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MINIMAL PROJECTIVE RESOLUTIONS

E. L. GREEN, Ø. SOLBERG, AND D. ZACHARIA

Dedicated to Helmut Lenzing for his 60th birthday

ABSTRACT. In this paper, we present an algorithmic method for computing a projective resolution of a module over an algebra over a field. If the algebra is finite dimensional, and the module is finitely generated, we have a computational way of obtaining a minimal projective resolution, maps included. This resolution turns out to be a graded resolution if our algebra and module are graded. We apply this resolution to the study of the Ext-algebra of the algebra; namely, we present a new method for computing Yoneda products using the constructions of the resolutions. We also use our resolution to prove a case of the "no loop" conjecture.

INTRODUCTION

In the study of homological properties of rings and modules, projective resolutions are a basic tool. Such resolutions occur naturally in commutative ring theory, the representation theory of finite dimensional algebras, group representation theory, algebraic geometry, and algebraic topology [A, AG, F, H1, HZ]. On the other hand, with the introduction of computers, computational and algorithmic techniques have grown in importance [Ba, FGKK]. Both theoretical and practical results are needed. This paper presents a new method of constructing projective resolutions in a broad setting which has both theoretical and computational implications. In particular, in the graded and finite dimensional cases, our results provide a recursive procedure for computing minimal projective resolutions.

The class of algebras studied in this paper consists of quotients of path algebras. We fix a field K for the remainder of this paper. If Q is a finite directed graph, which we call a *quiver*, then the *path algebra*, KQ, is the K-algebra with K-basis consisting of finite directed paths in Q. Thus, elements of KQ consist of K-linear combinations of paths in Q. The multiplicative structure on basis elements p and q is defined by concatenation pq if the terminus of p equals the origin of q, and by 0 otherwise. We view the vertices as paths of length 0 with multiplication given as follows. If v and w are vertices and p is a path, we let $v \cdot w$ be v if $v = w$ and 0 otherwise. We let $v \cdot p = p$ if v is the origin of p and 0 otherwise, and we define $p \cdot w$ similarly. The multiplication on paths is extended linearly to arbitrary elements of KQ. Note that the free associative K-algebra on n noncommuting variables is

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isomorphic to the path algebra KQ where Q has one vertex and n loops. We let Q_0 denote the vertex set of Q.

Let Q be a quiver and I be a (two-sided) ideal in the path algebra KQ. Let Λ denote KQ/I for the remainder of the introduction. The algebras in this class include all affine (that is, finitely generated) associative K-algebras. Every finite dimensional K-algebra is Morita equivalent to an algebra in this class if K is algebraically closed. Furthermore, this class includes graded K-algebras $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ where Λ_0 is a product of a finite number of copies of K, each Λ_i is a finite dimensional K-vector space and Λ is generated in degrees 0 and 1; that is, for $i, j \geq 0$, $\Lambda_i \Lambda_j = \Lambda_{i+j}$.

Let M be a $\Lambda = KQ/I$-module. Let $F \to M \to 0$ be an exact sequence of KQ-modules with $F = \coprod_{v \in Q_0} vKQ$. A main theme of the paper is the construction of a filtration of F by KQ-submodules which contains all the information needed to construct the Λ-projective resolution of M, the Λ-syzygies and the Yoneda product of extensions of Λ-modules. In particular, we find a filtration

$$\cdots \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0,$$

such that $F = F^0$, $M = F^0/F^1$ and

$$\cdots F^n/F^nI \to F^{n-1}/F^{n-1}I \to \cdots \to F^1/F^1I \to F^0/F^0I \to M \to 0$$

is a Λ-projective resolution of M with the maps induced by the inclusions of the filtration. For the basic construction, we do not assume that Λ is finite dimensional, or even noetherian. Furthermore, we do not assume that the Λ-module M is finitely generated. For our minimality results, M will be finitely generated and Λ either finite dimensional or graded.

We provide a recursive formula to compute F^n as a KQ-submodule of F^{n-1} from the previously obtained $F^{n-1} \subset F^{n-2}$. To explicitly find F^n from our formula, one must write an intersection of certain submodules of a projective KQ-module as a direct sum of cyclic submodules. A method for finding the generators of these cyclic submodules employs the theory of right Gröbner bases, and will appear elsewhere.

Our construction resembles earlier resolutions of Bongartz, Butler, Eilenberg, Eilenberg-Nagao-Nakayama and Gruenberg [Bo, E, ENN]. Their resolutions are almost never minimal in the finite dimensional case, and deal only with resolutions of semisimple modules. We recall their resolution. Let J denote the ideal of KQ generated by the arrows of Q. Furthermore, assume that $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geq 2$. Then we have the filtration

$$\cdots \subset JI^n \subset I^n \subset \cdots \subset I^3 \subset JI^2 \subset I^2 \subset JI \subset I \subset J \subset KQ.$$

Note that J/I is the Jacobson radical of Λ and that $\Lambda/(J/I)$ is isomorphic to KQ/J. One gets the following Λ-projective resolution of KQ/J

$$\cdots \to I^n/I^{n+1} \to JI^{n-1}/JI^n \to \cdots \to I/I^2 \to J/JI \to KQ/I \to KQ/J \to 0,$$

where the maps are induced by the inclusions.

The paper is organized as follows. In the first section, we give a general construction of a projective resolution of an arbitrary Λ-module M, where Λ is a quotient of a path algebra. We show that if Λ is a right noetherian algebra and M is a finitely generated Λ-module, then the resolution is finitely generated. If Λ is graded and M is a graded module, we show how to modify the construction to obtain a graded projective resolution.

In the second section, we provide algorithmic techniques to adjust the construction to obtain minimal projective resolutions in both the finite dimensional and the graded cases. The section ends with explicit computations of syzygies and Ext-groups.

Section 3 deals with Ext-algebras. If Λ_0 denotes Λ modulo its radical in the finite dimensional case, or, Λ modulo its graded radical in the graded case, then we study the algebraic structure of

$$E(\Lambda) = \coprod_{n \geq 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0).$$

Furthermore, if M is either a finite dimensional Λ-module or a graded Λ-module, we investigate the $E(\Lambda)$-module structure of $E(M) = \coprod_{n \geq 0}(M, \Lambda_0)$. A major result of the paper is that this module structure is included in the information obtained in the construction of the resolution. In particular, one need not "lift maps" to find the Yoneda products.

We apply our techniques to prove one case of the No Loop Conjecture in section four. Namely, we prove that if Λ is a finite dimensional K-algebra and a is a loop at the vertex v such that a^n is the first power of a belonging to the ideal I but a^n is not in $JI + IJ$, then $\operatorname{Ext}^n_\Lambda(S, S) \neq (0)$, for all $n \geq 1$, where S is the simple Λ-module corresponding to the vertex v.

In the final section we investigate the influence of the characteristic of the ground field K on the structure of projective resolutions. Other than some new examples, we show that if the global dimension of Λ is bounded by 2 in one characteristic, then the global dimension will be finite in all characteristics. We also provide an example of an algebra that has infinite global dimension in only one characteristic.

Finally, we note that all modules will be right modules unless otherwise stated. We also introduce some terminology. We say that an element x in the path algebra KQ is *right uniform*, if $x \neq 0$ and there is a vertex v such that $xv = x$. Note that if $x \neq 0$ is an element of KQ, then $x = \sum_{v \in Q_0} xv$. Hence, every nonzero element of KQ is a sum of right uniform elements. From a different point of view, $KQ = \coprod_{v \in Q_0} KQv$ as left modules. Hence, every nonzero element is a sum of right uniform elements in a unique way. An element is right uniform if and only if it is nonzero and a linear combination of paths ending at a single vertex. Finally, note that if x is a right uniform element with $xv = x$ for some $v \in Q_0$, then xKQ is a right projective KQ-module isomorphic to vKQ.

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1. The resolution

Let Q be a finite quiver, and let $R = KQ$ denote the path algebra of Q over a field K. Let I be a two-sided ideal in R such that $I \subseteq J^2$, where J denotes the ideal of R generated by the arrows of the quiver Q. Let $\Lambda = R/I$ be the quotient algebra, and let M be a right Λ-module. In this section we construct, in an algorithmic way, a projective resolution $(\mathcal{P}, \delta)$ of M over Λ. This resolution need not be finitely generated in general, but it is when Λ is noetherian and I is finitely generated as a right ideal in R. In particular, if I is an admissible ideal of R, that is, $J^N \subseteq I \subseteq J^2$ for some $N > 1$, then Λ is a finite dimensional K-algebra and the resolution $(\mathcal{P}, \delta)$ becomes a finitely generated resolution. In the next section we also show how, in this case, we can adjust $\mathcal{P}$ in an algorithmic way to obtain a minimal projective resolution of M_Λ. Moreover, if Λ is graded by the natural grading induced from the length grading on R, then the resolution constructed for a graded Λ-module is also graded.

We shall use the following well-known properties of the path algebra $R = KQ$: (a) for every x in R, the R-module xR is projective, and, (b) for each R-submodule Y of $\amalg_i x_i R$ with x_i in R, we have $Y = \amalg_j y_j R$ for some y_j in $\amalg_{i=1}^k x_i R$ (of course, if Y is finitely generated, then we can write $Y = \amalg_{j=1}^t y_j R$ for some finite set $\{y_1, \dots, y_t\}$ in $\amalg_i x_i R$, [G]). We now introduce the notation that will be needed in defining the resolution $(\mathcal{P}, \delta)$ of M, and, throughout this paper.

Choose a family $\{f_i^0\}_{i \in A}$ of elements of R such that the projective Λ-module $\amalg_{i\in A} f_i^0 R / \amalg_{i\in A} f_i^0 I$ maps onto M. Without loss of generality we choose the family to consist of vertices in R (repetitions allowed). We have

$$0 \to \Omega_R^1(M) \to \amalg_{i\in A} f_i^0 R \to M \to 0,$$

and, we then choose a set $\{f_i^{1*}\}$ of elements of $\amalg_{i\in A} f_i^0 R$ such that $\Omega_R^1(M) = \amalg_i f_i^{1*} R$. Discard all the elements f_i^{1*} that are in $\amalg_{i\in A} f_i^0 I$ and denote by $\{f_i^1\}$ those f^{1*}'s that are not elements of $\amalg_{i\in A} f_i^0 I$. Assume that we have constructed families of elements of $\amalg_{i\in A} f_i^0 R$: $\{f_i^k\}_i$ for each $k = 0, \dots, n$. We now construct the family $\{f_i^{n+1}\}_i$ as follows. We consider the intersection $(\amalg_i f_i^n R) \cap (\amalg_j f_j^{n-1} I)$. We stop if the intersection is zero, and we set it equal to some $\amalg_l f_l^{n+1*} R$ otherwise. Discard all the elements of the form f^{n+1*} that are in $\amalg_i f_i^n I$, and denote the remaining ones by $\{f_i^{n+1}\}_i$. If each element of the form f^{n+1*} is in $\amalg_i f_i^n I$, we again stop at this stage of the construction. Note that we may assume that for each n, each element f_i^n can be chosen to be right uniform, that is, there is a vertex v (dependent on f_i^n) such that $f_i^n v = f_i^n$. An element of R is *uniform* if it is a linear combination of paths in R, all starting at one vertex, and, all ending at one vertex. We also note that, for each $n > 0$, we have a representation of f_k^n in $\amalg f_i^{n-1} R$ as follows:

$$f_k^n = \sum_i f_i^{n-1} h_{i,k}^{n-1,n}$$

for scalars $h_{i,k}^{n-1,n}$ in R. Note that for each k, all but a finite number of $h_{i,k}^{n-1,n}$ are zero. It is convenient to encode this information in the matrix $(h_{i,k}^{n-1,n})$. Furthermore, since the f_k^n's and the f_i^{n-1}'s are right uniform, it follows that each $h_{i,k}^{n-1,n}$ is uniform.

Setting $F^n = \amalg_i f_i^n R$, from our construction, we have the following filtration of the right projective R-module F^0:

$$\cdots \subseteq F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^2 \subseteq F^1 \subseteq F^0.$$

Definition 1.1. For each $n \geq 0$ let $P_n = \amalg_i f_i^n R / \amalg_i f_i^n I$, and let $\delta^n \colon P_n \to P_{n-1}$ be the homomorphism induced by the inclusion $\amalg f_i^n R \subset \amalg f_j^{n-1} R$. We also define the matrix $(\overline{h}_{i,k}^{n-1,n})$ where $\overline{h}$ denotes the image in Λ of the element h in R.

Note that the boundary maps δ^n are, in fact, determined by multiplication by the matrix $(\overline{h}^{n-1,n})$, which gives a formula for the coordinates.

We can now state our first result.

Theorem 1.2. $(\mathcal{P},\delta)\colon\ \cdots \to P_n \xrightarrow{\delta^n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta^1} P_0 \to M \to 0$ *is a projective resolution of* M *over* Λ.

Proof. It is clear that for each $n \geq 0$, the modules P_n are projective Λ-modules. From the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \amalg_i f_i^0 I & = & \amalg_i f_i^0 I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_R(M) & \longrightarrow & \amalg_i f_i^0 R & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^1_\Lambda(M) & \longrightarrow & \amalg_i f_i^0 R/\amalg_i f_i^0 I & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

it follows that we have exactness at P_0. We now show that for each $n > 0$, we have $\delta^n\delta^{n+1} = 0$. We must show that $(\overline{h}_{i,k}^{n-1,n})(\overline{h}_{k,l}^{n,n+1})$ is the zero matrix, or, equivalently that for each i and l, the sum $\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}$ is in I. But $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1})$ is an element of $\amalg f_i^{n-1}R$, and we also have

$$\begin{aligned}\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}) &= \sum_k (\sum_i f_i^{n-1} h_{i,k}^{n-1,n}) h_{k,l}^{n,n+1} \\ &= \sum_k f_k^n h_{k,l}^{n,n+1} = f_l^{n+1}\end{aligned}$$

which lies in $\amalg f_i^{n-1}I$. We infer from the uniqueness of the representations as elements of direct sums that for each i and l, the element $\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}$ is in I.

It remains to show that for each n, $\operatorname{Ker}\delta^n \subseteq \operatorname{Im}\delta^{n+1}$. Let $(\overline{x}_k)_k$ be in the kernel of δ^n. Therefore, $\sum_k \overline{h}_{i,k}^{n-1,n}\overline{x}_k = \overline{0}$ for all i, or, equivalently $\sum_k h_{i,k}^{n-1,n} x_k$ is in the ideal I for all i. On the other hand,

$$\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k) = \sum_k (\sum_i f_i^{n-1} h_{i,k}^{n-1,n}) x_k = \sum_k f_k^n x_k$$

is an element of $\amalg_k f_k^n R$ and, since $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k)$ is also in $\amalg_i f_i^{n-1} I$, we have that the element $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k)$ is in $\amalg f_j^{n+1*} R$. Therefore, we can rewrite it as

$$\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k) = \sum_j f_j^{n+1}\gamma_j + u,$$

where γ_j is an element in R and u is an element of $\amalg f_k^n I$. We claim that we have $\delta^{n+1}((\overline{\gamma}_j)_j) = (\overline{x}_k)_k$, where $\overline{\gamma}_j$ denotes the image in Λ of the element γ_j in R. To prove this we have

$$\delta^{n+1}((\overline{\gamma}_j)_j) = (\overline{h}_{k,j}^{n,n+1})(\overline{\gamma}_j)_j = (\sum_j \overline{h}_{k,j}^{n,n+1}\overline{\gamma}_j)_k.$$

But, we have

$$\sum_k f_k^n(\sum_j h_{k,j}^{n,n+1}\gamma_j) = \sum_j f_j^{n+1}\gamma_j = \sum_k f_k^n x_k$$

modulo $\amalg f_k^n I$. Hence, we infer that for each k we get $\sum_j h_{k,j}^{n,n+1}\gamma_j = x_k$ modulo I. This proves that $\operatorname{Ker}\delta^n \subseteq \operatorname{Im}\delta^{n+1}$ and the proof is complete. □

We show next that the resolution constructed above is a finitely generated resolution if we assume in addition that Λ is noetherian and I is finitely generated as a right ideal in R.

Theorem 1.3. *Assume that Λ is noetherian and that I is finitely generated as a right ideal of $R = KQ$. Let M_Λ be finitely generated. Then the resolution $(\mathcal{P}, \delta)$ of M_Λ is finitely generated.*

Proof. First observe that we may choose $f_1^0, \dots, f_k^0$ in R such that $\amalg_{i=1}^k f_i^0 R / \amalg_{i=1}^k f_i^0 I$ maps onto M. To prove the theorem, it is enough to show that, for each $n > 0$, the direct sums $\amalg_i f_i^{n*} R$ are finite. We prove this first for $n = 1$. We have the exact sequence of R-modules,

$$0 \to \amalg_{i=1}^k f_i^0 I \to \Omega_R^1(M) \to \Omega_\Lambda^1(M) \to 0,$$

and, since both ends are finitely generated, then so is the middle term. But $\Omega_R^1(M) = \amalg_i f_i^{1*} R$ hence this sum must be finite. We show now by induction, that, for each $n > 1$ we have $\Omega_R^1(\Omega_\Lambda^{n-1}(M)) = \amalg_i f_i^{n*} R$ and that they are all finitely generated. (Here by $\Omega_\Lambda^k(M)$ we mean the kernel of the map $\amalg_i f_i^{k-1} R / \amalg_i f_i^{k-1} I \to \Omega_\Lambda^{k-1}(M)$.) If $n \geq 2$, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & (\amalg_i f_i^{n-2} I) \cap (\amalg_j f_j^{n-1} R) & \longrightarrow & \amalg_j f_j^{n-1} R & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \amalg_i f_i^{n-2} I & \longrightarrow & \Omega_R^1(\Omega_\Lambda^{n-2}(M)) & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \end{array}$$

which shows that $\Omega_R^1(\Omega_\Lambda^{n-1}(M)) = \amalg_t f_t^{n*} R$, and the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \amalg_j f_j^{n-1} I & = & \amalg_j f_j^{n-1} I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_R^1(\Omega_\Lambda^{n-1}(M)) & \longrightarrow & \amalg_j f_j^{n-1} R & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_\Lambda^n(M) & \longrightarrow & \amalg_j f_j^{n-1} R / \amalg_j f_j^{n-1} I & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

and, from the left vertical exact sequence, an easy induction argument shows that $\Omega_R^1(\Omega_\Lambda^{n-1}(M))$ is finitely generated. Thus the sum $\amalg_t f_t^{n*} R$ is finite. The proof of the theorem is now complete. $\square$

Remark 1.4. If I is an admissible ideal in R, then I is finitely generated as a right ideal in R and $\Lambda = R/I$ is a finite dimensional K-algebra, hence noetherian. Therefore, as a corollary of the above the resolution is finitely generated for any finitely generated Λ-module M_Λ when I is an admissible ideal.

The path algebra $R = KQ$ has a natural grading $R = \amalg_i (KQ)_i$ where, for each i, $(KQ)_i$ denotes the K-vector space spanned by the paths of Q of length i. Each $(KQ)_i$ is endowed with an obvious $(KQ)_0$-$(KQ)_0$-bimodule structure, and, $J = \amalg_{i \geq 1}(KQ)_i$ is the graded radical of R. If $I \subseteq J^2$ is a two-sided ideal generated by homogeneous elements, then $\Lambda = KQ/I$ has an induced grading. In this case we say that Λ is *length graded*. By a graded Λ-module M, we will always mean a graded Λ-module $M = \amalg_{i \in \mathbb{Z}} M_i$ such that, $M_i = (0)$ for sufficiently small i, and, each M_i is a finite dimensional K-vector space. In particular, Λ is a graded Λ-module. Given a graded module, it has a projective cover in the category of graded modules and degree zero maps, and, its kernel is again a graded module in our sense. Note also, that as a graded algebra, Λ is generated in degrees 0 and 1.

Proposition 1.5. *Let M_Λ be a graded Λ-module. Then, the projective resolution $(\mathcal{P}, \delta)$ of Definition 1.1 can be chosen to be a graded resolution of M_Λ.*

Proof. Since M_Λ is graded, we now take $\amalg f_i^0 R \to M \to 0$ as a degree 0 homomorphism with the f_i^0's homogeneous elements of R in the appropriate degrees. We have a sequence of R-modules

$$0 \to \Omega_R^1 M \to \amalg f_i^0 R \to M \to 0$$

which is exact in $\operatorname{gr} R$. Since $\Omega_R^1(M)$ is a graded submodule, we may take $\amalg f_i^{1*} R = \Omega_\Lambda^1(M)$ with the elements f_i^{1*} right uniform homogeneous elements of $\amalg_i f_i^0 R$. Since I is a homogeneous ideal of R, it follows that I is also a homogeneous right ideal of R. Thus $\amalg f_0 R$ is also a graded submodule of $\amalg f^0 R$. Therefore, we have that $\amalg f^{2*} R = (\amalg f^1 R) \cap (\amalg f^0 I)$ is also a graded submodule of $\amalg f^0 R$. We inductively construct a chain of graded R-submodules of $\amalg f^0 R$:

$$\cdots \subset \amalg f^n R \subset \amalg f^{n-1} R \subset \cdots \subset \amalg f^0 R.$$

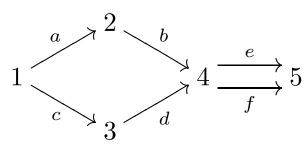
The result follows by taking quotients modulo I. □

2. Minimality

In this section we give an example showing that the projective resolution constructed in the previous section need not to be minimal when I is an admissible ideal. However, when I is an admissible ideal, we prove that the elements $\{f^n\}$ can be chosen such that the resolution is minimal. We also compare this resolution with the Bongartz-Butler-Gruenberg resolution in [Bo].

We start with an example showing that the resolution constructed in Definition 1.1 need not be minimal.

Example 2.1. Let R be the path algebra of the following quiver:



and let I be the ideal of R generated by $ab - cd$, bf and de. Let $\Lambda = R/I$, and let S_1 be the simple Λ-module corresponding to the vertex v_1. The ideal I is a 7-dimensional vector space with a basis given by the elements $ab - cd$, $abe - cde$, $abf - cdf$, abf, cde, bf and de. We construct now a projective resolution of S_1 over Λ using the resolution described in Definition 1.1.

We can take $f^0 = v_1$, so we have $0 \to \Omega^1_R S_1 \to v_1 R \to S_1 \to 0$ and we can decompose $\Omega^1_R S_1 = f^1_1 R \amalg f^1_2 R$, where $f^1_1 = a$ and $f^1_2 = c$. Next we note that $\amalg f^{2*} R = (aR \amalg bR) \cap v_1 I = v_1 I$, and a K-basis of $v_1 I$ is the set $\{ab - cd, abe - cde, abf - cdf, abf, cde\}$. We decompose $v_1 I$ as

$$v_1 I = (ab - cd)R \amalg abfR \amalg cdeR.$$

But abf and cde are in $\amalg f^1 I = aI \amalg cI$. So $f^2 = ab - cd$ and we have $f^2 R = (ab - cd)R$. Now we compute f^{3*}. We have that $\amalg f^{3*} R = (ab - cd)R \cap (aI \amalg cI)$, and it easy to check that $\amalg f^{3*} R = (0)$, so that we obtain the following projective Λ-resolution of S_1, which turns out to be minimal:

$$0 \to (ab - cd)R/(ab - cd)I \to aR/aI \amalg cR/cI \to v_1 R/v_1 I \to S_1 \to 0.$$

We remark that we could have decomposed $v_1 I$ also in the following way: $v_1 I = cdfR \amalg (ab - cd)R \amalg cdeR$ and $cdeR$ is contained in $aI \amalg cI$, but cdf and $(ab - cd)$ are not in $aI \amalg cI$. So we can write $f^2_1 = cdf$ and $f^2_2 = ab - cd$. We continue and get $\amalg f^{3*} R = (cdfR \amalg (ab - cd)R) \cap (aI \amalg cI) = abfR$. Finally, we get the following Λ-projective resolution of S_1, which is clearly not minimal:

$$0 \to abfR/abfI \to cdfR/cdfR \amalg (ab - cd)R/(ab - cd)I \\ \to aR/aI \amalg cR/cI \to v_1 R/v_1 I \to S_1 \to 0.$$

The next result shows, as in the above example, that we can always choose the elements $\{f^n\}$ in such a way that we obtain a minimal projective resolution of a finitely generated Λ-module when I is an admissible ideal.

Assume now that I is an admissible ideal, hence Λ is finite dimensional. Choose $\{f_i^0\}$ such that $\amalg_i f_i^0 R / \amalg_i f_i^0 I$ is a projective cover of M_Λ. We have the following.

Theorem 2.2. *In the resolution* $(\mathcal{P}, \delta) = (\amalg f_i^n R / f_i^n I, (\overline{h}^{n-1,n}))$ *the elements* $\{f_j^n\}$ *can be chosen in such a way that, for each* n*, no proper* K*-linear combination of a subset of* $\{f_j^n\}$ *is in* $\amalg f^{n-1} I + \amalg f^{n*} J$*.*

Moreover, there is a decomposition

$$\amalg f^{n*} R = (\amalg f_i^n R) \amalg (\amalg f_i^{n'} R),$$

where the elements $f_i^{n'}$ *can be chosen to be in* $\amalg f^{n-1} I$*.*

Proof. For each $n \geq 2$ we have the decomposition

$$\amalg f^{n*} R = (\amalg f^{n-1} R) \cap (\amalg f^{n-2} I) \tag{1}$$

Step 1: We show first that we can adjust the decomposition (1) to obtain a decomposition of the type

$$\amalg f^{n*} R = (f_1^n R \amalg \dots \amalg f_t^n R) \amalg (f_{t+1}^{n'} R \amalg \cdots \amalg f_k^{n'} R), \tag{2}$$

where each $f_j^{n'}$ is in $\amalg f^{n-1} I + \amalg f^{n*} J$, and, no proper K-linear combinations of a subset of $\{f_i^n\}$ is in $\amalg f^{n-1} I + \amalg f^{n*} J$. To prove this claim, start with the decomposition (1). If $x = \alpha_1 f_1^n + \cdots + \alpha_s f_s^n$ is a K-linear combination, where, say $\alpha_1 \neq 0$, then we have $f_1^n R \amalg \dots \amalg f_t^n R = xR \amalg f_2^n R \amalg \dots \amalg f_t^n R$. Thus, if x is in $\amalg f^{n-1} I + \amalg f^{n*} J$, we adjust our initial decomposition to the decomposition

$$\amalg f^{n*} R = (f_2^n R \amalg \dots \amalg f_t^n R) \amalg (xR \amalg f_{t+1}^{n'} R \amalg \dots \amalg f_k^{n'} R),$$

x thus becoming one of the $f^{n'}$'s. The element x may be assumed to be right uniform. We continue this process and the claim is proved.

Step 2: By the first step, we may assume that we have a decomposition of the type

$$\amalg f^{n*} R = (f_1^n R \amalg \dots \amalg f_t^n R) \amalg (f_{t+1}^{n'} R \amalg \dots \amalg f_k^{n'} R),$$

where each of the $f^{n'}$'s is in $\amalg f^{n-1} I + \amalg f^{n*} J$. We show now that we can further adjust this decomposition in such a way that each $f_i^{n'}$ is in fact in $\amalg_i f_i^{n-1} I$.

Let $y = f_j^{n'}$ be such that $f_j^{n'}$ is not in $\amalg f^{n-1} I$. We can write $y = a' + b'$ where a' is in $\amalg f^{n-1} I$ and b' is in $\amalg f^{n*} J$. But $\amalg f^{n-1} I$ is contained in $\amalg f^{n*} R = (\amalg f^{n-1} R) \cap (\amalg f^{n-2} I)$, so we can write $a' = ya - q$ and $b' = yb + q$ for some a in R, b in J and some q in $\amalg_{f^{n*} \neq y} f^{n*} R$.

We get $y = y(a + b)$ and, since y is right uniform with terminus w, we have $a + b = w$, so $a = w - b$. Let $z = (w + b)(w + b^2)(w + b^4) \cdots (w + b^{2^n})$. We multiply $a' = ya - q$ in $\amalg f^{n-1} I$ by z on the right and we obtain $y(w - b^{2^{n+1}}) - qz$ in $\amalg f^{n-1} I$ or $y - yb^{2^{n+1}} - qz$ in $\amalg f^{n-1} I$. Since I is admissible, for large enough n, $b^{2^{n+1}}$ is in I, so $yb^{2^{n+1}}$ is in $yI \subset \amalg f^{n-1} I$, so $y - qz$ is in $\amalg f^{n-1} I$.

We claim now that $(y - qz)R \amalg (\amalg_{f^{n*} \neq y} f^{n*} R) = \amalg f^{n*} R$. To show this we first observe that $\amalg f^{n*} R = (y - qz)R + (\amalg_{f^{n*} \neq y} f^{n*} R)$. It is obvious that the sum is direct.

In this way it is clear that we can adjust our decomposition, and that we may assume that each of the $f_i^{n'}$ is in fact in $\amalg f^{n-1} I$. The case where $n = 1$ is similar. □

We now give the analogous result for the graded (not necessarily noetherian) case.

Theorem 2.3. *Assume Λ is length graded and let M be a graded Λ-module. Then the resolution $(\mathcal{P}, \delta) = (\amalg_i f_i^n R / \amalg_i f_i^n I, (\overline{h}^{n-1,n}))$ can be chosen to be a graded resolution in such a way that for each n, the f_i^{n*}'s are homogeneous elements (and hence the f_i^n's), with no proper K-linear combination of a subset of $\{f_j^n\}$ is in $\amalg f^{n-1} I + \amalg f^{n*} J$. Moreover, there is a decomposition*

$$\amalg f^{n*} R = (\amalg f^n R) \amalg (\amalg f^{n'} R)$$

where the elements $f^{n'}$ can be chosen to be homogeneous elements in $\amalg f^n I$.

Proof. By Proposition 1.5 we begin with a graded resolution of M. For each $n \geq 2$, we have a decomposition

$$\amalg f^{n*} R = (\amalg f^{n-1} R) \cap (\amalg f^{n-2} I) \tag{3}$$

with the f^{n*}'s homogeneous.

Step 1: We show first that we may adjust the decomposition (3), to obtain a decomposition of the type

$$\amalg f^{n*} R = (\amalg f^n R) \amalg (\amalg f^{n'} R) \tag{4}$$

where each $f^{n'}$ is a homogeneous element in $\amalg f^{n-1} R + \amalg f^{n*} J$, and, no proper K-linear combination of a subset of $\{f^n\}$ is in $\amalg f^{n-1} I + \amalg f^{n*} J$. For each degree, there are only a finite number of f^{n*}'s in that degree, since each homogeneous component of M and Λ is finite dimensional. Fixing a degree, we obtain

$$\amalg f^{n*} R = (f_1^n R \amalg \cdots \amalg f_t^n R) \amalg (f_{t+1}^{n'} \amalg \cdots \amalg f_k^{n'} R) \tag{3'}$$

and we proceed, degree by degree, as in the proof of step 1 of Theorem 2.2.

Step 2: By the first step, we may assume that we have a collection of decompositions of the type

$$\amalg f^{n*} R = (f_1^n R \amalg \cdots \amalg f_t^n R) \amalg (f_{t+1}^{n'} \amalg \cdots \amalg f_k^{n'} R),$$

where all f^{n*}, f_i^n and $f_j^{n'}$ are homogeneous in the same degree, and, where each of the $f_j^{n'}$'s is in $\amalg f^{n-1} I + \amalg f^{n*} J$. We show now that we can adjust these decompositions in such a way that, each $f_j^{n'}$ is in fact in $\amalg f^{n-1} I$. Let $y = f_j^{n'}$ be such that y is not in $\amalg f^{n-1} I$, and of degree k. We can write $y = a' + b'$ where a' is in $\amalg f^{n-1} I$, b' is in $\amalg f^{n*} J$, and both are homogeneous of degree k. Since $\amalg f^{n-1} I$ is contained in $\amalg f^{n*} R$, we can write $a' = ya - q$ for some q in $\amalg_{f^{n*} \neq y} f^{n*} R$, homogeneous of degree s. Note that a must be a homogeneous element of $R_0 = (KQ)_0$. By right uniformity $ya = y$. Thus $y - q$ is in $\amalg f^{n-1} I$ and we also have

$$(y - q) R \amalg (\amalg_{f^{n*} \neq y} f^{n*} R) = \amalg f^{n*} R.$$

This completes the proof. □

We can now show that the adjusted resolution is minimal.

Theorem 2.4. *Let M be a Λ-module and let $(\mathcal{P}) = (\amalg f^n R / \amalg f^n I, (\overline{h}^{n-1,n}))$ be the projective resolution of M as in Theorem 1.2, where the representatives $\{f^n\}$ are chosen in such a way, that for each n, no proper K-linear combination of a subset of $\{f^n\}$ lies in $\amalg f^{n-1} I + \amalg f^{n*} J$. Then, the resolution $(\mathcal{P})$ is minimal.*

Proof. It is enough to show that for each n, the entries $h^{n-1,n}$ are in J, where $f^n = \sum f^{n-1} h^{n-1,n}$. We prove this for each n, the case $n = 1$ being obvious.

Assume that for some $n > 1$ we have a representative f_j^n written as $f_j^n = f_1^{n-1} h_1 + \cdots + f_t^{n-1} h_t$, where not all h_i are in J. Using the fact that the f^{n-1}'s are right uniform elements of R, we get an expression

$$f_j^n = f_1^{n-1}\alpha_1 + \cdots + f_t^{n-1}\alpha_t + f_1^{n-1} r_1 + \cdots + f_t^{n-1} r_t$$

where α_i is in K and r_i is in J for $i = 1, \ldots, t$ (not all necessarily nonzero). Let $x = \sum_{i=1}^t f_i^{n-1}\alpha_i$. Then we can write x as $x = f_j^n - \sum_{i=1}^t f_i^{n-1} r_i$ in $\amalg f^{n-2} I + \amalg f^{n-1} J$, which is a contradiction to the choice of the elements $\{f^{n-1}\}$. □

A projective resolution of $\Lambda/\underline{\mathrm{r}}$ and the groups $D\operatorname{Ext}^n_\Lambda(\Lambda/\underline{\mathrm{r}}, \Lambda/\underline{\mathrm{r}})$ are given in [Bo], where $D = \operatorname{Hom}_K(\ , K)$ is the usual duality. We now give some information on the elements f^n's, which enables us to give a connection between the Bongartz-Butler-Gruenberg resolution in [Bo] and our resolution.

Proposition 2.5. *Let M be a Λ-module and, for each $n \geq 0$, choose $\{f^n\}$ in such a way that the resolution $(\amalg f^n R/\amalg f^n I, \overline{h})$ is minimal as in Theorem 2.4.*

(a) *Let $n \geq 1$ and write $f^n = f_1^{n-1} r_1 + \cdots + f_t^{n-1} r_t$ with r_i in R. Then, the elements f^n can be chosen (adjusted) in such a way that each of the elements $r_1, \ldots, r_t$ are in $J \setminus I$.*

(b) *The elements f^{n+1} are in $(\amalg f^n J) \cap (\amalg f^{n-1} I)$ and*

$$(\amalg f^n J) \cap (\amalg f^{n-1} I) \subset \begin{cases} (\amalg f^0 J I^m) \cap (\amalg f^0 I^m J) & \text{if } n = 2m, \\ (\amalg f^0 I^{m+1}) \cap (\amalg f^0 J I^m J) & \text{if } n = 2m+1. \end{cases}$$

(c) $\Omega^n_\Lambda(M)/\Omega^n_\Lambda(M)\,\underline{\mathrm{r}} \simeq \amalg f^n R/((\amalg f^n R) \cap (\amalg f^{n-1} I) + \amalg f^n J)$ *and*

$$\amalg f^{n-1} I + \amalg f^n J \subset \begin{cases} \amalg f^0 J I^m + \amalg f^0 I^m J & \text{if } n = 2m, \\ \amalg f^0 I^{m+1} + \amalg f^0 J I^m J & \text{if } n = 2m+1. \end{cases}$$

(d) $D\operatorname{Ext}^n_\Lambda(M, \Lambda/\underline{\mathrm{r}}) \simeq \amalg f^n R/((\amalg f^n R) \cap (\amalg f^{n-1} I) + \amalg f^n J)$.

Proof. (a) It is clear that not all the r_i's can be in I. Write, say,

$$f_1^n = f_1^{n-1} r_1 + \cdots + f_k^{n-1} r_k + f_{k+1}^{n-1} r_{k+1} + \cdots + f_s^{n-1} r_s,$$

where $r_1, \ldots, r_k$ are not in I, but $r_{k+1}, \ldots, r_s$ are all in I.

Let $x_1 = f_1^{n-1} r_1 + \cdots + f_k^{n-1} r_k$. Then it is easy to check that $\amalg f_i^{n*} R = x_1 R \amalg (f_2^{n*} R \amalg \ldots \amalg f^{n*} R)$, since $x_1 = f_1^n - u$, where u is in $\amalg f^{n-1} I \subset (\amalg f^{n-1} R) \cap (\amalg f^{n-2} I) = \amalg f^{n*} R$. Furthermore, x_1 is not in $\amalg f^{n-1} I + \amalg f^{n*} J$; otherwise, f_1^n is. So we can replace f_1^n with x_1. Continue this process. The fact that the coefficients are in J follows from the minimality of the projective resolution.

(b) We have that $f^{n+1} = \sum f^n h^{n,n+1}$, where $h^{n,n+1}$ is in J, since the resolution is minimal. Therefore, f^{n+1} is in $\sum f^n J$. Moreover,

$$f^{n+1} = \sum f^n h^{n,n+1} = \sum f^{n-1} h^{n-1,n} h^{n,n+1},$$

where the last sum is in $\amalg f^{n-1} I$, since we have a resolution over $\Lambda = R/I$. It follows immediately from this that f^{n+1} is in $(\amalg f^n J) \cap (\amalg f^{n-1} I)$.

We saw above that f^i is in $\amalg f^{i-2} I$ for $i \geq 2$. This implies that f^{2m} is in $\amalg f^0 I^m$ and that f^{2m+1} is in $\amalg f^1 I^m \subset \amalg f^0 J I^m$. The last claim in (b) follows immediately from this.

(c) and (d) The second claim of (c) follows in a similar fashion to (b). Since the projective resolution of M is minimal and $\Lambda/\underline{\mathrm{r}}$ is semisimple, it follows immediately that

$$D\,\mathrm{Ext}^n_\Lambda(M,\Lambda/\underline{\mathrm{r}}) \simeq \mathrm{Tor}^\Lambda_n(M,\Lambda/\underline{\mathrm{r}}) \simeq \amalg f^n R/\amalg f^n J.$$

Furthermore, the minimality of the resolution, and the above imply that $\amalg f^n J = \amalg f^n J + (\amalg f^n R)\cap(\amalg f^{n-1} I)$. The result follows. □

It is shown in [Bo] that

$$D\,\mathrm{Ext}^n_\Lambda(\Lambda/\underline{\mathrm{r}},\Lambda/\underline{\mathrm{r}}) = \begin{cases} (I^m \cap JI^{m-1}J)/(JI^m + I^m J) & \text{for } n = 2m \geq 2, \\ (JI^m \cap I^m J)/(I^{m+1} + JI^m J) & \text{for } n = 2m+1 \geq 1. \end{cases}$$

We see that the formulas for $D\,\mathrm{Ext}^n_\Lambda(\Lambda/\underline{\mathrm{r}},\Lambda/\underline{\mathrm{r}})$ and for $D\,\mathrm{Ext}^n_\Lambda(M,\Lambda/\underline{\mathrm{r}})$ in Proposition 2.5 are very similar, when M is any finitely generated Λ-module.

3. Ext-algebras

Let $\Lambda = R/I$ where $R = KQ$ and $I \subseteq J^2$ is an admissible ideal of R, or a length homogeneous ideal. Let $\Lambda_0 = R/J$. Note that Λ_0 is a semisimple Λ-module. We recall that the Ext-algebra of Λ is the graded K-algebra $E(\Lambda) = \amalg_{n\geq 0}\mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)$ with the obvious addition and with multiplication given by the Yoneda product. Given a right Λ-module M, we have a graded left $E(\Lambda)$-module $E(M) = \amalg_{n\geq 0}\mathrm{Ext}^n_\Lambda(M,\Lambda_0)$. In this section we show how to use the minimal projective resolutions introduced in section 2, to effectively describe the $E(\Lambda)$ action on $E(M)$ (and thus, if $M = \Lambda_0$, the multiplicative structure of $E(\Lambda)$ itself), in an algorithmic way, by working at the level of the path algebra KQ.

We start by recalling the following interpretation of the Yoneda product, which will be used in this section. First, we observe that if M is a finitely generated Λ-module, in the admissible case, or M is graded in the length homogeneous case, we may consider a minimal Λ-resolution of M:

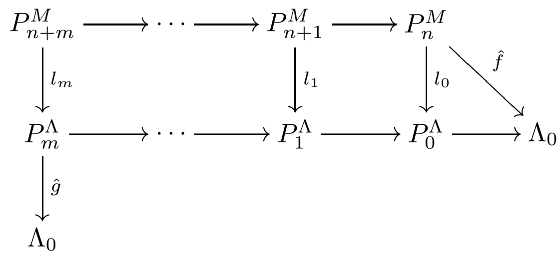
$$\cdots \to P_n \xrightarrow{\delta^n} P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

This resolution is minimal in the sense that, for each $n > 0$, we have $\mathrm{Im}\,\delta^n \subseteq P_{n-1}\,\underline{\mathrm{r}}$ where $\underline{\mathrm{r}}$ is the Jacobson radical of Λ in the case where I is admissible, and $\underline{\mathrm{r}}$ is the graded radical of Λ in the graded case which need not be finite dimensional. Since $\mathrm{Ext}^n_\Lambda(M,\Lambda_0)$ is the cohomology of the complex

$$0 \to \mathrm{Hom}_\Lambda(P_0,\Lambda_0) \xrightarrow{\delta_*} \mathrm{Hom}_\Lambda(P_1,\Lambda_0) \xrightarrow{\delta_*} \cdots,$$

and, since Λ_0 is semisimple, the boundary maps of this complex are all zero. It follows that, for each $n \geq 0$, we have $\mathrm{Ext}^n_\Lambda(M,\Lambda_0) = \mathrm{Hom}_\Lambda(P_n,\Lambda_0)$. Now let $\hat{f}$ be in $\mathrm{Ext}^n_\Lambda(M,\Lambda_0)$ and let $\hat{g}$ be in $\mathrm{Ext}^m_\Lambda(\Lambda_0,\Lambda_0)$. We have the following commutative diagram with exact rows:

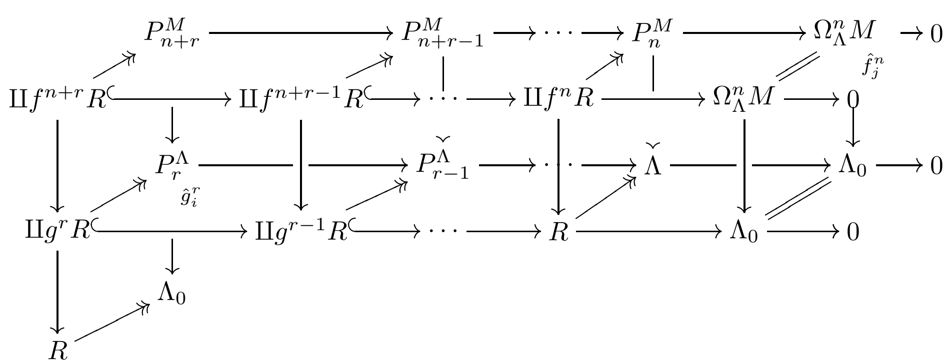
(5)



where the top row is part of a minimal projective resolution of M, the bottom sequence is part of a minimal projective resolution of Λ_0, and the vertical maps $l_0, l_1, \ldots, l_m$ are successive liftings (not necessarily unique) of $\hat{f}$. Then we have the following description of the $E(\Lambda)$ action on $E(M)$:

$$\hat{g} * \hat{f} = \text{the composition } \hat{g} \circ l_m.$$

It is well known that this action is well defined. We also remark that, since $P_n^M = \amalg_i (f_i^n R / f_i^n I)$, we have a *dual* basis of $\mathrm{Ext}^n_\Lambda(M, \Lambda_0) = \mathrm{Hom}_\Lambda(P_n^M, \Lambda_0)$ consisting of the maps $\{\hat{f}_i^n\}$ defined in the obvious way. In order to describe this action at the level of the path algebra KQ, we construct a commutative diagram of R-modules where, the first and the third rows are minimal projective Λ-resolutions of $\Omega^n_\Lambda M$ and, Λ_0 respectively, and the back face of the parallelepiped is the front face modulo the ideal I:



(Note that in order not to complicate our already rather complicated notation, we denote by $\hat{g}_i^r$ the map $\amalg g_i^r R \to R$ as well as its image modulo I, $\amalg \overline{g}_i^r \Lambda \to \Lambda_0$.)

Notation. Keeping the notation of section 2, let $(\amalg f_i^n R / \amalg\, f_i^n I, (\overline{h}^{n-1,n}))$ be a minimal projective resolution of M over Λ, and let $(\amalg g_i^n R / \amalg\, g_i^n I, (\overline{k}^{n-1,n}))$ be a minimal projective resolution of Λ_0 over Λ. Recall that the entries of the matrices $(\overline{h}^{n-1,n})$ and $(\overline{k}^{n-1,n})$ are given by the expressions

$$f_i^n = \sum_j f_j^{n-1} h_{j,i}^{n-1,n} \text{ and } g_i^n = \sum_j g_j^{n-1} k_{j,i}^{n-1,n}.$$

Recall also that for each $n \geq 2$ we have

$$(\amalg f^{n-1} R) \cap (\amalg f^{n-2} I) = (\amalg f^n R) \amalg (\amalg f^{n'} R)$$

where each $f^{n'}$ is in $\amalg f^{n-1}I$. We have similar statements and notation involving the g's, and, note that there are no $g^{0'}$'s and $g^{1'}$'s appearing.

We have the following lemma which is valid over an arbitrary ring S.

Lemma 3.1. *Let A, B, C and D be S-modules satisfying $B \subseteq C$ and $(A+C) \cap D = (0)$. Then we have*

$$(A \amalg B \amalg D) \cap (C \amalg D) \subseteq (A \cap C) \amalg B \amalg D.$$

Proof. We first observe that we have the inclusion $(A \amalg B \amalg C) \cap (C \amalg D) \subseteq [(A \amalg B) \cap C] \amalg D$ since, if $x = a+b+d = c+d'$, then $a+b-c = d'-d$ is in D, since b is also in C, we get that $d = d'$ and $a+b = c$. Therefore, x is in $[(A \amalg B) \cap C] + D$. But $D \cap (A+C) = (0)$ and $B \subseteq C$ so this sum is direct. The lemma follows now immediately since $B \subseteq C$ implies the well-known equality $(A \amalg B) \cap C = (A \cap C) \amalg B$. □

The following rather technical proposition is crucial to our description.

Proposition 3.2. (a) *For each $n \geq 0$, the matrix $(h^{n,n+1})$ has entries in $\amalg g^1 R$.*

(b) *For each $r \geq 2$ and $n \geq 0$, the product of matrices $(h^{n,n+1}) \cdots (h^{n+r-1,n+r})$ has entries in $(\amalg g^r R) \amalg (\amalg_{2 \leq i \leq r} g^{i'} R)$.*

Note that for simplicity we are using the following notation: the matrix $(h^{n,n+1})$ actually denotes the matrix $(h^{n,n+1}_{j,i})$ where the entries $h^{n,n+1}_{j,i}$ are obtained by expanding the elements of the form f^{n+1} in terms of the elements of the form f^n.

Proof. (a) From the minimality of the projective resolution of M it follows that each entry of the form $h^{n,n+1}_{j,i}$ is in J, so we can factor out the initial arrows, (note that the family $\{g^1\}$ is the set of arrows of $Q^!$), and we are done.

(b) We proceed in 2 steps.

Step 1: We show first that the sum is direct. Recall that, for each $p > 0$ we have $(\amalg g^p R) \amalg (\amalg g^{p'} R) \subseteq \amalg g^{p-1} R$. Assume that the sum $(\amalg g^r R) + (\amalg_{k \leq i \leq r} g^{i'} R)$ is direct. But this sum is a submodule of $\amalg g^{k-1} R$. Therefore its intersection with $\amalg g^{k-1'} R$ is zero. This proves that the sum $(\amalg g^r R) + (\amalg_{k-1 \leq i \leq r} g^{i'} R)$ is also direct, so it follows by induction that our sum is direct.

Step 2: We have the following:

$$\begin{aligned}&(h^{n,n+1}) \cdots (h^{n+r-1,n+r}) \\ &\quad = [(h^{n,n+1}) \cdots (h^{n+r-3,n+r-2})] \cdot [(h^{n+r-2,n+r-1})(h^{n+r-1,n+r})].\end{aligned}$$

By induction, the product of the first $r-2$ matrices has entries in $(\amalg g^{r-2} R) \amalg (\amalg_{2 \leq i \leq r-2} g^{i'} R)$, and it is clear that the product of the last two matrices has entries in I, hence the entire product has entries in $(\amalg g^{r-2} I) \amalg (\amalg_{2 \leq i \leq r-2} g^{i'} R)$. But, by looking at the product of the first $r-1$ matrices, we see that the entire product has entries also in $(\amalg g^{r-1} R) \amalg (\amalg_{2 \leq i \leq r-1} g^{i'} R)$. Hence, the entries of the product lie in the intersection

$$[(\amalg g^{r-1} R) \amalg (\amalg_{2 \leq i \leq r-1} g^{i'} R)] \cap [(\amalg g^{r-2} I) \amalg (\amalg_{2 \leq i \leq r-2} g^{i'} R)].$$

We now claim that we have the following inclusion:

$$\begin{aligned}&[(\amalg g^{r-1} R) \amalg (\amalg_{2 \leq i \leq r-1} g^{i'} R)] \cap [(\amalg g^{r-2} I) \amalg (\amalg_{2 \leq i \leq r-2} g^{i'} R)] \\ &\quad \subseteq [(\amalg g^{r-1} R) \cap (\amalg g^{r-2} I)] \amalg (\amalg_{2 \leq i \leq r-1} g^{i'} R)].\end{aligned}$$

Note that, since $(\amalg g^{r-1}R)\cap(\amalg g^{r-2}I)] = (\amalg g^r R)\amalg(\amalg g^{r'}R)$, the claim will indeed imply that the entries of our product of matrices are in the required sum. To prove this claim, let $A=\amalg g^{r-1}R$, $B=\amalg g^{r-1'}R$, $C=\amalg g^{r-2}I$, and $D=\amalg_{2\le i\le r-2}g^{i'}R$. With this notation we must prove that we have an inclusion:

$$(A\amalg B\amalg D)\cap(C\amalg D)\subseteq(A\cap C)\amalg B\amalg D.$$

This is precisely the statement of Lemma 3.1 once one shows that $B\subseteq C$ and that $(A+C)\cap D=(0)$. It is clear that $B\subseteq C$. We also have that A and C are included in $\amalg g^{r-2}R$ and $D\cap(\amalg g^{r-2}R)=(0)$; hence $D\cap(A+C)=(0)$. The proof is now complete. ☐

Definition 3.3. It follows from the previous proposition that each entry of the matrix product $(h^{n,n+1})\cdots(h^{n+r-1,n+r})$ is an element of the form $\sum g^r l^{r,n,n+r}+\sum_{2\le i\le r} g^{i'}s^{i,n,n+r}$. Let $(l^{r,n,n+r})$ be the corresponding matrix. (Note that the number of rows of this matrix is the number of elements in the family $\{g^r\}$, and that the number of columns is the number of elements in the family $\{f^{n+r}\}$.)

Proposition 3.4. *For each $n\ge 0$ and $r\ge 0$, the following diagram of R-module commutes modulo I:*

$$\begin{array}{ccc}\amalg f^{n+r}R & \xrightarrow{(h^{n+r-1,n+r})} & \amalg f^{n+r-1}R\\ \big\downarrow{\scriptstyle (l^{r,n,n+r})} & & \big\downarrow{\scriptstyle (l^{r-1,n,n+r-1})}\\ \amalg g^r R & \xrightarrow{(k^{r-1,r})} & \amalg g^{r-1}R\end{array}$$

Proof. We shall use a Sweedler-type notation in order to avoid multiple indices. Note that by Proposition 3.2 we have that the entries of the product of r consecutive $(h^{n+i,n+i+1})$-matrices have the form $X=\sum g^r l^{r,n,n+r}+\sum_{2\le i\le r}g^{i'}s^{i,n,n+r}$. We have the following:

$$\begin{aligned}X&=\sum g^r l^{r,n,n+r}+\sum g^{r'}s^{r,n,n+r}+\sum_{2\le i\le r-1}g^{i'}s^{i,n,n+r}\\&=\sum(\sum g^{r-1}k^{r-1,r})l^{r,n,n+r}\\&\quad+\sum(\sum g^{r-1}a^{r-1,r})s^{r,n,n+r}+\sum_{2\le i\le r-1}g^{i'}s^{i,n,n+r}\\&=\sum g^{r-1}(\sum k^{r-1,r}l^{r,n,n+r}+a^{r-1,r}s^{r,n,n+r})+\sum_{2\le i\le r-1}g^{i'}s^{i,n,n+r},\end{aligned}$$

where each $a^{r-1,r}$ is in the ideal I.

On the other hand, by expressing the matrix product as

$$[(h^{n,n+1})\cdots(h^{n+r-2,n+r-1})]\cdot(h^{n+r-1,n+r}),$$

we see that each entry of this product is an entry of the product

$$(\sum g^{r-1}l^{r-1,n,n+r-1}+\sum_{2\le i\le r-1}g^{i'}s^{i,n,n+r-1})\cdot(h^{n+r-1,n+r}).$$

Therefore, it has the form

$$\sum g^{r-1}(\sum l^{r-1,n,n+r-1}h^{n+r-1,n+r})+\sum_{2\le i\le r-1}g^{i'}(\sum s^{i,n,n+r-1}h^{n+r-1,n+r}).$$

From the uniqueness of the direct sum decomposition we obtain

$$\begin{aligned}&\sum g^{r-1}\Big(\sum l^{r-1,n,n+r-l}h^{n+r-1,n+r}\Big)\\ &\qquad=\sum g^{r-1}\Big(\sum k^{r-1,r}l^{r,n,n+r}+a^{r-1,r}s^{r,n,n+r}\Big)\end{aligned}$$

where we recall that each $a^{r-1,r}$ is in I. It is immediate that the above equality implies the proposition. □

We are ready to prove the main result of this section.

Theorem 3.5. *Let $\hat{f}_j^n$ be in $\operatorname{Ext}^n_\Lambda(M,\Lambda_0)$ and let $\hat{g}_i^r$ be in $\operatorname{Ext}^r_\Lambda(\Lambda_0,\Lambda_0)$ be two basis elements. Then, the element $\hat{g}_i^r * \hat{f}_j^n$ of $\operatorname{Ext}^{n+r}_\Lambda(M,\Lambda_0)$ is given by*

$$\hat{g}_i^r * \hat{f}_j^n = \sum_t \tilde{l}_{i,j,t}^{r,n,n+r}\hat{f}_t^{n+r}$$

where $\tilde{l}$ is the image in $\Lambda/\underline{\mathrm{r}}$ of the element l of R.

Proof. We infer from the previous proposition that for each $n \geq 0$, and $r > 1$, we have a commutative diagram of Λ-modules:

$$\begin{array}{ccc}\amalg\overline{f}^{n+r}\Lambda & \xrightarrow{(\overline{h}^{n+r-1,n+r})} & \amalg\overline{f}^{n+r-1}\Lambda\\ \Big\downarrow{\scriptstyle(\overline{l}^{r,n,n+r})} & & \Big\downarrow{\scriptstyle(\overline{l}^{r-1,n,n+r-1})}\\ \amalg\overline{g}^{r}\Lambda & \xrightarrow{(\overline{k}^{r-1,r})} & \amalg\overline{g}^{r-1}\Lambda\end{array}$$

This shows, as promised, that we have constructed a diagram (5) as in the beginning of this section, and, that the product $\hat{g}_i^r * \hat{f}_j^n$ is in fact the composition $\hat{g}_i^r \circ (\overline{l}^{r,n,n+r})$. The theorem now follows. □

4. The no loop conjecture

This section is devoted to applying the minimal projective resolution found in section 2 to the no loop conjecture. In particular, we show a special case of this conjecture.

Let K, Q, and R be as above, and let I be an admissible ideal in R. As before, denote R/I by Λ. The no loop conjecture says the following. If S is a simple Λ-module such that $\operatorname{Ext}^1_\Lambda(S,S) \neq (0)$, then $\operatorname{pd}_\Lambda S = \infty$. In [I], Igusa proved that the global dimension of Λ must be infinite, whenever there exists a simple Λ-module S with $\operatorname{Ext}^1_\Lambda(S,S) \neq (0)$. (See also [L].)

First we prove a small technical lemma.

Lemma 4.1. *Suppose a is an arrow in Q, where $a\colon v \to v$ for some vertex v in Q. Let S be the simple Λ-module corresponding to this vertex, and assume that the minimal resolution of S has been constructed finding the elements f^n, $f^{n'}$ and $h^{n-1,n}$. The residue class of a^n in $vI/v(JI+IJ)$ is nonzero if and only if*

$$a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i,$$

where the coefficients r_i and s_i are R, and where some r_i is not in J.

Proof. Assume that $a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i$ for r_i and s_i in R, where one of the r_i's is not in J. Assume that the residue class of a^n in $vI/v(JI+IJ)$ is zero. Since each $f_i^{2'}$ is in $\amalg f^1 I$, it is in vJI. Hence the set $\{f_i^2\}$ is linearly dependent in $vI/v(JI+IJ)$. This is a contradiction to the choice of f_i^2 (Theorem 2.4), so that the residue class of a^n in $vI/v(JI+IJ)$ is nonzero.

Conversely, assume that $a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i$ for some r_i and s_i in R, where all r_i are in J. Then clearly the residue class of a^n in $vI/v(JI+IJ)$ is zero. □

Now we show that if S is a simple Λ-module corresponding to a vertex with a loop a where a^n is in I and a^n has a nonzero residue class in $vI/v(JI+IJ)$, then $\operatorname{pd}_\Lambda S = \infty$.

Proposition 4.2. *Suppose a is an arrow in Q, where $a\colon v \to v$ for some vertex v in Q. Let S be the simple Λ-module corresponding to this vertex, and assume that a^n is in I for some $n \geq 2$ and that the residue class of a^n in $vI/v(JI+IJ)$ is nonzero. Then $\operatorname{Ext}^i_\Lambda(S,S) \neq (0)$ for all $i \geq 1$. In particular, $\operatorname{pd}_\Lambda S = \infty$.*

Proof. Note that the hypothesis implies that a^{n-1} is not in I. It is well-known that $\operatorname{Ext}^i_\Lambda(S,S) \neq (0)$ for $i = 1, 2$.

By the above lemma, we have that $a^n = \sum_i f_i^2 r_i^2 + \sum_i f_i^{2'} s_i^2$ for some r_i^2 and s_i^2 in R, where some r_i^2 is not in J. Let $z_2 = \sum_i f_i^{2'} s_i^2$. Then $b_2 = a^n - z_2$ is in $\amalg f^2 R$. We want to show that $b_2 a$ is in $\amalg f^2 R \cap \amalg f^1 I$. Since a^n is in I and z_2 is in $\amalg f^{2'} R$, the element z_2 is in $\amalg f^1 I$ and, therefore $b_2 a$ is in $\amalg f^2 R \cap \amalg f^1 I$.

Assume that $b_2 a$ is in $\amalg f^2 I$. Then there exist u_i in I such that $\sum_i f_i^2 r_i^2 a = \sum_i f_i^2 u_i$. Hence, $r_i^2 a$ is in I for all i. There exists an i_0 such that $r_{i_0}^2 = cv + h$ for c in K^* and h in J. Then

$$c^{-1}(v + (c^{-1}h)^{2^m}) \cdots (v + (c^{-1}h)^2)(v - (c^{-1}h))(cv+h)a = a - (c^{-1}h)^{2^{m+1}}$$

is in I. Since I is admissible and h is in J, for some large m, we get that a is in I. This is a contradiction, which shows that $b_2 a$ is not in $\amalg f^2 I$ and, $\operatorname{Ext}^3_\Lambda(S, \Lambda/\underline{\mathrm{r}}) \neq (0)$.

By the above, we can write $b_2 a$ as

$$b_2 a = \sum_i f_i^3 r_i^3 + \sum_i f_i^{3'} s_i^3$$

for some elements r_i^3 and s_i^3 in R. Expanding the two sides in the elements f^2, we get

$$\sum_i f_i^2 r_i^2 a = \sum_{t,i} f_t^2 h_{t,i}^{2,3} r_i^3 + \sum_{t,i} f_t^2 \alpha_{t,i}^3 s_i^3,$$

where each α_i^3 is in I. This implies that

$$r_t^2 a = \sum_i h_{t,i}^{2,3} r_i^3 + \sum_i \alpha_{t,i}^3 s_i^3.$$

There exists some t such that $r_t^2 a$ is not in J^2. If r_i^3 is in J for all i, then the right hand side is in J^2. Therefore, we infer that not all r_i^3 are in J. Since $b_2 a$ ends in v, there are also some f_i^3 such that $f_i^3 v \neq 0$. Hence, $\operatorname{Ext}^3_\Lambda(S,S) \neq (0)$. Let $z_3 = \sum_i f^{3'} s_i^3$ and $b_3 = b_2 a - z_3$.

Assume now that we have shown that

(a) $b_{2m+1} = b_{2m} a - z_{2m+1} = \sum_i f_i^{2m+1} r_i^{2m+1}$, where not all r_i^{2m+1} are in J,
(b) $b_{2m} = \sum_i f_i^{2m} r_i^{2m}$, where not all r_i^{2m} are in J,

(c) z_{2m+1} is in $\amalg f^{2m+1'}R$.

We want to show that $b_{2m+1}a^{n-1}$ represents some nonzero element in $\operatorname{Ext}_\Lambda^{2m+2}(S,S)$.

The element $b_{2m+1}a^{n-1}$ is clearly in $\amalg f^{2m+1}R$. Since b_{2m} is in $\amalg f^{2m}R$ and z_{2m+1} is in $\amalg f^{2m}I$, we have that $b_{2m+1}a^{n-1} = b_{2m}a^n + z_{2m+1}a^{n-1}$ is in $\amalg f^{2m}I$. Therefore, $b_{2m+1}a^{n-1}$ is in $\amalg f^{2m+1}R \cap \amalg f^{2m}I$.

Assume that $b_{2m+1}a^{n-1}$ is in $\amalg f^{2m+1}I$, that is, $b_{2m+1}a^{n-1} = \sum f_i^{2m+1}u_i^{2m+1}$, where u_i^{2m+1} is in I for all i. Expanding the elements in terms of the elements f^{2m} we get

$$\begin{aligned}\sum_i f_i^{2m} r_i^{2m} a^n &= \sum_i f_i^{2m+1}u_i^{2m+1} + z_{2m+1}a^{n-1} \\ &= \sum_t f_t^{2m} h_{t,i}^{2m,2m+1} u_i^{2m+1} \sum_t f_t^{2m}\beta_t^{2m+1}a^{n-1}\end{aligned}$$

where β_t^{2m+1} is in I for all t. For each t, we have that

$$r_t^{2m}a^n = \sum_i h_{t,i}^{2m,2m+1}u_i^{2m+1} + \beta_t^{2m+1}a^{n-1}.$$

The right hand side is in $JI + IJ$ for all t. For some t, the element r_t^{2m} is not in J. Using that I is an admissible ideal and similar arguments as before, we conclude that a^n is in $JI + IJ$. This is a contradiction, and therefore $b_{2m+1}a^{n-1}$ is not in $\amalg f^{2m+1}I$ and $\operatorname{Ext}_\Lambda^{2m+2}(S, \Lambda/\underline{\mathrm{r}}) \neq (0)$.

By the above, $b_{2m+1}a^{n-1}$ has a representation of the form

$$b_{2m+1}a^{n-1} = \sum_i f_i^{2m+2}r_i^{2m+2} + \sum_i f_i^{2m+2'}s_i^{2m+2}.$$

Expanding in terms of the elements f^{2m}, we get

$$\begin{aligned}\sum_i f_i^{2m}r_i^{2m}a^n &= \sum_i f_i^{2m+2}r_i^{2m+2} + \sum_i f_i^{2m+2'}s_i^{2m+2} + z_{2m+1}a^{n-1} \\ &= \sum f_t^{2m}h^{2m,2m+1}h^{2m+1,2m+2}r_i^{2m+2} \\ &\quad + \sum f_l^{2m+1}\alpha_{l,i}^{2m+2}s_i^{2m+2} + z_{2m+1}a^{n-1} \\ &= \sum f_t^{2m}h^{2m,2m+1}h^{2m+1,2m+2}r_i^{2m+2} \\ &\quad + \sum f_t^{2m}h^{2m,2m+1}\alpha_{l,i}^{2m+2}s_i^{2m+2} + z_{2m+1}a^{n-1}\end{aligned}$$

for some $\alpha_{l,i}^{2m+2}$ in I. We have that z_{2m+1} is in $\amalg f^{2m}I$, so that, if r_i^{2m+2} is in J for all i, then $r_i^{2m}a^n$ is in $JI + IJ$ for all i. Since there exists $r_{i_0}^{2m}$ which is not in J, we show as before that a^n is in $JI + IJ$. This is a contradiction, and consequently, not all r_i^{2m+2} are in J, and $\operatorname{Ext}_\Lambda^{2m+2}(S,S) \neq (0)$. Let $z_{2m+2} = \sum_i f^{2m+2'}s_i^{2m+2}$ and $b_{2m+2} = b_{2m+1}a^{n-1} - z_{2m+2}$. We can now move on to the following step.

Assume now that we have shown that

(a) $b_{2m} = b_{2m-1}a^{n-1} - z_{2m} = \sum_i f_i^{2m}r_i^{2m}$, where not all r_i^{2m} are in J,
(b) $b_{2m-1} = \sum_i f_i^{2m-1}r_i^{2m-1}$, where not all r_i^{2m-1} are in J,
(c) z_{2m} is in $\amalg f^{2m'}R$.

We want to show that $b_{2m}a$ represents some nonzero element in $\operatorname{Ext}_\Lambda^{2m+1}(S,S)$.

The element $b_{2m}a$ is clearly in $\amalg f^{2m}R$. Since b_{2m-1} is in $\amalg f^{2m-1}R$, z_{2m} is in $\amalg f^{2m-1}I$ and $b_{2m}a = b_{2m-1}a^n - z_{2m}a$, we have that $b_{2m}a$ is in $\amalg f^{2m}R \cap \amalg f^{2m-1}I$.

Assume that $b_{2m}a$ is in $\amalg f^{2m}I$. Then $b_{2m}a = \sum_i f_i^{2m} r_i^{2m} a = \sum_i f_i^{2m} u_i^{2m}$ with u_i^{2m} in I. Therefore, $r_i^{2m}a$ is in I for all i, and, as before, this would give the contradiction that a is in I. Thus $b_{2m}a$ is not in $\amalg f^{2m}I$ and $\operatorname{Ext}_\Lambda^{2m+1}(S, \Lambda/\underline{\mathrm{r}}) \neq (0)$.

By the above, the element $b_{2m}a$ has a representation of the form

$$b_{2m}a = \sum_i f_i^{2m+1} r_i^{2m+1} + \sum_i f^{2m+1'} s_i^{2m+1}.$$

Expanding in terms of the elements f^{2m}, we get

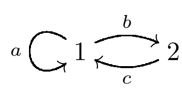
$$\begin{aligned}\sum_i f_i^{2m} r_i^{2m} a &= \sum f_i^{2m+1} r_i^{2m+1} + \sum_i f_i^{2m+1'} s_i^{2m+1} \\ &= \sum_t f_t^{2m} h_{t,i}^{2m,2m+1} r_i^{2m+1} + \sum_t f_t^{2m} \alpha_{t,i}^{2m+1} s_i^{2m+1}\end{aligned}$$

where $\alpha_{t,i}^{2m+1}$ is in I. If all r_i^{2m+1} are in J, then $r_i^{2m}a$ is in J^2 for all i. As before, this is a contradiction, and therefore, there exists some r_i^{2m+1} not in J. Using the same arguments as before, it follows that $\operatorname{Ext}_\Lambda^{2m+1}(S,S) \neq (0)$. Combining what we have shown so far, the proof of the proposition is complete. □

We point out that it follows from our proof of Proposition 4.2 that the residue class of a^{rn} is nonzero in $JI^{r-1}J \cap I^r/(JI^r + I^rJ)$ (which is isomorphic to $\operatorname{Tor}_{2r}^\Lambda(\Lambda/\underline{\mathrm{r}}, \Lambda/\underline{\mathrm{r}})$ by [Bo]), and that the residue class of a^{rn+1} is nonzero in $JI^r \cap I^rJ/(JI^rJ + I^{r+1})$ (which in turn is isomorphic to $\operatorname{Tor}_{2r+1}^\Lambda(\Lambda/\underline{\mathrm{r}}, \Lambda/\underline{\mathrm{r}})$).

However, when $\operatorname{Ext}_\Lambda^1(S,S) \neq (0)$, then $\operatorname{Ext}_\Lambda^i(S,S)$ is not in general nonzero for all $i \geq 1$. The following example was pointed out to us by Dieter Happel [H2].

Let Q be the quiver given by



Let I be the ideal in KQ generated by $(a^2 - bc, cab, cb)$. Denote by S the simple Λ-module corresponding to the vertex 1. As with all simple modules corresponding to a vertex with a loop, $\operatorname{Ext}_\Lambda^i(S,S) \neq (0)$ for $i = 1, 2$. However, $\operatorname{Ext}_\Lambda^3(S,S) = (0)$, and, since $\Omega_\Lambda^4(S) = S$, then $\operatorname{Ext}_\Lambda^i(S,S) = (0)$ if $i \equiv 3 \mod 4$, and nonzero otherwise for all $i \geq 1$.

5. Examples

Let Q be a finite quiver and let $I = \langle \rho_1, \ldots, \rho_n \rangle$ be an admissible ideal of KQ, where $\{\rho_1, \ldots, \rho_n\}$ is a minimal set of generators of the two-sided ideal I. Assume that for each $i = 1, \ldots, n$, the element ρ_i is a linear combination of paths in Q with coefficients $+1$ or -1. Let $\Lambda = KQ/I$ be the corresponding finite dimensional algebra. We know that if Λ is a monomial algebra, then the global dimension of Λ (more generally, the projective dimension of each simple Λ-module) is independent of the characteristic of K (see [GHZ]). If Λ is the incidence algebra of a partially ordered set, the global dimension of Λ, although always finite, can vary with the characteristic of K, unless $\operatorname{gldim} \Lambda \leq 2$ in which case it is again characteristic independent (see [C, IZ]). In this section, we give examples which show that the global dimension can fluctuate rather wildly according to the characteristic of the field. The proofs of these examples can be easily done using the minimal projective resolution constructed in the second section. We start, however, by showing that, if the global dimension is two in one characteristic, then it cannot be infinite in another. More specifically, we prove the following.

Theorem 5.1. *Let Q be a finite quiver and let E and F be two fields. Let $\rho_1, \dots, \rho_n$ be linear combinations of paths in Q, where the coefficients are ± 1, and let I_E (I_F respectively) denote the two-sided ideal of EQ (FQ respectively) generated by the elements $\rho_1, \dots, \rho_n$. Assume that I_E and I_F are both admissible ideals, and, that* $\operatorname{gldim} EQ/I_E \le 2$. *Then* $\operatorname{gldim} FQ/I_F < \infty$.

We will prove the theorem by induction on the number of vertices of Q, the case where we have only one vertex being obvious. We need a few preliminary results.

Lemma 5.2. *Let Λ be an artin algebra and let S_Λ be a simple Λ-module such that* $\operatorname{id}_\Lambda S \le 1$. *Let e be the primitive idempotent corresponding to S. Then,*

(i) $\operatorname{gldim}(1-e)\Lambda(1-e) \le \operatorname{gldim}\Lambda$,

(ii) *if* $\operatorname{gldim}\Lambda = \infty$, *then* $\operatorname{gldim}(1-e)\Lambda(1-e) = \infty$. *(Compare with* [Fu, *Proposition 2.5*].*)*

Proof. Since $\operatorname{id}_\Lambda S_\Lambda \le 1$, we have $\operatorname{pd}_{\Lambda^{\mathrm{op}}} DS \le 1$, where D is the usual duality, and we can use [Z] to infer (i). Assume now that $\operatorname{gldim}\Lambda = \infty$. Then there is a Λ-module M_Λ such that $\operatorname{pd} M_\Lambda = \infty$. Let $N = \Omega^2_\Lambda M$—the second syzygy of M. By [J], since $\operatorname{id} S \le 1$, we have that, if

$$\mathcal{P}\colon \cdots P_n \to \cdots \to P_0 \to N_\Lambda \to 0$$

is a minimal projective resolution of N, then no P_i has a summand isomorphic to $e\Lambda$. Thus, $\operatorname{Hom}_\Lambda((1-e)\Lambda, \mathcal{P})$ is a minimal projective resolution of the $(1-e)\Lambda(1-e)$-module $\operatorname{Hom}_\Lambda((1-e)\Lambda, N)$. Hence, $\operatorname{gldim}(1-e)\Lambda(1-e) = \infty$. □

The next result is well-known.

Lemma 5.3. *Let $\Lambda = KQ/I$ be a finite dimensional quotient of the path algebra KQ by an admissible ideal I. Let $\rho_1, \dots, \rho_n$ be a minimal set of generators of I in the sense that I cannot be generated by a proper subset of $\{\rho_1, \dots, \rho_n\}$. For a vertex v of Q the following are equivalent:*

(i) $\operatorname{id} S_v \le 1$.

(ii) $\rho_i v = 0$ *for each* $i = 1, \dots, n$.

Definition 5.4. Let Q be a finite quiver and let I be an admissible ideal of KQ. Let $\rho_1, \dots, \rho_n$ be a minimal set of generators of I and assume that $\rho_i v = 0$ for each i. It is clear that there are no loops at the vertex v since, otherwise, we would have that $\operatorname{Ext}^2_{KQ/I}(S_v, S_v) \ne (0)$, contradicting the fact that the injective dimension of S_v is less or equal to 1. We construct a new quiver Q^* as follows: the vertices of Q^* are all the vertices of Q different from v. The arrows of Q^* are obtained from the arrows of Q in the following way: each arrow of Q whose origin and terminus are both different from v, is also an arrow of Q^*, and, for each path ab in Q:

$$\underset{u}{\bullet} \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \underset{w}{\bullet}$$

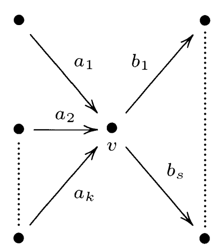
such that ab is not in I, we form a "new" arrow $C(a,b)$ in Q^*

$$\underset{u}{\bullet} \xrightarrow{C(a,b)} \underset{w}{\bullet}$$

The corresponding two-sided ideal I^* of KQ^* is defined in the following manner: if ρ_i is such that $v\rho_i = 0$, and if ρ_i is not a path of the form ab:

$$\bullet \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \bullet,$$

then we let ρ_i^* be the relation obtained from ρ_i, by replacing each occurrence of a path of the form $\bullet \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \bullet$ by the arrow $C(a,b)$. Assume now that ρ_i is a relation satisfying $v\rho_i \neq 0$. Let $a_1, \dots, a_k$ denote all the arrows into the vertex v, and let



$b_1, \dots, b_s$ denote all the arrows starting at v. We let ρ_{ij}^* be the relation obtained from $a_j\rho_i$ by replacing each path $a_j b_l$ not belonging to I, that appears in $a_j\rho_i$, by the arrow $C(a_j, b_l)$. We now define I^* to be the two-sided ideal of KQ^* generated by all the elements of type ρ^*.

Proposition 5.5. *There is a K-algebra isomorphism*

$$KQ^*/I^* \simeq (1-e)KQ/I(1-e),$$

where e is the primitive idempotent corresponding to the vertex v.

Proof. Let $\phi\colon KQ^*/I^* \to (1-e)KQ/I(1-e)$ be $\phi(w) = \overline{w}$ for each vertex w of Q^*, $\phi(a) = \overline{a}$ for each arrow a of Q^* inherited from Q, and also $\phi(C(a,b)) = \overline{ab}$ for each arrow of Q^* of the form $C(a,b)$ (here $\overline{x}$ means the image in KQ/I of the element x of KQ). It is clear that ϕ induces a K-algebra homomorphism which is onto, and, it is not hard to show that $\operatorname{Ker}\phi = I^*$. □

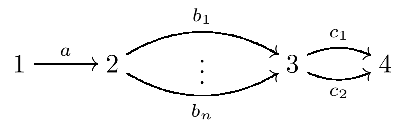
Remark 5.6. The generating set $\{\rho_i^*, \rho_{ik}^*\}$ of I^* need not be minimal. Indeed, it could happen that one of the ρ_{ik}^*'s can be written in terms of the remaining ρ^*'s. However, no element of the form ρ_i^* can be written in terms of the remaining ρ_i^*'s and ρ_{ik}^*'s. This means that, by dropping if necessary some of the elements of the form ρ_{ik}^*, we get a minimal generating set of I^*, and, by construction the coefficients involved are again ± 1.

Finally, we should point out that I^* need not be admissible, for instance, one of the minimal relations may be a sum of $C(a,b)$ and linear combinations of other paths. In this case, it is easy to see that KQ^*/I^* is isomorphic to an algebra KQ^{**}/I^{**}, where Q^{**} is obtained from Q^* by dropping the arrow $C(a,b)$, and I^{**} is obtained from I^* in the obvious way, that is, by replacing every occurrence of $C(a,b)$ by the given linear combination.

Proof of Theorem 5.1. Any artin algebra of global dimension n has at least one simple module with injective dimension equal to $n-1$ by an argument similar to the one in [Z]. The theorem follows from the remarks above by induction on the number of nonisomorphic simple modules. □

The following examples illustrate how global dimension can be affected by the characteristic of the ground field.

Example 5.7. Let Q be the following quiver:



where $n \geq 2$, and, let $I = \langle \sum_{i=1}^{n} ab_i, b_ic_1 + b_nc_2, \text{for } i = 1, \ldots, n \rangle$. Let $\Lambda = KQ/I$ and let S_1 be the simple Λ-module corresponding to the vertex 1.

We claim the following:

$$\operatorname{pd}_\Lambda S_1 = \begin{cases} 3, & \text{if } \operatorname{char} k \mid n, \\ 2, & \text{if } \operatorname{char} k \nmid n \end{cases}$$

and,

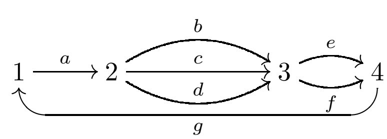
$$\operatorname{gldim} \Lambda = \begin{cases} 3, & \text{if } \operatorname{char} k \mid n, \\ 2, & \text{if } \operatorname{char} k \nmid n. \end{cases}$$

Proof. It is easy to see that $f^0 = v_1$ and $f^1 = a$. We can write $v_1I \cap aR = v_1I = \amalg_i f_i^{2*}R$ and we can easily see that $f^2 = \sum_{i=1}^{n} ab_i$, since for each i, $ab_ic_1 + ab_nc_2$ is an element of $aI = f^1I$. Next, in computing the intersection $f^2R \cap f^1I$, we must solve the equation

$$\sum_{i=1}^{n} (ab_ic_1 + ab_nc_2)\alpha_i = (\sum_{i=1}^{n} ab_ic_1)\beta + (\sum_{i=1}^{n} ab_ic_2)\gamma,$$

where $\alpha_1, \ldots, \alpha_n, \beta, \gamma$ are in the field K. By identifying corresponding coefficients, we see that for each $i = 1, \ldots, n$, $\alpha_i = \beta$, $\sum_{i=1}^{n} \alpha_i = \gamma$, and also that $\gamma = 0$. This implies that $n\beta = 0$, and $f^2R \cap f^1I = (\sum_{i=1}^{n} ab_ic_1\beta)R$. If $\operatorname{char} K \nmid n$, then $\beta = 0$. Thus $f^2R \cap f^1I = (0)$, and hence $\operatorname{pd} S_1 = 2$. If $\operatorname{char} K \mid n$, then we can take $\beta = 1$; we also see that $\sum_{i=1}^{n} ab_ic_1$ is not in f^2I, since c_1 is not in I. In this case, we have $f^3 = \sum_{i=1}^{n} ab_ic_1$. It is easy to show that $f^3R \cap f^2I = (0)$. The remaining statements are obvious. ☐

Example 5.8. A small adjustment of the previous example yields the following. Let Q be the quiver given by



and let $I = \langle ab + ac + ad, be + df, ce + df, de + df, eg, ga \rangle$. Let $\Lambda = KQ/I$ and let S_i be the simple Λ-module corresponding to the vertex i of Q. It is easy to see that Λ is finite dimensional. An easy calculation shows that Λ has finite global dimension if and only if the characteristic of $K \neq 3$. In characteristic different from 3 we have that $\operatorname{pd}_\Lambda S_1 = 2$, $\operatorname{pd}_\Lambda S_2 = 5$, $\operatorname{pd}_\Lambda S_3 = 4$ and $\operatorname{pd}_\Lambda S_4 = 3$. In characteristic equal to 3 all simple Λ-modules have infinite projective dimension. In fact, we have the following initial segments of the minimal projective resolutions of the simple Λ-modules.

In characteristic 3:

$$0 \to S_1 \to P_4 \to P_3 \to P_2 \xrightarrow{a} P_1 \to S_1 \to 0,$$
$$0 \to S_1^2 \to P_4^3 \to P_3^3 \to P_2 \to S_2 \to 0,$$
$$0 \to S_1 \to P_4^2 \to P_3 \to S_3 \to 0,$$
$$0 \to S_1 \to P_4 \to S_4 \to 0.$$

In characteristic different from 3:

$$0 \to P_3 \xrightarrow{b+c+d} P_2 \xrightarrow{a} P_1 \to S_1 \to 0$$

and as above $\Omega^3_\Lambda(S_2) \simeq S_1^2$, $\Omega^2_\Lambda(S_3) \simeq S_1$ and $\Omega_\Lambda(S_4) \simeq S_1$.

Appendix

The authors would like to thank M. C. R. Butler and the referee for kindly pointing out that we can also approach the construction of the projective resolutions more conceptually. One gains a notational simplicity but loses an explicit algorithmic method of constructing the resolution. We briefly sketch this approach and add some comments on how to reobtain our explicit constructions from it.

We keep the following notations from the paper. Let $R = KQ$ denote a path algebra, let I be an ideal in R, and $\Lambda = R/I$. We let J be the ideal of R generated by arrows of Q and let R_0 denote the K-span of the vertices of Q. Then R_0 is a semisimple K-algebra and $R = R_0 \oplus J$ as R_0-modules. A right projective R-module is isomorphic to $U \otimes_{R_0} R$ where U is a right R_0-module. See [BK, G]. Let M be a right R/I-module, and let F_0 be a semisimple R_0-module such that there is a surjection $F_0 \otimes_{R_0} R \to M$. Let $K_1 = \operatorname{Ker}(F_0 \otimes_{R_0} R \to M)$. Then $K_1 = F_1 \otimes_{R_0} R$ for some right R_0-module F_1. We note that F_1 can be chosen to be an R_0-complement of K_1J in K_1 which generates K_1. It should be remarked that one can always find an R_0-complement to K_1J in K_1. Because of the special structure of projective R-modules, F_1 can be chosen to generate K_1. An algorithmic method for constructing F_1 can be given by constructing a minimal right uniform Gröbner basis for K_1 in $F_0 \otimes_{R_0} R$. This Gröbner basis is an R_0-generating set for F_1 (see [G]). Fix F_1 with the desired properties. Then we get the beginning of a projective R/I-resolution of M.

$$F_1 \otimes_{R_0} R/F_1 \otimes_{R_0} I \to F_0 \otimes_{R_0} R/F_0 \otimes_{R_0} I \to M \to 0.$$

Next, proceed inductively as follows. Let $K_{n+1} = (F_n \otimes_{R_0} R) \cap (F_{n-1} \otimes_{R_0} I)$ in $F_0 \otimes_{R_0} R$. Let F^*_{n+1} be an R_0-complement to $K_{n+1}J$ in K_n which generates K_{n+1}. The same remarks made above about the existence and construction of F_1 apply to the existence and construction of F^*_{n+1}. Choose F^*_{n+1} with the desired properties. Thus $K_{n+1} = F^*_{n+1} \otimes_{R_0} R$. Decompose F^*_{n+1} as an R_0-module, $F^*_{n+1} = F_{n+1} \oplus F'_{n+1}$, with $F'_{n+1} \subseteq F_n \otimes_{R_0} I$. We get an R/I-projective resolution of M as follows.

$$\cdots \to F_2 \otimes_{R_0} R/F_2 \otimes_{R_0} I \to F_1 \otimes_{R_0} R/F_1 \otimes_{R_0} I$$
$$\to F_0 \otimes_{R_0} R/F_0 \otimes_{R_0} I \to M \to 0.$$

This method yields the construction of the paper by taking the f_i^n's to be an "R_0-basis" of F_n; namely, since F_n is a direct some of simple R_0-modules, the f_i^n choose one from each summand. As remarked above, the F_n's and the f_i^n's in particular, can be explicitly constructed using right Gröbner basis techniques.

In the construction in Section 1, we find the f^*_{n+1}'s and form F'_{n+1} by taking all f^*_{n+1}'s in $F_n \otimes_{R_0} I$. One does not need to remove all such f^*_{n+1}'s; that is, F'_{n+1} can be chosen to be *any* summand of F^*_{n+1} with the property that $F'_{n+1} \subseteq F_n \otimes_{R_0} I$. Of course, if one is attempting to construct a minimal projective resolution, F'_{n+1} must be taken as large as possible. On the other hand, always taking $F'_n = 0$ yields the following well known generalization of the Gruenberg resolution mentioned in the introduction. Let $P_0 = F_0 \otimes_{R_0} R$ and $P_1 = \mathrm{Ker}(P_0 \to M)$ (over R). Then, the above construction yields the long exact sequence

$$\to P_0I^2/P_0I^3 \to P_1I/P_1I^2 \to P_0I/P_0I^2 \to P_1/P_1I \to P_0/P_0I \to M \to 0,$$

coming from the filtration

$$\cdots \subseteq P_0I^2 \subseteq P_1I \subseteq P_0I \subseteq P_1 \subseteq P_0.$$

This immediately yields the following formulae. For $m \geq 1$,

$$\mathrm{Tor}^{\Lambda}_{2m}(M, R_0) = (P_1I^{m-1}J \cap P_0I^m)/(P_1I^m + P_0I^mJ),$$

and for $m \geq 0$,

$$\mathrm{Tor}^{\Lambda}_{2m+1}(M, R_0) = (P_1I^m \cap P_0I^mJ)/(P_1I^mJ + P_0I^{m+1}).$$

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Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061-0123
E-mail address: green@math.vt.edu

Institutt for matematiske fag, NTNU, Lade, N–7491 Trondheim, Norway
E-mail address: oyvinso@math.ntnu.no

Department of Mathematics, Syracuse University, Syracuse, New York 13244
E-mail address: zacharia@mailbox.syr.edu