, there are multiple non-isomorphic fixings of q positions in the last row of BC_7. In these cases, a search was performed for each non-isomorphic fixing. The table displays the search number and the elements that were fixed in each search. The last column gives the number of 11-sets found in each search that passed the requirements of Step 1. Note that these numbers are generally well below the size of the original search space.

Theorem 2: *The minimal critical set for BC_7 consists of 12 elements.*

Proof. Due to the cyclic nature of BC_7, completion of Step 1 above produces all 11-sets in BC_7 that are candidates for critical sets. From Lemma 1, the latin interchange I_w found in Step 2 provides proof that no 11-set can be a critical set.

Finally, we consider BC_9.

The back circulant latin square of order nine consists of nine subsquares of order three. From Lemma 5, recall that if C is a critical set for BC_9, then C must uniquely define each subsquare, and this requires at least two elements per subsquare. Lemma 6 provides a critical set of size twenty for BC_9, and hence, $18 \leq |C| \leq 20$. Thus, we must check whether a critical set exists of size eighteen or nineteen. Note that if a

‘FIND_11_SETS’ Algorithm

Description : ‘Find_11_sets’ produces sets of 11 positions in the back circulant latin square of order 7. Initially, ‘q’ positions in row 7 are fixed. The algorithm searches for all sets of (11-q) positions from the first 6 rows. Two inputs are required for the program: cnt1, an integer and s, a set. Counters ‘cnt1’ and ‘cnt2’ keep track of the 42 positions being searched for in rows 1 to 6. The algorithm is recursive and is initiated with cnt1=1 and s={ }.

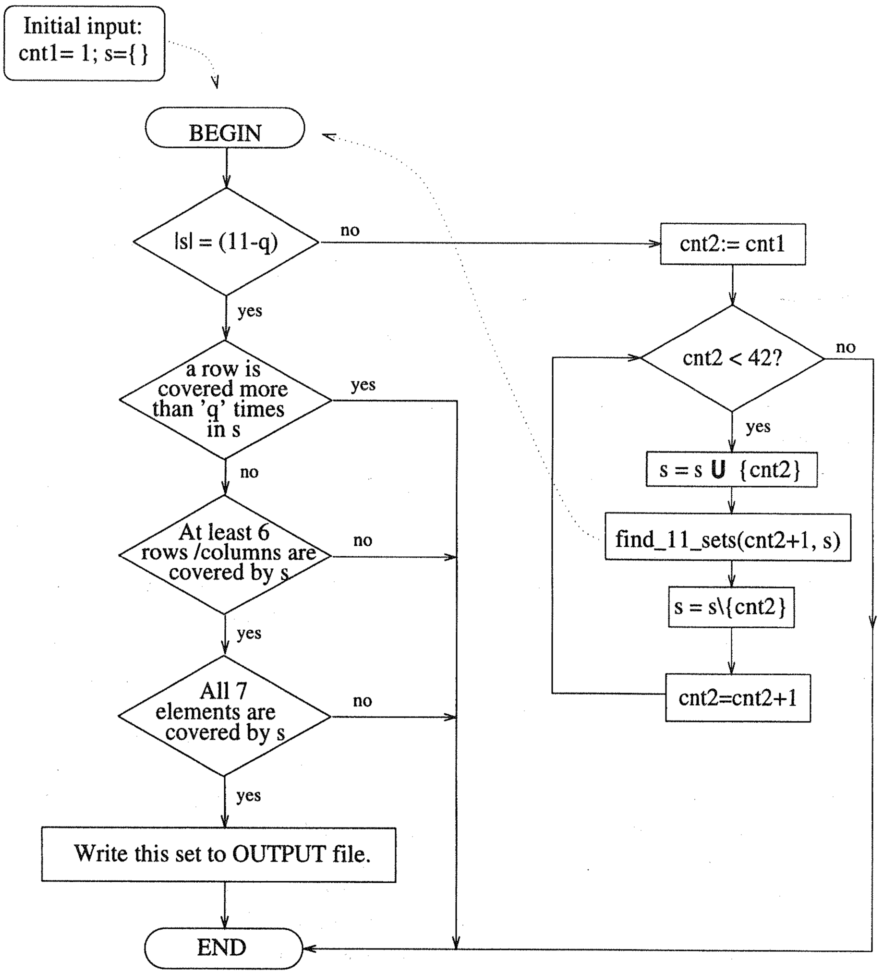

Figure 1: The search algorithm ‘Find_11_sets’

q	$\binom{42}{11-q}$	Search Number	columns fixed in row 7 of BC_7						Number of m-sets that pass Step 1
2	445,891,810	1a					5	6	14,083,838
		1b				4		6	14,083,838
		1c			3			6	14,083,838
3	118,030,185	2a				4	5	6	12,380,241
		2b			3		5	6	12,372,345
		2c		2			5	6	12,380,241
		2d		2		4		6	12,380,241
4	26,978,328	3a			3	4	5	6	2,744,768
		3b		2		4	5	6	2,742,288
		3c	1	2			5	6	2,744,768
		3d	1		3		5	6	2,744,768
5	5,245,786	4a		2	3	4	5	6	483,339
		4b	1		3	4	5	6	483,339
		4c	1	2		4	5	6	483,339
6	850,668	5	1	2	3	4	5	6	54,186

Table 5: Summary of searches performed.

UC set of size nineteen does not exist, then the minimal critical set for BC_9 must have size twenty.

The algorithm is adapted in order to reduce the search space and the task is to search for a UC set of size nineteen. Observe that in BC_9, a UC set of size nineteen must contain two elements from eight of the subsquares and three elements from the remaining subsquare. As each subsquare is isotopic to any other, we can fix the elements in the subsquare containing three elements. Any subsquare will do. There are four non-isotopic UC sets of size three for a latin square of order three, and hence four searches must be performed. Then, for each remaining subsquare, there are nine sets of two positions that form a critical set. Hence, for each of the four fixings of three elements, there are 8^9 ($\sim 10^8$) sets of sixteen elements that have UC within each subsquare. These sets combined with the set of three fixed elements are hence candidates for UC sets for BC_9. The first step of the modified algorithm produces all such sets of size nineteen. There are 486 latin interchanges of size ten (type 10b), as displayed in Table 6. Each position in BC_9 occurs in sixty of these. Any UC set must have non-zero intersection with each of these latin interchanges. Thus, the second step in the algorithm is to test whether all the potential UC sets of size nineteen found in the first step intersect with each of these 486 latin interchanges. The result is summarized in Theorem 3.

Theorem 3: *The minimal critical set for BC_9 consists of twenty elements.*

0	*	2	5
2	*	4	*
*	4	5	*
4	5	*	0

Table 6: An example of the latin interchange of size 10 and type 10b.

Proof. Adaptation of the general algorithm, as outlined above, produces all partial latin squares of size nineteen that satisfy the minimum requirements for a UC set. None of these satisfies Lemma 2, and hence none has UC. Thus, the critical set derived from Lemma 6 of size twenty must be minimal.

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