# MINIMAL PAIRS, INERTIA DEGREES, RAMIFICATION DEGREES AND IMPLICIT CONSTANT FIELDS 

ARPAN DUTTA


#### Abstract

An extension $(K(X) \mid K, v)$ of valued fields is said to be valuation transcendental if we have equality in the Abhyankar inequality. Minimal pairs of definition are fundamental objects in the investigation of valuation transcendental extensions. In this article, we associate a uniquely determined positive integer with a valuation transcendental extension. This integer is defined via a chosen minimal pair of definition, but it is later shown to be independent of the choice. Further, we show that this integer encodes important information regarding the implicit constant field of the extension $(K(X) \mid K, v)$.


## 1. Introduction

Throughout this article we will assume that $(\bar{K}(X) \mid K, v)$ is an extension of valued fields, where $\bar{K}$ is a fixed algebraic closure of $K$ and $X$ is an indeterminate. The extension $(K(X) \mid K, v)$ satisfies the famous Abhyankar inequality:

$$
\begin{equation*}
\text { rat rk } v K(X) / v K+\operatorname{trdeg}[K(X) v: K v] \leq 1, \tag{1}
\end{equation*}
$$

where $v K$ and $K v$ denote respectively the value group and residue field of $(K, v)$ and rat $\mathrm{rk} v K(X) / v K$ is the $\mathbb{Q}$-dimension of the divisible hull $\mathbb{Q} \otimes_{\mathbb{Z}}(v K(X) / v K)$. The above inequality is a consequence of [4, Chapter VI, $\S 10.3$, Theorem 1]. The extension $(K(X) \mid K, v)$ is said to be valuation transcendental if we have equality in (1). The extension is said to be value transcendental if we have rat $\operatorname{rk} v K(X) / v K=1$ and residue transcendental if $\operatorname{trdeg}[K(X) v: K v]=1$. Throughout this article, we will assume that the extension $(K(X) \mid K, v)$ is valuation transcendental.

Minimal pairs of definition have been used with great success in the study of valuation transcendental extensions [cf. 1, 2, 3, 6]. A pair $(a, \gamma) \in \bar{K} \times v \bar{K}(X)$ is said to be a minimal pair of definition for $v$ over $K$ if it satisfies the following conditions:
$($ MP1 $) v(X-a)=\gamma=\max v(X-\bar{K})$,
(MP2) $v(a-b) \geq \gamma \Longrightarrow[K(b): K] \geq[K(a): K]$ for all $b \in \bar{K}$,
where

$$
v(X-\bar{K}):=\{v(X-c) \mid c \in \bar{K}\} .
$$

A valuation transcendental extension always admits a minimal pair of definition. It has been observed in [8, Theorem 3.11] that $(K(X) \mid K, v)$ is value transcendental if and only if $\gamma \notin v \bar{K}$, that is, if and only if $\gamma$ is not a torsion element modulo $v K$. Further, it follows from [8, Lemma 3.3] that $(K(X) \mid K, v)$ is value (residue) transcendental if and only if $(L(X) \mid L, v)$ is also value (residue) transcendental, where $L$ is an arbitrary algebraic extension of $K$.

The goal of this article is twofold:

- associate a uniquely determined positive integer with a valuation transcendental extension,

[^0]- show that the said integer encodes important information regarding the implicit constant field of the extension (defined later).
We first prove the following result in Section 3:
Theorem 1.1. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Then

$$
\begin{equation*}
(v K(a, X): v K(X))[K(a, X) v: K(X) v]=j, \tag{2}
\end{equation*}
$$

where $v\left(a-a_{i}\right) \geq \gamma$ for exactly $j$ many conjugates $a_{i}$ of a over $K$, including a itself and counting multiplicities.

Given a minimal pair of definition $(a, \gamma)$ for $v$ over $K$, we will denote the integer $j$ as defined in Theorem 1.1 by $j(a, K, \gamma)$. When $K$ and $\gamma$ are implicitly understood, we will simply denote it by $j(a)$. We mention here that when the valuation $v$ is induced by a pseudo monotone sequence $M$ in $(K, v)$, the integer $j(a)$ is also referred to as the dominating degree of the minimal polynomial of $a$ over $K$ with respect to $M$ [cf. 10, 5 .

For any residue transcendental extension $(K(X) \mid K, v)$ with a minimal pair of definition $(a, \gamma)$, we construct a value transcendental extension $w$ of $v$ to $\bar{K}(X)$ with a minimal pair of definition $(a, \Gamma)$ such that $j(a, K, \gamma)=j(a, K, \Gamma)$. This observation is then used to obtain the following result:

Theorem 1.2. Take minimal pairs of definition $(a, \gamma)$ and $\left(a^{\prime}, \gamma\right)$ for $v$ over $K$. Then

$$
j(a)=j\left(a^{\prime}\right) .
$$

Theorem 1.2 illustrates that the integer $j(a)$ depends solely on the valued extension $(\bar{K}(X) \mid K, v)$ and is independent of the choice of the minimal pair of definition for $v$ over $K$. Hence the notation $j(v, K)$ would be more suited as it reflects the independence of $j$. However, for the sake of continuity we persist with the notation $j(a, K, \gamma)$ for the remainder of the article.

Theorem 1.1 is then applied to the computation of the implicit constant field of the extension $(K(X) \mid K, v)$. Given an extension of $v$ to $\overline{K(X)}$, the implicit constant field of the extension $(K(X) \mid K, v)$ is defined as

$$
I C(K(X) \mid K, v):=\bar{K} \cap K(X)^{h},
$$

where $K(X)^{h}$ is the henselization of $K(X)$ [cf. Section 2]. Implicit constant fields were introduced by Kuhlmann in [8] to construct extensions of $v$ to $K(X)$ with prescribed value groups and residue fields. The problem of the explicit computation of implicit constant fields for valuation transcendental extensions was considered in [6], where it was studied via minimal pairs of definition. Given a minimal pair of definition $(a, \gamma)$ for $v$ over $K$, it has been observed in [6, Theorem 1.1] that

$$
I C(K(X) \mid K, v) \subseteq K(a)^{h} .
$$

Under the additional assumptions that $a$ is separable over $K$ and there is a unique extension of $v$ from $K$ to $K(a)$, we observe in [6, Theorem 1.3] that

$$
I C(K(X) \mid K, v) \subsetneq K(a)^{h} \text { whenever } \gamma \leq \operatorname{kras}(a, K),
$$

where

$$
\operatorname{kras}(a, K):=\max \{v(a-\sigma a) \mid \sigma \in \operatorname{Gal}(\bar{K} \mid K) \text { and } \sigma a \neq a\}
$$

In this article, we observe that $j(a)$ is a more natural candidate for the investigation of $I C(K(X) \mid K, v)$ than kras $(a, K)$. Specifically, in Theorem 5.2 we show that

$$
j(a) \text { divides }\left[K(a)^{h}: I C(K(X) \mid K, v)\right] .
$$

This observation is independent of the separability of $a$ and also does not depend on the number of extensions of $v$ from $K$ to $K(a)$. When $(K, v)$ is a defectless field [cf. Section 2], then we obtain that

$$
j(a)=\left[K(a)^{h}: I C(K(X) \mid K, v)\right] .
$$

Given an extension of $v$ to $\overline{K(X)}$, we further observe in [6. Theorem 1.1] that

$$
\begin{equation*}
\left(K(a) \cap K^{r}\right)^{h} \subseteq I C(K(X) \mid K, v) \tag{3}
\end{equation*}
$$

when $v$ is value transcendental, and

$$
\begin{equation*}
\left(K(a) \cap K^{i}\right)^{h} \subseteq I C(K(X) \mid K, v) \tag{4}
\end{equation*}
$$

when $v$ is residue transcendental, where $K^{r}$ and $K^{i}$ denote the absolute ramification field and absolute inertia field of $(K, v)$ [cf. Section 2]. The fact that $K^{i} \subseteq K^{r}$ implies that (3) gives a tighter bound that (4). It is a natural question to inquire whether (3) also holds when $v$ is residue transcendental. We prove the following result using Theorem 1.1)

Theorem 1.3. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$ and fix an extension of $v$ to $\overline{K(X)}$. Then

$$
\left(K(a) \cap K^{r}\right)^{h} \subseteq I C(K(X) \mid K, v) \subseteq K(a)^{h} .
$$

## 2. Preliminaries

Throughout this article, we denote the value of an element $a$ by $v a$ and its residue by $a v$. The valuation ring of a valued field ( $K, v$ ) will be denoted by $\mathcal{O}_{K}$. The compositum of two fields $k_{1}$ and $k_{2}$ contained in some overfield $\Omega$ will be denoted by $k_{1} \cdot k_{2}$.

Fixing an extension $w$ of $v$ to $\bar{K}$, we define the following distinguished groups:

$$
\begin{aligned}
G^{d(w)} & :=\left\{\sigma \in \operatorname{Gal}(\bar{K} \mid K) \mid w \circ \sigma=w \text { on } K^{\text {sep }}\right\}, \\
G^{i(w)} & :=\left\{\sigma \in \operatorname{Gal}(\bar{K} \mid K) \mid w(\sigma a-a)>0 \text { for all } a \in \mathcal{O}_{K^{\text {sep }}}\right\}, \\
G^{r(w)} & :=\left\{\sigma \in \operatorname{Gal}(\bar{K} \mid K) \mid w(\sigma a-a)>w a \text { for all } a \in K^{\text {sep }} \backslash\{0\}\right\},
\end{aligned}
$$

where $K^{\text {sep }}$ denotes the separable-algebraic closure of $K$. The corresponding fixed fields in $K^{\text {sep }}$ will be denoted by $K^{d(w)}, K^{i(w)}$ and $K^{r(w)}$ and they are called the absolute decomposition field, absolute inertia field and the absolute ramification field of ( $K, v$ ). If the extension $w$ is clear from context, we will simply write them as $K^{d}, K^{i}$ and $K^{r}$. We have the following chain of inclusions:

$$
K^{d} \subseteq K^{i} \subseteq K^{r} .
$$

A valued field $(K, v)$ is said to be henselian if $v$ admits a unique extension to $\bar{K}$. Every valued field has a minimal separable-algebraic extension which is henselian. This extension is unique up to valuation preserving isomorphisms over $K$, and we can consider it to be the same as the absolute decomposition field. We will call this extension the henselization of ( $K, v$ ). Clearly, the henselization depends on the choice of the extension of $v$ to $\bar{K}$. When the extension of $v$ to $\bar{K}$ is tacitly understood, we will denote the henselization by $K^{h}$. The henselization of ( $K, v$ ) with respect to an extension $w$ of $v$ to $\bar{K}$ will be denoted by $K^{h(w)}$.

Henselization is an immediate extension, that is, $v K^{h}=v K$ and $K^{h} v=K v$. An algebraic extension of henselian valued fields is again henselian. For any algebraic extension $L$ of $K$, we have that $L^{h}=L . K^{h}$.

For a valued field $(K, v)$ admitting a unique extension of $v$ to a finite extension $L$, we have the Lemma of Ostrowski which states that

$$
[L: K]=(v L: v K)[L v: K v] p^{d} \text { for some } d \in \mathbb{N},
$$

where $p:=$ char $K v$ when char $K v>0$ and $p:=1$ otherwise. The number $p^{d}$ is said to be the defect of the extension $(L \mid K, v)$ and will be denoted by $d(L \mid K, v)$. The extension $(L \mid K, v)$ is said to be defectless if $d(L \mid K, v)=1$. Defect satisfies the following multiplicative property: if $L|F| K$ is a tower of fields such that $L \mid K$ is finite and $v$ admits a unique extension from $K$ to $L$, then

$$
d(L \mid K, v)=d(L \mid F, v) d(F \mid K, v)
$$

An arbitrary algebraic extension $(\Omega \mid K, v)$ is said to be defectless if $d(L \mid K, v)=1$ for every finite subextension $L \mid K$.

Observe that the Lemma of Ostrowski is applicable in particular to henselian valued fields. It is well-known that $\left(K^{r} \mid K, v\right)$ is a defectless extension for a henselian valued field $(K, v)$. A henselian field $(K, v)$ is said to be defectless if $d(L \mid K, v)=1$ for every finite extension $(L \mid K, v)$. We will say that an arbitrary valued field is defectless if its henselization is defectless.

## 3. Proof of Theorem 1.1

Lemma 3.1. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Then for any $\delta \in v K(a)$, there exists a polynomial $g(X) \in K[X]$ with $\operatorname{deg} g<[K(a): K]$ such that $v g=\delta$.

Proof. Set $n:=[K(a): K]$. The fact that $\delta \in v K(a)$ implies that we can take $c_{i} \in K$ such that $\delta=v \sum_{i=0}^{n-1} c_{i} a^{i}$. Define $g(X):=\sum_{i=0}^{n-1} c_{i} X^{i} \in K[X]$. Then $\operatorname{deg} g<[K(a): K]$. It now follows from [2, Theorem 2.1] and [6, Lemma 3.2] that $\delta=v g(a)=v g$.

Lemma 3.2. Assume that $(K(X) \mid K, v)$ is residue transcendental. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Let $E$ be the least positive integer such that $E \gamma \in v K(a)$. Take $f(X) \in K[X]$ with $\operatorname{deg} f<[K(a): K]$ such that $v f=-E \gamma$. Then

$$
K(a, X) v=K(a) v\left(f(X)(X-a)^{E} v\right) .
$$

Proof. Observe that $(K(a, X) \mid K(a), v)$ is a residue transcendental extension with minimal pair of definition $(a, \gamma)$. Take $d \in K(a)$ such that $v d=-E \gamma$. It follows from [2, Theorem 2.1] that $K(a, X) v=K(a) v\left(d(X-a)^{E} v\right)$. By Lemma 3.1, we can take $f(X) \in K[X]$ with $\operatorname{deg} f<[K(a): K]$ such that $v f=v d$. It now follows from [6, Lemma 6.1] that $\frac{d}{f} v \in K(a) v$. As a consequence,

$$
K(a, X) v=K(a) v\left(d(X-a)^{E} v\right)=K(a) v\left(\left(\frac{d}{f} v\right)\left(f(X)(X-a)^{E} v\right)\right)=K(a) v\left(f(X)(X-a)^{E} v\right)
$$

## We can now give a proof of Theorem 1.1.

Proof. Take the minimal polynomial $Q(X)$ of $a$ over $K$. Write

$$
Q(X)=(X-a)\left(X-a_{2}\right) \ldots\left(X-a_{j}\right) \ldots\left(X-a_{n}\right),
$$

where $v\left(a-a_{i}\right) \geq \gamma$ for $2 \leq i \leq j$ and $v\left(a-a_{i}\right)<\gamma$ for all $i>j$. By definition, the assumption that $(a, \gamma)$ is a pair of definition for $v$ over $K$ implies that $v(X-b)=\min \{\gamma, v(a-b)\}$ for all $b \in \bar{K}$. It follows that

$$
v Q=j \gamma+\alpha, \text { where } \alpha:=v\left(a-a_{j+1}\right)+\ldots v\left(a-a_{n}\right) \in v \bar{K}
$$

We first assume that $(K(X) \mid K, v)$ is a value transcendental extension, that is, $\gamma$ is not a torsion element modulo $v K$. It follows from [6, Remark 3.3] that

$$
v K(X)=v K(a) \oplus \mathbb{Z}(j \gamma+\alpha) \text { and } K(X) v=K(a) v .
$$

Further, we observe that $(a, \gamma)$ is a minimal pair of definition for $v$ over $K(a)$. Consequently,

$$
v K(a, X)=v K(a) \oplus \mathbb{Z} \gamma \text { and } K(a, X) v=K(a) v
$$

The facts that $j \gamma+\alpha \in v K(a) \oplus \mathbb{Z} \gamma$ and $\gamma$ is not a torsion element modulo $v K$ imply that $\alpha \in v K(a)$. Consequently, $v K(X)=v K(a) \oplus \mathbb{Z} j \gamma$. It follows that $(v K(a, X): v K(X))=j$ and hence

$$
(v K(a, X): v K(X))[K(a, X) v: K(X) v]=j .
$$

We now assume that $(K(X) \mid K, v)$ is residue transcendental, that is, $\gamma$ is a torsion element modulo $v K$. Set

$$
e:=(v K(X): v K(a)) \text { and } E:=(v K(a, X): v K(a)) .
$$

Hence $E=\lambda e$ where $\lambda:=(v K(a, X): v K(X))$. It follows from [2, Theorem 2.1] that $e$ is the least positive integer such that $e v Q \in v K(a)$. By Lemma 3.1, we can take $g(X) \in K[X]$ with $\operatorname{deg} g<\operatorname{deg} Q$ such that $v g=-e v Q$. We can then conclude from our observations in [6] Remark 3.1] that

$$
\begin{equation*}
K(X) v=K(a) v\left(g Q^{e} v\right) \tag{5}
\end{equation*}
$$

Observe that $(K(a, X) \mid K(a), v)$ is a residue transcendental extension with $(a, \gamma)$ as a minimal pair of definition and the corresponding minimal polynomial $X-a$. By definition, $v(X-a)=\gamma$. Similar arguments as above now imply that $E$ is the smallest positive integer such that $E \gamma \in v K(a)$. Take $f(X) \in K[X]$ with $\operatorname{deg} f<\operatorname{deg} Q$ such that $v f=-E \gamma$. Then by Lemma 3.2,

$$
\begin{equation*}
K(a, X) v=K(a) v\left(f(X)(X-a)^{E} v\right) . \tag{6}
\end{equation*}
$$

The fact that $v Q=j \gamma+\alpha$ implies that

$$
\lambda e v Q=E v Q=j E \gamma+E \alpha
$$

The fact that $e v Q, E \gamma \in v K(a)$ then implies that $E \alpha \in v K(a)$. Consequently, it follows from Lemma 3.1 that there exists $h(X) \in K[X]$ with $\operatorname{deg} h<\operatorname{deg} Q$ such that $v h=-E \alpha$. We have thus obtained that $\lambda v g-j v f-v h=0$, that is, $v \frac{g^{\lambda}}{f^{j} h}=0$. It follows from [6, Lemma 6.1] that

$$
\frac{g^{\lambda}}{f^{j} h} v \in K(a) v .
$$

As a consequence,

$$
K(a) v\left(\left(g Q^{e} v\right)^{\lambda}\right)=K(a) v\left(g^{\lambda} Q^{E} v\right)=K(a) v\left(\left(\frac{g^{\lambda}}{f^{j} h} v\right)\left(f^{j} h Q^{E} v\right)\right)=K(a) v\left(f^{j} h Q^{E} v\right)
$$

We observe from [2, Theorem 2.1] that $g Q^{e} v$ is transcendental over $K(a) v$. Consequently, $g^{\lambda} Q^{E} v=$ $\left(g Q^{e}\right)^{\lambda} v$ is also transcendental over $K(a) v$. Thus,

$$
\begin{equation*}
\left[K(X) v: K(a) v\left(f^{j} h Q^{E} v\right)\right]=\left[K(a) v\left(g Q^{e} v\right): K(a) v\left(g^{\lambda} Q^{E} v\right)\right]=\lambda \tag{7}
\end{equation*}
$$

Observe that $X-a$ divides $Q(X)$ over $K(a)$. Hence we have an expression of the form

$$
Q^{E}=\sum_{i=E}^{n E} c_{i}(X-a)^{i} \text { where } c_{i} \in K(a) .
$$

It follows from [2, Theorem 2.1] that $v c_{i}+i \gamma \geq E v Q$ for all $i$. Suppose that $v c_{i}+i \gamma=E v Q$ for some $i$ such that $E$ does not divide $i$. Then we can write $v c_{i}+i^{\prime} \gamma+t E \gamma=E v Q$ where $1 \leq i^{\prime} \leq E-1$. The fact that $v c_{i}, E \gamma, E v Q \in v K(a)$ then implies that $i^{\prime} \gamma \in v K(a)$, which contradicts the minimality of $E$. Hence,

$$
E \text { divides } i \text { whenever } v c_{i}+i \gamma=E v Q .
$$

We now consider the expression

$$
f^{j} h Q^{E}=\sum_{i=E}^{n E} f^{j} h c_{i}(X-a)^{i} .
$$

Observe that $v f^{j} h Q^{E}=0$. The preceding observations now imply that

$$
E \text { divides } i \text { whenever } v f^{j} h c_{i}(X-a)^{i}=0
$$

Taking residues, we obtain that

$$
f^{j} h Q^{E} v=\sum_{i=1}^{n} f^{j} h c_{i E}(X-a)^{i E} v=\sum_{i=1}^{n}\left(f^{j-i} h c_{i E}\right) v\left(f(X)(X-a)^{E}\right)^{i} v .
$$

It follows from [6, Lemma 6.1] that $\left(f^{j-i} h c_{i E}\right) v \in K(a) v$. As a consequence,

$$
\begin{equation*}
f^{j} h Q^{E} v \in K(a) v\left[f(X)(X-a)^{E} v\right] . \tag{8}
\end{equation*}
$$

The coefficient of $\left(f(X)(X-a)^{E}\right)^{i} v$ in $f^{j} h Q^{E} v$ is given by $f^{j-i} h c_{i E} v$, where $c_{i E}$ is the coefficient of $(X-a)^{i E}$ in $Q^{E}$. Hence,

$$
c_{i E}=(-1)^{n E-i E} \mathcal{E}_{n E-i E}\left(0, \ldots, 0, a_{2}-a, \ldots, a_{2}-a, \ldots, a_{n}-a, \ldots, a_{n}-a\right),
$$

where 0 and $a_{i}-a$ appear $E$ times for each $i$ and $\mathcal{E}_{n E-i E}\left(Y_{1}, \ldots, Y_{n E}\right)$ is the $(n E-i E)$-th elementary symmetric polynomial in the variables $Y_{1}, \ldots, Y_{n E}$. By definition, each contributing term in $\mathcal{E}_{n E-i E}\left(0, \ldots, 0, a_{2}-a, \ldots, a_{2}-a, \ldots, a_{n}-a, \ldots, a_{n}-a\right)$ is of the form $\left(a_{t_{1}}-a\right) \ldots\left(a_{t_{n E-i E}}-a\right)$, where $a_{t_{1}}, \ldots, a_{t_{n E-i E}} \in\left\{a, a_{2}, \ldots, a_{n}\right\}$. We first assume that $i>j$. The fact that $v\left(a-a_{i}\right)<\gamma$ for all $i>j$ now implies that

$$
v\left(\left(a_{t_{1}}-a\right) \ldots\left(a_{t_{n E-i E}}-a\right)\right)+(i E-j E) \gamma>E v\left(\left(a_{j+1}-a\right) \ldots\left(a_{n}-a\right)\right)=E \alpha
$$

Recall that $v h=-E \alpha$ and $v f=-E \gamma$. It follows that $v f^{j-i} h c_{i E}>0$ and as a consequence,

$$
\begin{equation*}
f^{j-i} h c_{i E} v=0 \text { whenever } i>j \tag{9}
\end{equation*}
$$

We now assume that $i=j$. The coefficient of $\left(f(X)(X-a)^{E}\right)^{j} v$ in $f^{j} h Q^{E} v$ is given by $h c_{j E} v$. Each contributing term in the expression of $c_{j E}$ is of the form $\left(a_{t_{1}}-a\right) \ldots\left(a_{t_{n E-j E}}-a\right)$, where $a_{t_{1}}, \ldots, a_{t_{n E-j E}} \in\left\{a, a_{2}, \ldots, a_{n}\right\}$. If $\left\{a_{t_{1}}, \ldots, a_{t_{n E-j E}}\right\}=\left\{a_{j+1}, \ldots, a_{n}\right\}$ with each term appearing $E$ times, then

$$
v\left(\left(a_{t_{1}}-a\right) \ldots\left(a_{t_{n E-j E}}-a\right)\right)=E v\left(\left(a_{j+1}-a\right) \ldots\left(a_{n}-a\right)\right)=E \alpha=-v h .
$$

Otherwise, there exists some $t_{k}$ such that $v\left(a_{t_{k}}-a\right) \geq \gamma$ and as a consequence,

$$
v\left(\left(a_{t_{1}}-a\right) \ldots\left(a_{t_{n E-j E}}-a\right)\right)>\operatorname{Ev}\left(\left(a_{j+1}-a\right) \ldots\left(a_{n}-a\right)\right) .
$$

It now follows from the triangle inequality that $v c_{j E}=-v h$. Consequently,

$$
\begin{equation*}
h c_{j E} v \neq 0 \tag{10}
\end{equation*}
$$

It follows from (10) and (9) that

$$
\begin{equation*}
\operatorname{deg}\left(f^{j} h Q^{E} v\right)=j \tag{11}
\end{equation*}
$$

As a consequence of (8) and (6) we then obtain that

$$
\begin{equation*}
\left[K(a, X) v: K(a) v\left(f^{j} h Q^{E} v\right)\right]=\left[K(a) v\left(f(X)(X-a)^{E} v\right): K(a) v\left(f^{j} h Q^{E} v\right)\right]=j . \tag{12}
\end{equation*}
$$

In light of the multiplicative property of degrees of field extensions, it now follows from (7) that $[K(a, X) v: K(X) v] \lambda=j$. Recall that $\lambda=(v K(a, X): v K(X))$. It follows that

$$
\begin{equation*}
[K(a, X) v: K(X) v](v K(a, X): v K(X))=j . \tag{13}
\end{equation*}
$$

We have thus proved the theorem.

## 4. Independence of $j$

Take any $a \in \bar{K}$ and $\gamma$ in some ordered abelian group containing $v \bar{K}$. Recall that any polynomial $f(X) \in \bar{K}[X]$ has a unique expression of the form $f(X)=\sum_{i=0}^{n} c_{i}(X-a)^{i}$, where $c_{i} \in \bar{K}$. Consider the map $v_{a, \gamma}: \bar{K}[X] \rightarrow v \bar{K}+\mathbb{Z} \gamma$ by setting

$$
v_{a, \gamma} f:=\min \left\{v c_{i}+i \gamma\right\} .
$$

Extend $v_{a, \gamma}$ canonically to $\bar{K}(X)$. Then $v_{a, \gamma}$ is a valuation transcendental extension of $v$ from $\bar{K}$ to $\bar{K}(X)$ [8, Lemma 3.10]. By definition, $v_{a, \gamma}\left(X-a^{\prime}\right):=\min \left\{\gamma, v\left(a-a^{\prime}\right)\right\}$ for any $a^{\prime} \in \bar{K}$. It follows that

$$
v_{a, \gamma}(X-a)=\gamma=\max v_{a, \gamma}(X-\bar{K}) .
$$

Lemma 4.1. Assume that $(K(X) \mid K, v)$ is a residue transcendental extension. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Consider the ordered abelian group $v \bar{K} \oplus \mathbb{Z}$ equipped with the lexicographic order. Embed $v \bar{K}$ into $(v \bar{K} \oplus \mathbb{Z})_{\text {lex }}$ by setting $\alpha \mapsto(\alpha, 0)$ for all $\alpha \in v \bar{K}$. Define

$$
\Gamma:=(\gamma,-1) .
$$

Take the extension $w:=v_{a, \Gamma}$ of $v$ to $\bar{K}(X)$. Then $(a, \Gamma)$ is a minimal pair of definition for $w$ over K. Further,

$$
j(a, K, \gamma)=j(a, K, \Gamma)
$$

Proof. It follows from our preceding discussions that

$$
w(X-a)=\Gamma=\max w(X-\bar{K}) .
$$

Take any $a^{\prime} \in \bar{K}$. By definition, $v\left(a-a^{\prime}\right) \geq \Gamma$ if and only if $\left(v\left(a-a^{\prime}\right), 0\right) \geq(\gamma,-1)$, which again holds if and only if $v\left(a-a^{\prime}\right) \geq \gamma$. We have thus obtained that

$$
\begin{equation*}
v\left(a-a^{\prime}\right) \geq \Gamma \text { if and only if } v\left(a-a^{\prime}\right) \geq \gamma . \tag{14}
\end{equation*}
$$

Recall that $(a, \gamma)$ is a minimal pair of definition for $v$ over $K$. As a consequence of (14), we now obtain that

$$
(a, \Gamma) \text { is a minimal pair of definition for } w \text { over } K .
$$

It further follows from (14) that

$$
j(a, K, \Gamma)=j(a, K, \gamma) .
$$

Lemma 4.2. Assume that $(K(X) \mid K, v)$ is value transcendental. Take minimal pairs of definition $(a, \gamma)$ and $\left(a^{\prime}, \gamma\right)$ for $v$ over $K$. Then $j(a)=j\left(a^{\prime}\right)$.

Proof. Take the minimal polynomial $Q(X)$ of $a$ over $K$ and the minimal polynomial $Q^{\prime}(X)$ of $a^{\prime}$ over $K$. Then $v Q=j(a) \gamma+\alpha$ and $v Q^{\prime}=j\left(a^{\prime}\right) \gamma+\alpha^{\prime}$, where $\alpha, \alpha^{\prime} \in v \bar{K}$. It follows from [6, Lemma 3.10] that $v Q=v Q^{\prime}$. Consequently, $\left(j(a)-j\left(a^{\prime}\right)\right) \gamma \in v \bar{K}$. The fact that $\gamma$ is not contained in the divisible group $v \bar{K}$ implies that $j(a)=j\left(a^{\prime}\right)$.

We can now give a proof of Theorem 1.2 ,
Proof. When the extension $(K(X) \mid K, v)$ is value transcendental, then the assertion of Theorem 1.2 is proved in Lemma 4.2. We now assume that $(K(X) \mid K, v)$ is residue transcendental. Consider the ordered abelian group $(v \bar{K} \oplus \mathbb{Z})_{\text {lex }}$. Embed $v \bar{K}$ into $(v \bar{K} \oplus \mathbb{Z})_{\text {lex }}$ by setting $\alpha \mapsto(\alpha, 0)$ for all $\alpha \in v \bar{K}$. Set $\Gamma:=(\gamma,-1)$. The fact that $(a, \gamma)$ and $\left(a^{\prime}, \gamma\right)$ are minimal pair of definition for $v$ over $K$ implies that $v\left(a-a^{\prime}\right) \geq \gamma$ and hence $v\left(a-a^{\prime}\right) \geq \Gamma$. It now follows from [1, Proposition 3] that $v_{a, \Gamma}=v_{a^{\prime}, \Gamma}$. Set $w:=v_{a, \Gamma}=v_{a^{\prime}, \Gamma}$. In light of Lemma 4.1, we observe that $(a, \Gamma)$ and $\left(a^{\prime}, \Gamma\right)$ are minimal pairs of definition of $w$ over $K$. Further, $j(a, K, \gamma)=j(a, K, \Gamma)$ and $j\left(a^{\prime}, K, \gamma\right)=j\left(a^{\prime}, K, \Gamma\right)$. The fact that $\Gamma \notin v \bar{K}$ implies that $w$ is a value transcendental extension of $v$ to $\bar{K}(X)$. It then follows from Lemma 4.2 that $j(a, K, \Gamma)=j\left(a^{\prime}, K, \Gamma\right)$. As a consequence, we conclude that

$$
j(a, K, \gamma)=j\left(a^{\prime}, K, \gamma\right)
$$

## 5. Implicit constant fields

Proposition 5.1. Assume that $L \mid K$ is an extension of fields such that $K$ is relatively algebraically closed in L. Take $a \in \bar{K}$. Then $K(a)$ and $L$ are linearly disjoint over $K$.

Proof. Take the minimal polynomial $Q(X)$ of $a$ over $L$. Write

$$
Q(X)=(X-a)\left(X-a_{2}\right) \ldots\left(X-a_{n}\right)=\sum_{i=0}^{n} c_{i} X^{i}
$$

Take the minimal polynomial $f(X)$ of $a$ over $K$. Then $Q$ divides $f$ over $L$ and hence each root of $Q$ is also a root of $f$. Thus each $a_{i}$ is a $K$-conjugate of $a$ and consequently $a_{i} \in \bar{K}$. Further, observe that each coefficient $c_{j}$ is a symmetric expression in the roots $a_{i}$ and hence $c_{j} \in \bar{K}$ for all $j$. Consequently, $c_{j} \in \bar{K} \cap L=K$, that is, $Q(X) \in K[X]$. It follows that

$$
[K(a): K]=[L(a): L],
$$

that is, $K(a)$ and $L$ are linearly disjoint over $K$.
For the rest of this section, we fix an extension of $v$ to $\overline{K(X)}$. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. We have observed in [6, Theorem 1.1] that $K^{h} \subseteq \operatorname{IC}(K(X) \mid K, v) \subseteq K(a)^{h}$. As a consequence,

$$
I C(K(X) \mid K, v)(a)=K(a)^{h} .
$$

By definition, $I C(K(X) \mid K, v)$ is relatively algebraically closed in $K(X)^{h}$. It is now a direct consequence of Proposition 5.1 that $K(a)^{h}$ and $K(X)^{h}$ are linearly disjoint over $I C(K(X) \mid K, v)$. Observe that $K(a)^{h} \cdot K(X)^{h}=K(X)^{h}(a)=K(a, X)^{h}$. Thus,

$$
\left[K(a)^{h}: I C(K(X) \mid K, v)\right]=\left[K(a, X)^{h}: K(X)^{h}\right] .
$$

From the Lemma of Ostrowski, we have that

$$
\left[K(a, X)^{h}: K(X)^{h}\right]=\left(v K(a, X)^{h}: v K(X)^{h}\right)\left[K(a, X)^{h} v: K(X)^{h} v\right] d\left(K(a, X)^{h} \mid K(X)^{h}, v\right) .
$$

Recall that henselization is an immediate extension. The following result now follows immediately from Theorem 1.1:

$$
\begin{equation*}
\left[K(a)^{h}: I C(K(X) \mid K, v)\right]=j(a) d\left(K(a, X)^{h} \mid K(X)^{h}, v\right) \tag{15}
\end{equation*}
$$

Assume that $(K, v)$ is a defectless valued field. The fact that $(K(X) \mid K, v)$ is a valuation transcendental extension implies that we can apply [9, Theorem 1.1] to this extension and obtain that $(K(X), v)$ is also a defectless field. By definition, $\left(K(X)^{h}, v\right)$ is a defectless field. We have thus arrived at the following result:

Theorem 5.2. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Fix an extension of $v$ to $\overline{K(X)}$. Then

$$
\left[K(a)^{h}: I C(K(X) \mid K, v)\right]=j(a) d\left(K(a, X)^{h} \mid K(X)^{h}, v\right)
$$

In particular,

$$
\left[K(a)^{h}: I C(K(X) \mid K, v)\right]=j(a) \text { whenever }(K, v) \text { is defectless. }
$$

Corollary 5.3. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$. Fix an extension of $v$ to $\overline{K(X)}$.
(i) A necessary condition for obtaining $I C(K(X) \mid K, v)=K(a)^{h}$ is that $j(a)=1$. If $(K, v)$ is defectless, then the condition is also sufficient.
(ii) Assume that $j(a)=[K(a): K]$. Then $I C(K(X) \mid K, v)=K^{h}$.

Proof. The first assertion is an immediate consequence of Theorem 5.2. It is thus enough to prove (ii). We assume that $j(a)=[K(a): K]$. It then follows from Theorem [5.2 that

$$
\left[K(a)^{h}: I C(K(X) \mid K, v)\right] \geq j(a)=[K(a): K]
$$

Recall that $K^{h}(a)=K(a)^{h}$. Consequently,

$$
[K(a): K] \geq\left[K(a)^{h}: K^{h}\right] \geq\left[K(a)^{h}: I C(K(X) \mid K, v)\right]
$$

It now follows that $\left[K(a)^{h}: I C(K(X) \mid K, v)\right]=\left[K(a)^{h}: K^{h}\right]$. As a consequence,

$$
I C(K(X) \mid K, v)=K^{h} .
$$

An immediate corollary is the following:
Corollary 5.4. ([6, Proposition 7.1]) Take a minimal pair of definition ( $a, \gamma)$ forv over $K$. Assume that $a$ is purely inseparable over $K$. Fix an extension of $v$ to $\overline{K(X)}$. Then $\operatorname{IC}(K(X) \mid K, v)=K^{h}$.

## 6. Proof of Theorem 1.3

Proof. When the extension $(K(X) \mid K, v)$ is value transcendental, the assertion is proved in [6, Theorem 1.1]. So we assume that $(K(X) \mid K, v)$ is residue transcendental.

We fix an extension of $v$ to $\overline{K(X)}$ and denote it again by $\bar{v}$. Observe that $K(a) \cap K^{r}$ is a finite separable extension, hence simple. Hence we can choose $b \in \bar{K}$ such that $K(a) \cap K^{r}=K(b)$. The inclusion $I C(K(X) \mid K, v) \subseteq K(a)^{h}$ follows from [6, Lemma 5.1]. It is left to show that $K(b)^{h} \subseteq$ $I C(K(X) \mid K, v)$. Recall that $I C(K(X) \mid K, v):=\bar{K} \cap K(X)^{h(v)}$. It is thus sufficient to show that $b \in K(X)^{h(v)}$, that is, $K(b, X)^{h(v)}=K(X)^{h(v)}$.

Applying the Lemma of Ostrowski to the extension $\left(K(a, X)^{h(v)} \mid K(X)^{h(v)}\right.$, v), we obtain the following relation in light of Theorem 1.1:

$$
\left[K(a, X)^{h(v)}: K(X)^{h(v)}\right]=j(a, K, \gamma) d\left(K(a, X)^{h(v)} \mid K(X)^{h(v)}, v\right) .
$$

Take $a^{\prime} \in \bar{K}$ such that $v\left(a-a^{\prime}\right) \geq \gamma$. By definition, $[K(a): K] \leq\left[K\left(a^{\prime}\right): K\right]$. The fact that $b \in K(a)$ then implies that

$$
[K(a, b): K]=[K(a): K] \leq\left[K\left(a^{\prime}\right): K\right] \leq\left[K\left(a^{\prime}, b\right): K\right] .
$$

Consequently,

$$
[K(a, b): K(b)] \leq\left[K\left(a^{\prime}, b\right): K(b)\right] .
$$

It follows that $(a, \gamma)$ is also a minimal pair of definition for $v$ over $K(b)$. It now follows from Theorem 1.1 and the Lemma of Ostrowski that

$$
\left[K(a, X)^{h(v)}: K(b, X)^{h(v)}\right]=j(a, K(b), \gamma) d\left(K(a, X)^{h(v)} \mid K(b, X)^{h(v)}, v\right)
$$

As a consequence,

$$
\left[K(b, X)^{h(v)}: K(X)^{h(v)}\right]=\frac{j(a, K, \gamma)}{j(a, K(b), \gamma)} d\left(K(b, X)^{h(v)} \mid K(X)^{h(v)}, v\right)
$$

It follows from [7. Theorem 3] that the condition that $b \in K^{r}$ implies that $K(b, X) \subseteq K(X)^{r(v)}$. Consequently, $\left(K(b, X)^{h(v)} \mid K(X)^{h(v)}, v\right)$ is a defectless extension. We have thus obtained that

$$
\begin{equation*}
\left[K(b, X)^{h(v)}: K(X)^{h(v)}\right]=\frac{j(a, K, \gamma)}{j(a, K(b), \gamma)} . \tag{16}
\end{equation*}
$$

We now consider the valuation $w$ as constructed in the statement of Lemma 4.1. Fix an extension of $w$ to $\overline{K(X)}$ and denote it again by $w$. Similar arguments as above yield that

$$
\left[K(b, X)^{h(w)}: K(X)^{h(w)}\right]=\frac{j(a, K, \Gamma)}{j(a, K(b), \Gamma)} .
$$

Observe that $\Gamma \notin v \bar{K}$ and hence $(K(X) \mid K, w)$ is a value transcendental extension. It then follows from [6. Theorem 1.1] that $K(b)^{h} \subseteq I C(K(X) \mid K, w)$. As a consequence, $b \in K(X)^{h(w)}$ and hence $K(b, X)^{h(w)}=K(X)^{h(w)}$. In light of the preceding discussions, we conclude that

$$
j(a, K, \Gamma)=j(a, K(b), \Gamma)
$$

It follows from Lemma 4.1 that $j(a, K, \gamma)=j(a, K, \Gamma)$. Observe that $(a, \gamma)$ is also a minimal pair of definition for $v$ over $K(b)$. Applying Lemma 4.1 to the extension $(K(b, X) \mid K(b), v)$, we obtain that $j(a, K(b), \Gamma)=j(a, K(b), \gamma)$. As a consequence,

$$
j(a, K, \gamma)=j(a, K, \Gamma)=j(a, K(b), \Gamma)=j(a, K(b), \gamma) .
$$

It now follows from (16) that $K(b, X)^{h(v)}=K(X)^{h(v)}$. Consequently, $b \in K(X)^{h(v)}$ and hence $K(b)^{h} \subseteq I C(K(X) \mid K, v)$. We have thus proved the theorem.

Corollary 6.1. Take a minimal pair of definition $(a, \gamma)$ for $v$ over $K$ and fix an extension of $v$ to $\overline{K(X)}$. Assume that $j(a)=\left[K(a): K(a) \cap K^{r}\right]$. Then $I C(K(X) \mid K, v)=\left(K(a) \cap K^{r}\right)^{h}$.

Proof. Take $b \in \bar{K}$ such that $K(b)=K(a) \cap K^{r}$. Recall that $j(a)$ divides $\left[K(a)^{h}: I C(K(X) \mid K, v)\right]$. In light of Theorem 1.3 we have the following chain of relations:

$$
j(a) \leq\left[K(a)^{h}: I C(K(X) \mid K, v)\right] \leq\left[K(a)^{h}: K(b)^{h}\right] \leq[K(a): K(b)] .
$$

The assumption that $j(a)=[K(a): K(b)]$ now implies that each of the inequalities in the above expression is an equality. As a consequence, we obtain that

$$
I C(K(X) \mid K, v)=K(b)^{h}
$$

## References

[1] V. Alexandru and A. Zaharescu, Sur une classe de prolongements à $K(x)$ d'une valuation sur une corp K, Rev. Roumaine Math. Pures Appl., 5 (1988), 393-400.
[2] V. Alexandru, N. Popescu and A. Zaharescu, A theorem of characterization of residual transcendental extensions of a valuation, J. Math. Kyoto University, 28 (1988), 579-592.
[3] V. Alexandru, N. Popescu and A. Zaharescu, Minimal pairs of definition of a residual transcendental extension of a valuation, J. Math. Kyoto University, 30 (1990), 207-225.
[4] N. Bourbaki, Commutative Algebra, Hermann, Paris (1972).
[5] A. Dutta, On the ranks and implicit constant fields of valuations induced by pseudo monotone sequences, https://arxiv.org/abs/2107.10570.
[6] A. Dutta, Minimal pairs, minimal fields and implicit constant fields, Journal of Algebra, 588 (2021), 479-514.
[7] A. Dutta and F.-V. Kuhlmann, Eliminating tame ramification: generalizations of Abhyankar's lemma, Pacific Journal of Mathematics, 307(1) (2020), 121-136.
[8] F.-V. Kuhlmann, Value groups, residue fields and bad places of rational function fields, Trans. Amer. Math. Soc., 356 (2004), 4559-4600.
[9] F.-V. Kuhlmann, Elimination of Ramification I: The Generalized Stability Theorem, Trans. Amer. Math. Soc., 362(11) (2010), 5697-5727.
[10] G. Peruginelli and D. Spirito, Extending valuations to the field of rational functions using pseudomonotone sequences, Journal of Algebra, 586 (2021), 756-786.

Department of Mathematics, IISER Mohali, Knowledge City, Sector 81, Manauli PO, SAS Nagar, Punjab, India, 140306.

Email address: arpan.cmi@gmail.com


[^0]:    Date: November 29, 2021.
    2010 Mathematics Subject Classification. 12J20, 13A18, 12J25.
    Key words and phrases. Valuation, minimal pairs, valuation transcendental extensions, ramification theory, implicit constant fields, extensions of valuation to rational function fields.

    This work was supported by the Post-Doctoral Fellowship of the National Board of Higher Mathematics, India.

