# Minimal Model Holography

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based on
MRG, R. Gopakumar, arXiv:1011.2986
MRG, T. Hartman, arXiv:1101.2910
MRG, R. Gopakumar, arXiv:1205.2472 ⟵

# Generalities

Duality between Vasiliev higher spin theory and CFT is `simplified' version of AdS/CFT correspondence which may lead to insights into

- conceptual understanding of the AdS/CFT correspondence
- quantum gravity from dual field theory point of view

# Higher spin -- CFT duality

Some years ago, a concrete proposal for such a duality was made:

[Klebanov-Polyakov]
[Sezgin-Sundell]

higher spin theory on AdS4 $\longleftrightarrow$ 3d O(N) vector model in large N limit

Actually different versions, depending on whether vector model fields are bosons or fermions and on whether one considers free or interacting fixed point.

# Checks of the proposal

Recently impressive checks of the duality have been performed, in particular

[Giombi, Yin]
see also Yin's talk

3-point functions of HS fields on AdS4

have been matched to

3-point functions of HS currents in O(N) model to leading order in 1/N.

Also, generalisations to a family of parity-violating theories have now been proposed.

[Giombi et.al.], [Aharony et.al.]

# AdS3 / CFT2

Here: describe 3d/2d CFT version of this duality.

Lower dimensional version interesting

- 2d CFTs well understood
- Higher spin theories simpler in 3d

Also, 3d conformal field theories with unbroken higher spin symmetry and finite number of d.o.f. (finite N) are necessarily free, but this is not the case in 2d.

[Maldacena, Zhiboedov]

# 3d proposal

The 3d/2d proposal takes the form

[MRG, Gopakumar]

**AdS3:**
higher spin theory
with a complex
scalar of mass M

$\longleftrightarrow$

**2d CFT:**
$\mathcal{W}_{N,k}$ minimal models
in large N 't Hooft limit
with coupling $\lambda$

where $\lambda = \dfrac{N}{N+k}$ and $M^2 = -(1-\lambda^2)$

# Scalars

In original version of conjecture there were two scalars.

Given our more detailed understanding of the symmetries (see below), it now seems that one of the scalars should be rather thought of as a non-perturbative state.

cf. also [Chang, Yin]

[This new point of view resolves also some puzzles regarding the structure of the correlation functions.]

[Papadodimas, Raju]
[Chang, Yin]

# Outline

In the rest of the talk I want to explain the proposal in more detail and indicate which consistency checks have been performed.

- The HS theory in 3d
- $\longrightarrow$ Matching the symmetries
- The spectrum
- Conclusions

# The HS theory on AdS3

The AdS3 HS theory can be described very simply.

Recall that pure gravity in AdS3: Chern-Simons theory based on

$$sl(2,\mathbb{R})$$

[Achucarro, Townsend]
[Witten]

Higher spin description: replace

$$sl(2,\mathbb{R}) \rightarrow \mathrm{hs}[\lambda]$$

[Vasiliev]

one hs gauge field for each spin $s = 2, 3, \ldots$

# Higher spin algebra

The higher spin algebra $\mathrm{hs}[\lambda]$ is an infinite dimensional Lie algebra that can be thought of as

$$\mathrm{hs}[\lambda] \equiv sl(\lambda, \mathbb{R})$$

[Bordemann et.al.]
[Bergshoeff et.al.]
[Pope, Romans, Shen]
[Fradkin, Linetsky]

since

$$\mathrm{hs}[\lambda]\Big|_{\lambda=N} / \chi_N \cong sl(N, \mathbb{R})$$ for integer N.

# Asymptotic symmetries

For these higher spin theories asymptotic symmetry algebra can be determined following Brown & Henneaux, leading to classical

$$\mathcal{W}_\infty[\lambda] \ \text{ algebra}$$

[Henneaux & Rey]
[Campoleoni et al]
[MRG, Hartman]

Extends algebra `beyond the wedge':

pure gravity: $sl(2,\mathbb{R}) \quad \rightarrow \quad \text{Virasoro}$

higher spin: $\text{hs}[\lambda] \quad \rightarrow \quad \mathcal{W}_\infty[\lambda]$ [Figueroa-O'Farrill et.al.]

(generated by one Virasoro primary for each spin $s = 2, 3, ..$)

# Dual CFT

By the usual arguments, dual CFT should therefore have

$$\mathcal{W}_\infty[\lambda] \text{ symmetry.}$$

Basic idea (see below):

$$\mathcal{W}_\infty[\lambda] \; = \; \lim_{N\to\infty} \mathcal{W}_N \qquad \text{with} \qquad \lambda = \frac{N}{N+k} \; .$$

‘t Hooft limit of 2d CFT!

# The minimal models

The minimal model CFTs are the cosets

$$\mathcal{W}_{N,k}: \qquad \frac{su(N)_k \oplus su(N)_1}{su(N)_{k+1}}$$

e.g. Ising model (N=2, k=1)
tricritical Ising (N=2, k=2)
3-state Potts (N=3,k=1),..

with central charge

$$c_N(k) = (N-1)\Big[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\Big] \ .$$

General N: higher spin analogue of Virasoro minimal models. [Spin fields of spin s=2,3,..,N.]

# Relation of symmetries

On the face of it, the two symmetries

$$\mathcal{W}_\infty[\lambda] \qquad \text{vs} \qquad \lim_{N,k\to\infty} \mathcal{W}_{N,k}$$

appear to be quite different. However, the asymptotic symmetry analysis only determines the classical symmetry algebra, i.e. the commutative Poisson algebra.

In order to understand above relation, we need to understand the quantum version of this algebra.

# Quantum symmetry

The full structure of the quantum algebra can actually be determined completely. [MRG, Gopakumar]

There are two steps to this argument. To illustrate them consider an example. Naive quantisation of classical algebra leads to

$$[W^3_m, W^3_n] = 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{8N_3}{c}(m-n)\,(LL)_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}$$

spin-3 field

non-linear term

# Jacobi identity

$$[W^3_m, W^3_n] = 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n}$$
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Jacobi identity determines quantum correction

$$[W^3_m, W^3_n] = 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n}$$
$$+\frac{8N_3}{c+\frac{22}{5}}(m-n)\,\Lambda^{(4)}_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}$$

where

$$\Lambda^{(4)}_n = \sum_p :L_{n-p}L_p: + \tfrac{1}{5}x_n L_n$$

Similar considerations apply for the other commutators.

# Structure constants

The second step concerns structure constants. The fields can be rescaled so that

$$W^3 \cdot W^3 \ \sim \ \tfrac{c}{3} \cdot \mathbf{1} + 2 \cdot L + \tfrac{32}{(5c+22)} \cdot \Lambda^{(4)} + 4 \cdot W^4$$

but then coupling constant

$$W^3 \cdot W^4 \ \sim \ C^4_{33} \cdot W^3 + \cdots$$

characterises algebra. Classical analysis determines

[MRG, Hartman]

$$\left(C^4_{33}\right)^2 = \frac{64}{5} \, \frac{\lambda^2 - 9}{\lambda^2 - 4} + \mathcal{O}(\tfrac{1}{c}) \ .$$

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Requirement that representation theory agrees for $\lambda = N$ with $\mathcal{W}_N$ :

$$\left(C_{33}^{4}\right)^{2} = \frac{64\,(c+2)\,(\lambda-3)\,\big(c(\lambda+3)+2(4\lambda+3)(\lambda-1)\big)}{(5c+22)\,(\lambda-2)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)} .$$

[Note: $\mathrm{hs}[\lambda]\big|_{\lambda=N} \cong sl(N,\mathbb{R})$ implies $\mathcal{W}_\infty[\lambda]|_{\lambda=N} = \mathcal{W}_N$.]

# Higher Structure Constants

Similarly, higher structure constants can be determined

[Blumenhagen, et.al.] [Hornfeck]

$$C_{33}^4 C_{44}^4 = \frac{48\big(c^2(\lambda^2-19)+3c(6\lambda^3-25\lambda^2+15)+2(\lambda-1)(6\lambda^2-41\lambda-41)\big)}{(\lambda-2)(5c+22)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)}$$

$$(C_{34}^5)^2 = \frac{25(5c+22)(\lambda-4)\big(c(\lambda+4)+3(5\lambda+4)(\lambda-1)\big)}{(7c+114)(\lambda-2)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)}$$

$$\begin{aligned} C_{45}^5 = & \frac{15}{8(\lambda-3)(c+2)(114+7c)\big(c(\mu+3)+2(4\lambda+3)(\lambda-1)\big)}\, C_{33}^4 \\ & \times \Big[c^3(3\lambda^2-97)+c^2(94\lambda^3-467\lambda^2-483)+c(856\lambda^3-5192\lambda^2+4120) \\ & \quad +216\lambda^3-6972\lambda^2+6756\Big] \,. \end{aligned}$$

# Higher Structure Constants

Actually, all of them can be rewritten more simply as

[MRG, Gopakumar]

$$C_{44}^{4} = \frac{9(c+3)}{4(c+2)}\,\gamma - \frac{96(c+10)}{(5c+22)}\,\gamma^{-1}$$

$$(C_{34}^{5})^2 = \frac{75(c+7)(5c+22)}{16(c+2)(7c+114)}\,\gamma^2 - 25$$

$$C_{45}^{5} = \frac{15\,(17c+126)(c+7)}{8\,(7c+114)(c+2)}\,\gamma - 240\frac{(c+10)}{(5c+22)}\,\gamma^{-1}$$

where

$$\gamma^2 \equiv \left(C_{33}^{4}\right)^2$$

These structure constants (and probably all) are actually determined in terms of $\gamma^2$ by Jacobi identity.

[Candu, MRG, Kelm, Vollenweider, to appear]

# Quantum algebra

Thus full quantum algebra characterised by two free parameters: central charge c and [MRG, Gopakumar]

$$\gamma^2 \ (\text{rather than } \lambda)$$

But

$$(C_{33}^4)^2 \equiv \gamma^2 = \frac{64(c+2)(\lambda-3)\big(c(\lambda+3)+2(4\lambda+3)(\lambda-1)\big)}{(5c+22)(\lambda-2)\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)} \ .$$

Thus there are three roots that lead to the same algebra:

$$\mathcal{W}_\infty[\lambda_1] \cong \mathcal{W}_\infty[\lambda_2] \cong \mathcal{W}_\infty[\lambda_3] \qquad \text{at fixed } c$$

\`Triality’

# Triality

In particular,

$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty[\tfrac{N}{N+k}] \cong \mathcal{W}_\infty[-\tfrac{N}{N+k+1}] \quad \text{at } c = c_{N,k}$$

minimal model

asymptotic symmetry algebra of hs theory

This is even true at finite N and k, not just in the 't Hooft limit!

This triality generalises level-rank duality of coset models of [Kuniba, Nakanishi, Suzuki] and [Altschuler, Bauer, Saleur].

# Higher spin symmetry

Since $\mathcal{W}_\infty[\lambda]$ algebra is non-linear, it does not contain higher spin algebra $\mathrm{hs}[\lambda]$ as a subalgebra at finite c:

Finite c: hs-symmetry is `broken', but non-linear deformation remains true symmetry.

cf. [Maldacena, Zhiboedov]

NB. Exception for $\lambda = 1$ --- free theory.

# Symmetries

So the symmetries suggest that we should have

HS on AdS3 $\longleftrightarrow$ 2d CFT with

=CS with $\mathrm{hs}[\lambda]$ $\mathcal{W}_\infty[\lambda]$ symmetry

$$\lambda = \frac{N}{N+k}$$

=

minimal model $\mathcal{W}_{N,k}$

Semiclassical limit: take c large --- ‘t Hooft limit!

# Spectrum

Higher spin fields themselves correspond only to the vacuum representation of the W-algebra!

To see this, calculate partition function of massless spin s field on thermal AdS3

[MRG, Gopakumar, Saha]

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \ . \qquad q = \exp\left(-\frac{1}{k_{\rm B}T}\right)$$

[Generalisation of Giombi, Maloney & Yin calculation to higher spin, using techniques developed in David, MRG, Gopakumar.]

# 1-loop partition function

The higher spin fields therefore contribute

$$Z_{\rm hs} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \ .$$

MacMahon function!

This reproduces precisely contribution to the partition function of dual CFT in ‘t Hooft limit coming from the vacuum representation

--- not a consistent CFT by itself.....

# Representations

Indeed, the full CFT also has the representations labelled by

$$(\rho, \mu; \nu)$$

rep of $\quad su(N)_k \quad su(N)_1 \quad su(N)_{k+1}$

Compatibility constraint: $\rho + \mu - \nu \in \Lambda_R(su(N))$

fixes $\mu$ uniquely: label representations by $(\rho; \nu)$ .

# Simple representations

Simplest reps that generate all W-algebra reps upon fusion: (0;f) and (f;0) (& conjugates).

't Hooft limit: $\boxed{h(\mathrm{f};0)} = \frac{1}{2}(1+\lambda)$ $\boxed{h(0;\mathrm{f})} = \frac{1}{2}(1-\lambda)$

semiclassical: (for fixed $N$) $\boxed{h(\mathrm{f};0)} = \frac{1}{2}(1-N)$ $\boxed{h(0;\mathrm{f})} = -\frac{c}{2N^2}$

dual to perturbative scalar

non-perturbative

# Proposal

Contribution from all representations of the form (*;0) is accounted for by adding to the hs theory a complex scalar field of the mass

[MRG, Gopakumar]

$$-1 \leq M^2 \leq 0 \quad \text{with} \quad M^2 = -(1-\lambda^2)\ .$$

[Compatible with hs symmetry since hs theory has massive scalar multiplet with this mass.]

[Vasiliev]

Corresponding conformal dimension then

$$M^2 = \Delta(\Delta - 2) \ \Rightarrow \ \Delta = 1 + \lambda\ .$$

(standard quantisation)

# Checks of proposal

Main evidence from 1-loop calculation:

Contribution of single real scalar to thermal partition function is

[Giombi, Maloney & Yin]

$$Z^{(1)}_{\text{scalar}} = \prod_{l=0,l'=0}^{\infty} \frac{1}{(1-q^{h+l}\bar{q}^{h+l'})} \ ,$$

where

$$h = \frac{1}{2}\Delta = \frac{1}{2}(1+\lambda) \ .$$

# Total 1-loop partition function

The total perturbative 1-loop partition function of our AdS theory is then

$$Z^{(1)}_{\rm pert} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2} \times \prod_{l,l'=0}^{\infty}\frac{1}{(1-q^{h+l}\bar{q}^{h+l'})^2}$$

We have shown analytically that this agrees exactly with CFT partition function of (*;0) representations in 't Hooft limit!

[MRG, Gopakumar]
[MRG, Gopakumar, Hartman, Raju]

**Strong consistency check!**

# Non-perturbative states

The remaining states, i.e. those of the form

$$(*; \nu) \qquad \text{with } \nu \neq 0$$

seem to correspond to conical defect solutions (dressed with perturbative scalar excitations).

[Castro, Gopakumar, Gutperle, Raeymaekers]
[MRG, Gopakumar]

# Generalisations

Various generalisations of the duality have also been proposed and tested, in particular

- supersymmetric version

  [Creutzig, Hikida, Ronne]
  [Candu, MRG]
  [Henneaux, Gomez, Park, Rey]
  [Hanaki, Peng]
  [Ahn]

- orthogonal (instead of unitary) groups

  [Ahn], [MRG, Vollenweider]

# Classical solutions

Another very interesting development concerns the classical solutions of the HS theory.

[Gutperle, Kraus, et.al.]

Very interesting lessons (that are maybe applicable more generally): because of large HS gauge symmetry, usual GR tensors are not gauge invariant any longer!

Characterisation of regular classical solutions is therefore subtle!

# Black holes

However CS description allows for HS gauge invariant formulation. Using this point of view, black hole solutions for these theories have been constructed.

[Gutperle, Kraus, et.al.]

Their entropy can be matched to dual CFT description.

[Kraus, Perlmutter]
[MRG, Hartman, Jin]

# Conclusions

Given strong evidence for duality between

**AdS3:**
higher spin theory
with a complex
scalar of mass M

$\longleftrightarrow$

**2d CFT:**
$\mathcal{W}_{N,k}$ minimal models
in large N 't Hooft limit
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where $\lambda = \dfrac{N}{N+k}$ and $M^2 = -(1-\lambda^2)$

# Conclusions

- The duality is non-supersymmetric.
- It allows for detailed precision tests: spectrum, correlation functions, etc.
- Can shed maybe interesting light on conceptual aspects of quantum gravity.

# Main challenges

- Prove the duality in the 't Hooft limit. [Main challenge: understand scalars from CS point of view.]
- Understand phase structure of partition function.

  cf. [Shenker, Yin], [Banerjee, Hellerman, Maltz, Shenker]
- Reproduce calculable quantum corrections of CFT from Higher Spin Quantum Gravity on AdS3.
- Embed hs theories on AdS3 into string theory, e.g. into the D1-D5 system.

  cf. [Giombi et.al.], [Chang, Minwalla, Sharma, Yin]