Minimal k-arc Connected Graph

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SUMMARY

A graph is k-arc-connected if it is necessary to remove at least k arcs in order to disconnect the graph. This paper solves the problem of determining the least number of arcs required in a k-arc-connected graph on n nodes by describing constructions that produce such graphs having $\frac{kn}{2}$ arcs (for kn even) or $\frac{kn+1}{2}$ arcs (for kn odd). These results have application to the practical problem of synthesizing minimum cost, "k-reliable" communication networks.

MINIMAL k-ARC-CONNECTED GRAPHS*

1. INTRODUCTION

In considering the synthesis of reliable communication networks with respect to link failure, the following question seems a natural one to raise. Suppose given the complete, unoriented graph G on n nodes $N = \{x,y,z,\ldots\}$, and let each arc (x,y) of G have associated with it a nonnegative number $c(x,y)$, to be thought of as the cost of installing a communication link between stations x and y. For each $k = 1, 2, \ldots, n-1$, find a minimum cost k-arc-connected spanning subgraph of G. Here the cost of a subgraph H is the sum of the numbers $c(x,y)$ corresponding to arcs of H, a spanning subgraph of G is a subgraph that has the same node set N as G does, and a k-arc-connected graph is one in which at least k arcs must be suppressed in order to disconnect the graph. Thus k might be thought of as the "reliability level" of the communication network, and the practical problem is to minimize cost subject to achieving a stipulated reliability level.

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*This paper was written in 1961 but never published because the authors became aware that Harary was preparing a paper which solved the more general problem of determining the least number of arcs required for k-node-connectivity. Harary's paper later on appeared under the title "The Maximum Connectivity of a Graph" in Proc. Nat. Acad. Sci. 48 (1962), 1142-1146. The authors of the present paper feel that the method of proof, which is quite different from Harary's proof of the more general result, may be of some interest.

For $k = 1$, the problem becomes that of finding a minimum cost spanning subtree of G; there are simple methods known for doing this [2,3]. But for $k > 1$, the situation seems to be quite different. Here we need only mention the fact that, with $k = 2$ and all arc costs 1 or ∞, the problem includes that of determining whether a given graph (the subgraph of unit cost arcs) contains a Hamiltonian cycle. Even with all arc costs unity, an interesting graph-theoretic problem emerges: to determine the minimum number of arcs required for a k-arc-connected graph on n nodes. Here, for $k \geq 2$, there is an obvious lower bound for the number of arcs needed, namely $\frac{kn}{2}$ (for even kn) or $\frac{kn+1}{2}$ (for odd kn), and it is reasonable to ask if this bound is always achieved. We answer this affirmatively by describing two constructions that produce graphs having the minimum number of arcs; one of these constructions is applicable for even k, the other for odd k.

Similar problems arise if one considers k-connectedness* not with respect to arcs but rather with respect to nodes. Thus, for example, one can ask for the smallest number of arcs required in a k-node-connected graph on n nodes. The lower bound mentioned above is unchanged, but very little appears to be known about the problem for nodes (cf. [1], Appendix IV, Problem 11). Since a graph that is k-connected with respect to nodes is k-connected with respect to arcs,

[redacted]

*In the literature on graphs, the phrase "k-connected graph" refers to nodes, see [1,4].

[redacted]

but not always conversely, the fact that the lower bound is achievable in the arc problem is a weaker assertion than the corresponding one for nodes.*

2. CUT SETS OF ARCS

Throughout this and the following sections, a graph is an unoriented one without 1 or 2-circuits, that is, at most one arc joins a pair of nodes and all arcs join distinct nodes. We write $G = [N; \mathcal{A}]$ to mean that the graph G has node set N and arc set $\mathcal{A}$. Nodes are denoted by $x, y, z, \ldots$, and arcs by unordered pairs of nodes, (x,y), $(x,z), \ldots$.

Let $G = [N; \mathcal{A}]$ be a graph on n nodes, $n \geq 2$. A subset $\mathcal{K} \subseteq \mathcal{A}$ is a <u>cut set of arcs</u> in G provided that the graph $G' = [N; \mathcal{A} - \mathcal{K}]$ obtained from G by supressing arcs of $\mathcal{K}$ is disconnected. A graph G is k-<u>arc-connected</u> if every cut set of arcs in G has at least k members; here $0 \leq k < n$. In dealing with k-arc-connectedness, attention can be restricted to cut sets of arcs of the following kind. Let X and $\bar{X} = N - X$ be a partition of the nodes of G into two non-empty sets, and let $(X,\bar{X})$ denote the set of arcs in G that have one end in X, the other end in $\bar{X}$. Thus $(X,\bar{X})$ is a cut set of arcs in G separating the nodes in X from those in $\bar{X}$. Moreover, given any cut set $\mathcal{K} \subsetneq \mathcal{A}$, one can determine $X \subseteq N$ so that $(X,\bar{X}) \subseteq \mathcal{K}$ by the recursive rule:

(a) select a node x and put x in X;

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*See the first footnote.

(b) if x is in X and (x,y) is in $\mathcal{A} - \mathcal{K}$, then put y in X.

The set $\bar{X} = N - X$ thus defined cannot be empty, since $\mathcal{K}$ is a cut set of arcs; it is also clear that $(X,\bar{X}) \subseteq \mathcal{K}$. Thus it suffices to consider cut sets of the form $(X,\bar{X})$, and we shall do this.

3. THE CASE k EVEN

We give a simple construction which furnishes an inductive proof on n, for fixed even k, that there are k-arc-connected graphs on n nodes having $\frac{kn}{2}$ arcs.

Lemma 3.1 below will be used in the construction. Call a set of arcs of G <u>independent</u> if no two arcs of the set have a node in common. The <u>degree</u> of a node x in G is the number of arcs on x.

<u>Lemma 3.1</u>. <u>If each node of a graph G has degree at least $k > 0$ then any arc of G is contained in a set of $\left[\frac{k+1}{2}\right]$ independent arcs.</u>

Here $\left[\frac{k+1}{2}\right]$ denotes the biggest integer in $\frac{k+1}{2}$. A proof can be made by induction on k. The conclusion is obviously valid for $k = 1, 2$. Suppose G is a graph each of whose nodes has degree $\geq k > 2$. Select an arc (x,y) arbitrarily in G, then suppress nodes x,y, and their arcs, to obtain G'. Now each node of G' has degree $\geq k - 2 > 0$. Since $k > 2$, G' contains at least one arc; hence by the induction assumption, G' contains a set of $\left[\frac{k-1}{2}\right]$ independent arcs. The arc (x,y) of G, together with these, gives a set of $\left\lceil \frac{k+1}{2} \right\rceil$ independent arcs of G.

The conclusion of Lemma 3.1 is very weak, but suffices for our purposes in this section. In treating odd k, a strengthened form of Lemma 3.1 will be used. The version given here has the advantage that the construction implicit in its proof is extremely simple: any maximal set of independent arcs will do.

Theorem 3.2. *Let n be a positive integer and k an even integer satisfying* $2 \le k < n$. *Then there is a graph on n nodes that is k-arc-connected and has* $\frac{kn}{2}$ *arcs.*

Let $k = 2p$. If $n = k + 1$, the complete graph on n nodes serves. We now proceed by induction on n, holding k fixed. Thus let G be a k-arc-connected graph on n nodes having np arcs. Then each node of G has degree k and hence by Lemma 3.1 G contains p independent arcs, say

$$(3.1) \qquad (x_1,y_1),\ (x_2,y_2),\ \ldots,\ (x_p,y_p).$$

Now let G' be the graph on n + 1 nodes obtained from G by deleting the arcs (3.1), then adding node z and the arcs

$$(3.2) \qquad (z,x_1),\ \ldots,\ (z,x_p),\ (z,y_1),\ \ldots,\ (z,y_p).$$

The graph G' has $np + 2p - p = (n + 1)p$ arcs. We assert that G' is k-arc-connected. For suppose not, and let $(X,\bar{X})$ by a cut set of arcs in G' containing $k - 1$ or fewer arcs. We may suppose z is in X. If X consists of the single node z, then $(X,\bar{X})$ has k members. Thus X contains a node of G. The cut $(X,\bar{X})$ in G' then produces a cut $(Y,\bar{Y})$ in G, by taking $Y = X - \{z\}$. But the number of arcs in $(Y,\bar{Y})$ is less than or equal to the number in $(X,\bar{X})$, since to each arc (x_i,y_i) of the deleted set (3.1) that is also in $(Y,\bar{Y})$, there corresponds at least one of the added arcs (3.2), either (z,x_i) or (z,y_i), which is in $(X,\bar{X})$. Thus $(Y,\bar{Y})$ has at most $k-1$ members, contradicting the fact that G is k-arc-connected. This proves Theorem 3.2.

4. THE CASE k ODD

For the case of odd k, say $k = 2p + 1$, the analogue of the above construction can fail. The difficulty comes in attempting to make the transition from odd n to even $n + 1$. Here one would start with a k-arc-connected graph G having $\left\lceil \frac{kn+1}{2} \right\rceil$ arcs, so that some node of G has degree $k + 1$, all others have degree k. Lemma 3.1 can be used to select $p + 1$ independent arcs, one of which is on the node x_1 of degree $k + 1$. If it could be shown that the graph G' obtained from G by deleting the independent arcs $(x_1,y_1), \ldots, (x_{p+1},y_{p+1})$, then adding node z and the arcs $(z,x_2), \ldots (z,x_{p+1}), (z,y_1), \ldots, (z,y_{p+1})$, were k-arc-connected, a proof for odd k would be obtained. But this is false, as the following example for $k = 3$ shows. Let G be the graph of Fig. 4.1 below; G has the minimum number of arcs and it can be checked that G is 3-arc-

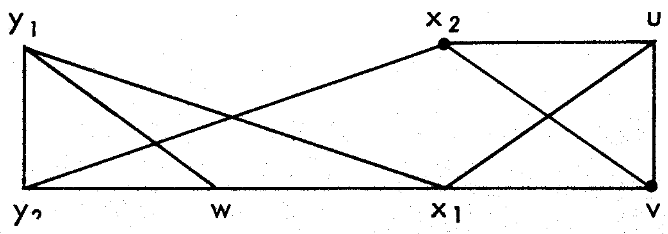

Fig. 4.1.

connected. Let (x_1, y_1), (x_2, y_2) be the candidates for elimination. One then obtains the graph G' of Fig. 4.2, which is only 2-arc-connected.

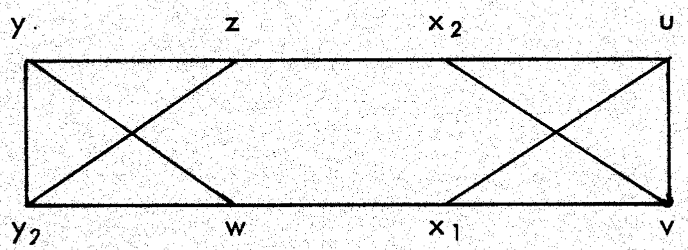

Fig. 4.2.

Fortunately, the troublesome feature exhibited by the preceding example can be avoided by employing a construction that adds two nodes to the graph at each step, instead of one. For this we first need to strengthen Lemma 3.1.

In Lemmas 4.1 and 4.2, k may be either even or odd, although we use them only for odd k.

<u>Lemma 4.1</u>. <u>If each node of a graph</u> G <u>on</u> n <u>nodes has degree at least</u> k, <u>then</u> G <u>contains at least min</u> $([\frac{n}{2}], k)$ <u>independent arcs</u>.

Let $\mathcal{M} \subseteq \mathcal{A}$ be a maximum set of independent arcs in $G = [N;\mathcal{A}]$, i,e., one of maximum cardinality. Say that x in N is <u>covered</u> in $\mathcal{M}$ if x is the end of some member of $\mathcal{M}$, <u>uncovered</u> otherwise. If G contains at most one uncovered node, then $\mathcal{M}$ has $[\frac{n}{2}]$ members. Suppose that G has at least two uncovered nodes, and let u_1, u_2 be two such. Since $\mathcal{M}$ is a maximum independent set, each node x that neighbors an uncovered node must be covered. Let u_i have degree $k_i \geq k$, $i = 1,2$, and separate the k_i neighbors of u_i into two types: (a) those joined together in pairs by arcs of $\mathcal{M}$, (b) those not so joined. Let x be a neighbor of u_1 of type (a), and let y be the neighbor of u_1 for which (x,y) is in $\mathcal{M}$. Then x cannot neighbor u_2, for otherwise the set $\mathcal{M}'$ of arcs obtained from $\mathcal{M}$ by deleting (x,y) and adding the arcs (u_1, y), (x, u_2) is independent and contains more members than $\mathcal{M}$, a contradiction. Hence if m_i is the number of arcs of $\mathcal{M}$ that join type (a) neighbors of u_i, $i = 1,2$, then $\mathcal{M}$ contains at least $m_1 + m_2 + \max(k_1-2m_1, k_2-2m_2) \geq k$ members. Thus, in any event, $\mathcal{M}$ has at least $\min([\frac{n}{2}], k)$ members, proving Lemma 4.1.

We need one other preliminary lemma before proceeding to the proof of Theorem 4.3.

Lemma 4.2. If each node of a graph G *on* n *nodes has degree at least* $k \geq \frac{n}{2}$, *then* G *is* k-*arc-connected. Hence for such* k,n *there are* k-*arc-connected graphs on* n *nodes having* $\left\lceil \frac{kn+1}{2} \right\rceil$ *arcs.*

To prove Lemma 4.2, let $(X,\bar{X})$ be a cut in G. Let X have h members. We have $1 \leq h$ and may assume $h \leq \frac{n}{2}$. Hence by hypothesis, $1 \leq h \leq k$. It follows that [redacted] $(k-h)(h-1) \geq 0$, and hence

$$kh - h(h-1) \geq k. \tag{4.1}$$

But the number of arcs in $(X,\bar{X})$ is greater than or equal to the left hand side of (4.1), since each node of X has degree at least k, and hence at least $k-h+1$ arcs joining it to members of $\bar{X}$. This proves the first part of the lemma. To prove the second part, we need only establish the existence of graphs on n nodes having $\frac{kn}{2}$ arcs (for kn even) or $\frac{kn+1}{2}$ arcs (for kn odd), with each node having degree $\geq k$. This can be accomplished in various ways. For example, the construction of the preceding section does this for even k, and an entirely analogous construction works for odd k.

Theorem 4.3. Let n *be a positive integer and* k *an odd integer satisfying* $3 \leq k < n$. *Then there is a graph on* n *nodes that is* k-*arc-connected and has* $\left\lceil \frac{kn+1}{2} \right\rceil$ *arcs.*

It follows from Lemma 4.2 that for n in the range $k+1 \leq n \leq 2k$, Theorem 4.3 is valid. The construction described below increases n by two at each step. If $n > 2k$ is even, we may start the induction at $2k$ in order to reach n; if $n > 2k$ is odd, we may start at $2k - 1$. We now describe the inductive step.

Suppose $n \geq 2k - 1$ and let G be a k-arc-connected graph on n nodes having the minimum number of arcs. By Lemma 4.1, G contains at least $k - 1$ independent arcs, say

(4.2) $$(x_1,y_1), \ldots, (x_p,y_p), (u_1,v_1), \ldots, (u_p,v_p).$$

Here $k = 2p + 1$. Now form G' by deleting the arcs (4.2), then adding two nodes z,w together with the arcs

(4.3) $$(z,x_1), \ldots, (z,x_p), (z,y_1), \ldots, (z,y_p),$$

(4.4) $$(w,u_1), \ldots, (w,u_p), (w,v_1), \ldots, (w,v_p),$$

(4.5) [redacted] $$(z,w).$$

Observe that G' has [redacted]k more arcs than G does, so that the arc count has gone up appropriately. The proof that G' is k-arc-connected is similar to that given in the proof of Theorem 3.2. Let $(X,\overline{X})$ be a cut set of arcs in G' and suppose, contrary to what we wish to show, that $(X,\overline{X})$ has $k-1$ or fewer members. If both nodes z and w are on one side of this cut, say z and w are in X, then X must surely contain nodes of G. As before, the cut $(Y,\overline{Y})$ in G induced by taking $Y = X - \{z,w\}$ can have at most $k - 1$ members, a contradiction. If z and w are on opposite sides of the cut, say z is in X, w in $\overline{X}$, then both X and $\overline{X}$ contain nodes of G, since z and w each have degree k in G'. Again the cut $(Y,\overline{Y})$ induced in G by defining $Y = X - \{z\}$, $\overline{Y} = \overline{X} - \{w\}$, has no more arcs than does $(X,\overline{X})$, and we have a contradiction. This completes the proof of Theorem 4.3.

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