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Minimal Fine Limits for a Class of Potentials

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Abstract. We consider potentials $G_k\mu$ associated with the Weinstein equation with parameter k in $\mathbb{R}$, $\sum_{j=1}^{n}(\partial^2 u/\partial x_j^2) + (k/x_n)(\partial u/\partial x_n) = 0$, on the upper half space in $\mathbb{R}^n$. We show that if the representing measure μ satisfies the growth condition $\int y_n^{\omega}/(1+|y|)^{n-k} < \infty$, where $\max(k, 2-n) < \omega \leqslant 1$, then $G_k\mu$ has a minimal fine limit of 0 at every boundary point except for a subset of vanishing $(n-2+\omega)$ dimensional Hausdorff measure. We also prove this exceptional set is best possible.

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1. Introduction and Main Results

Let $\mathbb{R}_+^n = \{x = (x', x_n) : x_n > 0, x' = (x_1, \ldots, x_{n-1})\}$ denote the upper half space in $\mathbb{R}^n$, $n \geqslant 2$, with $\mathbb{R}^{n-1}$ its boundary. Generalising the classical results on the boundary behavior of potentials by Littlewood and subsequently by Privalov, L. Carleson [3] proved the following result.

THEOREM A. *Let $p = G\mu$ be a potential on $\mathbb{R}_+^n$ such that $\int (y_n^{\omega}/(1+|y|)^n)\,\mathrm{d}\mu(y) < \infty$, where $0 < \omega \leqslant 1$. Then p has a perpendicular limit of zero at all points of $\mathbb{R}^{n-1}$ except for a set of vanishing $n-2+\omega$ dimensional Hausdorff measure.*

The result of Privalov (the case $\omega = 1$ in Theorem A) was generalised by Doob [5] to the setting of any Green space and its Martin boundary, and this was generalised to the following result by the first author.

THEOREM B (Fatou–Naïm–Doob theorem). *Let Ω be a Brelot harmonic space with a positive potential. Let $u > 0$ be a harmonic function with corresponding representing measure μ_u on the minimal part of the Martin boundary. Then, for* *every potential p, the minimal fine limit of p/u is zero for μ_u-almost every point of the boundary.*

In this article we generalise the above result in the spirit of the result of Carleson. Accordingly we consider a class of Weinstein potentials (for $k \neq 1$) whose representing measures satisfy a growth condition on $\mathbb{R}^n_+$. The particulars concerning the Weinstein equation, the Hausdorff measure etc. are described in the next section. We prove the following two main results in this article.

THEOREM 1. *Let μ be a positive Radon measure on $\mathbb{R}^n_+$ and k the parameter in the Weinstein equation.*

(a) *If* $\max(k, 2-n) < \omega \leqslant 1$ *and* $\int (y_n^{\omega}/(1+|y|)^{n-k})\,\mathrm{d}\mu(y) < \infty$, *then the minimal fine limit of $G_k\mu = 0$ at each boundary point of $\mathbb{R}^{n-1}$ except for a set $E \subset \mathbb{R}^{n-1}$ having zero $(n-2+\omega)$dimensional Hausdorff measure.*
(b) *If* $\max(2-k, 2-n) < \omega \leqslant 1$ *and* $\int (y_n^{k+\omega-1}/(1+|y|)^{n+k-2})\,\mathrm{d}\mu(y) < \infty$, *then the minimal fine limit of $x_n^{k-1}G_k\mu(x) = 0$ at each boundary point of $\mathbb{R}^{n-1}$ except for a set $E \subset \mathbb{R}^{n-1}$ having zero $(n-2+\omega)$ dimensional Hausdorff measure.*

In the above statement, G_k is the Green function corresponding to the Weinstein operator [9]. The next theorem shows that the exceptional sets are best possible.

THEOREM 2. *Let* $\max(k, 2-n) < \omega \leqslant 1$. *Let $E \subset \mathbb{R}^{n-1}$ be of $n-2+\omega$ dimensional Hausdorff measure zero. Then there exists a Weinstein potential $G_k\mu$ satisfying the growth condition $\int (y_n^{\omega}/(1+|y|)^{n-k})\,\mathrm{d}\mu(y) < \infty$ such that the minimal fine* limsup *of $G_k\mu(x) = \infty$ at each point of E. A similar result is valid for the case when* $\max(2-k, 2-n) < \omega \leqslant 1$.

We recall that the case where the Weinstein parameter $k = 0$ corresponds to the Laplace equation and the classical Green potentials on $\mathbb{R}^n_+$. We believe that our result is new even in this classical case when $\omega < 1$. We also remark that when $k = 0$ and $\omega = 1$, the result is a special case of the result of Doob on Green spaces.

2. Preliminaries

The Weinstein equation with parameter k is $L_k(f) = 0$ where

$$L_k(f) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} + \frac{k}{x_n}\frac{\partial f}{\partial x_n}.$$

The C^2-functions which satisfy the Weinstein equation form a Brelot harmonic space satisfying the Domination Principle [13]. We recall that the Green function

$G_k(x, y)$ is given by

$$G_k(x, y) = a_{n,k}\, x_n^{1-k}\, y_n \int_0^{\pi} \frac{\sin^{1-k} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n-k)/2}}\, \mathrm{d}t, \quad \text{for } k \leqslant 1,$$

and

$$G_k(x, y) = a_{n,2-k}\, y_n^{k} \int_0^{\pi} \frac{\sin^{k-1} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n+k-2)/2}}\, \mathrm{d}t, \quad \text{for } k \geqslant 1,$$

where

$$a_{n,k} = \frac{\Gamma\left(\frac{n-k}{2}\right)}{2\pi^{n/2}\Gamma\left(\frac{2-k}{2}\right)}, \quad \text{for } k \leqslant 1.$$

The function $G_k\mu(x) = \int G_k(x, y)\, \mathrm{d}\mu(y)$ is a potential if and only $G_k\mu(x) < \infty$ for at least one x and this happens for $k < 1$ if and only if $\int (y_n/(1+|y|)^{n-k})\, \mathrm{d}\mu(y) < \infty$. We shall prove all the results only in the case $k < 1$. For the other case when $k > 1$, the results are deduced in a simple way by using the fact that

$$\left(\frac{x_n}{y_n}\right)^{k-1} G_k(x, y) = G_{2-k}(x, y).$$

We remark that $L_k(G_k(\cdot, y)) = -\delta_y$ in the sense of distribution. We shall make use of the resulting fact that $L_k(G_k\mu) = -\mu$ in the sense of distribution. We have also the estimates for G_k, viz. if $k < 1$,

$$c_1 \frac{x_n^{1-k} y_n}{|x-\overline{y}|^{n-k}} \leqslant G_k(x, y) \leqslant c_2 \frac{x_n^{1-k} y_n}{|x-y|^{n-k}} \tag{1}$$

where $\overline{y}$ is the reflection of $y \in \mathbb{R}^n_+$ in the hyperplane boundary. All of these facts are verified in [9].

We recall the Domination Principle: Let $G_k\mu$ be a locally bounded potential and v a positive L_k superharmonic function on $\mathbb{R}^n_+$. If $v \geqslant G_k\mu$ on the support of μ, then $v \geqslant G_k\mu$ everywhere ([1, p. 129], [13, p. 436]).

For the sake of notational convenience, from now on we shall refer to the associated potential theoretic terminology and concepts in classical terms suppressing k. These include terms such as harmonic, superharmonic, potential, polar set, reduced functions, balayage, quasi-everywhere, minimal fine limits, etc. We shall hereon use the term measure to denote a positive Radon measure. We denote by $B(x, r)$ the ball of the dimension of the point x with radius r. We will use the same notation for balls contained in $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$.

Finally we remark that we need many constants in our estimates which vary with each step. However, when it does not in any way depend on anything other than the fixed quantities n, k, and ω, we simply use the same ‘c’ for such constants. Occasionally we do give other constants to motivate some steps.

3. A Capacity

In this section we define a set function on the class of all compact subsets of $\mathbb{R}^n_+$ and prove that it is a strong capacity in the sense of Choquet [1], [4], [12]. The definition of this capacity was inspired by the work of Essen and Jackson [7]. This capacity plays a crucial role in the proof of the main result. We start with a reciprocity result which is similar to a result for classical Green potentials [12, p. 111] [6, p. 227].

LEMMA 1 (Reciprocity Lemma). *Let μ and ν be two measures on $\mathbb{R}^n_+$ such that $G_k\mu$ and $G_k\nu$ are potentials. Then*

$$\int x_n^k \, G_k\mu(x) \, \mathrm{d}\nu(x) = \int x_n^k \, G_k\nu(x) \, \mathrm{d}\mu(x). \tag{2}$$

Proof. We note that

$$x_n^k \, G_k\mu(x) = a_{n,k} \int_{\mathbb{R}^n_+} \int_0^\pi \frac{x_n y_n \sin^{1-k} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n-k)/2}} \, \mathrm{d}t \, \mathrm{d}\mu(y).$$

A simple application of Fubini’s theorem gives us that the value of the two integrals in the lemma is equal to

$$a_{n,k} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \int_0^\pi \frac{x_n y_n \sin^{1-k} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n-k)/2}} \, \mathrm{d}t \, \mathrm{d}\mu(y) \, \mathrm{d}\nu(x). \quad \square$$

LEMMA 2. *Let μ be a measure supported by a compact subset E of $\mathbb{R}^n_+$ such that $G_k\mu$ is a locally bounded potential. Let $F \subset E$ be polar. Then $\mu(F) = 0$.*

Proof. For every positive integer j, let $E_j = \{x \in E : G_k\mu(x) \leqslant j\}$ and let μ_j be the restriction of μ to E_j. Clearly, $G_k\mu_j \leqslant G_k\mu$ and hence it is locally bounded. Further, on the support of $G_k\mu_j$, the constant (superharmonic) function j is greater or equal to $G_k\mu_j$ and so by the Domination Principle, $G_k\mu_j \leqslant j$ on the whole space. Let V be a compact neighborhood of E. We may, if necessary by passing to the reduced function, choose a potential $G_k\nu$ with support in V such that $G_k\nu \equiv \infty$ on F. Now, by the Reciprocity Lemma we have

$$\left(\inf_{x \in E \cap F} x_n^k G_k\nu(x)\right) \mu_j(E \cap F) \leqslant \int x_n^k G_k\nu(x) \, \mathrm{d}\mu_j(x)$$

$$= \int x_n^k G_k\mu_j(x) \, \mathrm{d}\nu(x) < c \; j \; \nu(V) < \infty.$$

However, $G_k \nu(x) \equiv \infty$ on the set F and hence the above inequality is possible only if $\mu_j(E \cap F) = 0$. This is true for every j and hence

$$\mu(F) = \lim_{j\to\infty} \mu(E_j \cap F) = \lim_{j\to\infty} \mu_j(E_j \cap F) = 0. \qquad \square$$

For the rest of this section, ω and k are real numbers such that $1 > \omega > k$. We observe that $L_k(x_n^{\omega-k}) = (\omega - k)(\omega - 1)x_n^{\omega-k-2}$ is negative everywhere, hence $x \mapsto x_n^{\omega-k}$ is superharmonic. The choice of our Green's function G_k which verifies $L_k(G_k(\cdot, y)) = -\delta_y$ lets us conclude that the representing measure of this function is given by the density $-L_k(x_n^{\omega-k})$ relative to the Lebesgue measure. In view of the fact that $x_n^{\omega-k}$ tends to zero uniformly at every boundary point and cannot possibly minorise x_n^{1-k} (which is a multiple of the minimal harmonic function corresponding to ∞ [2]) on all of $\mathbb{R}^n_+$, we conclude that $x \mapsto x_n^{\omega-k}$ is a potential.

Let E be a compact subset of $\mathbb{R}^n_+$. Let λ_E and λ'_E be the representing measures corresponding to the potentials $\hat{R}^E_{x_n^{\omega-k}}$ and $\hat{R}^E_1$ respectively. It is clear these measures are supported by E. In view of the fact that $\hat{R}^E_{x_n^{\omega-k}} = R^E_{x_n^{\omega-k}}$ and $\hat{R}^E_1 = R^E_1$ except on a polar subset of E [1], we conclude by Lemma 2 that this polar set is of λ_E and λ'_E measure zero. Define

$$\mathbf{C}(E) = \int x_n^k \, \mathrm{d}\lambda_E = \int x_n^\omega \, \mathrm{d}\lambda'_E.$$

Note that the above equality is a consequence of the Reciprocity Lemma. The set function $\mathbf{C}(E)$ is really dependent on the parameters ω and k. However, for brevity we have suppressed the parameters.

LEMMA 3. *The set function* $\mathbf{C}(E)$ *defines a strong capacity on the class of compact sets* [1], [4].

We need to show that $\mathbf{C}$ is monotone increasing, strongly subadditive and continuous on the right. All the three properties are proved using Lemma 1 and corresponding properties of the reduced functions. The procedure is really classical and we omit the details.

We extend this strong capacity using standard method to an outer capacity (which we will continue to denote by $\mathbf{C}$). We will use the fact that this set function, defined for all subsets of $\mathbb{R}^n_+$, has the property that if $A_1 \subset A_2 \subset \cdots$, then $\mathbf{C}(\cup A_l) = \lim_l \mathbf{C}(A_l)$.

LEMMA 4. *For every compact set* E*,*

$$\begin{aligned} \mathbf{C}(E) &= \inf\left\{\int x_n^\omega \, \mathrm{d}\mu(x) : G_k\mu \geqslant 1 \textit{ q.e. on } E\right\} \\ &= \inf\left\{\int x_n^k \, \mathrm{d}\mu(x) : G_k\mu \geqslant x_n^{\omega-k} \textit{ q.e. on } E\right\} \end{aligned}$$

$$= \sup\left\{\int_E x_n^k \,\mathrm{d}\mu(x) : \mathrm{supp}(\mu) \subset E,\ G_k\mu \leqslant x_n^{\omega-k}\right\}$$

$$= \sup\left\{\int_E x_n^\omega \,\mathrm{d}\mu(x) : \mathrm{supp}(\mu) \subset E,\ G_k\mu \leqslant 1\right\}.$$

Proof. In this article we use only the first relation which we shall prove now. The rest of the proof is similar.

Suppose $G_k\mu \geqslant 1$ q.e. on E. Then $G_k\mu \geqslant \hat{R}_1^E$. Hence

$$\begin{aligned}\mathbf{C}(E) &= \int x_n^k\ \hat{R}_1^E(x)\,\mathrm{d}\lambda_E(x)\\ &\leqslant \int x_n^k\ G_k\mu(x)\,\mathrm{d}\lambda_E(x)\\ &= \int x_n^k\ G_k\lambda_E(x)\,\mathrm{d}\mu(x) \ \leqslant \int x_n^\omega\,\mathrm{d}\mu(x).\end{aligned}$$

Now the definition of $\mathbf{C}(E)$ $(= \int x_n^\omega \ d\lambda'_E(x))$ shows that $\mathbf{C}(E)$ is the above infimum. □

4. Proof of Theorem 1

We recall that we are dealing with L_k-superharmonic functions for the Weinstein operator L_k with $k < 1$.

LEMMA 5. *Let $G_k\mu$ be a potential. Fix $x \in \mathbb{R}_+^n$ and let $|z - x| < x_n/2$. If*

$$\int_{|x-y|>x_n/2} G_k(x, y)\,\mathrm{d}\mu(y) \geqslant \tfrac{1}{2},$$

then

$$\int_{|x-y|>x_n/2} G_k(z, y)\,\mathrm{d}\mu(y) \geqslant c,$$

where c depends only on the dimension n and the Weinstein parameter k.

Proof. Choose a y such that $|x - y| \geqslant x_n/2$. We have

$$\frac{|x-\overline{y}|^2}{|x-y|^2} = 1 + \frac{4x_n y_n}{|x-y|^2}.$$

If $y_n \leqslant 2x_n$, then $1 + (4x_n y_n/|x-y|^2) \leqslant 33$. If $y_n \geqslant 2x_n$, then $|x-y| \geqslant y_n - x_n \geqslant y_n/2$, so $1 + (4x_n y_n/|x-y|^2) \leqslant 9$. Thus

$$\frac{|x-\overline{y}|}{|x-y|} \leqslant \sqrt{33} \text{ for all } y \text{ such that } |x-y| \geqslant \frac{x_n}{2}. \tag{3}$$

Let x, z be as in the statement of the lemma, y as above. Then by (1),

$$\begin{aligned}
\frac{G_k(x,y)}{G_k(z,y)} &\leqslant c\frac{x_n^{1-k}}{z_n^{1-k}}\frac{|z-\overline{y}|^{n-k}}{|x-y|^{n-k}} \\
&\leqslant c\left(\frac{x_n}{z_n}\right)^{1-k}\frac{\left[|z-x|+|x-\overline{y}|\right]^{n-k}}{|x-y|^{n-k}} \\
&\leqslant c\frac{|z-x|^{n-k}+|x-\overline{y}|^{n-k}}{|x-y|^{n-k}}
\end{aligned} \tag{4}$$

since $(x_n/z_n) \leqslant 2$ and $(a+b)^m \leqslant 2^{m-1}(a^m+b^m)$ for $m \geqslant 1$. It follows from (3) and (4) that $G_k(x,y)/G_k(z,y) \leqslant c$ where c depends only on n and k. The result is an immediate consequence. □

Let ν be a measure on $\mathbb{R}^{n-1}$ with the property that the measure of every ball of radius r is at most $r^{n-2+\omega}$, for all $r > 0$. Such a measure is referred to as a test measure for $H_{n-2+\omega}$. Let $h(x)$ be the (positive) harmonic function on $\mathbb{R}^n_+$ with ν as the representing measure, i.e.,

$$h(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n^{1-k}}{|x-y|^{n-k}}\, \mathrm{d}\nu(y) \quad ([2]).$$

LEMMA 6. *For h as above,* $h(x) \leqslant c\, x_n^{\omega-1}$ *for all* $x \in \mathbb{R}^n_+$.

Proof. We have

$$h(x) = \int_{\mathbb{R}^{n-1}} \frac{x_n^{1-k}}{\left[|x'-y|^2+x_n^2\right]^{(n-k)/2}}\, \mathrm{d}\nu(y) = I_0 + \sum_{j=1}^{\infty} I_j$$

where I_0 is the part of the integral taken over the set where $|x'-y| \leqslant x_n$ and for each positive integer j, I_j is the integral over the set where $2^{j-1}x_n \leqslant |x'-y| \leqslant 2^j x_n$. Now

$$I_0 \leqslant \frac{x_n^{1-k}}{x_n^{n-k}}\nu(B(x',x_n)) \leqslant x_n^{1-n}\, x_n^{n-2+\omega} = x_n^{\omega-1}, \tag{5}$$

and for every positive integer j,

$$\begin{aligned}
I_j &\leqslant \frac{x_n^{1-k}}{(2^{j-1}x_n)^{n-k}}\, \nu(B(x',2^j x_n)) \\
&\leqslant \frac{1}{2^{(j-1)(n-k)}} x_n^{1-n}\, (2^j x_n)^{n-2+\omega} \\
&= c\, \frac{x_n^{\omega-1}}{2^{(2-k-\omega)j}}.
\end{aligned} \tag{6}$$

Since $2 - k - \omega > 0$, we have $\sum_j 1/(2^{2-k-\omega})^j < \infty$ and the proof is complete from (5) and (6). □

Before proceeding to the proof of Theorem 1, we the state the theorem of Frostman which gives a convenient way to decide if a given compact set has zero Hausdorff measure. The proof is in [11, Lemma 5.4].

THEOREM C. *Let K be a compact subset of $\mathbb{R}^{n-1}$ having positive $(n-2+\omega)$-dimensional Hausdorff measure. Then there exists a regular Borel measure ν on K that is a test measure for $H_{n-2+\omega}$ such that $\nu(K) > 0$.*

Proof of Theorem 1. Let μ be a measure on $\mathbb{R}^n_+$ satisfying the growth condition of the theorem. Suppose E is a compact subset of $\mathbb{R}^n$. The potential of the restriction of the measure μ to the complement of E tends to zero uniformly as $x \in \mathbb{R}^n_+$ tends to x' (in the Euclidean topology) in a convenient open subset of $E \cap \mathbb{R}^{n-1}$. Also, the countable union of sets of $H_{n-2+\omega}$-measure 0 has the same property. Using the above two properties, it suffices to prove that the minimal fine limit of $G_k\mu$ is zero for all points of $\mathbb{R}^{n-1}$ with the required exceptional set for $G_k\mu$ where μ has compact support in $\mathbb{R}^n$. In this case, we may further replace the growth condition on μ by

$$\int_{\mathbb{R}^n_+} y_n^\omega \, \mathrm{d}\mu(y) < \infty.$$

In order to prove that $G_k\mu$ has minimal fine limit zero for all $x' \in \mathbb{R}^{n-1}$ with the exception of a set E with $H_{n-2+\omega}(E) = 0$, it suffices to prove that the set M of all $x' \in \mathbb{R}^{n-1}$ such that $A = \{x \in \mathbb{R}^n_+ : G_k\mu(x) > 1\}$ is not minimally thin at x' has $H_{n-2+\omega}$-measure zero. Since M is analytic [8, Theorem II.1], it is enough to prove that every compact subset of M has $H_{n-2+\omega}$-measure zero [14, Corollary 7]. We also observe that the Euclidean neighbourhoods of points in $\mathbb{R}^{n-1}$ intersected with $\mathbb{R}^n_+$ are minimal fine neighbourhoods (see Lemma 7). Hence, we may assume that A itself is in a bounded part of $\mathbb{R}^n$. However, by Theorem C, $H_{n-2+\omega}(M) = 0$ if we can prove that $\nu(M) = 0$ for all test measures ν for $H_{n-2+\omega}$. Accordingly, let ν be a test measure and let $h(x)$ be the harmonic function on $\mathbb{R}^n_+$ with ν as the representing measure, as in Lemma 6. We recall that the representing measure of the greatest harmonic minorant of R_h^A is precisely the restriction of ν to the set M [8, p. 327]. Hence it suffices to prove that R_h^A is a potential on $\mathbb{R}^n_+$. Further, by Lemma 6, this will be the case if we can produce a potential which majorises $x_n^{\omega-1}$ on A. We will in fact show that there are potentials majorising $x_n^{\omega-1}$ on A_1 and A_2 where

$$A_1 = \left\{ x \in \mathbb{R}^n_+ : \int_{|x-y| \leqslant x_n/2} G_k(x, y) \, \mathrm{d}\mu(y) \geqslant \tfrac{1}{2} \right\} \cap A,$$

and

$$A_2 = \left\{ x \in \mathbb{R}^n_+ : \int_{|x-y| \geqslant x_n/2} G_k(x, y) \, \mathrm{d}\mu(y) \geqslant \tfrac{1}{2} \right\} \cap A.$$

First consider A_1. Define the measure $\mathrm{d}\mu_1(y) = \mathrm{d}\mu(y)/y_n^{1-\omega}$. Then μ_1 has compact support in $\mathbb{R}^n$ and $\int y_n \, \mathrm{d}\mu_1(y) = \int y_n^\omega \, \mathrm{d}\mu(y) < \infty$, hence $G_k\mu_1$ is a potential. Let $x \in A_1$. Then

$$G_k\mu_1(x) \geqslant \int_{|x-y| \leqslant x_n/2} G_k(x, y) \frac{\mathrm{d}\mu(y)}{y_n^{1-\omega}} \geqslant c \frac{1}{x_n^{1-\omega}}.$$

Hence $(1/c)G_k\mu_1$ is a potential which majorises $x_n^{\omega-1}$ on A_1.

We now turn to A_2. Consider $\{\overline{B(x, x_n/3)}\}_{x \in A_2}$. By the covering theorem of Besicovitch [10, p. 5], we can find $\mathcal{F}_1, \ldots, \mathcal{F}_J$ a finite number of mutually disjoint countable subcollections of this family of closed balls such that the union of these subcollections for $j = 1$ to J covers A_2. Here we note that J depends only on the dimension n. We will construct a potential corresponding to each subcollection $\mathcal{F}_j$ which majorises $x_n^{\omega-1}$ on the union of the sets in $\mathcal{F}_j$. Then clearly, the sum of these potentials $j = 1$ to J, is a potential and has the required property.

Let A_0 be a countable set of $\mathbb{R}^n$ contained in A_2 such that $\{\overline{B(x, x_n/3)} : x \in A_0\}$ is pairwise disjoint. Let $E_p = \cup \overline{B(x, x_n/4)}$ where the union is taken over the first p elements of A_0 in any convenient enumeration of A_0. It follows from Lemma 5 that for some c independent of p, $G_k\mu \geqslant c$ on E_p. It follows from Lemma 4 that

$$\mathbf{C}(E_p) \leqslant (1/c) \int y_n^\omega \, \mathrm{d}\mu(y). \tag{7}$$

However, $\mathbf{C}(E_p) = \int y_n^k \, \mathrm{d}\lambda_{E_p}(y)$ where λ_{E_p} is the representing measure of $\hat{R}^{E_p}_{x_n^{\omega-k}}$. Also, λ_{E_p} is $-L_k(\hat{R}^{E_p}_{x_n^{\omega-k}})$ in the sense of distribution. However, $-L_k(\hat{R}^{E_p}_{x_n^{\omega-k}})$ is zero outside E_p, since the function is harmonic outside E_p. Also on the interior of E_p, $\hat{R}^{E_p}_{x_n^{\omega-k}}(y) = y_n^{\omega-k}$ which is a C^∞ function and gives $-L_k(\hat{R}^{E_p}_{x_n^{\omega-k}})(y) = (\omega - k)(1 - \omega)y_n^{\omega-k-2}$. Hence $\mathrm{d}\lambda_{E_p}(y) \geqslant (\omega - k)(1 - \omega) \sum \chi_{B(x, x_n/4)} y_n^{\omega-k-2} \, \mathrm{d}y$ (χ denotes the characteristic function), the sum taken over the first p elements of A_0. It follows that

$$\begin{aligned} \mathbf{C}(E_p) = \int y_n^k \, \mathrm{d}\lambda_{E_p}(y) &\geqslant c \sum \int_{B(x, x_n/4)} y_n^k y_n^{\omega-k-2} \, \mathrm{d}y \\ &= c \sum \int_{B(x, x_n/4)} y_n^{\omega-2} \, \mathrm{d}y \\ &\geqslant c \sum (3x_n/4)^{\omega-2} x_n^n. \end{aligned} \tag{8}$$

This is true for every such finite sum and we conclude using the inequalities (7) and (8) that

$$\sum_{x\in A_0} x_n^{n+\omega-2} \leqslant c\int y_n^{\omega}\,\mathrm{d}\mu(y) < \infty. \tag{9}$$

Now consider the measure $\eta = \sum_{x\in A_0} x_n^{n-3+\omega}\delta_x$ where δ_x is the Dirac measure concentrated at the point x. The fact that η is supported by a bounded set of $\mathbb{R}^n$ and the inequality $\int y_n\,\mathrm{d}\eta < \infty$ (by (9)) imply that $G_k\eta$ is a potential. Let $x \in A_0$ and $z \in B(x, x_n/2)$.

$$G_k\eta(z) \;\geqslant\; G_k(z,x)x_n^{n-3+\omega} \;\geqslant\; c\frac{z_n^{1-k}x_n}{|z-\overline{x}|^{n-k}}x_n^{n-3+\omega}.$$

Since $1/2 \leqslant z_n/x_n \leqslant 3/2$ and $|z-\overline{x}| \leqslant 5z_n$ we conclude that

$$G_k\eta(z) \;\geqslant\; c\frac{z_n^{1-k}z_n z_n^{n-3+\omega}}{z_n^{n-k}} \;=\; cz_n^{-1+\omega}. \tag{10}$$

We apply the above argument to each of the collections $\mathcal{F}_j$ and obtain a corresponding potential $G_k\eta_j$. Let $p = \{\sum G_k\eta_j : j = 1 \text{ to } J\}$. Then p is a potential. Further, every $z \in A_2$ belongs to some ball $\overline{B(x, x_n/3)} \subset B(x, x_n/2)$, x in some $\mathcal{F}_j$, and by inequality (10) we get $p(z) \geqslant G_k\eta_j(z) \geqslant cz_n^{-1+\omega}$. This completes the proof. □

5. Proof of Theorem 2

The proof of the next result is implicitly contained in [8, Chapter IV] and [2]. We include a direct proof for the sake of completeness.

LEMMA 7. *Let $y' \in \mathbb{R}^{n-1}$ and let $W = \{x \in \mathbb{R}^n_+ : |x-(y',0)| < \varepsilon\}$ for some ε between 0 and 1. Then $W^c = \{x \in \mathbb{R}^n_+ : |x-(y',0)| \geqslant \varepsilon\}$ is minimally thin at y'.*

Proof. Let $p(x) = 1$ if $x_n \geqslant 1$ and $p(x) = x_n^{1-k}$ if $x_n < 1$. It follows from [1, Theorem 4, p. 72] that p is superharmonic. We may thus write p as $G_k\mu + h$ with $G_k\mu$ a potential and h nonnegative harmonic. Since $p(x) \leqslant x_n^{1-k}$ on $\mathbb{R}^n_+$ and the latter function is minimal [2], $h(x) = cx_n^{1-k}$ for some constant c. However, $p(x) = G_k\mu(x) + cx_n^{1-k} = 1$ on $\{x_n > 1\}$. Letting $x_n \to \infty$, we see this is possible only if $c = 0$. It follows that p is a potential.

Let $p_{y'}(x) = x_n^{1-k}/|x-(y',0)|^{n-k}$ denote the minimal harmonic function corresponding to y'. On $W^c \cap \{x_n \leqslant 1\}$ we have $p_{y'}(x) \leqslant x_n^{1-k}/\varepsilon^{n-k} = p(x)/\varepsilon^{n-k}$ and on $W^c \cap \{x_n \geqslant 1\}$ we have $p_{y'}(x) \leqslant x_n^{1-k}/x_n^{n-k} = 1/x_n^{n-1} \leqslant 1 = p(x) \leqslant p(x)/\varepsilon^{n-k}$. Thus the potential $p(x)/\varepsilon^{n-k}$ majorizes $p_{y'}$ on W^c. □

The following result can be proved easily using Lemma 7 and the Harnack property.

LEMMA 8. *Let $A \subset \mathbb{R}^n_+$. Let $y' \in \mathbb{R}^{n-1}$. For each $j \in \mathbb{Z}^+$, let $W_j = \{x \in \mathbb{R}^n_+ : |x-(y',0)| < 1/j\}$. Then A is minimally thin at y' if and only if* $\lim_{j\to\infty} \hat{R}^{A\cap W_j}_{p_{y'}}(z) = 0$ *for every $z \in \mathbb{R}^n_+$.* □

LEMMA 9. *Let $\{t_j\}$ be a sequence of positive numbers that decreases to* 0. *Then for each $y' \in \mathbb{R}^{n-1}$, $\cup_j B((y',t_j),t_j/4)$ is not minimally thin at y'.*

Proof. It is easy to see a set A is minimally thin at $0'$ if and only if $A+\{(y',0)\}$ is minimally thin at y', so we may assume $y'=0'$. Notice that $\cup_j B((0',t_j),t_j/4) = \cup_j t_j B((0',1),1/4)$. Denote this union by B. For any set $A \subset \mathbb{R}^n_+$,

$$R^{tA}_{p_{0'}(\cdot)}(z) = R^{A}_{p_{0'}(t\cdot)}(z/t).$$

Let $W_\varepsilon = \{x \in \mathbb{R}^n_+ : |x-(0',0)| < \varepsilon\}$, where $0<\varepsilon<1/2$. Then $R^{B\cap W_\varepsilon}_{p_{0'}}(0',1) \geqslant R^{t_jB((0',1),1/4)}_{p_{0'}}(0',1)$, where j is chosen so that $t_jB((0',1),1/4) \subset W_\varepsilon$. By Lemma 8, we will be done if we can show $R^{t_jB((0',1),1/4)}_{p_{0'}}(0',1)$ is bounded away from 0 by a constant that is independent of j. We have

$$R^{t_jB((0',1),1/4)}_{p_{0'}(\cdot)}(0',1) = R^{B((0',1),1/4)}_{p_{0'}(t_j\cdot)}(0',1/t_j) = t_j^{1-n} R^{B((0',1),1/4)}_{p_{0'}}(0',1/t_j).$$

But $R^{B((0',1),1/4)}_{p_{0'}}$ is a potential , $G_k\mu$, with support in $B((0',1),1/4)$. Thus

$$\begin{aligned} t_j^{1-n} &\int_{B((0',1),1/4)} G_k((0',1/t_j),x)\,\mathrm{d}\mu(x) \\ &\geqslant ct_j^{1-n}\int_{B((0',1),1/4)} \frac{(1/t_j)^{1-k}x_n}{|(0',1/t_j)-\overline{x}|^{n-k}}\,\mathrm{d}\mu(x) \\ &\geqslant c\int_{B((0',1),1/4)} \frac{t_j^{k-n}t_j^{n-k}}{|(0',1)-t_j\overline{x}|^{n-k}}\,\mathrm{d}\mu(x) \\ &\geqslant c, \end{aligned}$$

where c is independent of j. This completes the proof. □

Proof of Theorem 2. We may assume that E is a bounded subset of $\mathbb{R}^n$. Let $H_{n-2+\omega}(E)=0$. For each $m \in \mathbb{Z}^+$ there exist balls $\{B(x'_{m,j},r_{m,j})\}_j$ in $\mathbb{R}^{n-1}$ such that $E \subset \cup_j B(x'_{m,j},r_{m,j})$, and $\sum_j r^{n-2+\omega}_{m,j} < 2^{-m}$. Let μ be the measure

$$\mu = \sum_{m,j} m r^{n-2}_{m,j}\,\delta_{(x'_{m,j},4r_{m,j})},$$

where $\delta_{(x'_{m,j}, 4r_{m,j})}$ is the Dirac measure concentrated at the point $(x'_{m,j}, 4r_{m,j})$ in $\mathbb{R}^n_+$. Then μ has bounded support in $\mathbb{R}^n$,

$$\int y_n^{\omega}\, \mathrm{d}\mu(y) = \sum_{m,j} m r_{m,j}^{n-2} (4r_{m,j})^{\omega} < \infty,$$

and for any $x \in B((x'_{m,j}, 4r_{m,j}), 2r_{m,j})$,

$$G_k\mu(x) \geqslant G_k(x, (x'_{m,j}, 4r_{m,j})) m r_{m,j}^{n-2} \geqslant c \frac{(2r_{m,j})^{1-k}(4r_{m,j}) m r_{m,j}^{n-2}}{(10r_{m,j})^{n-k}} = cm.$$

Let $x' \in E$. Then for each $m \in \mathbb{Z}^+$ there exists $j(m) \in \mathbb{Z}^+$ such that $x' \in B(x'_{m,j(m)}, r_{m,j(m)})$. Note that $B((x', 4r_{m,j(m)}), r_{m,j(m)}) \subset B((x'_{m,j(m)}, 4r_{m,j(m)}), 2r_{m,j(m)})$. Let $B = \cup B((x', 4r_{m,j(m)}), r_{m,j(m)})$. Then $G_k\mu$ has a limsup of ∞ as we approach x' within B, and by Lemma 9, B is not minimally thin at x'. This completes the proof. □

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