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MINIMAL REEB VECTOR FIELDS ON ALMOST COSYMPLECTIC MANIFOLDS

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Abstract

We show that the Reeb vector field of an almost cosymplectic three-manifold is minimal if and only if it is an eigenvector of the Ricci operator. Then, we show that Reeb vector field ξ of an almost cosymplectic three-manifold M is minimal if and only if M is (κ, μ, ν)-space on an open dense subset. After, using the notion of strongly normal unit vector field introduced in [8], we study the minimality of ξ for an almost cosymplectic $(2n+1)$-manifold. Finally, we classify a special class of almost cosymplectic three-manifold whose Reeb vector field is minimal.

1. Introduction

Let (M, g) be a Riemannian manifold and (T^1M, g_S) its unit tangent sphere bundle equipped with the Sasaki metric g_S induced by the Riemannian metric g. A unit vector field V on M determines an immersion $V : M \to (T^1M, g_S)$. When M is compact, the volume of V is the volume of the corresponding submanifold (M, V^*g_S) of (T^1M, g_S). This gives a functional defined on the set $\mathfrak{X}^1(M)$ of all unit vector fields on (M, g). A unit vector field V is said to be a *minimal vector field* if it is a critical point for the volume functional $F : \mathfrak{X}^1(M) \to \mathbf{R}$. This functional has been studied in [4] where similar notion is introduced when M is also non-compact. One remarkable fact is that V is a minimal unit vector field if and only if the submanifold (M, V^*g_S) is minimal, that is, the mean curvature vector field vanishes. The study of the minimal unit vector fields is motivated from the work of Gluck-Ziller [6] where they considered the problem of determining those unit vector fields V which have minimal volume. In particular, Gluck-Ziller [6] proved that on the unit sphere $\mathbf{S}^3$ these optimal unit vector fields are the Hopf vector fields (see also [13] for a different proof). In in the last fifteen years, many papers have been published containing

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examples and general results on minimal unit vector fields in different geometrical situations (see, for example, [4], [5], [8], [9], [12], [13], [14]).

An interesting geometrical situation, in which a distinguished vector field appears in a natural way, is given by an almost contact metric manifold where we have the Reeb vector field ξ, also called the characteristic vector field. It is a unit field and plays a fundamental role in the study of the Riemannian geometry of an almost contact metric manifold [1]. The purpose of this paper is to study, mainly in dimension three, almost cosymplectic manifolds whose Reeb vector field is minimal. In Section 2 we give some results on the geometry of an almost cosymplectic manifold. In Section 3, we show that the Reeb vector field of an almost cosymplectic three-manifold is minimal if and only if it is an eigenvector of the Ricci operator. In particular the minimality condition for the Reeb vector field of an almost cosymplectic three-manifold is invariant for a D-homothetic deformation. In Section 4 we explicitly the Ricci tensor of an almost cosymplectic three-manifold M, then we show that Reeb vector field ξ of M is a minimal if and only if M is (κ, μ, ν)-space on an open dense subset. After, using the notion of strongly normal unit vector field introduced in [8], we study the minimality of ξ for an almost cosymplectic $(2n+1)$-manifold. Finally, we classify a special class of almost cosymplectic three-manifolds whose Reeb vector field is minimal.

2. Almost cosymplectic manifolds

An *almost contact structure* (ξ, ϕ, η) on a differentiable manifold M consists of a tensor field ϕ of type $(1,1)$, a tangent vector field ξ (called the *Reeb vector field* or the *characteristic vector field*), and a differential 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

As a consequence, the dimension of M is odd $(= 2n+1)$, $\phi(\xi) = 0$ and $\eta \circ \phi = 0$. Given an almost contact structure (ϕ, ξ, η) on M, an *associated metric* is a Riemannian metric g on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \mathfrak{X}(M)$, and then $\eta(X) = g(\xi, X)$. Associated metrics are known to exist (cf. [1], p. 34). The extended object (ϕ, ξ, η, g) is an *almost contact metric structure*. The 2-form Φ defined by

$$\Phi(X, Y) = g(X, \phi Y) \quad \text{for any } X, Y \in \mathfrak{X}(M)$$

is called the *fundamental* 2-*form*.

Note that an almost contact metric structure on an orientable $(2n+1)$-dimensional manifold M may be regarded as a reduction of the structure group of M to $U(n) \times 1$. If an almost contact metric structure satisfies in addition the *contact condition* $(d\eta)(X, Y) = \Phi(X, Y)$, then (ϕ, ξ, η, g) is called a *contact metric structure*.

For a given Riemannian manifold (M, g), we denote by ∇ the Levi-Civita connection, by R the corresponding Riemann curvature tensor given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

by *Ric* the Ricci tensor and by Q the corresponding Ricci operator defined by $g(QX, Y) = Ric(X, Y)$.

Following S.I. Goldberg and K. Yano [7], an almost contact metric manifold (M, ϕ, ξ, η, g) is said to be an *almost cosymplectic* manifold if both the fundamental 2-form Φ and the 1-form η are closed, that is,

$$d\Phi = 0 \quad \text{and} \quad d\eta = 0.$$

The identity $d\eta = 0$ shows that the distribution $\ker \eta = 0$ is integrable and its (maximal) integral submanifolds are hypersurfaces of M. The restrictions of Φ and η to the associated foliation are closed forms, so that any leave is an almost Kaehler submanifold. An almost cosymplectic manifold M is *cosymplectic* if the underlying almost contact metric structure is normal, that is, $[\phi, \phi] = 0$, where $[\phi, \phi]$ is the Nijenhuis tensor of the tensor field ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any $X, Y \in \mathfrak{X}(M)$. A cosymplectic manifolds has Kaehlerian leaves, however there are almost cosymplectic manifolds with Kaehlerian leaves which are not cosymplectic manifolds [11]. Besides, an almost contact metric manifold is cosymplectic if and only if $\nabla\phi = 0$. Normality is known to imply that ξ is parallel, that is, $\nabla\xi = 0$ (as a consequence of $\phi\xi = 0$ and $\nabla\phi = 0$). In dimension three an almost contact metric manifold is cosymplectic if and only if ξ is parallel (cf. [10], p. 248).

A cosymplectic manifold is locally the product of a Kähler manifold and an interval in $\mathbf{R}$. There are however examples of cosymplectic manifolds which aren't globally the product of a Kähler manifold and a real 1-dimensional manifold (cf. [1], p. 77). For an almost cosymplectic manifolds we have the following properties (cf. [2], [15]):

$$\nabla_\xi \phi = 0, \quad \nabla\xi = h\phi, \quad \text{where } h = (1/2)\mathscr{L}_\xi \phi, \tag{2.1}$$

$$h\phi = -\phi h, \quad h\xi = 0, \quad \operatorname{tr} h = 0, \quad \operatorname{div} \xi = 0 \quad \text{and} \tag{2.2}$$

$$\nabla_\xi h = h^2\phi + \phi\ell, \tag{2.3}$$

where ℓ is the *Jacobi operator* associated to the Reeb vector field: $\ell = R(\cdot, \xi)\xi$. From $\nabla\xi = h\phi$, we have that $h = 0$ if and only if ξ is parallel. Moreover, from

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = g(h\phi X, Y) + g(X, h\phi Y) = 2g(h\phi X, Y),$$

we get that $h = 0$, i.e. $\nabla\xi = 0$, if and only if ξ is Killing.

Next, let $(M, \eta, g, \xi, \varphi)$ be a three-dimensional almost cosymplectic manifold. Let $\mathscr{U}_1$ be the open subset of M where $h \neq 0$ and $\mathscr{U}_2$ the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. Then, $\mathscr{U}_1 \cup \mathscr{U}_2$ is an open dense subset of M. For any point $p \in \mathscr{U}_1 \cup \mathscr{U}_2$ there exists a local orthonormal basis

$\{\xi, e_1, e_2 = \phi e_1\}$ of smooth eigenvectors of h in a neighborhood of p. On $\mathcal{U}_1$ we put $he_1 = \lambda e_1$, where λ is a non-vanishing smooth function which we suppose to be positive. From (2.2), we have $he_2 = -\lambda e_2$. We note that the eigenvalue function λ is continuos on M and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$. Then we have

LEMMA 2.1. *On* $\mathcal{U}_1$ *we have*

$$(2.4)\quad \begin{cases} \nabla_\xi e_1 = ae_2, \quad \nabla_\xi e_2 = -ae_1, \quad \nabla_{e_1}\xi = -\lambda e_2, \quad \nabla_{e_2}\xi = -\lambda e_1, \\ \nabla_{e_1} e_1 = \dfrac{1}{2\lambda}\{e_2(\lambda) + \sigma(e_1)\}e_2, \quad \nabla_{e_2} e_2 = \dfrac{1}{2\lambda}\{e_1(\lambda) + \sigma(e_2)\}e_1, \\ \nabla_{e_1} e_2 = \lambda\xi - \dfrac{1}{2\lambda}\{e_2(\lambda) + \sigma(e_1)\}e_1, \quad \nabla_{e_2} e_1 = \lambda\xi - \dfrac{1}{2\lambda}\{e_1(\lambda) + \sigma(e_2)\}e_2, \end{cases}$$

$$(2.5)\qquad \ell e_1 = -\xi(\lambda)e_2 + (\lambda^2 + 2a\lambda)e_1, \quad \ell e_2 = -\xi(\lambda)e_1 + (\lambda^2 - 2a\lambda)e_2,$$

$$(2.6)\qquad \nabla_\xi h = \left(\frac{\xi(\lambda)}{\lambda} I + 2a\phi\right)h,$$

where a *is a smooth function and* σ *is the* 1*-form given by* $Ric(\xi, \cdot)$.

Proof. From (2.1) we obtain $\nabla_{e_1}\xi = h\phi e_1 = -\lambda e_2$ and $\nabla_{e_2}\xi = h\phi e_2 = -\lambda e_1$. Since $\nabla_\xi\xi = 0$, we have $\nabla_\xi e_1 = ae_2$ and $\nabla_\xi e_2 = -ae_1$, where a is a smooth function. Moreover $g(\nabla_{e_i}e_i, \xi) = -g(\nabla_{e_i}\xi, e_i) = g(\phi he_i, e_i) = 0$ gives

$$\nabla_{e_1} e_1 = \alpha e_2 \quad \text{and} \quad \nabla_{e_2} e_2 = \beta e_1,$$

where α, β are smooth functions. Besides,

$$\nabla_{e_1} e_2 = \lambda\xi - \alpha e_1 \quad \text{and} \quad \nabla_{e_2} e_1 = \lambda\xi - \beta e_2.$$

Using these formulas, we get

$$\begin{aligned} R(e_1, e_2)\xi &= -\nabla_{e_1}\nabla_{e_2}\xi + \nabla_{e_2}\nabla_{e_1}\xi + \nabla_{[e_1, e_2]}\xi \\ &= (e_1(\lambda) - 2\beta\lambda)e_1 - (e_2(\lambda) - 2\alpha\lambda)e_2 \end{aligned}$$

and hence

$$\begin{aligned} \sigma(e_1) &= Ric(\xi, e_1) = g(R(e_1, e_2)\xi, e_2) = 2\alpha\lambda - e_2(\lambda), \\ \sigma(e_2) &= Ric(\xi, e_2) = g(R(e_2, e_1)\xi, e_1) = 2\beta\lambda - e_1(\lambda). \end{aligned}$$

Then,

$$\alpha = \frac{e_2(\lambda) + \sigma(e_1)}{2\lambda} \quad \text{and} \quad \beta = \frac{e_1(\lambda) + \sigma(e_2)}{2\lambda}$$

This completes the proof of (2.4). From

$$\begin{aligned} \ell e_1 &= R(e_1, \xi)\xi = -\nabla_{e_1}\nabla_\xi\xi + \nabla_\xi\nabla_{e_1}\xi + \nabla_{[e_1, \xi]}\xi \\ \ell e_2 &= R(e_2, \xi)\xi = -\nabla_{e_2}\nabla_\xi\xi + \nabla_\xi\nabla_{e_2}\xi + \nabla_{[e_2, \xi]}\xi, \end{aligned}$$

using (2.4), we get (2.5). The formulas (2.6) follows from

$$(\nabla_\xi h)\xi = 0 = \left(\frac{\xi(\lambda)}{\lambda} I + 2a\phi\right) h\xi,$$

$$(\nabla_\xi h)e_1 = \xi(\lambda)e_1 - 2ahe_2 = \left(\frac{\xi(\lambda)}{\lambda} I + 2a\phi\right) he_1,$$

$$(\nabla_\xi h)e_2 = -\xi(\lambda)e_2 - 2ah\phi e_2 = \left(\frac{\xi(\lambda)}{\lambda} I + 2a\phi\right) he_2. \qquad \square$$

From (2.5) we deduce that

$$Ric(\xi, \xi) = -2\lambda^2 = -\mathrm{tr}\, h^2. \tag{2.7}$$

Proposition 2.1. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic manifold of dimension $2n+1$. Then, for any $X \in \ker \eta$, $\|X\| = 1$, the vertical sectional curvature satisfies the following properties*:

$$K(\xi, X) = -\|hX\|^2 - g((\nabla_\xi h)X, \phi X), \tag{2.8}$$

$$K(\xi, X) - K(\xi, \phi X) = -2g((\nabla_\xi h)X, \phi X). \tag{2.9}$$

In particular, $\nabla_\xi h = 0$ implies

$$K(\xi, X) = K(\xi, \phi X) \leq 0,$$

and $K(\xi, X) = K(\xi, \phi X) = 0$ if and only if $h = 0$.

Proof. From (2.3) we have

$$K(\xi, X) = R(\xi, X, \xi, X) = -g(\ell X, X) = g(\phi(\nabla_\xi h)X, \phi X) - g(h^2X, X)$$

and

$$K(\xi, \phi X) = R(\xi, \phi X, \xi, \phi X) = g((\nabla_\xi h)\phi X, \phi^2 X) - g(h^2 X, X),$$

Then, since $(\nabla_\xi h)\phi = -\phi\nabla_\xi h$, we get (2.9). $\square$

Proposition 2.2. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic three-manifold. Then,*

$$Ric(e, \phi e) = g((\nabla_\xi h)e, e), \tag{2.10}$$

$$Ric(e, e) = (r/2) + (\mathrm{tr}\, h^2/2) - g((\nabla_\xi h)e, \phi e), \tag{2.11}$$

$$Ric(\phi e, \phi e) = (r/2) + (\mathrm{tr}\, h^2/2) + g((\nabla_\xi h)e, \phi e). \tag{2.12}$$

for any $e \in \ker \eta$, $\|e\| = 1$.

Proof. Since

$$Ric(e, e) = R(e, \phi e, e, \phi e) + R(\xi, e, \xi, e),$$

$$Ric(\phi e, \phi e) = R(e, \phi e, e, \phi e) + R(\xi, \phi e, \xi, \phi e),$$

from (2.9) we get

$$Ric(\phi e, \phi e) - Ric(e, e) = 2g((\nabla_\xi h)e, \phi e). \tag{2.13}$$

Of course (2.13) holds for any $e \in \ker \eta$. Then, for any $e, e' \in \ker \eta$, using (2.13) and $\phi \nabla_\xi h = -(\nabla_\xi h)\phi$, we get

$$Ric(\phi e, \phi e') - Ric(e, e') = 2g((\nabla_\xi h)e, \phi e'). \tag{2.14}$$

If we put $e' = \phi e$, from (2.14) we obtain (2.10); (2.11) and (2.12) follow from (2.13) and (2.7) because the scalar curvature r is given by

$$\begin{aligned} r = \operatorname{tr} Ric &= Ric(e, e) + Ric(\phi e, \phi e) + Ric(\xi, \xi) \\ &= 2\, Ric(e, e) + 2g((\nabla_\xi h)e, \phi e) - \operatorname{tr} h^2. \end{aligned}$$

□

3. Minimality of ξ in dimension three

Let (M, g) be a Riemannian manifold and (T^1M, g_S) its unit tangent sphere bundle equipped with the Sasaki metric g_S. A unit vector field V on M determines an immersion $V : (M, g) \to (T^1M, g_S)$. When M is compact, the volume of V, that we denote by $F(V)$, is the volume of the Riemannian manifold $(M, V^* g_S)$. This gives a functional $F : \mathfrak{X}^1(M) \to \mathbf{R}$ defined on the set $\mathfrak{X}^1(M)$ of all unit vector fields on (M, g). The metric $V^* g_S$ is related to the metric g by the identity

$$(V^* g_S)(X, Y) = g(L_V X, Y),$$

where L_V is the tensor of type $(1, 1)$ defined by

$$L_V = I + (\nabla V)^t \circ \nabla V.$$

Then

$$F(V) := \int_M v_{V^* g_S} = \int_M f(V) v_g,$$

where $f(V) = \sqrt{\det L_V}$. Consider the 1-form ω_V defined by

$$\omega_V(X) = \operatorname{tr}(Y \mapsto (\nabla_Y K_V)X),$$

where K_V is the tensor of type $(1, 1)$ defined by

$$K_V = f(V)[L_V^{-1}(\nabla V)^t].$$

The unit vector field V is called a *minimal vector field* if it is critical for the volume functional F defined on the set $\mathfrak{X}^1(M)$. The corresponding critical point condition

$$\omega_V(A) = 0 \quad \text{for any } A \in V^\perp,$$

has been determined in [4], where similar notion is introduced when M is also non-compact. One remarkable fact is that *V is a minimal unit vector field if* *and only if* $V : (M, g) \to (T^1M, g_S)$ *is a minimal immersion*, that is, the mean curvature vector field is zero. Unit Killing vector fields on a manifold of constant sectional curvature are minimal [4]. Hopf vector fields on $\mathbf{S}^{2n+1}$ and Reeb vector fields of K-contact manifolds are minimal with respect to the Sasaki metric g_S ([4], [8]) and, more in general, with respect to a class of g-natural metric of Kaluza-Klein type [14].

Now we use Lemma 2.1 to derive a minimality condition for the Reeb vector field ξ of an almost cosymplectic three-manifold.

THEOREM 3.1. *Let* (M, ξ, ϕ, η, g) *be an almost cosymplectic three-manifold. Then, the* 1*-form* ω_ξ *is given by*

$$\omega_\xi = Ric(\xi, \cdot). \tag{3.1}$$

So, ξ *is minimal if and only if* ξ *is an eigenvector of the Ricci operator.*

Proof. We recal that $\mathscr{U}_1 \cup \mathscr{U}_2$ is an open dense subset of M. For any point $p \in \mathscr{U}_1 \cup \mathscr{U}_2$ there exists a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of smooth eigenvectors of h in a neighborhood of p. On $\mathscr{U}_1$ we put $he_1 = \lambda e_1$, where λ is a non-vanishing smooth function which we suppose to be positive, and $he_2 = -\lambda e_2$. Now, on $\mathscr{U}_1$ we determine 1-form ω_ξ, which is defined by

$$\omega_\xi(X) = \mathrm{tr}(Y \mapsto (\nabla_Y K_\xi)X).$$

From (2.1), we get

$$L_\xi = I + (\nabla\xi)^t(\nabla\xi) = I + h^2$$

and so

$$L_\xi\xi = \xi, \quad L_\xi e_1 = (1+\lambda^2)e_1, \quad L_\xi e_2 = (1+\lambda^2)e_2.$$

Now, we determine the tensor

$$K_\xi = f(\xi)L_\xi^{-1}(\nabla\xi)^t = f(\xi)L_\xi^{-1}h\phi, \quad \text{where } f(\xi) = \sqrt{\det L_\xi} = (1+\lambda^2).$$

Since

$$L_\xi^{-1}\xi = \xi, \quad L_\xi^{-1}e_i = (1/(1+\lambda^2))e_i \quad (i = 1, 2),$$

we find

$$K_\xi\xi = 0, \quad K_\xi e_i = -\lambda e_i \quad (i = 1, 2).$$

Moreover, using Lemma 2.1, we find

$$(\nabla_\xi K_\xi)e_1 = 2a\lambda e_1 - \xi(\lambda)e_2, \quad (\nabla_\xi K_\xi)e_2 = -\xi(\lambda)e_1 - 2a\lambda e_1,$$
$$(\nabla_{e_1} K_\xi)e_1 = -\lambda^2\xi + (\sigma(e_1) + e_2(\lambda))e_1 - e_1(\lambda)e_2,$$
$$(\nabla_{e_1} K_\xi)e_2 = -e_1(\lambda)e_1 - (\sigma(e_1) + e_2(\lambda))e_2,$$

$$(\nabla_{e_1} K_\xi)\xi = -\lambda^2 e_1, \quad (\nabla_{e_2} K_\xi)\xi = -\lambda^2 e_2,$$
$$(\nabla_{e_2} K_\xi)e_1 = -(\sigma(e_2) + e_1(\lambda))e_1 - e_2(\lambda)e_2,$$
$$(\nabla_{e_2} K_\xi)e_2 = -\lambda^2\xi - e_2(\lambda)e_1 + (\sigma(e_2) + e_1(\lambda))e_2.$$

All these formulas imply that

$$\omega_\xi(e_1) = g((\nabla_\xi K_\xi)e_1, \xi) + g((\nabla_{e_1} K_\xi)e_1, e_1) + g((\nabla_{e_2} K_\xi)e_1, e_2) = \sigma(e_1),$$
$$\omega_\xi(e_2) = g((\nabla_\xi K_\xi)e_2, \xi) + g((\nabla_{e_1} K_\xi)e_2, e_1) + g((\nabla_{e_2} K_\xi)e_2, e_2) = \sigma(e_2),$$
$$\omega_\xi(\xi) = g((\nabla_\xi K_\xi)\xi, \xi) + g((\nabla_{e_1} K_\xi)\xi, e_1) + g((\nabla_{e_2} K_\xi)\xi, e_2) = Ric(\xi, \xi),$$

Therefore, $\omega_\xi = Ric(\xi, \cdot)$ on $\mathscr{U}_1$. If the set $\mathscr{U}_2$ is not empty, then the restriction of the almost cosymplectic structure on $\mathscr{U}_2$ is cosympletic, that is , $\nabla\xi = 0$. In such case, we get $\omega_\xi = 0 = Ric(\xi, \cdot)$. Then, $\omega_\xi = Ric(\xi, \cdot)$ on $\mathscr{U}_1 \cup \mathscr{U}_2$ and so on M because the open set $\mathscr{U}_1 \cup \mathscr{U}_2$ is dense in M and the tensors ω_ξ and $Ric(\xi, \cdot)$ are continuos on M. □

Remark 3.1. The minimality condition for the Reeb vector field of an almost cosymplectic three-manifold is invariant for a D-homothetic deformation of type

$$\phi' = \phi \quad \xi' = (1/\beta)\xi, \quad \eta' = \beta\eta, \quad g' = tg + (\beta^2 - t)\eta \otimes \eta$$

where t is a positive constant, β is a smooth function with $\beta(p) \neq 0$ for any $p \in M$ and $d\beta \wedge \eta = 0$. In fact, in [16] we proved that for a such deformation ξ' is an eigenvector of the Ricci operator Q' if and only if ξ is an eigenvector of the Ricci operator Q.

4. Almost cosymplectic (κ, μ, ν)-spaces and minimality

We start this section with the following

Proposition 4.1. *The Ricci tensor of an almost cosymplectic three-manifold is given (locally) by*

$$Q = \alpha I + \beta\eta \otimes \xi + \phi\nabla_\xi h - \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_1 + \sigma(e_2)\eta \otimes e_2 \tag{4.1}$$

where $\alpha = (r + \operatorname{tr} h^2)/2$ *and* $\beta = -(r + 3 \operatorname{tr} h^2)/2$.

Proof. Let $\{\xi, e_1, e_2 = \phi e_1\}$ be a local orthonormal ϕ-basis. We put

$$Q_1 = Q - \alpha I - \beta\eta \otimes \xi,$$

and

$$\tilde{Q}_1 = \phi\nabla_\xi h - \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_1 + \sigma(e_2)\eta \otimes e_2,$$

where $\alpha = (r + \operatorname{tr} h^2)/2$ and $\beta = -(r + 3 \operatorname{tr} h^2)/2$. Using (2.7) and $(\nabla_\xi h)(\xi) = 0$, we get

$$\begin{aligned} Q_1\xi &= Q\xi - (\alpha+\beta)\xi \\ &= Ric(\xi,\xi)\xi + \sigma(e_1)e_1 + \sigma(e_2)e_2 + (\operatorname{tr} h^2)\xi \\ &= \sigma(e_1)e_1 + \sigma(e_2)e_2 \\ &= \tilde{Q}_1\xi. \end{aligned}$$

Moreover, using (2.10) and (2.11), we have

$$\begin{aligned} Q_1e_1 &= Qe_1 - \alpha e_1 \\ &= \sigma(e_1)\xi + Ric(e_1,e_1)e_1 + Ric(e_1,e_2)e_2 - \alpha e_1 \\ &= \sigma(e_1)\xi - g((\nabla_\xi h)e_1, \phi e_1)e_1 + g((\nabla_\xi h)e_1, e_1)e_2 \\ &= \sigma(e_1)\xi + g(\phi(\nabla_\xi h)e_1, e_1)e_1 + g(\phi(\nabla_\xi h)e_1, e_2)e_2 \\ &= \sigma(e_1)\xi + \phi(\nabla_\xi h)e_1 - g(\phi(\nabla_\xi h)e_1, \xi)\xi \\ &= \sigma(e_1)\xi + \phi(\nabla_\xi h)e_1 \\ &= \tilde{Q}_1e_1. \end{aligned}$$

Analogously, we get $Q_1e_2 = \tilde{Q}_1e_2$. Therefore, $Q_1 = \tilde{Q}_1$ and hence we obtain (4.1). □

Now, we recall the following

Definition 4.1. An almost cosymplectic $(2n+1)$-manifold (M,ξ,ϕ,η,g) is said to be a (κ,μ,ν)-space if the curvature tensor satisfies the following condition

$$\begin{aligned} R(X,Y)\xi = {} & \kappa(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX) \\ & + \nu(\eta(X)\phi hY - \eta(Y)\phi hX), \end{aligned} \tag{4.2}$$

where κ, μ, ν are smooth functions. Such definition was introduced in [2] with the additional condition that $\kappa,\mu,\nu \in \mathcal{R}_\eta(M)$, where $\mathcal{R}_\eta(M)$ is the subring of the smooth functions f on M for which $df \wedge \eta = 0$, or equivalently $df = \xi(f)\eta$.

Theorem 4.1. *Let (M,ξ,ϕ,η,g) be an almost cosymplectic three-manifold. If M is a (κ,μ,ν)-space, then ξ is a minimal unit vector field. Conversely, if ξ is minimal, then M is a (κ,μ,ν)-space on an open dense subset of M.*

Proof. Let us suppose that M is a (κ,μ,ν)-space. From (4.2) we have $R(X,Y)\xi = 0$ for any $X, Y \in \ker\eta$ and hence $Ric(X,\xi) = 0$ for any $X, Y \in \ker\eta$. Then $Q\xi = Ric(\xi,\xi)\xi$ and by (2.7) we get $Q\xi = -(\operatorname{tr} h^2)\xi$. So, by Theorem 3.1, ξ is minimal. Vice versa, we suppose that ξ is minimal, that is, ξ is an eigenvector of the Ricci operator Q. From now, we use the notations introduced in Lemma 2.1. If the open set $\mathcal{U}_2$ is non-empthy, then it inherits the almost cosymplectic structure of M. In particular such structure is cosympletic, and since

$$R(X, Y)\xi = (\nabla_Y \nabla \xi)X - (\nabla_X \nabla \xi)Y,$$

we get $R(X, Y)\xi = 0$. Then M is a (κ, μ, ν)-space with $\kappa = \mu = \nu = 0$. Next, let $\mathcal{U}_1$ be a non-empthy set and let $\{\xi, e_1, e_2\}$ be the local ϕ-basis described in Lemma 2.1. Since ξ is minimal, the 1-form $\sigma = 0$ and by Proposition 4.1 we have

$$Q = \alpha I + \beta\eta \otimes \xi + \phi \nabla_\xi h$$

from which using (2.6) we obtain

$$Q = \alpha I + \beta\eta \otimes \xi + \frac{\xi(\lambda)}{\lambda}\phi h - 2ah. \tag{4.3}$$

On the other hand, for a tree-dimensional Riemannian manifold the curvature tensor is completely determined by the Ricci operator. In our case, we have

$$R(X, Y)\xi = \eta(X)QY - \eta(Y)QX - g(QY, \xi)X + g(QX, \xi)Y - \frac{r}{2}(\eta(X)Y - \eta(Y)X).$$

So, using (4.3) we get

$$R(X, Y)\xi = (-\lambda^2)(\eta(X)Y - \eta(Y)X) - 2a(\eta(X)hY - \eta(Y)hX) + \frac{\xi(\lambda)}{\lambda}(\eta(X)\phi hY - \eta(Y)hX)$$

which is the formulas (4.2) with $\kappa = -\lambda^2$, $\mu = -2a$ and $\nu = \xi(\lambda)/\lambda$ on the open set $\mathcal{U}_1$. Therefore, the almost cosymplectic structure defines a (κ, μ, ν)-space on $\mathcal{U}_1 \cup \mathcal{U}_2$. □

Remark 4.1. Let (M, ξ, ϕ, η, g) be a (κ, μ, ν)-almost cosymplectic three-manifold. Then, from the proof of Theorem 4.1 we get

$$Q\xi = -(\operatorname{tr} h^2)\xi, \quad \kappa = -\lambda^2 \leq 0, \quad \mu = -2a \quad \text{and} \quad \lambda\nu = \xi(\lambda).$$

We recall that a unit vector field is said to be *a harmonic vector field* if it satisfies the critical point condition for the energy functional $E(V) = (1/2)\int_M \|dV\|^2 = (m/2)\operatorname{vol}(M) + (1/2)\int_M \|\nabla V\|^2 v_g$ restricted to the space of all unit vector fields, where $m = \dim M$. We refer to the recent monograph [3] for more information about harmonic vector fields. In [16] we study the harmonicity of the Reeb vector field for locally conformal almost cosymplectic manifolds. In particular, we have the following (which is also implicit in Goldberg and Yano's work [7]).

Proposition 4.2. *Let (M, ϕ, ξ, η, g) be an almost cosymplectic three-manifold. Then, ξ is a harmonic vector field if and only if it is an eigenvector of the Ricci operator.*

Then, Theorem 3.1 and Proposition 4.2 give the following

Theorem 4.2. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic three-manifold. Then, ξ is a minimal unit vector field if and only if ξ is a harmonic unit vector field.*

In [8], the authors introduced the notion of *strongly normal unit vector field*. A unit vector field V on a Riemannian manifold is called strongly normal if

$$g((\nabla_X \nabla V)Y, Z) = 0 \quad \text{for any } X, Y, Z \perp V.$$

Most of the results obtained in [8] are based on this notion because a strongly normal unit vector field is minimal. Now, we show the following

Theorem 4.3. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic $(2n+1)$-manifold. If M is a (κ, μ, ν)-space, then ξ is strongly normal and hence minimal, with $X(\operatorname{tr} h^2) = 0$ for any $X \in \ker \eta$.*

Proof. Let $\mathcal{U}_1$ be the open subset of M where $h \neq 0$ and $\mathcal{U}_2$ the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. Then, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open dense subset of M. If $\mathcal{U}_2$ is not empty, then the restriction of the almost contact metric structure to $\mathcal{U}_2$ is cosymplectic and in this case $\nabla\xi = 0$ and $h = 0$. So, on $\mathcal{U}_2$, ξ is strongly normal and $h = 0$. Next, let $\mathcal{U}_1$ be non-empty. On $\mathcal{U}_1$, from (4.2) we get

$$\ell X = R(X, \xi)\xi = \kappa(\eta(X)\xi - X) - \mu hX - \nu\phi hX,$$

$$\ell\phi X = R(\phi X, \xi)\xi = -\kappa\phi X - \mu h\phi X - \nu hX,$$

and hence

$$\phi\ell X + \ell\phi X = -2\kappa\phi X.$$

Moreover, from (3.2) of [15] we have $\phi\ell X + \ell\phi X = 2h^2\phi X$. Then,

$$h^2 = \kappa\phi^2, \quad \text{where } \kappa < 0. \tag{4.4}$$

For an arbitrary almost cosymplectic manifold, the following curvature identity is well known [10]

$$R(X, Y, \phi Z, \xi) - R(\phi X, \phi Y, \phi Z, \xi) - R(\phi X, Y, Z, \xi) - R(X, \phi Y, Z, \xi)$$
$$= -2(\nabla_{\phi hZ}\Phi)(X, Y)$$

On the other hand, $R(X, Y)\xi = 0$ for any $X, Y \in \ker \eta$ and hence

$$(\nabla_{\phi hZ}\Phi)(X, Y) = 0 \quad \text{for any } X, Y \in \ker \eta.$$

Replacing Z by ϕhZ in this formula, and taking into account of (4.4), we get

$$(\nabla_Z\Phi)(X, Y) = 0 \quad \text{for any } X, Y \in \ker \eta,$$

that is

$$g((\nabla_Z\phi)Y, X) = 0 \quad \text{for any } X, Y \in \ker \eta,$$

which is equivalent to

$$(\nabla_Z \phi) Y = g((\nabla_Z \phi) Y, \xi)\xi = -g((\nabla_Z \phi)\xi, Y)\xi,$$

that is

$$(\nabla_Z \phi) Y = g(hZ, Y)\xi, \tag{4.5}$$

for Z arbitrary and $Y \in \ker \eta$. From (4.4) and (4.5), we have

$$(\nabla_Z h) h Y = \kappa g(hZ, \phi Y)\xi. \tag{4.6}$$

Since $\kappa < 0$ on $\mathcal{U}_1$, from (4.5) and (4.6) we get that $(\nabla_X h) Y$ and $(\nabla_X \phi) Y$ are proportional to ξ for any $X, Y \in \ker \eta$. Then, since $\nabla \xi = h\phi$, we get

$$(\nabla_X \nabla \xi) Y = (\nabla_X h)\phi Y + h(\nabla_X \phi) Y \quad \text{for any } X, Y \in \ker \eta$$

which shows that ξ is strongly normal (and hence minimal) on $\mathcal{U}_1$. Since ξ is strongly normal on $\mathcal{U}_1 \cup \mathcal{U}_2$, we get that $g((\nabla_X (\nabla \xi) Y, Z) = 0$ for any $X, Y, Z \in \ker \eta$. Therefore ξ is strongly normal on M. Now, let E be a unit eigenvector of h: $hE = \lambda E$ and $h\phi E = -\lambda \phi E$, $\lambda = \sqrt{-\kappa}$. Since $(\nabla_E \nabla \xi) E = (\nabla_E h)\phi E + h(\nabla_E \phi) E$ is proportional to ξ, we get $E(\lambda) = 0$. Similarly we find $(\phi E)(\lambda) = 0$, and so $X(\operatorname{tr} h^2) = 0$ for any $X \in \ker \eta$. □

In dimension three, we get

Proposition 4.3. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic three-manifold. Then, the following statements are equivalent.*

a) *ξ is a strongly normal unit vector field*;

b) *ξ is minimal and $X(\operatorname{tr} h^2) = 0$ for any $X \in \ker \eta$*;

c) *M is an almost cosymplectic (κ, μ, ν)-space on an open dense subset of M.*

Proof. a) $\Rightarrow$ b). If ξ is strongly normal, from [8] we have that ξ is minimal. Moreover, if $\{\xi, e_1, e_2 = \phi e_1\}$ is a local orthonormal ϕ-basis of eigenvector of h, using Lemma 2.1 we find

$$\begin{cases} (\nabla_{e_1} \nabla \xi) e_1 = -\lambda^2 \xi + e_2(\lambda) e_1 - e_1(\lambda) e_2, \\ (\nabla_{e_1} \nabla \xi) e_2 = (\nabla_{e_2} \nabla \xi) e_1 = -e_1(\lambda) e_1 - e_2(\lambda) e_2, \\ (\nabla_{e_2} \nabla \xi) e_2 = -(\nabla_{e_1} \nabla \xi) e_1 - 2\lambda^2 \xi, \end{cases} \tag{4.7}$$

and so ξ strongly normal implies $e_1(\lambda) = e_2(\lambda) = 0$, that is, $X(\operatorname{tr} h^2) = 0$ for any $X \in \ker \eta$. b) $\Rightarrow$ c). Follows from Theorem 4.1. c) $\Rightarrow$ a). If M is an almost cosymplectic (κ, μ, ν)-space on an open dense subset $\mathcal{U}$ of M, then ξ is strongly normal on the open dense subset $\mathcal{U}$, that is $g((\nabla_X \nabla \xi) Y, Z) = 0$ for any $X, Y, Z \in \ker \eta$ on $\mathcal{U}$ and hence on M. □

Using the invariant $\bar{p} := \|\nabla_\xi h\| - \sqrt{2}\|h\|^2$, we get the following

Theorem 4.4. *Let (M, ξ, ϕ, η, g) be an almost cosymplectic three-manifold. Then, the following statements are equivalent.*

a) *ξ is a strongly normal unit vector field with $\|\nabla_\xi h\|$ and $\|h\|$ constant along the integral curves of ξ;*
b) *ξ is minimal with $\|\nabla_\xi h\|$ and $\|h\|$ constant;*
c) *M is cosymplectic or is locally isometric to a simply connected unimodular Lie group $\tilde{G}$ equipped with a left invariant almost cosymplectic structure. More precisely:*
   - *if $\bar{p} > 0$, $\tilde{G}$ is the group $\tilde{E}(2)$, universal covering of the group of rigid motions of the Euclidean 2-space;*
   - *if $\bar{p} = 0$, $\tilde{G}$ is the Heisenberg group H^3;*
   - *if $\bar{p} < 0$, $\tilde{G}$ is the group $E(1,1)$ of the rigid motions of the Minkowski 2-space.*

Proof. From Proposition 4.3, we get that ξ is a strongly normal unit vector field with $\xi(\|\nabla_\xi h\|) = \xi(\|h\|) = 0$ if and only if ξ is minimal with $\|h\|$ constant and $\xi(\|\nabla_\xi h\|) = 0$. Now, we show that $\|\nabla_\xi h\|$ is constant. We use notations of Lemma 2.1. If $\mathscr{U}_2$ is not empty, then the restriction of the almost contact metric structure to $\mathscr{U}_2$ is cosymplectic and in this case $\|\nabla_\xi h\| = \|h\| = const. = 0$. Next, let U_1 be non-empty and let (ξ, e_1, e_2) be a local ϕ-basis on $\mathscr{U}_1$ as in Lemma 2.1. In this case $\|\nabla_\xi h\|^2 = 8\lambda^2 a^2$. Since ξ is minimal and λ is constant, from (4.1), using (2.6), we get

$$\begin{cases} Q\xi = -2\lambda^2\xi, \\ Qe_1 = \left(\dfrac{r}{2} + \lambda^2 - 2a\lambda\right)e_1, \\ Qe_2 = \left(\dfrac{r}{2} + \lambda^2 + 2a\lambda\right)e_2, \end{cases}$$

from which we easily get

$$\begin{cases} (\nabla_\xi Q)\xi = 0, \\ (\nabla_{e_1} Q)e_1 = \left(e_1\left(\dfrac{r}{2}\right) - 2\lambda e_1(a)\right)e_1, \\ (\nabla_{e_2} Q)e_2 = \left(e_2\left(\dfrac{r}{2}\right) + 2\lambda e_2(a)\right)e_2. \end{cases}$$

Then, using the formula

$$\frac{1}{2}X(r) = \sum_i g((\nabla_{E_i} Q)E_i, X)$$

where $\{E_i\}$ is an local orthonormal basis, we get

$$\begin{cases} e_1\left(\dfrac{r}{2}\right) = e_1\left(\dfrac{r}{2}\right) - 2\lambda e_1(a), \\ e_2\left(\dfrac{r}{2}\right) = e_2\left(\dfrac{r}{2}\right) + 2\lambda e_2(a). \end{cases}$$

So, $e_1(a) = e_2(a) = 0$ and hence, since $\xi(\|\nabla_\xi h\|) = 0$ gives $\xi(a) = 0$, we obtain that a is locally constant on $\mathcal{U}_1$. Since λ is continuous, it follows that $M = \mathcal{U}_1$ and hence λ and a are globally constant. Now, we show b) $\Rightarrow$ c). If M is not cosymplectic, as before we get that λ and a are globally constant on M, and Lemma 2.1 gives

$$[\xi, e_1] = c_2 e_2, \quad [\xi, e_2] = c_1 e_1 \quad \text{and} \quad [e_1, e_2] = 0,$$

where $c_1 = \lambda - a$ and $c_2 = \lambda + a$ are constant. From this we obtain that M is locally isometric to a unimodular Lie group with a left-invariant almost cosymplectic structure (see [[17], p. 10] and Theorem 4.1 of [15]). In [15] (see Theorem 4.1) we classify the simply connected homogeneous almost cosymplectic three-manifolds using, in the unimodular case, the sign of the invariant $p = \|\mathcal{L}_\xi h\| - 2\|h\|^2$. On the other hand, by Lemma 2.1, we find

$$\|\mathcal{L}_\xi h\|^2 - 4\|h\|^4 = \|\nabla_\xi h\|^2 - 2\|h\|^4.$$

Then, we can replace the invariant p by the invariant $\bar{p} := \|\nabla_\xi h\| - \sqrt{2}\|h\|^2$, and the classification of c) follows from Theorem 4.1 of [15]. Of course, if M is cosymplectic or a Lie group listed in c), Theorem 4.1 of [15] gives that ξ is an eigenvector of the Ricci operator, and so it is minimal, with $\|\nabla_\xi h\|$ and $\|h\|$ constant. □

Corollary 4.1. *Let M be an almost cosymplecyic three-manifold with ξ minimal. If M has constant vertical sectional curvature, then it is cosymplectic or is locally isometric to the Lie group $E(1,1)$ equipped with a left invariant almost cosymplectic structure of negative vertical sectional curvature.*

Proof. We consider the notations of Lemma 2.1. If $\mathcal{U}_1$ is empty, the structure is cosymplectic and in this case the vertical sectional curvature vanishes. Now, we suppose that the open set $\mathcal{U}_1$ is not empty. Since the vertical sectional curvature is constant, and the 1-form $\sigma = 0$, from (2.5) we have

$$-\lambda^2 - 2a\lambda = K(\xi, e_1) = \text{const.} = K(\xi, e_2) = -\lambda^2 + 2a\lambda$$

from which we get $a = 0$ and $\lambda = \text{const.}$ on $\mathcal{U}_1$. Since λ is continuos, it follows that $M = \mathcal{U}_1$ and thus a and λ are globally constant. In particular, the functions $\|\nabla_\xi h\|$ and $\|h\|$ are constant and the invariant $\bar{p} := \|\nabla_\xi h\| - \sqrt{2}\|h\|^2 = -\sqrt{2}\|h\|^2 < 0$. Then, Theorem 4.4 gives that M is locally isometric to the Lie group $E(1,1)$, of the rigid motions of the Minkowski 2-space, equipped with a left invariant almost cosymplectic structure. In such case, for any unit vector field $X \in \ker \eta$, the vertical sectional curvature $K(\xi, X) = const. = -\lambda^2 < 0$. Indeed, if $X = a_1 e_1 + a_2 e_2$, from (2.5) one gets

$$\begin{aligned} K(\xi, X) &= a_1^2 K(\xi, e_1) + a_2^2 K(\xi, e_2) - 2a_1 a_2 g(\ell e_1, e_2) \\ &= -a_1^2 g(\ell e_1, e_1) - a_2^2 g(\ell e_2, e_2) = -\lambda^2 < 0. \end{aligned}$$

□

Remark 4.2. The Lie groups listed in c) of Theorem 4.4 are examples of (κ, μ, ν)-spaces with κ, μ constant and $\nu = 0$. Moreover, since by Theorem 4.2 the minimality condition of ξ is equivalent (for almost cosymplectic three-manifolds) to the harmonicity condition, Theorem 4.4 and Corollary 4.1 give a partial answer to a question posed in [16].

The following is an example of *non-homogeneous almost cosymplectic three-manifold whose Reeb vector field is minimal.*

Example 4.1. Let $M = \mathbf{R}^3$ with the cartesian coordinates (x, y, z). We consider the Riemannian metric

$$g = d^2x + d^2y - 2y(f_1(z)/f_3(z))\, dxdz - 2x(f_2(z)/f_3(z))\, dydz + \bar{f}(z)\, d^2z, \tag{4.8}$$

and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = yf_1(z)\frac{\partial}{\partial x} + xf_2(z)\frac{\partial}{\partial y} + f_3(z)\frac{\partial}{\partial z},$$

where $f_1(z)$, $f_2(z)$, $f_3(z)$ are arbitray smooth functions of the variable z, with $f_3(z) \neq 0$ for any $z \in \mathbf{R}$, and $\bar{f}(z) = ((y^2 f_1^2(z) + x^2 f_2^2(z) + 1)/f_3^2(z))$. We get that the vector fields e_1, e_2, e_3 are orthonormal with respect to the metric g in each point, and satisfy

$$[e_1, e_2] = 0, \quad [e_1, e_3] = f_2(z)e_2, \quad [e_2, e_3] = f_1(z)e_1. \tag{4.9}$$

We define the vector field ξ, the 1-form η and the tensor ϕ of type $(1,1)$ by

$$\xi = e_3, \quad \eta = g(\xi, \cdot), \quad \phi e_3 = 0, \quad \phi e_1 = e_2, \quad \phi e_2 = -e_1.$$

Then, (g, ξ, η, ϕ) is an almost contac metric structure on M. Moreover, we easily get that the 1-form η and the 2-form $\Phi(X, Y) = g(X, \phi Y)$ are closed. So, the structure is almost cosymplectic. Using (4.8), (4.9) and the Levi-Civita equation, we find

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & -\frac{f_1 + f_2}{2} e_3 & \frac{f_1 + f_2}{2} e_2 \\ -\frac{f_1 + f_2}{2} e_3 & 0 & \frac{f_1 + f_2}{2} e_1 \\ \frac{f_1 - f_2}{2} e_2 & \frac{f_2 - f_1}{2} e_1 & 0 \end{pmatrix}. \tag{4.10}$$

Then, using (4.10), by a direct calculation we find

$$Ric(\xi, \xi) = -(f_1 + f_2)^2/2, \quad Ric(\xi, e_1) = Ric(\xi, e_2) = 0. \tag{4.11}$$

From (4.11) and Theorem 3.1, we get that ξ is a minimal unit vector field. From (4.10) we have that $\operatorname{tr} h^2 = (f_1 + f_2)^2/2$ is not constant, and so the structure is not homogeneous. Moreover, $e_1(\operatorname{tr} h^2) = e_2(\operatorname{tr} h^2) = 0$ and thus, by Proposition 4.3, ξ is strongly normal. Moreover, the three-manifold is a (κ, μ, ν)-space where κ, μ, ν are not constant.

Remark 4.3. Let (M, g, η, ϕ, ξ) be an almost cosymplectic three-manifold. In [15] (see Theorem 4.2) we proved that $\xi : (M, g) \to (T^1M, g_S)$ is a harmonic map if and only if ξ is a harmonic vector field and $\xi(\mathrm{tr}\, h^2) = 0$. Then, Theorem 4.2 gives that $\xi : (M, g) \to (T^1M, g_S)$ is a harmonic map if and only if $\xi : (M, \xi^* g_S) \to (T^1M, g_S)$ is a minimal immersion and $\xi(\mathrm{tr}\, h^2) = 0$. So, in all the examples listed in Theorem 4.4 the Reeb vector field ξ determines a minimal immersion and a harmonic map. Recall that, in general, an isometric immersion $f : (M_1, g_1) \to (M_2, g_2)$ is minimal if and only if it is a harmonic map. Moreover, a unit vector field V determines an isometric immersion $V : (M, g) \to (T^1M, g_S)$, that is $V^* g_S = g$, if and only if $\nabla V = 0$ (see, for example, [3]). Therefore, only in the cosymplectic case the Reeb vector field of an almost cosymplectic three-manifold determines an isometric immersion.

Remark 4.4. A submanifold N of a contact metric manifold $(\tilde{M}, \tilde{g}, \tilde{\eta}, \tilde{\phi}, \tilde{\xi})$ is said to be an *invariant submanifold* if $\tilde{\phi}(T_pN) \subset T_pN$ for every $p \in N$. The invariance implies that $\tilde{\xi}$ is tangent to N at each of its points, and an invariant submanifold inherits a contact metric structure from the ambient manifold. Moreover, we have that *an invariant submanifold of a contact metric manifold is minimal* ([1], p. 122). Now, let (M, g, η, ϕ, ξ) be an almost cosymplectic manifold and let $(\tilde{g}, \tilde{\eta}, \tilde{\phi}, \tilde{\xi})$ the standard contact metric structure on the unit tangent sphere bundle T^1M, where $\tilde{\xi}_{(p,u)} = 2u^H$ is the geodesic flow and $\tilde{g} = (1/4)g_S$. If $\xi(M)$ is an invariant submanifold, then ξ is minimal. However, from Theorem 4.1 of [14] we get that $\xi(M)$ is an invariant submanifold of $(T^1M, \tilde{g}, \tilde{\eta}, \tilde{\phi}, \tilde{\xi})$ if and only if $(\nabla\xi)^2 = -I$ on $\ker \eta$. Since $(\nabla\xi)^2 = (h\phi)^2 = h^2$ on $\ker \eta$, we conclude that $\xi(M)$ can not be an invariant submanifold of $(T^1M, \tilde{g}, \tilde{\eta}, \tilde{\phi}, \tilde{\xi})$. This remark corrects the result of Theorem 4.2 in [12].

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