Minimal dominating sets in graph classes: combinatorial bounds and enumeration

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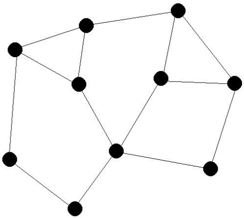
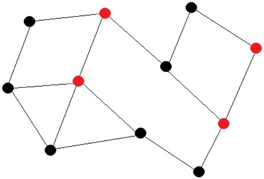
$G = (V, E)$ simple undirected graph.
V its vertex set.
E its edge set.



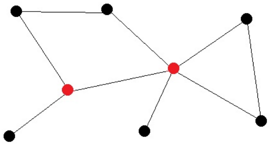
A set D is a *dominating set* of the graph $G = (V, E)$, if $\forall v \in V$:

- either $v \in D$
- or $\exists x \in D$ such that $vx \in E$



Minimum dominating set

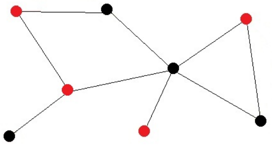
- Input : graph $G = (V, E)$
- Output : minimum cardinality of a dominating set D of G



This problem is NP-complete.
The best known exact algorithm runs in $O^*(1.4957^n)$ [J. van Rooij].

A set D is a *minimal dominating set* of the graph $G = (V, E)$ if D is a dominating set, and $\forall x \in D$

- either x has no neighbour in D
- or $\exists$ a neighbour $y \in V \setminus D$ of x such that y has no neighbour in $D \setminus \{x\}$. y is called a *private neighbour* of x.



Inclusion minimal dominating set

- Input : graph $G = (V, E)$
- Output : an inclusion minimal dominating set D of G.

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What if one minimal dominating set is not enough ?

Enumerating all minimal dominating sets

- Input : graph $G = (V, E)$
- Output : all minimal dominating sets of G.

Enumerating all minimal dominating sets allows immediate solution of corresponding NP-hard optimisation and counting problems.

Combinatorial Question

How many minimal dominating sets may a graph on n vertices have ? Not more than 2^n but ...

What is the maximum number of minimal dominating sets in a graph on n vertices ?

An upper bound was given in 2008 by F. V. Fomin, F. Grandoni, A. V. Pyatkin, and A. A. Stepanov.

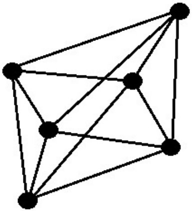
The number of minimal dominating sets in a graph on n vertices is at most 1.7159^n.

What is the maximum number of minimal dominating sets in a graph on n vertices ?

Fomin et al. also give a lower bound.

There is a graph on n vertices with $15^{n/6}$ minimal dominating sets.

This gives a lower bound of 1.5704^n for the maximum number of minimal dominating sets.



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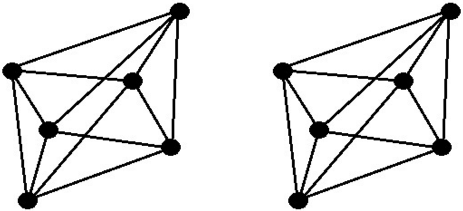

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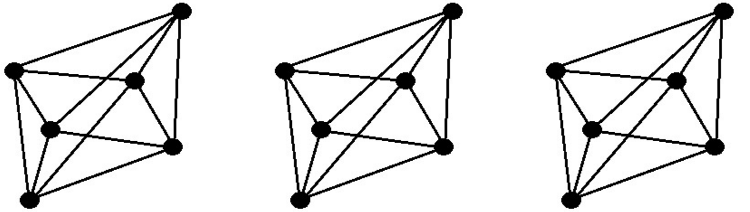

Not tight !

There is a huge gap between the lower bound 1.5704^n and the upper bound 1.7159^n.

No improvements have been achieved until today.

Graph classes

Our work is dealing with some well-known graph classes. The goal is to find corresponding lower and upper bounds.

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Why graph classes ?

We attempt to exploit the particular structure of various graph classes to achieve better bounds, preferably even *matching* upper and lower bounds.

Reminder : general case

We have alredy mentioned

Lower bound	Upper bound
1.5704^n	1.7159^n

In the following we summarize our results :

Some graph classes and the corresponding bounds :

Graph Class	Lower Bound	Upper Bound
chordal	1.4422^n	1.6181^n
split	1.4422^n	1.4656^n
proper interval	1.4422^n	1.4656^n
trivially perfect*	1.4422^n	1.4423^n

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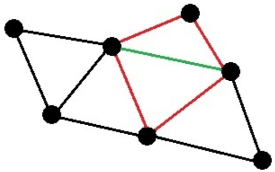

A graph is a split graph if its vertex set can be partitioned in an independent set and a clique.

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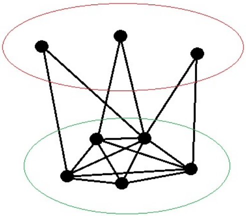

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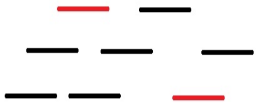

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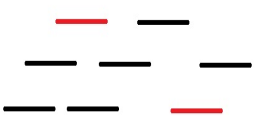

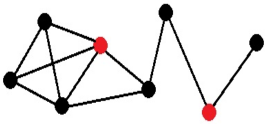

A graph is trivially perfect if it has neither P_4 nor C_4 as induced subgraph.

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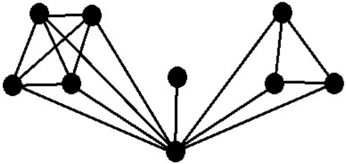

In this table, all graph classes have the same lower bound.
The 1.4422^n lower bound is achieved by two types of graphs on n vertices, both having $3^{\frac{n}{3}}$ minimal dominating sets.

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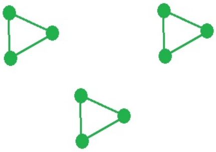

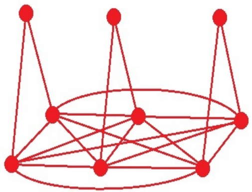

More lower and upper bounds on the maximum number of minimal dominating sets in a graph on n vertices in certain graph classes :

Graph Class	Lower Bound	Upper Bound
cobipartite	1.3195^n	1.5875^n
cograph*	1.5704^n	1.5705^n
threshold*	$\omega(G)$	$\omega(G)$
chain*	$\lfloor n/2 \rfloor + m$	$\lfloor n/2 \rfloor + m$
forest	1.4142^n	1.4656^n

... can we see an algorithm now ?

Why an algorithm ?

The execution of a branching algorithm can be represented by a search tree.

If the algorithm enumerate all solution, when the execution is finished, all solution is contain in a leaf of the search tree.

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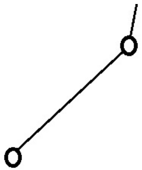

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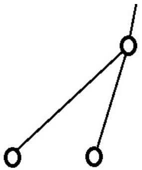

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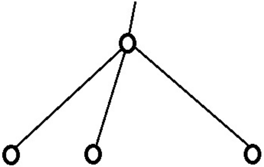

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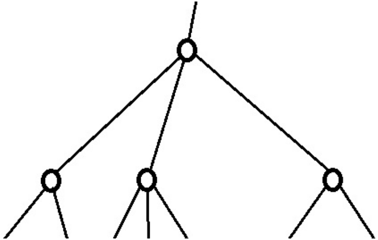

If the algorithm enumerate all solution, when the execution is finished, all solution is contain in a leaf of the search tree.

The number of leaf in the search tree is an upper bound !

If we can bound the number of leaves in the search tree, we bound at the same time the number of solutions of the problem !
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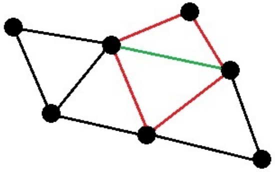
And bound the number of leaves in the search tree is exactly what an estimation of execution time does.

Be careful, it is an upper bound !

Every solution is in a leaf, but every leaf does not have a solution.

Chordal graphs

A graph is chordal if every cycle of length at least 4 has a chord.



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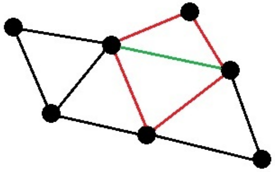

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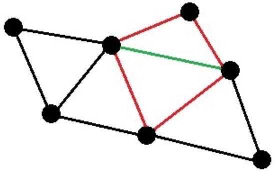
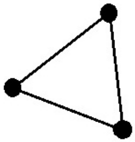


Every chordal graph has a simplicial vertex.
A vertex x is *simplicial* if its neighbourhood $N(x)$ is a clique.

A lower bound of 1.4422^n

Take a disjoint union of $n = 3t$ triangles.
This chordal graph has $3^{\frac{n}{3}}$ minimal dominating sets.



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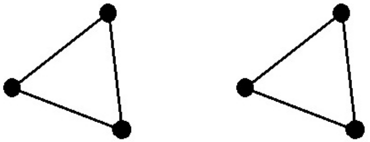

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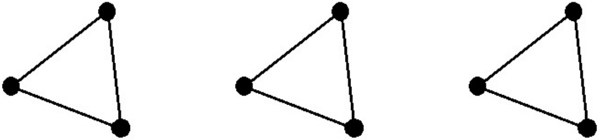
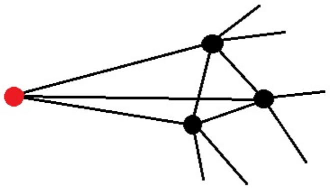


Our algorithm to enumerate all minimal dominating sets of a chordal graph always chooses a simplicial vertex x to branch on. There are three different types of branchings.

Case 1 : x is already dominated.

- $x \in D$. Since x is simplicial and needs a private neighbour in $N(x)$, we can delete x and all its neighbours.
- $x \notin D$. Since it is already dominated, it is safe to delete x.

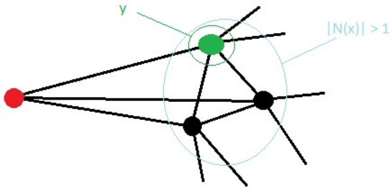


Thus the branching vector is $(2, 1)$.

Case 2 : x is not already dominated and $|N(x)| \geq 2$

Let y be a neigbour of x.

- $y \in D$. Since x is simplicial, all neighbours of x are dominated by y. We delete x and y.
- $y \notin D$. Since $y \in N(x)$, any vertex we select later to dominate x will also dominate y. Thus we can delete y.

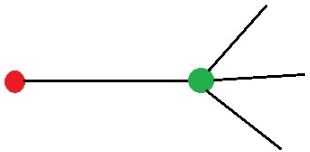


Thus the branching vector is $(2, 1)$.

Case 3 : x is not already dominated and $|N(x)| \leq 1$.

Let y be the neighbour of x.

- $x \in D$. Since y is the private neighbour of x, we can delete x and y.
- $x \notin D$. The only way to dominate x is to take y into D. Hence $y \in D$ and we can delete x and y.



Thus the branching vector is $(2, 2)$.

Running time of algorithm

Our three branching rules have branching vectors $(2, 1)$, $(2, 1)$ and $(2, 2)$.
The worst case is due to the branching vector $(2, 1)$. This implies that the enumeration algorithm has a running time of $O^*(1.6181^n)$.

Upper bound

This also implies an upper bound of $O^*(1.6181^n)$ for the number of minimal dominating sets in a chordal graph on n vertices.

Lower bound

Recall that the lower bound for chordal graphs is 1.4422^n.

Cographs

A graph G is a *cograph* if it can be constructed from isolated vertices by the operations *disjoint union* and *join*.
This construction can be represented by a *cotree*.

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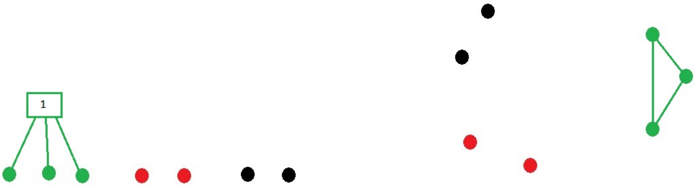

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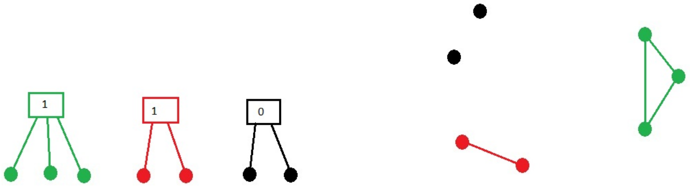

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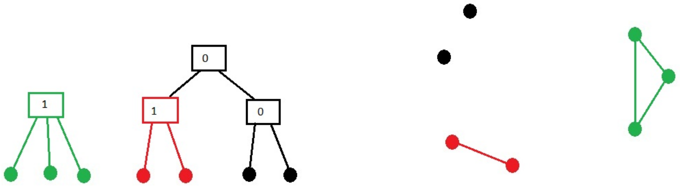

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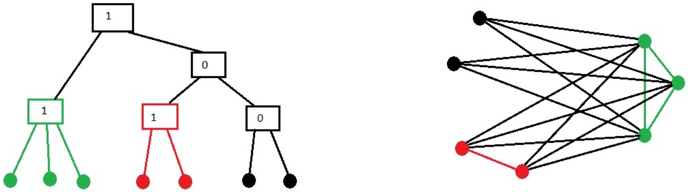

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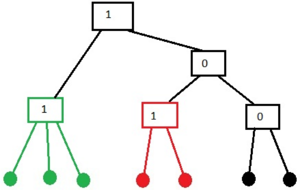
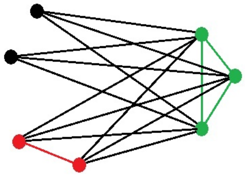
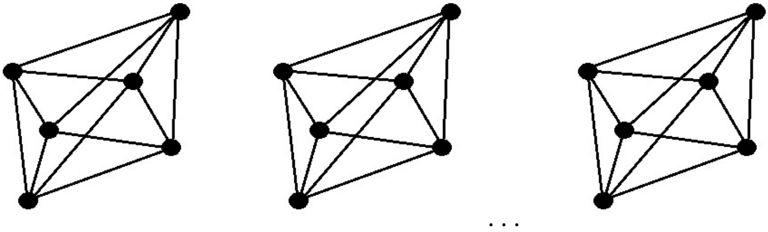




A graph is a cograph iff it has no P_4 as induced subgraph.

Lower bound.

The lower bound graph for the general case is indeed a cograph.



There is a cograph with $15^{\frac{n}{6}}$ minimal dominating sets.

Theorem

Every cograph has at most $15^{\frac{n}{6}}$ minimal dominating sets.

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Proof by induction.

It is not difficult to enumerate all the possible cographs with $n \leq 6$ vertices and to verify that each has at most $15^{\frac{n}{6}}$ minimal dominating sets.

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Proof by induction.

It is not difficult to enumerate all the possible cographs with $n \leq 6$ vertices and to verify that each has at most $15^{\frac{n}{6}}$ minimal dominating sets.

Assume the theorem is true for all cographs with less than n vertices ...

Let $G = (V, E)$ be a cograph.

Every cograph can be constructed from isolated vertices by disjoint union and by join operation.
Hence G can be partitioned into graphs G_1 with n_1 vertices and G_2 with n_2 vertices such that :

- if G is a disjoint union of G_1 and G_2, then there is no edge between G_1 and G_2.
- if G is a join of G_1 and G_2, then all the edges with one endpoint in G_1 and one in G_2 are present in G.

Note that $n = n_1 + n_2$.

Let $\mu(G)$ be the number of minimal dominating sets in G.

Case 1 : G is a disjoint union of G_1 and G_2.

Since every minimal dominating set D of G is the union of a minimal dominating set D_1 of G_1 and a minimal dominating set D_2 of G_2, we have :

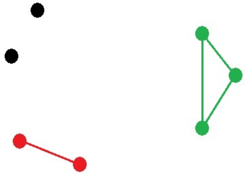
$$\mu(G) = \mu(G_1) \cdot \mu(G_2)$$

Using induction hypothesis for G_1 and G_2, we obtain that the number of minimal dominating sets in G is at most $15^{\frac{n_1}{6}} \cdot 15^{\frac{n_2}{6}} = 15^{\frac{n}{6}}$.

Case 2 : G is a join of G_1 and G_2.

Since for each vertex x_1 of G_1 and for each vertex x_2 of G_2, there is an edge x_1x_2 in G, there are three types of minimal dominating sets of G.

- a minimal dominating set D_1 of G_1,
- a minimal dominating set D_2 of G_2, and
- $\{x_1, x_2\}$ for all vertices x_1 of G_1 and all vertices x_2 of G_2.



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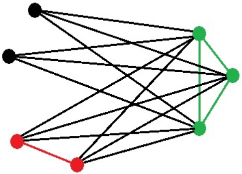

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Consequently :

$$\mu(G) = \mu(G_1) + \mu(G_2) + n_1 \cdot n_2$$

Using induction hypothesis for G_1 and G_2 and the fact that $n \geq 7$, we obtain that the number of minimal dominating sets in G is at most $15^{\frac{n_1}{6}} + 15^{\frac{n_2}{6}} + n_1 \cdot n_2 \leq 15^{\frac{n}{6}}$.

Lower bound matches upper bound

$15^{\frac{n}{6}}$ is a tight upper bound for the maximum number of minimal dominating sets in a cograph on n vertices.

Future work

- Various bounds are not tight. Improving bounds for general graphs might be hard. Improving bounds for some graph classes might be easier.
- Output sensitive approach to enumeration : constructing output polynomial or even polynomial delay algorithms to enumerate all minimal dominating sets.
- Could our enumeration algorithms be used to establish fast exact exponential algorithms solving the NP-hard problems Domatic Number and Connected Dominating Set on split and chordal graphs ?

Thank you !

F. V. Fomin, F. Grandoni, A. V. Pyatkin, and A. A. Stepanov. Combinatorial bounds via measure and conquer : Bounding minimal dominating sets and applications. ACM Trans. Algorithms 5(1) : (2008).