

MINIMAL HYPERSURFACES OF UNIT SPHERE

HUAFEI SUN AND KOICHI OGIUE

(Received March 4, 1996, revised July 8, 1996)

Abstract. In this paper, a pinching theorem for minimal hypersurfaces in a unit sphere is given and Peng-Terng's Theorem is improved.

Let S^{n+1} be an $(n+1)$ -dimensional unit sphere and M a closed minimally immersed hypersurface in S^{n+1} . S denotes the square of the length of the second fundamental form of M . It is well known that if $0 \leq S \leq n$, then $S \equiv 0$ or $S \equiv n$ (cf. [1]). Chern-do Carmo-Kobayashi [2] and Lawson [3] independently proved that the Clifford tori are the only minimal hypersurfaces with $S \equiv n$. In [2] Chern conjectured that for minimal hypersurfaces in the unit sphere S^{n+1} with constant scalar curvature, the values S are discrete. On this conjecture, Peng and Terng [5] made a breakthrough, and proved:

THEOREM A. *Let M be an n -dimensional closed minimally immersed hypersurface in S^{n+1} with constant scalar curvature. Then there exists a constant $\varepsilon(n)$ such that if $n \leq S \leq n + \varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

It is natural to consider the problem without assuming that the scalar curvature is constant. For $n \leq 5$, Peng and Terng [4] proved:

THEOREM B. *Let M be an n -dimensional closed minimally immersed hypersurface in S^{n+1} , $n \leq 5$. Then there exists a constant $\varepsilon(n) = (6 - 1.13n)/(5 + \sqrt{17})$ such that if $n \leq S \leq n + \varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

In this paper, we remove Peng-Terng's assumption $n \leq 5$ and improve Theorem B. Our result is:

THEOREM. *Let M be an n -dimensional closed minimally immersed hypersurface in S^{n+1} . Then there exists a constant $\varepsilon(n) = 2n^2(n+4)/[3(n+2)^2]$ such that if $n \leq S \leq n + \varepsilon(n)$, then $S \equiv n$ so that M is a Clifford torus.*

It is clear that our Theorem is better than Theorem B and this $\varepsilon(n)$ is the same as that in Cheng [6].

The authors would like to express their thanks to the referee for valuable suggestions.

1. Fundamental Formulas. We use the same notation and terminologies as in [4], [5] unless otherwise stated.

Let S^{n+1} be an $(n+1)$ -dimensional unit sphere and M a closed minimally immersed hypersurface in S^{n+1} . $\{e_1, \dots, e_n, e_{n+1}\}$ is an orthonormal frame field on S^{n+1} , $\{e_1, \dots, e_n\}$ is tangent to M , and let $\{\omega_1, \dots, \omega_{n+1}\}$ be the dual frame field. We use of the following convention on the range of the indices:

$$A, B, \dots = 1, \dots, n+1; i, j, \dots = 1, \dots, n.$$

Then the structure equations of S^{n+1} are given by

$$\begin{aligned} d\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \\ K_{ABCD} &= \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}. \end{aligned}$$

Restricting these forms to M , we get

$$\begin{aligned} \omega_{n+1} &= 0, \quad \omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}, \\ d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l, \end{aligned}$$

where

$$R_{ijkl} = K_{ijkl} + h_{ik} h_{jl} - h_{il} h_{jk}.$$

We call $h = \sum_{ij} h_{ij} \omega_i \omega_j$ and $H = (1/n) \sum_i h_{ii}$ the second fundamental form and the mean curvature of the immersion, respectively. If $H \equiv 0$, then M is said to be minimal. Denote by $S = \sum_{ij} h_{ij}^2$ the square of the length of h . For a minimal hypersurface M in S^{n+1} , we have

$$S = n(n-1) - R,$$

where R is the scalar curvature of M . It is clear that S is constant if and only if so is R . Define h_{ijk} , h_{ijkl} and h_{ijklm} respectively by

$$\begin{aligned} \sum_{ijk} h_{ijk} \omega_k &= dh_{ij} - \sum_m h_{mj} \omega_{mi} - \sum_m h_{im} \omega_{mj}, \\ \sum_l h_{ijkl} \omega_l &= dh_{ijk} - \sum_m h_{mjk} \omega_{mi} - \sum_m h_{imk} \omega_{mj} - \sum_m h_{ijm} \omega_{mk}, \\ \sum_m h_{ijklm} \omega_m &= dh_{ijkl} - \sum_m h_{mjkl} \omega_{mi} - \sum_m h_{imkl} \omega_{mj} - \sum_m h_{ijml} \omega_{mk} - \sum_m h_{ijkm} \omega_{ml}. \end{aligned}$$

Here

$$h_{ijk} = h_{ikj},$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl},$$

and

$$h_{ijkln} - h_{ijknl} = \sum_m h_{mjk} R_{miln} + \sum_m h_{imk} R_{mjln} + \sum_m h_{ijm} R_{mkl n}.$$

We can choose a local frame field $\{e_1, \dots, e_n\}$ such that

$$h_{ij} = h_{ii} \delta_{ij}.$$

From now on, we assume that M is minimal. By calculation we have (cf. [4], [5])

(1.1)
$$\Delta h_{ij} = (n - S)h_{ij},$$

(1.2)
$$\int_M |\nabla h|^2 = \int_M S(S - n),$$

(1.3)
$$\int_M \sum_{ijkl} h_{ijkl}^2 = \int_M \left[(S - 2n - 3)|\nabla h|^2 + 3(A - 2B) + \frac{3}{2}|\nabla S|^2 \right],$$

where

$$A = \sum_{ijk} h_{ijk}^2 h_{ii}^2, \quad B = \sum_{ijk} h_{ijk}^2 h_{ii} h_{jj}.$$

(1.4)
$$\sum_{ijkl} h_{ijkl}^2 \geq \sum_i h_{iiii}^2 + 3 \sum_{i \neq j} h_{ijij}^2.$$

Set

$$g_3 = \sum_{ijk} h_{ij} h_{jk} h_{ki}, \quad g_4 = \sum_{ijkl} h_{ij} h_{jk} h_{kl} h_{li}.$$

Then

(1.5)
$$\int_M (A - 2B) = \int_M \left(Sg_4 - g_3^2 - S^2 - \frac{1}{4}|\nabla S|^2 \right).$$

(1.6)
$$\int_M |\nabla S|^2 = 2 \int_M [S^2(S - n) - S|\nabla h|^2].$$

2. Proof of the Theorem. In order to prove our Theorem, we need the following:

LEMMA. *Let M be an n -dimensional closed minimally immersed hypersurface in S^{n+1} . If $S \geq n$, then we have*

$$\sum_{ijkl} h_{ijkl}^2 \geq \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2).$$

PROOF. Since M is minimal, we have $\sum_i h_{ii} = 0$ and

$$\sum_{ij} h_{iijj} h_{jj} = 0.$$

From (1.1) we get $\Delta h_{ii} = (n-S)h_{ii}$ and

$$\sum_{ij} h_{iijj} h_{ii} = S(n-S).$$

Let $f_{ij} = h_{iijj}$. We consider $f = \sum_i f_{ii}^2 + 3 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{ij} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}$ as a function of f_{ij} . Solve the following problem for the conditional extremum:

$$(2.1) \quad F = \sum_i f_{ii}^2 + 3 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{ij} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \lambda \left(\sum_{ij} f_{ij} h_{ii} - S(n-S) \right) + \mu \sum_{ij} f_{ij} h_{jj},$$

where λ and μ are the Lagrange multipliers. It is clear that the critical point of F is the minimum point of f . Taking derivatives of F with respect to f_{ij} , we get

$$(2.2) \quad F_{f_{ii}} = 2f_{ii} + \lambda h_{ii} + \mu h_{ii} = 0, \quad i = j,$$

$$(2.3) \quad F_{f_{ij}} = 6f_{ij} + 6(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) + \lambda h_{ii} + \mu h_{jj} = 0, \quad i \neq j,$$

and they satisfy

$$(2.4) \quad \sum_{ij} h_{jj} f_{ij} = 0, \quad \sum_{ij} h_{ii} f_{ij} = S(n-S), \quad \sum_i h_{ii}^2 = S, \quad \sum_i h_{ii} = 0.$$

From (2.2) and (2.3) we get

$$(2.5) \quad \sum_i f_{ii} F_{f_{ii}} = 2 \sum_i f_{ii}^2 + (\lambda + \mu) \sum_i f_{ii} h_{ii} = 0,$$

$$(2.6) \quad \sum_{i \neq j} f_{ij} F_{f_{ij}} = 6 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \lambda \sum_{i \neq j} f_{ij} h_{ii} + \mu \sum_{i \neq j} f_{ij} h_{jj} = 0,$$

$$(2.7) \quad \sum_{i \neq j} h_{ii} F_{f_{ij}} = 6 \sum_{i \neq j} h_{ii} f_{ij} + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ii} + \lambda \sum_{i \neq j} h_{ii}^2 + \mu \sum_{i \neq j} h_{ii} h_{jj} = 0,$$

$$(2.8) \quad \sum_{i \neq j} h_{jj} F_{f_{ij}} = 6 \sum_{i \neq j} h_{jj} f_{ij} + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{jj} + \lambda \sum_{i \neq j} h_{ii} h_{jj} + \mu \sum_{i \neq j} h_{jj}^2 = 0,$$

$$(2.9) \quad \sum_i h_{ii} F_{f_{ii}} = 2 \sum_i h_{ii} f_{ii} + \lambda \sum_i h_{ii}^2 + \mu \sum_i h_{ii}^2 = 0.$$

From (2.5) and (2.6) we get

$$2\left(\sum_i f_{ii}^2 + 3 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}\right) - 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \lambda \sum_{ij} h_{ii} f_{ij} + \mu \sum_{ij} f_{ij} h_{jj} = 0$$

and so, in view of (2.4),

$$(2.10) \quad \begin{aligned} &\sum_i f_{ii}^2 + 3 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} \\ &= 3 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} - \frac{1}{2} \lambda \sum_{ij} h_{ii} f_{ij} \\ &= 3 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \frac{\lambda}{2} S(S-n). \end{aligned}$$

Using (2.4), from (2.8) and (2.9) we get

$$(2.11) \quad -4 \sum_i h_{ii} f_{ii} + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{jj} + n\mu S = 0,$$

and from (2.7), (2.9) we get

$$(2.12) \quad -4 \sum_i h_{ii} f_{ii} + 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ii} + n\lambda S + 6S(n-S) = 0.$$

Notice that $S \geq n$. Thus by (2.11) and (2.12) we obtain

$$n(\mu - \lambda)S = 6S(n-S) + 12S^2,$$

that is,

$$(2.13) \quad \mu - \lambda = \frac{6}{n}(n-S) + \frac{12}{n}S.$$

On the other hand, (2.11) + (2.12) + 4 × (2.9) shows

$$4(\lambda + \mu)S + n(\lambda + \mu)S + 6S(n-S) = -6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{jj} - 6 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ii} = 0,$$

that is,

$$(2.14) \quad \lambda + \mu = \frac{6}{4+n}(S-n).$$

Combining (2.13) with (2.14), we get

$$(2.15) \quad \lambda = \frac{6(n+2)}{n(n+4)}(S-n) - \frac{6}{n}S.$$

(2.10) and (2.15) show that

$$\begin{aligned} & \sum_i f_{ii}^2 + 3 \sum_{i \neq j} f_{ij}^2 + 6 \sum_{i \neq j} (h_{jj}^2 h_{jj} - h_{ii}^2 h_{jj}) f_{ij} \\ &= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}, \end{aligned}$$

and so,

$$\begin{aligned} (2.16) \quad \sum_i h_{iii}^2 + 3 \sum_{i \neq j} h_{ijj}^2 &\geq \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) - 3 \sum_{i \neq j} (h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) h_{ijj} \\ &= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2). \end{aligned}$$

Combining (1.4) with (2.16), we get the Lemma.

PROOF OF THE THEOREM. Using (1.3) and the Lemma we get

$$\begin{aligned} (2.17) \quad & \int_M \left[(S-2n-3)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 \right] \\ & \geq \int_M \left[\frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(Sg_4 - g_3^2 - S^2) \right]. \end{aligned}$$

Noting (1.5), from (2.17) we obtain

$$\begin{aligned} & \int_M \left[(S-2n-3)|\nabla h|^2 + 3(A-2B) + \frac{3}{2}|\nabla S|^2 \right] \\ & \geq \int_M \left[\frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3(A-2B) + \frac{3}{4}|\nabla S|^2 \right], \end{aligned}$$

namely,

$$(2.18) \quad \int_M \left[(S-2n-3)|\nabla h|^2 - \frac{3(n+2)}{n(n+4)} S(S-n)^2 + \frac{3}{n} S^2(S-n) + \frac{3}{4}|\nabla S|^2 \right] \geq 0.$$

Substituting (1.6) into (2.18), we get

$$\begin{aligned} & \int_M \left[(S-2n-3)|\nabla h|^2 - \frac{3(n+2)}{n(n+4)} S(S-n)^2 + \frac{3}{n} S^2(S-n) \right. \\ & \quad \left. + \frac{3}{2} S^2(S-n) - \frac{3}{2} S|\nabla h|^2 \right] \geq 0, \end{aligned}$$

that is,

$$(2.19) \quad \int_M \left[\left(-\frac{1}{2} S - 2n - 3 \right) |\nabla h|^2 + \frac{3(n+2)^2}{2n(n+4)} S^2(S-n) + \frac{3(n+2)}{n+4} S(S-n) \right] \geq 0.$$

Now, we assume that $n \leq S \leq n + \varepsilon$. Then using (1.2) we get

$$(2.20) \quad \int_M S^2(S-n) \leq (n+\varepsilon) \int_M S(S-n) = (n+\varepsilon) \int_M |\nabla h|^2.$$

By (2.20), (2.19) yields

$$(2.21) \quad \int_M \left[-\frac{1}{2}S - \frac{1}{2}n + \frac{3(n+2)^2}{2n(n+4)}\varepsilon \right] |\nabla h|^2 \geq 0.$$

Taking $\varepsilon = 2n^2(n+4)/[3(n+2)^2]$ in (2.21), we get

$$\int_M \left(-\frac{1}{2}S + \frac{1}{2}n \right) |\nabla h|^2 \geq 0.$$

Since $S \geq n$, we see that $S \equiv n$ so that M is a Clifford torus. This completes the proof of the Theorem.

REFERENCES

- [1] J. SIMONS, Minimal varieties in a Riemannian manifold, *Ann. of Math.* 88 (1968), 62–105.
- [2] S. S. CHERN, M. DO CARMO AND S. KOBAYASHI, *Functional analysis and related fields*, Springer, Berlin, Heidelberg, New York, 1970.
- [3] H. B. LAWSON, Local rigidity theorem for minimal hypersurfaces, *Ann. of Math.* 89 (1969), 187–197.
- [4] C. K. PENG AND C. L. TERNG, The scalar curvature of minimal hypersurfaces in sphere, *Math. Ann.* 266 (1983), 105–113.
- [5] C. K. PENG AND C. L. TERNG, Minimal hypersurfaces of sphere with constant scalar curvature, *Ann. of Math. Studies* 103 (1983), 177–198.
- [6] Q. M. CHENG, The rigidity of Clifford torus $S^1(1/\sqrt{n}) \times S^{n-1}(\sqrt{(n-1)/n})$, *Comment. Math. Helvetici* 71 (1996), 60–69.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
HACHIOJI, TOKYO, 192-03
JAPAN

