Notre Dame Journal of Formal Logic
Volume 36, Number 2, Spring 1995

Minimal Non-contingency Logic

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Abstract Simple finite axiomatizations are given for versions of the modal logics **K** and **K4** with non-contingency (or contingency) as the sole modal primitive. This answers two questions of I. L. Humberstone.

Modal logic is supposed to be the study of principles of reasoning involving necessity, possibility, impossibility, contingency, non-contingency, and related notions. It has become customary to construct systems in which necessity alone, or necessity and possibility, are treated as primitive connectives. In most such systems the modal concepts mentioned are all interdefinable, so that these systems can be regarded as systematizing, at least indirectly, reasoning involving all of them. Nevertheless, systems in which contingency or non-contingency are treated as primitive connectives have certain technical and philosophical interest (see Montgomery and Routley [5]). Such systems have been investigated in Montgomery and Routley [5], [6], [7], and Mortensen [4]. (See also Brogan [1] for a discussion of Aristotle's logic of contingency.) The investigations were facilitated by the observation that necessity is definable in the systems considered. For example, in extensions of the system **T**, *necessarily A* is equivalent to *A and not contingently A*. Cresswell [2] provides examples of systems not containing **T** in which necessity is otherwise definable. In the contingency version of the "minimal" modal system **K**, however, necessity is not definable, and so a general account of the logic of contingency has not emerged so quickly. Humberstone [3] solves this problem by showing how to modify standard completeness arguments for necessity systems to a system in which non-contingency is primitive. The axiomatization in Section 3 of [3], however, contains a somewhat unwieldy rule schema, and the author asks whether a finite axiomatization is possible. This note answers that question affirmatively by presenting a considerably simpler completeness proof that does not require the unwieldy schema. It also solves another problem raised in [3], axiomatizing the non-contingency version of **K4**.

Our base language is that of classical propositional logic with $\vee$ and $\neg$ as primitive connectives. We add two "modal" connectives, Δ and ∇, for contingency

Received March 31, 1995; revised May 9, 1995

and non-contingency, respectively. To facilitate comparison with [3], we take non-contingency as primitive and define contingency by the condition: $\nabla A = \neg\Delta A$. A Kripke model is a structure (W, R, V), where W is a nonempty set (the "worlds"), R is a binary relation ("accessibility") on W, and V is a function from sentence letters to sets of worlds. The notion *A is true in $\mathcal{M}$ at w* (written $(\mathcal{M}, w) \models A$) is defined in the standard way. The clause for Δ is as follows:

> $(\mathcal{M}, w) \models \Delta B$ if and only if either, for all $v \in W$ such that wRv, $(\mathcal{M}, v) \models B$, or, for all $w \in W$ such that wRv, not $(\mathcal{M}, v) \models B$.

As usual, A is false in $\mathcal{M}$ at w (written $(\mathcal{M}, w) \not\models A$) iff it is not true in $\mathcal{M}$ at w. A formula that is true at every world in a model $\mathcal{M}$ is said to be *true in $\mathcal{M}$*. If it is true in all the members of some class C of models, it is said to be *valid in C*. If it is valid in all Kripke models, it is said to be *valid*. *Minimal non-contingency logic* is the set of all formulas valid according to this definition.

Now let **K4**Δ be the set of all formulas provable in the following axiom system.

PL All substitution instances of tautologies

A1 $\Delta\neg A \to \Delta A$

A2 $\Delta A \wedge \nabla(A \wedge B) \to \nabla B$

A3 $\Delta A \wedge \nabla(A \vee B) \to \Delta(\neg A \vee C)$

RΔ If $\vdash A$ then $\vdash \Delta A$

RE If $\vdash A \leftrightarrow B$ then $\vdash \Delta A \leftrightarrow \Delta B$

MP If $\vdash A$ and $\vdash A \to B$ then $\vdash B$.

Note that all these schemas except **PL** are finite, and **PL** can be replaced by any finite set of axiom schemas that generate the tautologies using **MP**.

In the remainder of this paper we assume some familiarity with [3]. We establish first that **K4**Δ is contained in the system **NC** of [3]. Indeed every theorem of **K** can be proved using only **PL**, **MP**, $\Delta\neg$, and $(\text{NCR})_i$ for $i = 0, 1, 2$. The rule **R**Δ is just $(\text{NCR})_0$ and the rule **RE** is derived (under the label *Rcong*) in Section 2 of [3]. Similarly, axiom schema **A1** is just $\Delta\neg$ and **A2** (in the presence of **PL** and **MP**) is interderivable with the schema $\Delta A \wedge \Delta B \to \Delta(A \wedge B)$. This is an instance of the Principle 2.2 that, by an argument in Section 3 of [3] is provable from $(\text{NCR})_2$. It remains only to prove **A3**. By **PL**, $\vdash A \to (A \vee B)$ and $\vdash \neg A \to (\neg A \vee C)$, and so by $(\text{NCR})_1$, $\vdash \Delta A \to (\Delta(A \vee B) \vee \Delta(\neg A \vee C))$. By **PL** and **MP**, $\vdash \Delta A \wedge \nabla(A \vee B) \to \Delta(\neg A \vee C)$ i.e., **A3** is provable.

The principal result of this note can be expressed as follows.

Theorem 1 (completeness of **K4**Δ) **K4**Δ = *minimal non-contingency logic.*

The soundness (i.e., the "$\subseteq$") half of the theorem follows from the observation above that **K4**Δ is contained in **NC** and the proof in [3] of the soundness of **NC**. To prove the sufficiency (i.e., the "$\supseteq$") half of the theorem, we show that every nontheorem is false in some model. In fact, as is often the case, we can show that there is a single "canonical" model which falsifies all the nontheorems (and which satisfies all consistent sets). Our construction of the canonical model uses an auxiliary function, λ (playing the same role as its namesake, constructed in Section 3 of [3]). If x is a maximal consistent set of formulas, then $\lambda(x) = \{A : \text{for every formula } B, \Delta(A \vee B) \in x\}$.

If x is maximal consistent, then the following properties are satisfied.

Property P1 $\lambda(x)$ is nonempty.

Proof: Take a tautology A, then for any formula $B, \vdash A \vee B$. By $R\Delta, \vdash \Delta(A \vee B)$. Since x is maximal consistent, $\Delta(A \vee B) \in x$. Hence $A \in \lambda(x)$.

Property P2 If $A \in \lambda(x)$ and $\vdash A \rightarrow B$ then $B \in \lambda(x)$.

Proof: Take an arbitrary formula C. We must show $\Delta(B \vee C) \in x$. Since $A \in \lambda(x)$, $\Delta(A \vee (B \vee C)) \in x$. Since $\vdash A \rightarrow B, \vdash (A \vee (B \vee C)) \leftrightarrow (B \vee C)$, and so, by **RE**, $\Delta(B \vee C) \in x$, as was to be proved.

Property P3 If $\Delta A \in x$ and $A \notin \lambda(x)$ then $\neg A \in \lambda(x)$.

Proof: Suppose $\Delta A \in x$ and $A \notin \lambda(x)$ but $\neg A \notin \lambda(x)$. Then there is some formula B such that $\Delta(\neg A \vee B) \notin x$. Since $A \notin \lambda(x)$, there is also a C such that $\Delta(A \vee C) \notin x$. By definition of ∇ and maximal consistency of x, $\nabla(A \vee C) \in x$. Since $\Delta A \in x$, this implies $\Delta A \wedge \nabla(A \vee C) \in x$. By **A3**, $\Delta(\neg A \vee B) \in x$. This contradicts the earlier conclusion, and so the supposition is false, and the claim is true.

Property P4 If $A \in \lambda(x)$ and $B \in \lambda(x)$ then $(A \wedge B) \in \lambda(x)$.

Proof: Suppose $A \in \lambda(x)$ and $B \in \lambda(x)$, but $(A \wedge B) \notin \lambda(x)$. Then there is some formula C such that $\Delta((A \wedge B) \vee C) \notin x$. By definition of ∇, $\nabla((A \wedge B) \vee C) \in x$. By **RE**, $\nabla((A \vee C) \wedge (B \vee C)) \in x$. Since $A \in \lambda(x)$, $\Delta(A \vee C) \in x$. Since x is maximal consistent, $\Delta(A \vee C) \wedge \nabla((A \vee C) \wedge (B \vee C)) \in x$. By **A2**, $\nabla(B \vee C) \in x$, and so $\Delta(B \vee C) \notin x$. But this contradicts the assumption that $B \in \lambda(x)$, so the supposition is false, and the claim is true.

Let W be the set of all maximal consistent sets of formulas. For all $u, v \in W$, let uRv iff $\lambda(u) \subseteq v$ and, for all sentence letters q, let $V(q) = \{w \in W : q \in w\}$. The canonical model is the model $\mathcal{M} = (W, R, V)$.

Lemma 2 *If $\mathcal{M} = (W, R, V)$ and $w \in W$, then $(\mathcal{M}, w) \models A$ iff $A \in w$.*

Proof: By induction on A. We do the case $A = \Delta B$. First, suppose $A \in w$. By **P3**, either B or $\neg B$ is in $\lambda(w)$. By the definition of R, then, either $\forall v(wRv \Rightarrow B \in v)$ or $\forall v(wRv \Rightarrow B \notin v)$. By induction hypothesis, either $\forall v(wRv \Rightarrow (\mathcal{M}, v) \models B)$ or $\forall v(wRv \Rightarrow (\mathcal{M}, v) \not\models B)$. By the truth definition, $(\mathcal{M}, w) \models A$, as required. Conversely, suppose $A \notin w$. Let $x_1 = \lambda(w) \cup \{B\}$ and let $x_2 = \lambda(w) \cup \{\neg B\}$. Both of these are consistent. For, if x_1 were not, we would have $\vdash C_1 \wedge \ldots \wedge C_n \rightarrow \neg B$, where $C_1, \ldots, C_n \in \lambda\ (w)$. (By **P1**, we may assume without loss of generality that $n \geq 1$.) By $n - 1$ applications of **P4**, $(C_1 \wedge \ldots \wedge C_n) \in \lambda(w)$, and therefore by **P2**, $\neg B \in \lambda(w)$. This implies $\Delta\neg B \in w$, which, by **A1**, implies $\Delta B \in w$, contradicting the supposition that $A \notin w$. The argument for x_2 is similar. Thus W contains maximal consistent sets u and v containing x_1 and x_2, respectively. By definition of R, wRu, and wRv. By induction hypothesis, $(\mathcal{M}, u) \models$B and $(\mathcal{M}, v) \not\models B$. By truth definition $(\mathcal{M}, w) \not\models \Delta B$, as required.

To prove the theorem it is sufficient to observe that, if A is a nontheorem, then $\neg A$ is consistent, and so $\{\neg A\}$ can be expanded to a maximal consistent set w. By the lemma above $(\mathcal{M}, w) \models \neg A$, and so $\mathcal{M}$ falsifies A, as required.

More generally, the argument here establishes that every extension of **K4**Δ is complete with respect to some class of (non-contingency) Kripke models. It can also be adapted to provide special completeness results for particular non-contingency logics. Consider, for example the question raised in Section 4 of [3] of axiomatizing the logic determined by the class of transitive models. Let **K4**Δ be the formulas provable in the axiom system obtained by adding the schema $\Delta A \rightarrow \Delta(\Delta A \vee B)$ to the system for **K4**Δ and let transitive non-contingency logic be the formulas valid in all transitive models. Then we can show the following.

Theorem 3 (completeness of **K4**Δ) **K4**Δ = *transitive non-contingency logic.*

Proof: To prove soundness it is sufficient to show that the new schema valid in the transitive models. Suppose there is a transitive model $\mathcal{M} = (W, R, V)$ and a world $w \in W$ such that $(\mathcal{M}, w) \not\models \Delta A \rightarrow \Delta(\Delta A \vee B)$. Then $(\mathcal{M}, w) \models \Delta A$ but $(\mathcal{M}, w) \not\models \Delta(\Delta A \vee B)$. The former condition implies that A is either true at all worlds accessible from w or false at all such worlds. The latter condition implies that for some v such that wRv, $(\mathcal{M}, v) \not\models \Delta A \vee B$, which implies that $(\mathcal{M}, v) \not\models \Delta A$. Thus there is a world u_1 accessible from v at which A is true and a world u_2 accessible from v at which A is false. Since $\mathcal{M}$ is transitive, however, u_1 and u_2 are both accessible from w, contradicting our earlier conclusion. Since $\mathcal{M}$ and w are arbitrary the new axiom is valid.

To prove sufficiency, we may show that the canonical model (defined as above) is transitive. Suppose u and v are worlds in the canonical model such that uRv and vRw and suppose that, for every formula B, $\Delta(A \vee B) \in u$. Then by the new schema we have that, for every formulas B and C, $\Delta(\Delta(A \vee B) \vee C) \in u$. Since uRv, $\Delta(A \vee B) \in v$ for every formula B. Since vRw, $A \in w$. Thus, we have shown that $\Delta(A \vee B) \in u$ for all formulas B implies $A \in w$, which is exactly the condition required for uRw.

Acknowledgments I have benefited from comments of Lloyd Humberstone.

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