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Minimal Free Resolution of Curves of Degree 6 or Lower in the 3-Dimensional Projective Space

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Let $C \subset \mathbf{P}^3$ be a nondegenerate smooth irreducible space curve of degree d and genus g over an algebraically closed field K. It is called n-regular if

$$H^1(\mathbf{P}^3, \mathcal{I}(n-1)) = H^2(\mathbf{P}^3, \mathcal{I}(n-2)) = 0\,,$$

where $\mathcal{I}$ is the ideal sheaf of C. Let

$$0 \to \bigoplus_{k\geq 4} S^{c_k}(-k) \to \bigoplus_{j\geq 3} S^{b_j}(-j) \to \bigoplus_{i\geq 2} S^{a_i}(-i) \to I \to 0$$

be a minimal free resolution of the defining ideal

$$I = \bigoplus_{l\geq 0} H^0(\mathbf{P}^3, \mathcal{I}(l)) \subset S = K[x_0, x_1, x_2, x_3]\,.$$

If C is n-regular, then $a_i = b_j = c_k = 0$ for $i \geq n+1$, $j \geq n+2$ and $k \geq n+3$. (See [1].) We call $(a_2, \cdots, a_n \mid b_3, \cdots, b_{n+1} \mid c_4, \cdots, c_{n+2})$ the Betti sequence of C, which is the most important information of the minimal free resolution. When we fix degree and genus, the number of the Betti sequences is finite. But listing them is very difficult. See [2] when C lies in a smooth cubic surface. In this paper, we shall show the following.

THEOREM 1. *If $d \leq 6$, then the Betti sequence of C is as follows.*

(d, g)	*Betti sequence*	*type*			
(3, 0)	(3|2|0)	(1)*	(6, 0)	(1, 0, 0, 5|0, 0, 0, 8|0, 0, 0, 3)	(8)
(4, 0)	(1, 3|0, 4|0, 1)	(2)		(0, 1, 6|0, 0, 9|0, 0, 3)	(9)
(4, 1)	(2, 0|0, 1|0, 0)	(3)*		(0, 2, 2, 1|0, 0, 4, 2|0, 0, 1, 1)	(10)
(5, 0)	(1, 0, 4|0, 0, 6|0, 0, 2)	(4)	(6, 1)	(0, 2, 3|0, 0, 6|0, 0, 2)	(11)
	(0, 4, 1|0, 3, 2|0, 0, 1)	(5)	(6, 2)	(0, 3, 1|0, 1, 3|0, 0, 1)	(12)
(5, 1)	(0, 5|0, 5|0, 1)	(6)	(6, 3)	(1, 0, 3|0, 0, 4|0, 0, 1)	(13)
(5, 2)	(1, 2|0, 2|0, 0)	(7)*		(0, 4|0, 3|0, 0)	(14)*
			(6, 4)	(1, 1, 0|0, 0, 1|0, 0, 0)	(15)*

*Here * means S/I is Cohen-Macaulay ring.*

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These cases actually occur. See Remark at the end.

The following theorem is essential to the proof.

THEOREM 2 (Gruson-Lazarsfeld-Peskine [4]). *Let C be a nondegenerate reduced irreducible space curve of degree d. Then C is $(d-1)$-regular. Moreover if $d \geq 5$, C is not $(d-2)$-regular if and only if C is a smooth rational curve with a $(d-1)$-secant line.*

The following two lemmas are useful in the calculation of the Betti numbers.

LEMMA 3. *Let C be a nondegenerate space curve which may not be irreducible, of degree d and arithmetic genus g. If C is n-regular, then we put $\beta_2 = a_2$, $\beta_3 = a_3 - b_3$ and $\beta_l = a_l - b_l + c_l$ for $4 \leq l \leq n+2$. Then the integers β_l satisfy the following four equations*:

$$\sum_{l=2}^{n+2} \beta_l = 1\,, \quad \sum_{l=2}^{n+2} l\beta_l = 0\,, \quad \sum_{l=2}^{n+2} l^2\beta_l = -2d \quad \text{and} \quad \sum_{l=2}^{n+2} l^3\beta_l = -12d - 6g + 6\,.$$

PROOF. Let

$$0 \to \bigoplus_{k=4}^{n+2} S^{c_k}(-k) \to \bigoplus_{j=3}^{n+1} S^{b_j}(-j) \to \bigoplus_{i=2}^{n} S^{a_i}(-i) \to I \to 0$$

be a minimal free resolution of I. We denote by $P_I(z)$ the Hilbert polynomial of I. By the minimal free resolution, we have

$$P_I(z) = \sum_{i=2}^{n} a_i \binom{z+3-i}{3} - \sum_{j=3}^{n+1} b_j \binom{z+3-j}{3} + \sum_{k=4}^{n+2} c_k \binom{z+3-k}{3}\,.$$

By the Riemann-Roch theorem, we have

$$P_I(z) = \binom{z+3}{3} - (dz + 1 - g)\,.$$

Comparing the coefficient of z^3, z^2, z and 1, we have the required equations. □

LEMMA 4. *Let C be a nondegenerate reduced irreducible space curve. We put*

$$m = \min\{l \in \mathbf{Z} \mid H^0(\mathbf{P}^3, \mathcal{I}(l)) \neq 0\}\,.$$

Let

$$0 \to H \to G \to F \to I \to 0$$

be a minimal free resolution of I, where

$$\begin{aligned} F &= S^{a_m}(-m) \oplus S^{a_{m+1}}(-m-1) \oplus \cdots\,, \\ G &= S^{b_{m+1}}(-m-1) \oplus S^{b_{m+2}}(-m-2) \oplus \cdots\,, \\ H &= S^{c_{m+2}}(-m-2) \oplus S^{c_{m+3}}(-m-3) \oplus \cdots\,. \end{aligned}$$

(1) $b_m = c_{m+1} = 0$.
(2) *If* $a_m = 1$, *then* $b_{m+1} = c_{m+2} = 0$.
(3) *If* $a_m = 1$ *and* $a_{m+1} = 0$, *then* $b_{m+1} = b_{m+2} = c_{m+2} = c_{m+3} = 0$.

(4) *If* $a_m = 1$ *and* $a_{m+1} = a_{m+2} = 0$, *then* $b_{m+1} = b_{m+2} = b_{m+3} = c_{m+2} = c_{m+3} = c_{m+4} = 0$.

(5) *If* $a_m = 2$, *then* $b_{m+1} = c_{m+2} = 0$.

(6) *If* $a_m = a_{m+1} = 1$, *then* $b_{m+1} = b_{m+2} = c_{m+2} = c_{m+3} = 0$.

(7) *If* $a_m = 3$ *and* $m \geq 3$, *then* $b_{m+1} \leq 1$ *and* $c_{m+2} = 0$.

PROOF. (1), (2), (3) and (4) follow immediately from the minimality.

(5) Let f_1 and f_2 be a basis of $H^0(\mathbf{P}^3, \mathcal{I}(m))$. If $b_{m+1} \neq 0$, then by minimality there is a linear relation $h_1 f_1 + h_2 f_2 = 0$ in S_{m+1}, where h_1 and h_2 are non-zero elements of S_1. Since S is the unique factorization domain, we have $h_1 | f_2$ and $h_2 | f_1$. Since f_1 and f_2 are irreducible and reduced, this is a contradiction.

(6) It can be proved by the same argument as (5).

(7) Let f_1, f_2 and f_3 be a basis of $H^0(\mathbf{P}^3, \mathcal{I}(m))$. If $b_{m+1} \geq 2$ then there are two independent linear relations $h_1 f_1 + h_2 f_2 + h_3 f_3 = 0$ and $h_4 f_1 + h_5 f_2 + h_6 f_3 = 0$ in S_{m+1}, where $h_1, \cdots, h_6 \in S_1$. This implies

$$f_1 : f_2 : f_3 = \begin{vmatrix} h_2 & h_3 \\ h_5 & h_6 \end{vmatrix} : \begin{vmatrix} h_3 & h_1 \\ h_6 & h_4 \end{vmatrix} : \begin{vmatrix} h_1 & h_2 \\ h_4 & h_5 \end{vmatrix} .$$

Then f_1, f_2 and f_3 are reducible, this is a contradiction. Therefore, we have $b_{m+1} \leq 1$, which implies $c_{m+2} = 0$. □

PROOF OF THEOREM 1. By Castelnuovo's bound

$$g \leq \begin{cases} \dfrac{1}{4}d^2 - d + 1 & \text{if } d \text{ is even} \\ \dfrac{1}{4}(d^2 - 1) - d + 1 & \text{if } d \text{ is odd} \end{cases}$$

([5, Chap. IV, Th. 6.4]), the possible values of (d, g) are as in the table. In each case of (d, g), $C \subset \mathbf{P}^3$ is either $(d-1)$-regular and sometimes even $(d-2)$-regular by Theorem 2. If $d \geq 5$, then $a_2 \leq 1$, therefore we have the Table A of the values of β by Lemma 3. Here p and q are integers. In the second case of $(d, g) = (6, 3)$, C is 3-regular since $h^1(\mathbf{P}^3, \mathcal{I}(2)) = h^2(\mathbf{P}^3, \mathcal{I}(1)) = 0$. The second case of $(d, g) = (6, 4)$ does not occur since $\beta_6 = c_6 = -1$. Therefore the Betti sequences are determined as the table in Theorem 1 by Lemma 4, except for $(d, g) = (5, 0)$, $(6, 0)$ and $(6, 2)$. We shall prove the cases $(d, g) = (6, 0)$ and $(6, 2)$. The case $(5, 0)$ is similar to the case $(6, 0)$.

First, we consider the case $(d, g) = (6, 2)$. Let

$$0 \to H \to G \to F \to I \to 0$$

be a minimal free resolution of I. In this case, C is 4-regular by Theorem 2, therefore we have

$$F = S^{a_2}(-2) \oplus S^{a_3}(-3) \oplus S^{a_4}(-4)\,,$$
$$G = S^{b_3}(-3) \oplus S^{b_4}(-4) \oplus S^{b_5}(-5)\,,$$
$$H = S^{c_4}(-4) \oplus S^{c_5}(-5) \oplus S^{c_6}(-6)\,.$$

TABLE A.

(d, g)	regularity	β_2	β_3	β_4	β_5	β_6	β_7
(3, 0)	2-regular	3	-2	0			
(4, 0)	3-regular	1	3	-4	1		
(4, 1)	3-regular	2	0	-1	0		
(5, 0)	4-regular	1	0	4	-6	2	
	4-regular	0	4	-2	-2	1	
(5, 1)	3-regular	0	5	-5	1		
(5, 2)	3-regular	1	2	-2	0		
(6, 0)	5-regular	1	p	$-4p$	$6p+5$	$-4p-8$	$p+3$
	5-regular	0	$q+1$	$-4q+6$	$6q-9$	$-4q+3$	q
(6, 1)	4-regular	1	-2	9	-10	3	
	4-regular	0	2	3	-6	2	
(6, 2)	4-regular	1	-1	6	-7	2	
	4-regular	0	3	0	-3	1	
(6, 3)	4-regular	1	0	3	-4	1	
	4-regular	0	4	-3	0	0	
(6, 4)	4-regular	1	1	0	-1	0	
	4-regular	0	5	-6	3	-1	

By (2) of Lemma 4, we have $a_2 = 0$. This implies $b_3 = c_4 = 0$. Castelnuovo ([3, Chap. 2, Sect. 5]) proved that the number of 4-secant lines of C is

$$\frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2 - 7d + 13 - g)\,,$$

unless C has 1-dimensional family of 4-secant lines. In our case, this number is one. Therefore, C has a 4-secant line, and this implies $a_4 \geq 1$. By (7) of Lemma 4, we have $b_4 \leq 1$ and $c_5 = 0$. Therefore, the Betti sequence of C is of type (12) by the table of values of β.

We consider the case $(d, g) = (6, 0)$. Let

$$0 \to H \to G \to F \to I \to 0$$

be a minimal free resolution of I. By Theorem 2, we have

$$F = S^{a_2}(-2) \oplus S^{a_3}(-3) \oplus S^{a_4}(-4) \oplus S^{a_5}(-5)\,,$$
$$G = S^{b_3}(-3) \oplus S^{b_4}(-4) \oplus S^{b_5}(-5) \oplus S^{b_6}(-6)\,,$$
$$H = S^{c_4}(-4) \oplus S^{c_5}(-5) \oplus S^{c_6}(-6) \oplus S^{c_7}(-7)\,.$$

Assume that $a_2 = 1$ and let Q be the quadric surface containing C. Q is smooth since the genus of a smooth sextic curve in a quadric cone is four. Therefore C is of bidegree (1, 5) in $Q \cong \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. Hence $a_3 = a_4 = 0$. By the table of β and (4) of Lemma 4, the Betti sequence of C is of type (8).

If $a_2 = 0$, then

$$a_3 = h^0(\mathbf{P}^3, \mathcal{I}_C(3)) \geq h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)) - h^0(C, \mathcal{O}_C(3)) = 20 - 19 = 1\,,$$

$$h^1(\mathbf{P}^3, \mathcal{I}_C(3)) = a_3 - 1\,.$$

Case 1: $a_3 = 1$. C is 4-regular, because

$$h^1(\mathbf{P}^3, \mathcal{I}_C(3)) = 0 \quad \text{and} \quad h^2(\mathbf{P}^3, \mathcal{I}(2)) = h^1(C, \mathcal{O}_C(2)) = 0\,.$$

Therefore, we have $a_5 = b_6 = c_7 = 0$. By the table of β and (2) of Lemma 4, the Betti sequence of C is of type (9).

Case 2: $a_3 = 2$. In this case, C is not 4-regular because $h^1(\mathbf{P}^3, \mathcal{I}_C(3)) = 1$. Thus, by Theorem 2, C has a 5-secant line l. Now we consider the union $C' = C \cup l$. We denote by D the intersection divisor of C and l. D is of degree five, since $\deg C = 6$. Then we have

$$H^1(C, \mathcal{O}_C(n) \otimes \mathcal{O}_C(-D)) = 0 \quad \text{for } n \geq 1\,.$$

Therefore, the restriction map

$$H^0(C, \mathcal{O}_C(n)) \to H^0(D, \mathcal{O}_D(n))$$

is surjective. By the Mayer-Vietoris exact sequence

$$\begin{aligned} 0 \to H^0(C', \mathcal{O}_{C'}(n)) &\to H^0(C, \mathcal{O}_C(n)) \oplus H^0(l, \mathcal{O}_l(n)) \\ &\to H^0(D, \mathcal{O}_D(n)) \to H^1(C', \mathcal{O}_{C'}(n)) \to 0\,, \end{aligned}$$

we have

$$h^1(C', \mathcal{O}_{C'}) = 4 \quad \text{and} \quad h^1(C', \mathcal{O}_{C'}(n)) = 0 \quad \text{for } n \geq 1\,.$$

Now C' is 4-regular because

$$h^1(\mathbf{P}^3, \mathcal{I}_{C'}(3)) = h^0(\mathbf{P}^3, \mathcal{I}_{C'}(3)) - h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3)) + h^0(C', \mathcal{O}_{C'}(3)) = 2 - 20 + 18 = 0\,,$$

$$h^2(\mathbf{P}^3, \mathcal{I}_{C'}(2)) = h^1(C', \mathcal{O}_{C'}(2)) = 0\,.$$

If C is contained in a surface of degree less than five, then l is also contained in it since there are five points in common. Therefore the Betti sequences of C and C' coincide except for a_5, b_6 and c_7. Therefore by Lemma 3 and (5) of Lemma 4, the Betti sequence of C' is $(0, 2, 2|0, 0, 4|0, 0, 1)$. Hence the Betti sequence of C is of type (10).

Case 3: $a_3 \geq 3$. We consider the multiplication map

$$\Psi : H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) \otimes V \to H^0(\mathbf{P}^3, \mathcal{I}(4))\,,$$

where $V \subset H^0(\mathbf{P}^3, \mathcal{I}(3))$ is a 3-dimensional subspace. Since C is 5-regular, $h^0(\mathbf{P}^3, \mathcal{I}(4)) = 10$. Therefore the multiplication map has at least 2-dimensional kernel. But this is impossible by (7) of Lemma 4.

REMARK. There exists a curve of degree d and genus g for every (d, g) in the table of Theorem 1. (See [5, Chap. IV Sect. 6.]) We have to show the existence for the types (4), (5), (8), (9), (10), (13) and (14)

$(d, g) = (5, 0)$. Let C_1 be the image of the morphism $\mathbf{P}^1 \to \mathbf{P}^3$ defined by $(s, t) \mapsto (s^5, s^4t, st^4, t^5)$. C_1 is contained in the smooth quadric $Q : x_0x_3 - x_1x_2 = 0$ and bidegree $(1, 4)$ in $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$. This curve has 1-dimensional family of 4-secant lines. The Betti sequence of C_1 is of type (4).

Let C_2 be the rational curve in $\mathbf{P}^3$ defined by $(s,t) \mapsto (s^5, s^4t+s^3t^2, st^4, t^5)$. This is not contained in a quadric. Hence the Betti sequence of C_2 is of type (5).

$(d,g)=(6,0)$. Let C_3 be the rational curve in $\mathbf{P}^3$ defined by $(s,t) \mapsto (s^6, s^5t, st^5, t^6)$. C_3 is bidegree $(1,5)$ in the smooth quadric $x_0x_3 - x_1x_2 = 0$. This curve has 1-dimensional family of 5-secant lines. The Betti sequence of C_3 is of type (8).

Let C_4 be the rational curve in $\mathbf{P}^3$ defined by $(s,t) \mapsto (s^6, s^5t - s^4t^2, s^2t^4 + st^5, t^6)$ when $\mathrm{ch}(K) \neq 2$. If $\mathrm{ch}(K) = 2$, then define C_4' by $(s,t) \mapsto (s^6, s^5t + s^3t^3, s^2t^4 + st^5, t^6)$. C_4 is contained in the unique cubic

$$x_1{}^2x_2 + x_0x_2{}^2 - x_1x_2{}^2 - x_0{}^2x_3 + x_0x_1x_3 + x_1{}^2x_3 - x_0x_2x_3 - x_0x_3{}^2 = 0$$

and C_4' is contained in the unique cubic

$$x_1{}^2x_2 + x_0x_2{}^2 + x_1x_2{}^2 + x_2{}^3 + x_0{}^2x_3 + x_0x_1x_3 + x_1{}^2x_3 + x_1x_2x_3 + x_0x_3{}^2 + x_1x_3{}^2 = 0\,.$$

Therefore the Betti sequences of C_4 and C_4' are of type (9).

Let C_5 be the rational curve in $\mathbf{P}^3$ defined by $(s,t) \mapsto (s^6, s^5t + s^4t^2, st^5, t^6)$. C_5 is contained in two cubics, therefore the Betti sequence of C_5 is of type (10).

$(d,g) = (6,3)$ (cf. [6]). Every curve with $g=3$ can be embedded in $\mathbf{P}^3$ by a very ample line bundle of degree 6. In this case, C is hyperelliptic if and only if C has a 4-secant line. If C is hyperelliptic, then the Betti sequence is of type (13). Otherwise, it is of type (14). $\square$

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