Minimal Projections on Hyperplanes in Sequence Spaces.

J. BLATTER - E. W. CHENEY (Austin, Tex.) (*)

Summary. - *The projection constants of hyperplanes in the classical sequence spaces* (c_0) *and* (l_1) *are determined, together with the projections of minimum norm.*

1. - Introduction.

If Y is a closed linear subspace in a Banach space X, then a *projection* of X onto Y is a bounded linear map $P: X \to Y$ such that $Py = y$ for all $y \in Y$. If such a projection exists, then Y is said to be *complemented* in X. In this case, there is some interest in discovering whether there exist projections of minimal norm, and if so, what their properties are.

Many applications of projections occur in numerical analysis and approximation theory, for Px can be regarded as an approximation to x in Y. The quality of this approximation relative to the *best* approximation is governed by the inequality

$$\|x - Px\| \leqslant \|I - P\| \cdot \operatorname{dist}(x, Y) .$$

In some previous work [1, 2, 3, 4] we have drawn attention to the problem of determining projections of minimal norm, assuming that a pair $Y \subset X$ has been prescribed. Many interesting open problems remain in this area of endeavor; for example, the minimal projections of $C[0,1]$ onto the subspace of polynomials of degree $\leqslant n$ are still unknown!

The present paper has the modest goal of studying projections of minimal norm onto hyperplanes in the classical sequence spaces (c_0) and (l_1). We give formulas for projection constants, identify cases in which there exist projections of norm 1, and so forth. In the course of the investigation we encounter a number of unusual extremal problems which must be solved to yield minimal projections. The situation as regards hyperplanes in (c_0) is reasonably simple. See, for example, Theorem 2 below. In (l_1), however, the description of the projection constants is suprisingly complex and seems to require the consideration of a number of cases. See, for example, Theorems 7, 8, and 9.

(*) Entrata in Redazione l'11 dicembre 1972.

We summarize here a few definitions:

(1) (c_0) is the space of all real sequences $x=(x_1, x_2, \dots)$ such that $\lim_{n\to\infty} x_n = 0$. The norm is $\|x\|_\infty = \sup_n |x_n|$.

(2) (l_1) is the space of sequences for which $\sum_{n=1}^{\infty} |x_n| < \infty$, the norm being $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$.

(3) (l_∞) is the space of sequences for which $\sup |x_n| < \infty$, the norm being $\|x\|_\infty = \sup_n |x_n|$.

As is well-known, (l_1) is isometrically isomorphic to $(c_0)^*$ and (l_∞) is isometrically isomorphic to $(l_1)^*$; in both cases the functionals are given by the formula $(f, x) = \sum_{n=1}^{\infty} f_n x_n$.

A *hyperplane* (in any normed space) X is defined here to be a set of the form

$$f^{-1}(0) = \{x \in X : f(x) = 0\} \qquad (f \in X^*, f \neq 0) .$$

The *relative projection constant* of a complemented subspace Y in a Banach space X is the number

$$p(Y) = \inf \{\|P\| : P \text{ projects } X \text{ onto } Y\} .$$

The subspace Y is termed an $\mathcal{E}$-*space* in X if the infimum $\inf \{\|x-y\| : y \in Y\}$ is attained for each $x \in X$.

LEMMA 1. – *Let X be a normed linear space and let f be a continuous nonzero linear functional on X. Each projection of X onto the hyperplane $f^{-1}(0)$ is of the following form, for some $z \in f^{-1}(1)$:*

$$(1) \qquad P_z = I - f \otimes z \qquad \text{i.e.} \qquad P_z x = x - f(x) z .$$

PROOF. – If P is a projection of X onto $f^{-1}(0)$, select $v \in f^{-1}(1)$ and put $z = v - Pv$. Then $f(z) = f(v) - f(Pv) = f(v) = 1$. Hence X is the direct sum of $f^{-1}(0)$ and the subspace generated by z. On $f^{-1}(0)$, P and P_z agree since $P_z x = x - f(x) z = x = Px$ if $x \in f^{-1}(0)$. On the subspace generadet by z, P and P_z agree since $P_z z = z - f(z) z = 0 = P(v - Pv) = Pz$. ∎

2. – The space (c_0).

LEMMA 2. – *Let f be an element of norm 1 in (l_1). Every projection of (c_0) onto $f^{-1}(0)$ is of the form $P_z = I - f \otimes z$ for some $z \in f^{-1}(1)$, and moreover*

$$\|P_z\| = \sup_i \{|1 - z_i f_i| + |z_i|(1 - |f_i|)\} .$$

PROOF. – Writing $\sup_x$ for the supremum as x ranges over the unit ball of (c_0), we have

$$\|P_z\| = \sup_x \sup_i |(P_z x)_i| = \sup_x \sup_i |x_i - f(x) z_i|$$

$$= \sup_i \sup_x |\sum_{j=1}^{\infty} (\delta_{ij} - f_j z_i) x_j| = \sup_i \sum_{j=1}^{\infty} |\delta_{ij} - f_j z_i|$$

$$= \sup_i \{|1 - f_i z_i| + \sum_{\substack{i=1 \\ j \neq i}}^{\infty} |f_j z_i|\}$$

$$= \sup_i \{|1 - f_i z_i| + |z_i|(1 - |f_i|)\} . \quad \blacksquare$$

THEOREM 1. – *Let $f \in (l_1)$ and $\|f\| = 1$. In order that there exist a projection of norm 1 from (c_0) onto the hyperplane $f^{-1}(0)$ if is necessary and sufficient that $|f_i| \geqslant \frac{1}{2}$ for some i. In order that $f^{-1}(0)$ have a unique projection of norm 1 it is necessary and sufficient that $|f_i| \geqslant \frac{1}{2}$ for exactly one index i.*

PROOF. – By Lemma 1, a necessary and sufficient condition for the existence of a norm-1 projection is that there exist a point z in (c_0) such that

$$(1) \qquad |1 - f_i z_i| + |z_i|(1 - |f_i|) \leqslant 1 \qquad \text{for all } i$$

$$(2) \qquad \sum_{i=1}^{\infty} f_i z_i = 1 .$$

From inequality (1) it is clear that $\operatorname{sgn} z_i = \operatorname{sgn} f_i$ for all i. Hence (1) implies

$$1 - f_i z_i + |z_i| - z_i f_i \leqslant 1$$

and

$$|z_i|(1 - 2|f_i|) \leqslant 0 .$$

For each i satisfying $|f_i| < \frac{1}{2}$ we must therefore have $z_i = 0$. Since $f(z) = 1$, there must exist at least one index j such that $|f_j| \geqslant \frac{1}{2}$. On the other hand, this condition is sufficient, because if $|f_j| \geqslant \frac{1}{2}$ then z can be defined so that $z_j = 1/f_j$ and $z_i = 0$ for $i \neq j$. Then $f(z) = 1$ and $\|P_z\| = 1$, as can be easily verified from (1). If exactly one index j exists for which $|f_j| \geqslant \frac{1}{2}$ then z is uniquely determined in the above manner. If two components of f exist such that $|f_i| = \frac{1}{2} = |f_j| (i \neq j)$, then two projections exist as described above, and all convex linear combinations of these two projections have norm 1. $\blacksquare$

REMARK. – In (c_0) there exist hyperplanes which are not $\mathcal{E}$-spaces but which nevertheless possess minimal projections. In the constructions above, take for example $f = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $z = 2, 0, 0, \ldots)$.

THEOREM 2. – *Let $f \in (l_1)$ and $\|f\|_1 = 1$. The relative projection constant of $f^{-1}(0)$ in (c_0) is 1 if $\|f\|_\infty \geq \frac{1}{2}$, and otherwise it is $1 + \left(\sum_{i=1}^{\infty} |f_i|/(1-2|f_i|)\right)^{-1}$.*

PROOF. – The case $\|f\|_\infty \geq \frac{1}{2}$ has been considered above. Assume that $\|f\|_\infty < \frac{1}{2}$. Then the numbers $1-2|f_i|$ are bounded away from zero and we put $\lambda_n = \left(\sum_{i=1}^{n} |f_i|/ /(1-2|f_i|)\right)^{-1}$, n being fixed. We shall define a projection of norm $1+\lambda_n$. Let $z_i = \lambda_n(\operatorname{sgn} f_i)/(1-2|f_i|)$ for all $i \leq n$ and let $z_i = 0$ for all $i > n$. Then $f(z) = \sum f_i z_i = 1$. Since $0 \leq z_i f_i \leq \sum z_\nu f_\nu = 1$, we have (for $i \leq n$)

$$|1-z_i f_i| + |z_i|(1-|f_i|) = 1 - z_i f_i + |z_i| - z_i f_i$$
$$= 1 + |z_i|(1-2|f_i|) = 1 + \lambda_n .$$

The corresponding expression reduces to 1 when $i > n$. Therefore by Lemma 2, $\|P_z\| = 1 + \lambda_n$. The projection constant is thus at most $1 + \lambda = 1 + \lim \lambda_n$.

In order to show that the projection constant is not less than $1+\lambda$, suppose on the contrary that for some $z \in (c_0)$ we have $f(z) = 1$ and

$$|1-z_i f_i| + |z_i|(1-|f_i|) < 1 + \lambda . \tag{3}$$

Since $1-|z_i f_i| \leq 1 - z_i f_i \leq |1-z_i f_i|$, a consequence of (3) is that

$$1 - |z_i f_i| + |z_i| - |z_i f_i| < 1 + \lambda ,$$

whence $|z_i|(1-2|f_i|) < \lambda$. Then the following contradiction arises:

$$1 = f(z) = \sum_{i=1}^{\infty} f_i z_i \leq \sum_{i=1}^{\infty} |f_i z_i| < \lambda \sum_{i=1}^{\infty} \frac{|f_i|}{1-2|f_i|} = 1 .$$

COROLLARY. – *Let $f \in (l_1)$ and $\|f\|_1 = 1$. If $\|f\|_\infty \geq \frac{1}{2}$, then the hyperplane $f^{-1}(0)$ in (c_0) has a minimal projection. In the case $\|f\|_\infty < \frac{1}{2}$, however, $f^{-1}(0)$ has a minimal projection if and only if at most a finite number of f_i are different from 0.*

COROLLARY. – *The totality of projection constants for all the hyperplanes in (c_0) is precisely the interval* $[1, 2)$.

PROOF. – Let $\lambda \in [1, 2)$. For $\lambda = 1$, Theorem 1 describes the hyperplanes having projection constant λ. If $\lambda \in (1, 2)$, select n so that $1 < \lambda \leq 2 - 2n^{-1}$. Define $f \in (l_1)$ by putting $f_1 = \ldots = f_n = r$, $f_{n+1} = 1 - nr$, and $f_{n+2} = f_{n+3} = \ldots = 0$. Here r is chosen in the interval $(1/2n, 1/n]$ so that

$$\frac{nr}{1-2r} + \frac{1-nr}{2nr-1} = (\lambda - 1)^{-1} . \tag{4}$$

Such a value of r exists because the left side of (4) is a continuous function of r in the prescribed interval, and its range is an interval $[n/(n-2), \infty)$ which contains $(\lambda-1)^{-1}$. Observe that $f \geqslant 0$ and that $\|f\|_1 = 1$. By Theorem 2, the projection constant of $f^{-1}(0)$ is

$$1+\left(\sum_{i=1}^{n+1}\frac{f_i}{1-2f_i}\right)^{-1} = 1+\left[\frac{nr}{1-2r}+\frac{1-nr}{2nr-1}\right]^{-1} = \lambda .$$

Now we remark that by a theorem of Levin and Petunin [5], every hyperplane has projection constant at most 2. Hence in the present situation we only must rule out 2 as a possible value. If $\|f\|_1 = 1$ and $\|f\|_\infty < \frac{1}{2}$ then

$$\sum |f_i|(1-2|f_i|)^{-1} = \sum |f_i| + 2\sum |f_i|^2(1-2|f_i|)^{-1} > 1 .$$

Hence by Theorem 2, $p[f^{-1}(0)] < 2$. ∎

3. - The space (l_1).

LEMMA 3. - *Let f be an element of norm 1 in (l_∞). Every projection of (l_1) onto the hyperplane $f^{-1}(0)$ is of the form $P_z = I - f \otimes z$ for some $z \in f^{-1}(1)$, and moreover,*

$$\|P_z\| = \sup_n \{|1-f_n z_n| + |f_n|(\|z\| - |z_n|)\} . \tag{1}$$

PROOF. - If $x \in (l_1)$ and $\|x\| \leqslant 1$ then

$$\begin{aligned}\|P_z x\| &= \|x - f(x)z\| = \sum_{i=1}^{\infty}|x_i - f(x)z_i| = \sum_{i=1}^{\infty}\Big|\sum_{j=1}^{\infty}(\delta_{ij} - f_j z_i)x_j\Big| \\ &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\delta_{ij} - f_j z_i|\,|x_j| = \sum_{j=1}^{\infty}|x_j|\sum_{i=1}^{\infty}|\delta_{ij} - f_j z_i| \\ &\leqslant \sum_{j=1}^{\infty}|x_j|\cdot\sup_n \sum_{i=1}^{\infty}|\delta_{in} - f_n z_i| \\ &= \sup_n \sum_{i=1}^{\infty}|\delta_{in} - f_n z_i| = \sup_n \Big\{|1-f_n z_n| + \sum_{\substack{i=1\\ i\neq n}}^{\infty}|f_n z_i|\Big\} \\ &= \sup_n \{|1-f_n z_n| + |f_n|(\|z\| - |z_n|)\} .\end{aligned}$$

For the reverse inequality, fix an index n and define $x \in (l_1)$ by $x_i = \delta_{in}$. Then

$$\begin{aligned}\|P_z\| \geqslant \|P_z x\| = \|x - f(x)z\| &= \sum_{i=1}^{\infty}|x_i - f_n z_i| \\ &= |1-f_n z_n| + |f_n|(\|z\| - |z_n|) .\end{aligned}$$

By taking a supremum in n, we complete the proof of Eq. (1). ∎

If $f \geqslant 0$ and $z \geqslant 0$ then the condition $(f, z) = 1$ implies that $0 \leqslant f_i z_i \leqslant 1$ for each i. Hence Eq. (1) can be simplified in this case to read

$$(2) \qquad \|P_z\| = 1 + \sup_i f_i(\|z\| - 2z_i) .$$

THEOREM 3. *Let $0 \neq f \in (l_\infty)$. The hyperplane $f^{-1}(0)$ in (l_1) has a projection of norm 1 if and only if at most two components of f are different from zero. The hyperplane has a unique projection of norm 1 if and only if exactly two components of f are different from 0.*

PROOF. – We assume that $\|f\|_\infty = 1$. In order that $f^{-1}(0)$ have a projection of norm 1 it is necessary and sufficient that there exist $z \in (l_1)$ such that $f(z) = 1$ and

$$(3) \qquad |1 - f_i z_i| + |f_i|(\|z\| - |z_i|) \leqslant 1 \qquad (\text{all } i) .$$

It is clear that this implies $f_i z_i \geqslant 0$ for all i. Hence $1 = \sum f_i z_i \geqslant f_i z_i \geqslant 0$. Inequality (3) is now equivalent to

$$|f_i|\{\|z\| - 2|z_i|\} \leqslant 0 .$$

If $f_i \neq 0$ then $\|z\| \leqslant 2|z_i|$. There can be at most two indices for which $f_i \neq 0$ since $\|z\| = \sum |z_i| \geqslant \sum_{f_i \neq 0} |z_i| \geqslant \sum_{f_i \neq 0} \frac{1}{2}\|z\|$.

Now suppose that f has exactly one nonzero component, say $f_j = 1$ and $f_i = 0$ for $i \neq j$. Let k be any index different from j, and define z as follows $z_j = 1$, $z_k \in [-1, 1]$, and $z_i = 0$ for all remaining indices i. One verifies easily that $(f, z) = 1$ and $\|P_z\| = 1$. The non-uniqueness is plain, since there are many choices for k.

Finally, suppose that f has exactly two nonzero components, f_j and f_k. We may suppose that $1 = |f_j| \geqslant |f_k| > 0$. Then the arguments above show that $|z_j| = |z_k| = \frac{1}{2}\|z\|$. Hence for all i, $z_i = \frac{1}{2}\|z\| \operatorname{sgn} f_i$. This condition, together with the equation $f(z) = 1$, fixes z uniquely. ∎

LEMMA 4. – *Let $f \in (l_\infty)$, $\|f\|_\infty = 1$, and $f \geqslant 0$. Let $z \in (l_1)$, $f(z) = 1$ and $\min z_i < 0$. Then there exists an $x \in (l_1)$ such that $f(x) = 1$, $x \geqslant 0$, and $\|P_x\| < \|P_z\|$.*

PROOF. – Define $x_i = 0$ if $z_i \leqslant 0$ and $x_i = \theta z_i$ if $z_i > 0$. Here $\theta = (\sum' f_i z_i)^{-1}$, the summation symbol denoting the sum for $z_i > 0$. This choice of θ ensures that $f(x) = 1$. Since $1 = \sum f_i z_i \leqslant \sum' f_i z_i$, we have $0 < \theta \leqslant 1$. We now prove that $\|P_x\| < \|P_z\|$. By the remark after Lemma 3, $\|P_x\| = \max r_i(x)$, with $r_i(x) = 1 + f_i(\|x\| - 2x_i)$. We distinguish several cases.

Case 1, $z_i < 0$. Then $x_i = 0$ and $r_i(x) = 1 + f_i\|x\| = 1 + f_i \sum' \theta z_j \leqslant 1 + \theta f_i \|z\| \leqslant 1 + f_i\|z\| < \|P_z\|$, by Eq. (1) in Lemma 3.

Case 2, $0 \leqslant f_i z_i \leqslant 1$. Then $r_i(x) = 1 + f_i(\|x\| - 2x_i) = 1 + f_i(\sum' \theta z_j - 2\theta z_i) = 1 + \theta f_i(\sum' z_j - 2z_i) \leqslant 1 + \theta f_i(\|z\| - 2z_i)$. If $\|z\| \geqslant 2z_i$ then $r_i(x) \leqslant 1 + f_i(\|z\| - 2z_i) < \|P_z\|$, by Eq. (1). If $\|z\| < 2z_i$ then $r_i(x) \leqslant 1 < \|P_z\|$.

Case 3, $f_i z_i > 1$. Then by Eq. (1) $-1 + f_i \|z\| \leqslant \|P_z\|$, whence $f_i \|z\| \leqslant 1 + \|P_z\|$. As in Case 2, we have then $r_i(x) \leqslant 1 + \theta f_i(\|z\| - 2z_i) \leqslant 1 + \theta(1 + \|P_z\|) - 2\theta = 1 + \theta(\|P_z\| - 1) \leqslant 1 + (\|P_z\| - 1) = \|P_z\|$. ∎

In the remainder of the paper, we shall assume that $f \geqslant 0$. This involves no loss of generality since $f^{-1}(0)$ and $g^{-1}(0)$ have the same projection constant if $|f| = |g|$. Indeed, if J is any subset of $\{1, 2, ...\}$, then the mapping of (l_1) into (l_1) defined by $x_i' = -x_i$ if $i \in J$ and $x_i' = x_i$ if $i \notin J$ is an isometry. In accordance with the above lemma, we may then restrict our search for minimal projections to those projections P_z for which $z \in (l_1)$, $f(z) = 1$, and $z \geqslant 0$.

We introduce now the following definitions:

$$a_n = \sum_{i=1}^{n} f_i$$

$$b_n = \sum_{i=1}^{n} f_i^{-1}$$

$$\beta_n = \frac{1}{n-2} b_n \qquad (n > 2) .$$

LEMMA 5. - *Let* $f \in (l_\infty)$, $\|f\|_\infty = 1$, $f \geqslant 0$, $z \in (l_1)$, $f(z) = 1$, *and* $z \geqslant 0$. *Define* $I = \{i : z_i = 0\}$ *and* $J = \{i : r_i(z) = \|P_z\|\}$, *where* $r_i(z) = 1 + f_i(\|z\| - 2z_i)$. *If* P_z *is not a minimal projection of* (l_1) *onto the byperplane* $f^{-1}(0)$, *then there exists an element* $u \in (l_1)$ *such that* $f(u) = 0$, $u_i \geqslant 0$ *for* $i \in I$, *and* $\sum_{i=1}^{\infty} u_i < 2 \inf_{i \in J} u_i$.

PROOF. - If P_z is not minimal, then there exists an $x \in (l_1)$ such that $f(x) = 1$ and $\|P_x\| < \|P_z\|$. By Lemma 4 we may assume that $x \geqslant 0$. Put $u = x - z$. Then $f(u) = 0$ and $u_i \geqslant 0$ for all $i \in I$ and $\varepsilon = \|P_z\| - \|P_x\|$ then

$$1 + f_i(\|x\| - 2x_i) = r_i(x) \leqslant \|P_x\| = \|P_z\| - \varepsilon = r_i(z) - \varepsilon = 1 + f_i(\|z\| - 2z_i) - \varepsilon .$$

Consequently, $\|x\| - 2x_i \leqslant \|z\| - 2z_i - \varepsilon f_i^{-1} \leqslant \|z\| - 2z_i - \varepsilon$. Equivalently, $\sum x_j - 2x_i \leqslant \sum z_j - 2z_i - \varepsilon$, whence $\sum u_j \leqslant 2u_i - \varepsilon$. ∎

THEOREM 5. - *Let* $1 = f_1 \geqslant f_2 \geqslant ... \geqslant f_n \geqslant f_i \geqslant 0$ $(i > n)$ *and assume that* $n > 2$, $f_n > 0$, $f_n^{-1} \leqslant \beta_n$, $a_{n-1} \geqslant n - 3$, *and* $a_n < n - 2$. *Then the projection constant of the hyperplane* $f^{-1}(0)$ *in* (l_1) *is* $1 + [2a_n\beta_n - n + (\beta_n - f_n^{-1})(n - 2 - a)]^{-1}$.

PROOF. - Put $k = b_n - (n-2)f_n^{-1}$, $u = 2[k - n + a_n f_n^{-1}]^{-1}$, and $v = (1 - k)u$. Define $x \in (l_1)$ by putting $x_1 = \frac{1}{2}u(f_n^{-1} - 1 + k)$, $x_i = \frac{1}{2}u(f_n^{-1} - f_i^{-1})$ for $2 \leqslant i \leqslant n$, and $x_i = 0$ for $i > n$.

Observe that $k - n + a_n f_n^{-1} > 0$. Indeed, $a_n f_n^{-1} - n \geqslant 0$ since $a_n f_n^{-1} = (f_1 + ... + f_n) f_n^{-1} \geqslant n f_n f_n^{-1}$. Also $k \geqslant 0$ because $f_n^{-1} \leqslant \beta_n = (n-2)^{-1} b_n$. Now equality cannot occur simultaneously in these two inequalities because if $n = a_n f_n^{-1}$ then $f_1 = ... = f_n = 1$, and in that event, $b_n = n > (n-2) f_n^{-1}$.

Observe next that $x \geqslant 0$. Indeed, $u > 0$ and $f_n^{-1} \geqslant f_i^{-1}$ for $i = 2, \ldots, n$. Also, $k \geqslant 0$ as proved above, and $f_n^{-1} - 1 \geqslant 0$.

Next we prove that $(f, x) = 1$. Indeed, $\sum_{i=1}^{n} f_i x_i = \sum_{i=1}^{n} f_i \frac{1}{2} u(f_n^{-1} - f_i^{-1}) + \frac{1}{2} u k = \frac{1}{2} u[k + \sum_1^n (f_i f_n^{-1} - 1)] = \frac{1}{2} u[k + f_n^{-1} a_n - n] = 1$.

Next we prove that $\|x\|_1 = u f_n^{-1}$. Indeed $\sum_1^n x_i = \sum_1^n \frac{1}{2} u(f_n^{-1} - f_i^{-1}) + \frac{1}{2} uk = \frac{1}{2} u[n f_n^{-1} - b_n + k] = \frac{1}{2} u[n f_n^{-1} - (n-2) f_n^{-1}] = u f_n^{-1}$.

Observe next that $u \geqslant v$. Indeed, this follows at once from the inequality $k \geqslant 0$ proved above.

Next we prove the equation

$$f_i(\|x\| - 2x_i) = \begin{cases} v & i = 1 \\ u & 2 \leqslant i \leqslant n \\ f_i f_n^{-1} u & n < i \end{cases}.$$

Indeed, for $i = 1$ we have $f_i(\|x\| - 2x_i) = u f_n^{-1} - u(f_n^{-1} - 1 + k) = u(1 - k) = v$. For $2 \leqslant i \leqslant n$ we have $f_i(\|x\| - 2x_i) = f_i[u f_n^{-1} - u(f_n^{-1} - f_n^{-1})] = u$. For $i > n$ we have $f_i(\|x\| - 2x_i) = f_i u f_n^{-1}$.

By Lemma 3, $\|P_x\| = 1 + \max_i (\|x\| - 2x_i) = 1 + u$.

Now we prove that P_x is a minimal projection. If it is not, then by Lemma 5, there exists a vector $\theta = (\theta_1, \theta_2, \ldots)$ in (l_1) having the following three properties:

(1) $(f, \theta) = 0$

(2) $\theta_i \geqslant 0$ for indices i such that $x_i = 0$

(3) $\sum_{i=1}^{\infty} \theta_i < 2 \min_{2 \leqslant i \leqslant n} \theta_i$

These conditions will lead to a contradiction. Let $q = \min_{2 \leqslant i \leqslant n} \theta_i$. Note that $\theta_i \geqslant 0$ for $i \geqslant n$. Hence $2q > \sum_{i=1}^{\infty} \theta_i = \sum_{i=1}^{\infty} (1 - f_i) \theta_i \geqslant \sum_{i=2}^{n} (1 - f_i) \theta_i \geqslant q \sum_{i=2}^{n} (1 - f_i) = q(n - a_n)$. Thus $q \neq 0$. If $q > 0$, then we obtain $2 > n - a_n$, contrary to hypotheses. If $q < 0$, we use the fact that $\theta_n \geqslant 0$ to obtain as above $2q > \sum_{i=2}^{n-1} (1 - f_i) \theta_i \geqslant q \sum_{i=2}^{n-1} (1 - f_i) = q(n - 1 - a_{n-1})$. Since $q < 0$, this implies that $2 < n - 1 - a_{n-1}$ or $a_{n-1} < n - 3$, contrary to hypotheses.

To obtain the formula in the theorem, we write $1 + u = 1 + 2[b_n - (n-2) f_n^{-1} - n + a_n f_n^{-1}]^{-1} = 1 + 2[(n-2)\beta_n - (n-2) f_n^{-1} - n + a_n f_n^{-1}]^{-1} = 1 + 2[a_n \beta_n - n + (\beta_n - f_n^{-1})(n - 2 - a_n)]^{-1}$. ∎

THEOREM 6. - *Let* $1 = f_1 \geqslant f_2 \geqslant \ldots \geqslant f_{n+1} \geqslant f_i \geqslant 0$ $(i > n)$. *Assume that* $n > 2$, $f_n > 0$, $f_n \geqslant \beta_n^{-1} \geqslant f_{n+1}$, *and that* $a_n \geqslant n - 2$. *Then the projection constant of the hyperplane* $f^{-1}(0)$ *in* (l_1) *is* $1 + 2(a_n \beta_n - n)^{-1}$.

PROOF – Define $u = 2(a_n\beta_n - n)^{-1}$. Define $x \in (l_1)$ by putting $x_i = \frac{1}{2}u(\beta_n - f_i^{-1})$ for $1 \leqslant i \leqslant n$, and $x_i = 0$ for $i > n$.

Observe first that u is well-defined and positive. Indeed, $a_n b_n = \sum_1^n f_i \sum_1^n f_i^{-1} = \sum_1^n f_i f_i^{-1} + \sum_{i<j} (f_i f_j^{-1} + f_j f_i^{-1}) \geqslant n + [\frac{1}{2}n(n-1)2 = n^2$. Hence $a_n\beta_n - n \geqslant n^2(n-2)^{-1} - n = 2n(n-2)^{-1} > 0$.

Observe next that $x \geqslant 0$. Indeed, by hypothesis $f_i^{-1} \leqslant f_n^{-1} \leqslant \beta_n$ for $i = 1, \ldots, n$.

Next we prove that $(f, x) = 1$. We have

$$\sum_1^n f_i x_i = \tfrac{1}{2}u \sum_i^n (f_i \beta n - 1) = \tfrac{1}{2}u(\beta_n a_n - n) = 1.$$

Next we prove that $\|x\| = u\beta_n$. Indeed,

$$\sum_1^n x_i = \tfrac{1}{2}u \sum_1^n (\beta_n - f_i^{-1}) = \tfrac{1}{2}u(n\beta_n - b_n) = \tfrac{1}{2}u(n\beta_n - (n-2)\beta_n) = u\beta_n .$$

Next we prove the equation $f_i(\|x\| - 2x_i) = u$ for $i = 1, \ldots, n$. Indeed, $f_i(\|x\| - 2x_i) = f_i(u\beta_n - u(\beta_n - f_i^{-1})) = u$.

Next we observe that $f_i(\|x\| - 2x_i) \leqslant u$ for $i > n$. Indeed, $f_i(\|x\| - 2x_i) = f_i u\beta_n \leqslant f_{n+1} u\beta_n \leqslant u$ by hypothesis.

By Eq. (2) following Lemma 3, $\|P_x\| = 1 + u$. If P_x is not a minimal projection then by Lemma 5, there exists a vector $\theta = (\theta_1, \theta_2, \ldots)$ in (l_1) having the properties:

(1) $(f, \theta) = 0$

(2) $\theta_i \geqslant 0$ for indices i such that $x_i = 0$

(3) $\sum_{i=1}^{\infty} \theta_i < 2 \min_{1 \leqslant i \leqslant n} \theta_i$.

These conditions will lead to a contradiction. Put $q = \min_{1 \leqslant i \leqslant n} \theta_i$. Then $2q > \sum_{i=1}^{\infty} \theta_i = \sum_{i=1}^{n} \theta_i + \sum_{i=n+1}^{\infty} \theta_i \geqslant \sum_{i=1}^{n} \theta_i \geqslant nq$. Hence $q < 0$. Now write

$$2q > \sum_{i=1}^{\infty} \theta_i = \sum_{i=1}^{\infty} (1 - f_i)\theta_i \geqslant \sum_{i=1}^{n} (1 - f_i)\theta_i \geqslant q \sum_{i=1}^{n} (1 - f_i) = q(n - a_n) .$$

Since $q < 0$, this yields $2 < n - a_n$ contrary to one of the hypotheses. ∎

LEMMA 6. – *Let* $1 = f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$, $f_3 > 0$, *and* $\lim f_n < 1$. *Let* A_n (*for* $n \geqslant 3$) *denote the assertion that* $f_n b_{n-1} \geqslant n - 3$ *and* $a_{n-1} \geqslant n - 3$. *Then* A_3 *is true, and there is a unique index* $n \geqslant 3$ *such that* A_n *is true and* A_{n+1} *is false.*

PROOF. – A_3 is true because $f_3b_2 \geqslant 0$ and $a_2 \geqslant 0$. Now we observe that if $a_n \geqslant n-2$ then $a_{n-1} \geqslant n-3$ because $a_{n-1} = a_n - f_n \geqslant n-2-f_n \geqslant n-2-1 = n-3$. Next we observe that if $f_{n+1}b_n \geqslant n-2$ then $f_nb_{n-1} \geqslant n-3$ because $f_nb_{n-1} = f_n(b_n - f_n^{-1}) = f_nb_n - 1 \geqslant f_{n+1}b_n - 1 \geqslant n-2-1 = n-3$.

The preceeding arguments show that A_{n+1} implies A_n. Hence either A_n is true for all $n = 3, 4, \ldots$ or there is an index n (necessarily unique) such that A_n is true and A_{n+1} is false.

We will show that A_n is eventually false. By hypothesis, there is a number θ such that $\lim f_n < \theta < 1$. Select m such that $f_m < \theta$ and $m(1-\theta) > 2$. Then $a_{2m} = (f_1 + \ldots + f_m) + (f_{m+1} + \ldots + f_{2m}) \leqslant m + m\theta = 2m - m(1-\theta) < 2m-2$. Thus A_{2m+1} is false, inasmuch as it involves the assertion that $a_{2m} \geqslant 2m-2$. ∎

LEMMA 7. – *If* $f \in (l_\infty)$, $f \geqslant 0$, *and* $\|f\| = 1$, *then the projection constant of* $f^{-1}(0)$ *in* (l_1) *is at least* $1 + (a_n - 2)n^{-1}$ *for all* n.

PROOF. – Let $z \in (l_1)$, $z \geqslant 0$, and $f(z) = 1$. Then $\|P_z\| = 1 + \sup f_i(\|z\| - 2z_i)$. Hence $\|P_z\| - 1 \geqslant f_i\|z\| - 2f_iz_i$. By summing for indices $i = 1, \ldots, n$ we obtain $n(\|P_z\| - 1) \geqslant a_n\|z\| - 2\sum f_iz_i \geqslant a_n - 2$. ∎

COROLLARY. – *If* $f \in (l_\infty)$, $\|f\| = 1$, *and* $\limsup |f_n| = 1$ *then the projection constant of* $f^{-1}(0)$ *in* (l_1) *is* 2.

PROOF. – Take a permutation π of the natural numbers such that $f_{\pi_i} \geqslant 1 - \varepsilon$ for $i = 1, \ldots, n$. Put $g_i = f_{\pi_i}$. Then $g^{-1}(0)$ and $f^{-1}(0)$ have the same projection constant. By Lemma 7, $g^{-1}(0)$ has projection constant at least $1 + [n(1-\varepsilon) - 2]n^{-1}$. This can be made arbitrarily close to 2. ∎

THEOREM 7. – *Let* $1 = f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$. *The relative projection constant* p *of the hyperplane* $f^{-1}(0)$ *in* (l_1) *is described thus:*

(1) *If* $\lim f_k = 1$ *then* $p = 2$.

(2) *If* $f_3 = 0$ *then* $p = 1$.

(3) *If* $\lim f_k < 1$ *and* $f_3 > 0$ *then let* n *be the unique index such that* $\min\{f_nb_{n-1}, a_{n-1}\} \geqslant n-3$ *and* $\min\{f_{n+1}b_n, a_n\} < n-2$. (*See Lemma* 6.) *Then*

$$p = 1 + 2[a_n\beta_n - n + (\beta_n - f_n^{-1})\max\{n-2-a_n, 0\}]^{-1}.$$

PROOF. – The case when $\lim f_k = 1$ (and thus $f_k = 1$ for all k) is governed by Lemma 7. In this case we take $z = (0, \ldots, 0, 1, 0, \ldots)$ to get a minimal projection. The case when $f_3 = 0$ is governed by Theorem 4.

The case when $\lim f_k < 1$, $f_3 > 0$, $f_nb_{n-1} \geqslant n-3$, $a_{n-1} \geqslant n-3$, and $a_n < n-2$ is governed by Theorem 5. We only need to verify that $f_n^{-1} \leqslant \beta_n$. This is true because

$$f_n\beta_n = f_nb_n(n-2)^{-1} = f_n(b_{n-1} + f_n^{-1})(n-1)^{-1} = (f_nb_{n-1} + 1)(n-2)^{-1} \geqslant (n-3+1)(n-2)^{-1} = 1.$$

The final case is when $\lim f_k < 1$, $f_3 > 0$, $f_n b_{n-1} \geqslant n-3$, $a_{n-1} \geqslant n-3$, $a_n \geqslant n-2$, and $f_{n+1} b_n < n-2$. This case is governed by Theorem 6. We must verify that $f_n \geqslant \beta_n^{-1} \geqslant f_{n+1}$. We have $f_n \geqslant \beta_n^{-1}$ by the argument of the preceding paragraph. We have $\beta_n^{-1} \geqslant f_{n+1}$ because $f_{n+1}\beta_n = f_{n+1} b_n (n-2)^{-1} < 1$. ∎

COROLLARY. – *If $1 = f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$ then the hyperplane $f^{-1}(0)$ in (l_1) possesses at least one minimal projection.*

LEMMA 8. – *Let $f \in (l_\infty)$, $f \geqslant 0$, $\|f\| = 1$. For each $n \in \boldsymbol{N}$ and for each $\varepsilon > 0$ there is a permutation $\pi: \boldsymbol{N} \to \boldsymbol{N}$ such that $f_{\pi_1} \geqslant f_{\pi_2} \geqslant \ldots \geqslant f_{\pi_n} \geqslant f_{\pi_i} - \varepsilon$ for all $i > n$.*

PROOF. – Let n and ε be given. For each $k \in \boldsymbol{N}$, define $J(k) = \{i: 1 \geqslant f_i \geqslant 1 - k\varepsilon\}$. Let k_0 denote the first element of N such that $J(k_0)$ contains at least n elements.

If $k_0 = 1$, then select integers $\pi_1, \ldots, \pi_n \in J(1)$. We rearrange these integers so that $f_{\pi_1} \geqslant \ldots \geqslant f_{\pi_n}$. Let the set $\boldsymbol{N} \diagdown \{\pi_1, \ldots, \pi_n\}$ be enumerated in any convenient order as $\pi_{n+1}, \pi_{n+2}, \ldots$. Clearly $f_{\pi_n} \geqslant 1 - \varepsilon \geqslant f_{\pi_i} - \varepsilon$ for $i > n$.

If $k_0 > 1$ then $J(k_0 - 1)$ contains fewer than n elements. Let them be emmerated as $\pi_1, \ldots, \pi_s$ with $f_{\pi_1} \geqslant \ldots \geqslant f_{\pi_s}$ and $s < n$. Since $J(k_0)$ contains at least n elements, we can select integers $\pi_{s+1}, \ldots, \pi_n \in J(k_0) \diagdown J(k_0 - 1)$. These too can be arranged so that $f_{\pi_{s+1}} \geqslant \ldots \geqslant f_{\pi_n}$. Now we have

$$f_{\pi_1} \geqslant \ldots \geqslant f_{\pi_s} \geqslant 1 - (k_0 - 1)\varepsilon > f_{\pi_{s+1}} \geqslant \ldots \geqslant f_{\pi_n} \geqslant 1 - k_0 \varepsilon .$$

Let the set $\boldsymbol{N} \diagdown \{\pi_1, \ldots, \pi_n\}$ be enumerated as $\pi_{n+1}, \pi_{n+2}, \ldots$. Then for $i > n$ we have $f_{\pi_i} < 1 - (k_0 - 1)\varepsilon = 1 - k_0\varepsilon + \varepsilon \leqslant f_{\pi_n} + \varepsilon$. ∎

LEMMA 9. – *If $0 < \varepsilon < 1$, if $f, g \in X^*$, if $\|f\| = \|g\| = 1$ and if $\|f - g\| < \varepsilon/72$ then the projection constants of the corresponding hyperplanes satisfy the inequality*

$$|p[f^{-1}(0)] - p[g^{-1}(0)]| < \varepsilon .$$

PROOF. – Select $z \in X$ so that $(f, z) = 1$ and so that $\|I - f \otimes z\| < p[f^{-1}(0)] + \varepsilon/2$. Since $p[f^{-1}(0)] \leqslant 2$, [5], we have

$$3 > 2 + \varepsilon/2 > \|I - f \otimes z\| \geqslant \|f \otimes z\| - 1 .$$

From this we conclude that $\|z\| = \|f \otimes z\| < 4$. If $\|f - g\| < \varepsilon/72$ then $|(g - f, z)| < \varepsilon/18 < \frac{1}{2}$. Hence $(g, z) > \frac{1}{2}$ and it is permissible to define $x = z/(g, z)$. We have now

$$\begin{aligned} \|x - z\| &= \|z\| \, |1 - (g, z)^{-1}| < 4|(g, z) - 1|(g, z)^{-1} \\ &< 8|(g - f, z)| < 4\varepsilon/9 \end{aligned}$$

$$\begin{aligned} \|(I - g \otimes x) - (I - f \otimes z\| &= \|f \otimes z - g \otimes x\| \\ &\leqslant \|(f - g) \otimes z\| + \|g \otimes (z - x)\| \leqslant \|f - g\| \, \|z\| + \|g\| \, \|z - x\| \\ &\leqslant 4\varepsilon/72 + 4\varepsilon/9 = \varepsilon/2 . \end{aligned}$$

Thus $p[g^{-1}(0)] \leqslant \|I - g \otimes x\| \leqslant \|I - f \otimes z\| + \varepsilon/2 < p[f^{-1}(0)] + \varepsilon$. Since the hypotheses involve g and f symmetrically, the inequality $p[f^{-1}(0)] < p[g^{-1}(0)] + \varepsilon$ must also be true. ∎

THEOREM 8. – *Let $f \in (l_\infty)$, $\|f\| = 1$, $f \geqslant 0$, and $\lambda \equiv \limsup f_n < 1$. If the set $\{i : f_i > \lambda\}$ is finite, say $i_1, \ldots, i_k$, then the projection constant of $f^{-1}(0)$ is the same as that of $g^{-1}(0)$, where $g = (f_{i_1}, f_{i_2}, \ldots, f_{i_k}, \lambda, \lambda, \lambda, \ldots)$.*

PROOF. – Select an index $m > k + 2(1-\lambda)^{-1}$. By our hypotheses and by Lemma 8, there exists for every $\varepsilon > 0$ a permutation π such that

$$1 = f_{\pi_1} \geqslant \ldots \geqslant f_{\pi_k} > \lambda > f_{\pi_{k+1}} \geqslant \ldots \geqslant f_{\pi_m} \geqslant f_{\pi_i} - \varepsilon \qquad (i > m)\,.$$

Since $\limsup f_n = \lambda$, $f_{\pi_m} \geqslant \lambda - \varepsilon$. Define $h \in (l_\infty)$ by putting

$$h_i = \begin{cases} f_{\pi_i} & i \leqslant m \\ \max(f_{\pi_i} - \varepsilon, 0) & i > m\,. \end{cases}$$

Then $1 \geqslant h_1 \geqslant \ldots \geqslant h_k \geqslant \lambda > h_{k+1} \geqslant \ldots \geqslant h_m \geqslant h_i$ $(i > m)$. Moreover, if we calculate the numbers a_n for h we find that

$$\begin{aligned} a_m &= (h_1 + \ldots + h_k) + (h_{k+1} + \ldots + h_m) \leqslant k + (m-k)\lambda \\ &= m - (m-k)(1-\lambda) < m - 2\,. \end{aligned}$$

Thus in applying Theorem 5 or 6 to h, we know that $n < m$. Hence the projection constant of $h^{-1}(0)$ depends only on $h_1, \ldots, h_m$. Letting $\varepsilon \to 0$ and using Lemma 9, we establish the theorem. ∎

THEOREM 9. – *Let $f \in (l_\infty)$, $\|f\| = 1$, $f \geqslant 0$, and $\lambda \equiv \limsup f_n < 1$. If the set $\{i : f_i > \lambda\}$ is infinite then there exist indices $m_1, m_2, \ldots, m_r$ such that $1 = f_{m_1} \geqslant f_{m_2} \geqslant$ $\geqslant \ldots \geqslant f_{m_r} \geqslant f_i$ for $i \notin \{m_1, \ldots, m_r\}$ and such that the projection constant of $f^{-1}(0)$ in (l_1) is the same as that of $g^{-1}(0)$, where*

$$g = (f_{m_1}, \ldots, f_{m_r}, 0, 0, \ldots)$$

PROOF. — Let the set $\{i : f_i > \lambda\}$ be enumerated as $m_1, m_2, \ldots$ in such a way that $f_{m_1} \geqslant f_{m_2} \geqslant \ldots$. Select k so that $f_{m_k} < \frac{1}{2}(1+\lambda)$. Select $r > k + 4(1-\lambda)^{-1}$. Define $g \in (l_\infty)$ as above.

If we compute the numbers a_n for the sequence g we find that $a_r = (g_1 + \ldots + g_k) +$ $+ (g_{k+1} + \ldots + g_r) \leqslant k + (r-k)\frac{1}{2}(1+\lambda) < r - 2$. Thus (by Theorem 7 and lemma 7) the projection constant of $g^{-1}(0)$ depends only on the set of numbers $\{g_1, \ldots, g_r\} =$ $= \{f_{m_1}, \ldots, f_{m_r}\}$.

Now by Lemma 8, there exists for each $\varepsilon > 0$ a permutation $\pi: \boldsymbol{N} \to \boldsymbol{N}$ such that $\pi_1 = m_1$, $\pi_2 = m_2$, ..., $\pi_r = m_r$ and

$$f_{\pi_1} \geqslant f_{\pi_2} \geqslant \dots \geqslant f_{\pi_r} \geqslant f_{\pi_i} - \varepsilon \qquad (i > r) .$$

Define $h \in (l_\infty)$ by letting $h_i = f_{\pi_i}$ for $i \leqslant r$ and $h_i = \max\{f_{\pi_i} - \varepsilon, 0\}$ for $i > n$. Then (by Lemma 9) the projection constants of $h^{-1}(0)$ and $f^{-1}(0)$ differ by at most 72ε. The projection constant of $h^{-1}(0)$ equals that of $g^{-1}(0)$. Since this is true for every $\varepsilon > 0$, the projection constants of $f^{-1}(0)$ and $g^{-1}(0)$ are equal.

REFERENCES

[1] J. BLATTER - E. W. CHENEY, *On the existence of extremal projections*, J. Approximation Theory, **6** (1972), pp. 72-79.

[2] E. W. CHENEY - K. H. PRICE, *Minimal projections*, in *Approximation Theory*, A. Talbot, ed., Academic Press, New York, pp. 261-289.

[3] E. W. CHENEY - K. H. PRICE, *Minimal interpolating projections*, in *Iterations-Verfahren Numerische Mathematik Approximationstheorie*, ISNM, vol. 15, Birkhauser Verlag, Basel, 1970, pp. 115-121.

[4] E. W. CHENEY, *Projections with finite carrier*, Research Paper CNA 28, Center for Numerical Analysis, The University of Texas at Austin, Texas, July 1971. To appear in Proceedings of a Conference on Numerical Methods in Approximation Theory, Oberwolfach, Germany, June 1971, ISMN vol. 16, Birkhauser-Verlag, Basel, 1972.

[5] A. JU. LEVIN - JU. I. PETUNIN, *Some problems related to the concept to orthogonality in a Banach space*, Uspehi Math. Nauk, **18** (1963), no. 3(111), pp. 167-170, MR 27-2833.