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CESAR CAMACHO
ALCIDES LINS NETO
PAULO SAD

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MINIMAL SETS OF FOLIATIONS ON COMPLEX PROJECTIVE SPACES

by C. CAMACHO, A. LINS NETO *and* P. SAD

Dedicated to René Thom
with admiration.

This paper is concerned with the study of global properties of codimension one holomorphic foliations on projective spaces. The exposition deals with the case of dimension two for reasons of simplicity; some of the results extend naturally to higher dimensions.

So let us consider a polynomial differential equation in $\mathbf{C}^2$

$$P(x,y)\,dy - Q(x,y)\,dx = 0, \tag{1}$$

where P and Q are relatively prime polynomials. Such an equation defines a foliation by complex curves, with a finite number of singularities, which extends itself to the projective space $\mathbf{C}P(2)$; this is done as follows: a point of $\mathbf{C}P(2)$ has three different affine coordinates (x,y), (u,v) and (z,w) related by

$$u = 1/x, \qquad v = y/x \quad \text{and} \quad z = 1/y, \qquad w = x/y. \tag{2}$$

Therefore (1) becomes in the coordinates $(u,v) \in \mathbf{C}^2$

$$\widetilde{P}(u,v)\,dv - \widetilde{Q}(u,v)\,du = 0, \tag{3}$$

where $\widetilde{P}(u,v) = u^{k+1}P(1/u, v/u)$ and $\widetilde{Q}(u,v) = u^k[vP(1/u, v/u) - Q(1/u, v/u)]$ where $k \in \mathbf{N}$ is the smallest integer necessary to grant a polynomial equation in (3). In the same way the equation (1) may be transformed to the coordinates $(z,w) \in \mathbf{C}^2$. Thus a foliation $\mathscr{F}$, with a finite set sing $(\mathscr{F})$ of singularities, is defined in $\mathbf{C}P(2)$; we are interested in the limit sets of the leaves of $\mathscr{F}$. A basic question refers to the existence of singularities of the foliation in all these subsets. Although we do not provide an answer to such a question, we present some results that may play a role in the understanding of the problem. Let us proceed to state them.

A *minimal set* of $\mathscr{F}$ is a nonempty closed invariant subset of $\mathbf{C}P(2)$ which contains no proper subset with these properties; if it is not a singularity, we say that the minimal set is nontrivial.

We start Section 1 by proving that *$\mathscr{F}$ has at most one nontrivial minimal set; also, under generic conditions imposed on the singularities, all leaves accumulate on that set.* The proof relies upon the application of the Maximum Principle for harmonic functions in order to analyse the distance between leaves. Although the question about the existence of a nontrivial minimal set is still unsettled, it is interesting to remark two facts: first, *among the foliations possessing only generic singularities, the subset of those without minimal leaves is nonempty and open in a suitable topology* (to be discussed later). Also, in the special situation when a one-dimensional invariant algebraic subvariety exists, a result of [3] implies that only this algebraic subvariety can possibly be the minimal set (a simpler proof is provided here), but this possibility is ruled out as a consequence of the appendix of [2]. In fact, if we try to push further the ideas of this appendix, we naturally realize that *the leaves of a nontrivial minimal set in projective spaces have exponential growth*; this is discussed in Section 2. More generally, it is shown that *a foliation with a nontrivial minimal set does not admit an invariant transverse measure with support in that set.* At this point, it is worth regarding the situation from the standpoint of real codimension one C^∞-foliations. It is known that an exceptional minimal set has a leaf of exponential growth (see [6]); the basic reason is the presence of a hyperbolic contraction in the holonomy group of some leaf of that set. In our case the real codimension is two and we have no information about holonomy groups; what allows us to prove the growth property is the very special geometry of projective spaces.

Now we are motivated to look for more geometrical information. Starting with the construction of a Hermitian metric which has strict negative curvature along leaves not accumulating on singularities, we immediately reach upon the fact that *all leaves in the minimal set are hyperbolic Riemann surfaces* (see Section 3). Even a stronger statement is true: *the family of uniformizations of leaves in the minimal set is a compact family*, a result derived after comparing complete conformal Riemannian metrics in the unit disc of $\mathbf{C}$ to the Poincaré metric. We close Section 4 with a byproduct of both geometrical and dynamical analysis, namely: *the leaves in the minimal set have no parabolic ends.* Let us mention here that the last two theorems we have stated above are related to the results obtained in [8] by A. Verjovski in a different context.

We acknowledge interesting conversations with I. Kupka.

1. Uniqueness of minimal sets

We start this section by proving a geometrical property of minimal sets.

Proposition **1.** — *Let $\mathscr{M}$ be a nontrivial minimal set of the foliation $\mathscr{F}$. Then $\mathscr{M}$ intersects every* 1*-dimensional algebraic subset of* $\mathbf{C}P(2)$.

Proof. — Let $S \subset \mathbf{CP}(2)$ be an algebraic set of dimension 1. Suppose that S has degree k and is given in the affine coordinate system $(x, y) = (x : y : 1)$ by $f(x, y) = 0$, where f is a polynomial of degree k. For $a, b, c > 0$, consider the function

$$\varphi_{a,b,c}(x, y) = \varphi(x, y) = \frac{|f(x,y)|^2}{(a + b\,|x|^2 + c\,|y|^2)^k}.$$

Since f has degree k, φ can be extended as a real analytic function in $\mathbf{CP}(2)$. For instance, in the coordinate system $(1 : v : u)$, $u = 1/x$, $v = y/x$, the expression of φ is

$$\varphi(u, v) = \frac{|\tilde{f}(u, v)|^2}{(a\,|u|^2 + b + c\,|v|^2)^k}, \qquad \tilde{f}(u, v) = u^k f(1/u, v/u).$$

From the above, it can be easily proved that $S = \varphi^{-1}(0)$. Suppose by way of contradiction that $\mathcal{M} \cap S = \emptyset$. In this case $\varphi \mid \mathcal{M}$ has a positive minimum, say $\varphi \mid \mathcal{M} \geqslant \varphi(p_0) = \alpha > 0$, where $p_0 \in \mathcal{M}$. Let $\psi : \mathcal{M} \to \mathbf{R}$ be defined by $\psi(p) = \log(\varphi(p))$, $p \in \mathcal{M}$. Clearly $\psi \geqslant \psi(p_0) = \log \alpha > -\infty$. On the other hand ψ is superharmonic along the leaves of $\mathcal{F}$ contained in $\mathcal{M}$. In fact, if $z \mapsto p(z) = (x(z), y(z))$ is a local holomorphic parametrization of a leaf $L \subset \mathcal{M}$, then $\psi(p(z)) > -\infty$ and

$$\begin{aligned}
\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\psi(p(z)) &= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log|f(p(z))| \\
&\qquad - k\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log(a + b\,|x(z)|^2 + c\,|y(z)|^2) \\
&= -k(a + b\,|x(z)|^2 + c\,|y(z)|^2)^{-2} \\
&\qquad .\,(ab\,|x'(z)|^2 + ac\,|y'(z)|^2 + bc\,|x(z)\,y'(z) - y(z)\,x'(z)|^2) \\
&\leqslant 0.
\end{aligned}$$

In particular, this implies that ψ, and also φ, are constant along the leaf L of $\mathcal{F}$ through p_0. Since this leaf is dense in $\mathcal{M}$, it follows that φ is constant in $\mathcal{M}$. This fact is independent of a, b, c, so that, for any triple $(a, b, c) \in (0, +\infty)^3$ there exists $\alpha > 0$ such that $|f(p)|^{2/k} = \alpha(a + b\,|x|^2 + c\,|y|^2)$ for every $p = (x, y) \in \mathcal{M}$. It is not difficult to see that this is impossible. Hence $\mathcal{M} \cap S = \emptyset$. ∎

As a consequence, if $A \subset \mathbf{CP}(2)$ is a 1-dimensional invariant algebraic subvariety, then $\mathcal{M} = A$, which is impossible as we will see in Section 2.

Let us now proceed to prove the main result of this Section; another proof of it was shown to us by H. Rosenberg.

Theorem **1**. — *The foliation $\mathcal{F}$ has at most one nontrivial minimal set.*

Proof. — Let $\mathcal{M}$ be a nontrivial minimal set of $\mathcal{F}$, and L a (nonsingular) leaf of $\mathcal{F}$, such that $\bar{L} \cap \operatorname{sing}(\mathcal{F}) = \emptyset$. It is enough to prove that $\bar{L} \cap \mathcal{M} \neq \emptyset$. Suppose on the contrary that $\bar{L} \cap \mathcal{M} = \emptyset$. Let $(x, y) = (x : y : 1)$ be an affine coordinate system on $\mathbf{CP}(2)$. Consider in $\mathbf{C}^2 = \{(x : y : 1); x, y \in \mathbf{C}\} \subset \mathbf{CP}(2)$ the euclidean metric

$$d_e((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Let $\mathcal{M}' = \mathcal{M} \cap \mathbf{C}^2$ and $F = \bar{L} \cap \mathbf{C}^2$. Then $\mathcal{M}'$ and F are closed subsets of $\mathbf{C}^2$. Set $d = d_e(F, \mathcal{M}') = \inf\{d_e(p, q); p \in F, q \in \mathcal{M}'\}$. We assert that $d > 0$ and that there are points $p_0 \in F$ and $q_0 \in \mathcal{M}'$ such that $d = d_e(p_0, q_0)$.

In fact, let ρ be the Fubini-Study metric in $\mathbf{CP}(2)$, induced by the Hermitian metric whose quadratic form is

$$ds^2 = \frac{|dx|^2 + |dy|^2 + |x\,dy - y\,dx|^2}{(1 + |x|^2 + |y|^2)^2}.$$

It is not difficult to check that, given $p_0 \neq p_1 \in \mathbf{C}^2$, then

$$\rho(p_0, p_1) \leqslant \frac{d_e(p_0, p_1)}{1 + (\delta(p_0, p_1))^2} \leqslant d_e(p_0, p_1), \tag{4}$$

where $\delta(p_0, p_1)$ is the euclidean distance between the origin and the line segment which joins p_0 and p_1. Now let $(p_n)_{n \geqslant 1}$ and $(q_n)_{n \geqslant 1}$ be sequences in F and $\mathcal{M}'$ such that $\lim_{n \to \infty} d_e(p_n, q_n) = d$. Since $\mathcal{M}$ and $\bar{L}$ are compact, we can suppose that $p_n \to p_0 \in \bar{L}$ and $q_n \to q_0 \in \mathcal{M}$. If $p_0, q_0 \in \mathbf{C}^2$, we have $d = d_e(p_0, q_0) > 0$, and we are done. Suppose by way of contradiction that this is not the case. Then, $p_0, q_0 \in L_\infty = \mathbf{CP}(2) - \mathbf{C}^2$, because $(d_e(p_n, q_n))_{n \geqslant 1}$ is bounded. This implies also that $\lim_{n \to \infty} \delta(p_n, q_n) = +\infty$ and therefore $\rho(p_0, q_0) = 0$, and $p_0 = q_0$, a contradiction. This proves the assertion.

After a rotation and a translation in $\mathbf{C}^2$ we can suppose that $p_0 = (0, 0)$ and $q_0 = (0, y_0)$, $|y_0| = d$. Suppose first that p_0 is not a singularity of $\mathcal{F}$. Let L_1 and L_2 be the leaves of $\mathcal{F}$ through p_0 and q_0 respectively. Since $d = d_e(\mathcal{M}', F) = d_e(p_0, q_0)$, the y-axis is normal to L_1 at p_0 and to L_2 at q_0. This implies that we can parametrize L_1 and L_2 locally as $y = y_1(x)$ and $y = y_2(x)$ respectively, where y_1 and y_2 are analytic in a disc $D = \{x; |x| < \varepsilon\}$. Let $\varphi : D \to \mathbf{R}$ be defined by $\varphi(x) = \log|y_2(x) - y_1(x)|$. Since $d = d_e(\mathcal{M}', F) > 0$, we have $|y_2(x) - y_1(x)| \geqslant d$ for all $x \in D$ and $|y_2(0) - y_1(0)| = d$. This implies that φ is a harmonic function in D with a minimum at $x = 0$. Therefore $\varphi \equiv d$ in D, and $y_2(x) = y_1(x) + y_0$ for all $x \in D$. By analytic continuation, it follows that $L_2 \cap \mathbf{C}^2 = T(L_1 \cap \mathbf{C}^2)$, where $T(x, y) = (x, y + y_0)$. From Proposition 1, it follows that there exists a sequence $q_n = (x_n, y_n) \in L_2$ such that $x_n \to \infty$. The sequence $p_n = (x_n, y_n - y_0)$ is in L_1 and $d_e(p_n, q_n) = |y_0|$. It follows from (4) that $\rho(p_n, q_n) \to 0$, and so $\bar{L}_1 \cap \mathcal{M} \neq \emptyset$. Since $L_1 \subset \bar{L}$, we also have $\bar{L} \cap \mathcal{M} \neq \emptyset$, a contradiction. ∎

There is a case where we still have $\bar{L} \cap \mathcal{M} \neq \emptyset$ even if $\bar{L} \cap \operatorname{sing}(\mathcal{F}) \neq \emptyset$, namely when the following condition holds: (*) *whenever a singularity of $\mathcal{F}$ belongs to $\bar{L}$, at least one of its local separatrices is contained in $\bar{L}$ as well (a local separatrix of $p_0 \in \operatorname{sing}(\mathcal{F})$ is an analytic immersion $s : D^* \to \mathbf{CP}(2)$ such that $s(D^*)$ is contained in a leaf of $\mathcal{F}$ and $\lim_{t \to 0} s(t) = p_0$).*

In order to prove this, let us keep the notation of the above proof, but this time assuming that $p_0 \in F \cap \operatorname{sing}(\mathcal{F})$.

Observe that the y-axis is normal to L_2 and q_0, and so L_2 can be parametrized locally as $y = y_2(x)$, where $y_2(0) = y_0, y_2'(0) = 0$. From the hypothesis, $\bar{L}$ contains some local analytic separatrix of p_0, say S. This separatrix has a Puiseux's parametrization of the form $\beta(T) = (T^n, y_1(T))$, $|T| < \varepsilon$, where $y_1(T) = T^m \alpha(T)$, $\alpha(0) \neq 0$, $m, n \geqslant 1$. Consider the harmonic function

$$\varphi(T) = \log(d_e((T^n, y_1(T)), (T^n, y_2(T^n)))) = \log |y_2(T^n) - y_1(T)|.$$

As before, φ has a minimum at $T = 0$, and so $y_2(T^n) = y_1(T) + y_0$, $|T| < \varepsilon$. This implies that the parametrization of S can be written as $T \mapsto (T^n, y_2(T^n) - y_0)$. Since the Puiseux parametrization is injective, it follows that $n = 1$ and $\beta(T) = (T, y_2(T) - y_0)$. From this and analytic continuation, it is possible to construct sequences $p_n = (x_n, y_n) \in L_1$ and $q_n = (x_n, y_n + y_0) \in L_2$, where $\lim_{n\to\infty} x_n = \infty$ and L_1 is the leaf of $\mathscr{F}$ which contains S. As before, this implies that $\bar{L} \cap \mathscr{M} \neq \emptyset$. ∎

Condition (*) is satisfied, for example, when any singularity of $\mathscr{F}$ has, as linearized equation (in suitable coordinates), the 1-form

$$\lambda x\, dy - \mu y\, dx = 0$$

where $\lambda . \mu \neq 0$ and $\lambda/\mu \notin \mathbf{R}$. In this case we say that the singularity belongs to the *Poincaré domain.* In fact, in this situation (which is frequent as we will see in a moment) we have a slight improvement.

Proposition **2.** — *Suppose all singularities of $\mathscr{F}$ belong to the Poincaré domain. Let* L_1 *and* L_2 *be non singular leaves of $\mathscr{F}$; then* $\bar{L}_1 \cap \bar{L}_2 \neq \emptyset$.

In particular, if there is a minimal set $\mathscr{M}$ for $\mathscr{F}$ (which is unique), then any non-singular leaf accumulates on it.

Before closing this Section, we discuss briefly how to introduce a topology in the space $\mathscr{X}$ of foliations of $\mathbf{CP}(2)$ (see [5]). In this topology, $\mathscr{X}$ is the disjoint union of connected components $\mathscr{X}_n$ containing the foliations of *degree* $n \in \mathbf{N}$. The *degree* of $\mathscr{F} \in \mathscr{X}$ is the number of tangencies of $\mathscr{F}$ with a generic complex line of $\mathbf{CP}(2)$. In affine coordinates, $\mathscr{F} \in \mathscr{X}_n$ is written as

$$P\, dy - Q\, dx = 0$$

where $P = P_0 + P_1 + \dots + P_n + xg$, $Q = Q_0 + Q_1 + \dots + Q_n + yg$; P_j, Q_j, $0 \leqslant j \leqslant n$, are homogeneous polynomials of degree $j \in \mathbf{N}$, and g is a homogeneous polynomial of degree n. A neighborhood of $\mathscr{F}$ in $\mathscr{X}_n$ contains the foliations, whose equations involve homogeneous polynomials with coefficients close to the ones of $\mathscr{F}$.

It is not difficult to check that the *subset* $\mathscr{P}_n \subset \mathscr{X}_n$ *of foliations, all of whose singularities belong to the Poincaré domain, is an open dense subset of* $\mathscr{X}_n$. The local topology of $\mathscr{F} \in \mathscr{P}_n$ at $p \in \operatorname{sing}(\mathscr{F})$ is such that the leaves cross transversely any small sphere placed around $p \in \mathbf{CP}(2)$.

Proposition **3.** — *The subset of foliations of* $\mathscr{P}_n$ *without nontrivial minimal sets is open and nonempty.*

2. Growth of the leaves of minimal sets

Let us now introduce the Fubini-Study metric in $\mathbf{C}P(2)$, namely

$$ds^2 = \frac{|\,dx\,|^2 + |\,dy\,|^2 + |\,x\,dy - y\,dx\,|^2}{(1 + |\,x\,|^2 + |\,y\,|^2)^2}.$$

What makes it particularly interesting is that the associated 2-form

$$\Omega = i\,\frac{dx \wedge d\bar{x} + dy \wedge d\bar{y} + (\bar{x}\,dy - \bar{y}\,dx) \wedge (x\,d\bar{y} - y\,d\bar{x})}{(1 + |\,x\,|^2 + |\,y\,|^2)^2}$$

is given by the formula

$$\Omega = i\,\bar{\partial}\,\partial \log(1 + |\,x\,|^2 + |\,y\,|^2)$$

and the area of a standard domain $D \subset F$ (F is a leaf) is

$$A(D) = \frac{1}{2\pi}\int_D \Omega\,|_F.$$

Let us define $\sigma = -\,\partial \log(1 + |\,x\,|^2 + |\,y\,|^2)$, so that $\Omega = i\,\bar{\partial}\sigma$.

Suppose there is a minimal set $\mathcal{M} \subset \mathbf{C}P(2)$ of $\mathcal{F}$ and assume that

$$(\tilde{x}, \tilde{y}) = (0, \infty) \in \mathbf{C}P(2)$$

does not belong to $\mathcal{M}$. The first projection $p(x,y) = x$ induces on $E = \mathbf{C}P(2)\backslash(\tilde{x},\tilde{y})$ the structure of a fibre bundle over $\mathbf{C}P(1) = \overline{(y = 0)}$. Without loss of generality we can assume that the fiber through $x = \infty$, $y = 0$ is transverse to $\mathcal{M}$.

Let $\eta = \dfrac{\partial}{\partial y}\left(\dfrac{P}{Q}\right) dx$ be defined outside the algebraic curve $Q(x,y) = 0$.

Lemma. — $\tau = \sigma + \eta$ *is a well defined* 1*-form on* $E\backslash\overline{\{(x,y) \in \mathbf{C}^2;\, P(x,y) = 0\}}$.

Proof. — Consider the 1-forms

$$\tilde{\sigma} = -\,\partial \log(1 + |\,u\,|^2 + |\,v\,|^2), \qquad \tilde{\eta} = \frac{\partial}{\partial v}\left(\frac{\tilde{P}}{\tilde{Q}}\right) du, \qquad \tilde{\tau} = \tilde{\sigma} + \tilde{\eta}$$

where $\tilde{P}$, $\tilde{Q}$ are as in (3) (remember that (1) changes to (3) under

$$G(x,y) = (u, v) = (1/x, y/x)).$$

Let us prove that $G^*\tilde{\tau} = \tau$. Since $\tilde{\sigma} = -\,\partial \log(1 + |\,u\,|^2 + |\,v\,|^2)$, we find that

$$G^*\tilde{\sigma} = \frac{dx}{x} + \sigma.$$

Also

$$\tilde{\eta} = \frac{\partial}{\partial v}\left[\frac{vP(1/u, v/u) - Q(1/u, v/u)}{uP(1/u, v/u)}\right] du$$
$$= \frac{\partial}{\partial v}\left[\frac{v}{u} - \frac{Q(x,y)}{uP(x,y)}\right] du = \frac{du}{u} - \frac{1}{u}\frac{\partial}{\partial v}\left(\frac{Q(x,y)}{P(x,y)}\right) du,$$

so that

$$G^*\tilde{\eta} = -\frac{dx}{x} - x^2\frac{\partial}{\partial y}\left(\frac{Q}{P}\right)\left(-\frac{dx}{x^2}\right)$$

$$= -\frac{dx}{x} + \frac{\partial}{\partial y}\left(\frac{Q}{P}\right)dx = -\frac{dx}{x} + \eta.$$

Therefore $G^*\tilde{\tau} = \tau$. ∎

Suppose now that $p = (x_0, y_0) \in E$ is a point where $P(x_0, y_0) = 0$ and $Q(x_0, y_0) \neq 0$. The leaf F through p can be written locally as the curve

$$x - x_0 = \sum_{j=m}^{\infty} a_j(y - y_0)^j, \qquad a_m \neq 0.$$

Definition. — *The integer* $O(F, p) = m - 1 \in \mathbf{N}$ *is the* order of tangency *of* F *with the fiber of* E *through* $p \in E$.

It is easily verified that

$$O(F, p) = -\operatorname{Res}(\eta|_F, p).$$

Consider now a standard domain (in the sense of Whitney) $D \subset F$; assume that all points $p_1, \ldots, p_k$ of tangency between D and the fibers of E lie in int D and assume that none of these points belongs to the fiber $x = \infty$.

The following result plays the same role in our context of the First Fundamental Theorem in Nevanlinna's theory for analytic curves in projective spaces.

Theorem. — $A(D) = \dfrac{i}{2\pi}\displaystyle\int_{\partial D}\tau - \Sigma_{j=1}^k O(F, p_j).$

Proof. — The formula is a consequence of Stokes' Theorem. Consider small discs $D_1, \ldots, D_k$ contained in D around the points $p_1, \ldots, p_k$; we then have

$$\int_{\partial(D\setminus\bigcup_{j=1}^k D_j)}\tau = \int_{D\setminus\bigcup_{j=1}^k D_j} d\tau$$

or

$$\int_{\hat{D}} d\tau = \int_{\partial D}\tau - \sum_{j=1}^{k}\int_{\partial D_j}\tau, \quad \text{where } \hat{D} = D\setminus\bigcup_{j=1}^{k} D_j.$$

Since $\tau = \sigma + \eta$, we get $d\tau|_{\hat{D}} = \bar{\partial}\sigma|_{\hat{D}} = \dfrac{1}{i}\Omega|_{\hat{D}}$ and $A(\hat{D}) = \dfrac{1}{2\pi}\displaystyle\int_{\hat{D}}\Omega|_F = \dfrac{i}{2\pi}\displaystyle\int_{\hat{D}} d\tau.$

From the above Definition it follows that

$$\int_{\partial D_j}\tau = \int_{\partial D_j}\sigma - 2\pi i O(F, p_j).$$

Therefore

$$A(\hat{D}) = \frac{i}{2\pi}\int_{\partial D}\tau - \sum_{j=1}^{k}\frac{1}{2\pi i}\int_{\partial D_j}\sigma - \sum_{j=1}^{k} O(F, p_j).$$

As $D_j \to p_j$ we get:

$$A(D) = \frac{i}{2\pi}\int_{\partial D} \tau - \sum_{j=1}^{k} O(F, p_j). \quad \blacksquare$$

Remark. — We can apply the above formula to show that there are no compact leaves. Indeed, if F is a compact leaf, it has to be a smooth algebraic subvariety of $\mathbf{C}P(2)$ (otherwise its singularities would also be singularities of the foliation). Now we just consider $D = F$ in the formula and get $A(F) \leqslant 0$, a contradiction.

Let us now proceed to prove the growth property for a leaf F contained in the minimal set $\mathcal{M}$. Consider the function

$$\varphi(r) = A(D_r),$$

where D_r is the disc of radius $r > 0$ centered at some fixed point $p_0 \in F$. According to [6] it is enough to show that

$$\liminf_{r\to\infty} \frac{\ell(\partial D_r)}{A(D_r)} > 0,$$

where $\ell(\partial D_r)$ is the lenght of D_r and the variable $r \in \mathbf{R}_+$ avoids a countable subset of $\mathbf{R}$.

Let $\{p_j\}_{j=1}^{\infty}$ be the set of tangencies of F with the fibers of E. Since $\operatorname{int}\mathcal{M} = \emptyset$, we may assume, without loss of generality, that

(α_1) $\{p_j\}_{j=1}^{\infty} \cap E_\infty = \emptyset$, where E_∞ is the fiber $x = \infty$,

(α_2) $O(F, p_j) = 1$, $1 \leqslant j < \infty$ (which simply means that F crosses $P(x,y) = 0$ at smooth points).

We may then choose closed discs $F \supset D_j \ni p_j$ such that

(α_3) $D_i \cap D_j = \emptyset$ if $i \neq j$,

(α_4) $D_i \cap E_\infty = \emptyset$, $1 \leqslant i < \infty$,

(α_5) each D_j projects univalently onto a disc $\hat{D}_j$ contained in the fiber of E through p_j, with radius $\frac{\delta}{2} \leqslant r_j < \delta$, where $\delta \in \mathbf{R}$ will be chosen below.

Now, take a sequence $r_n \to \infty$ such that $\partial D_{r_n} \cap (E_\infty \cup \{p_j\}_{j=1}^{\infty}) = \emptyset$, $n = 1, 2 \ldots$ From the Theorem we have:

$$A(D_{r_n}) + \sum_{D_j \subset \operatorname{int} D_{r_n}} O(F, p_j)$$
$$= \frac{i}{2\pi}\int_{\partial D_{r_n} \setminus \bigcup_j D_j} \tau + \frac{i}{2\pi}\sum_j \int_{\partial D_{r_n} \cap D_j \neq \emptyset} \tau - \sum_{D_j \cap \partial D_{r_n} \neq \emptyset} O(F, p_j).$$

Let us set

$$B_n = \sum_{j \in N_1} \left[O(f, p_j) + \frac{1}{2\pi i}\int_{\partial D_{r_n} \cap D_j} \eta\right]$$

and

$$C_n = \sum_{j \in N_2} \frac{1}{2\pi i}\int_{\partial D_{r_n} \cap D_j} \eta,$$

where $N_1 = \{j \in N; D_j \cap D_{r_n} \neq \emptyset$ and $p_j \in \operatorname{int} D_{r_n}\}$ and $N_2 = \{j \in \mathbf{N}; D_j \cap \partial D_{r_n} \neq \emptyset$ and $p_j \notin \operatorname{int} D_{r_n}\}$. It follows that

$$(*)\qquad A(D_{r_n}) + \sum_{p_j \in \operatorname{int} D_{r_n}} O(F, p_j) + B_n + C_n = \frac{i}{2\pi}\left[\int_{\partial D_{r_n} \setminus \bigcup_j D_j} \tau - \sum_j \int_{D_j \cap \partial D_{r_n} \neq \emptyset} \sigma\right].$$

Lemma. — *For* $\delta > 0$ *(see condition* (α_5) *above) small enough, we have* $\operatorname{Re} B_n \geqslant 0$ *and* $\operatorname{Re} C_n \geqslant 0$.

Proof. — We have to look to $O(F, p_j) + \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_{r_n} \cap D_j} \eta = \dfrac{1}{2\pi i}\displaystyle\int_{(b,a)} -\eta$.

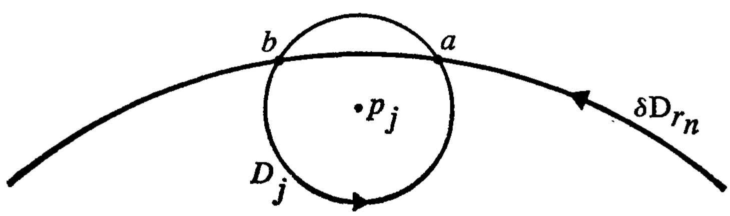

FIG. 1

The disc D_j can be parametrized by $y \mapsto (x(y), y)$, where $y \in \hat{D}_j$ (see (α_2) above). Let $\hat{f}(y) = -\dfrac{Q}{P}(x(y), y) = f(x(y), y)$. Thus

$$\int_{(b,a)} -\eta = \int_{(b,a)} \frac{\partial x}{\partial y}\, dx = \int_{(b,a)} \frac{1}{f}\frac{\partial f}{\partial y}\, dy$$
$$= \int_{(\hat{b},\hat{a})} \frac{1}{\hat{f}}\frac{\partial f}{\partial y}(x(y), y)\, dy,$$

where $(\hat{b}, \hat{a}) \subset \partial \hat{D}_j$. Now, if $p_j = (x_j, y_j)$, we have

$$\frac{1}{\hat{f}}\frac{\partial f}{\partial y}(x(y), y) = \frac{1}{y - y_j} + \varphi_j(x - x_j, y - y_j),$$

where $|\varphi_j(x - x_j, y - y_j)| \leqslant 2\pi\gamma$ $(\gamma > 0)$ for all $j \in \mathbf{N}$ provided that $\delta > 0$ is small enough. It follows that

$$\operatorname{Re}\frac{1}{2\pi i}\int_{(\hat{b},\hat{a})} \varphi_j(x - x_j, y - y_j)\, dy \leqslant \frac{1}{2\pi}\int_{(\hat{b},\hat{a})} |\varphi_j|\,|dy| \leqslant \gamma \ell(\hat{b}, \hat{a}),$$

where $\ell(\hat{b}, \hat{a})$ is the length of $(\hat{b}, \hat{a})$ (see Figure 1), and

$$\operatorname{Re} \frac{1}{2\pi i} \int_{(b,a)} - \eta \geqslant \operatorname{Re} \frac{1}{2\pi i} \int_{(\hat{b},\hat{a})} \frac{dy}{y - y_j} - \gamma \ell(\hat{b}, \hat{a})$$
$$= \theta(\hat{b}, \hat{a}) - \gamma \ell(\hat{b}, \hat{a}),$$

where $\theta(\hat{b}, \hat{a})$ is the angular variation from $\hat{b}$ to $\hat{a}$ in the positive sense. Finally we see that, for $\delta > 0$ small enough,

$$\operatorname{Re} \frac{1}{2\pi i} \int_{(b,a)} - \eta \geqslant 0,$$

so that $B_n \geqslant 0$.

Similarly we can prove that $C_n \geqslant 0$.

Now it is easy to see that there exists $c > 0$ such that

$$\left| \int_{\partial D_{r_n} \setminus \bigcup_j D_j} \tau \right| \leqslant c . \ell(D_{r_n})$$

and

$$\left| \sum_j \int_{D_j \cap \partial D_{r_n} \neq \varnothing} \sigma \right| \leqslant c . \ell(D_{r_n}),$$

so that from (*) we find

$$A(D_{r_n}) + \Sigma O(F, p_j) + \operatorname{Re} B_n + \operatorname{Re} C_n \leqslant \frac{2c}{2\pi} \ell(\partial D_{r_n}),$$

which implies

$$\liminf_{r_n \to \infty} \frac{\ell(\partial D_{r_n})}{A(D_{r_n})} > 0.$$

Another proof of the growth property follows (see [7]) from the next

Theorem **2.** — *Let $\mathscr{M}$ be a nontrivial minimal set of $\mathscr{F}$. Then there is no $\mathscr{F}$-transverse invariant measure with support on $\mathscr{M}$.*

Proof. — Suppose on the contrary that ν is such a measure. Then this implies the existence of a current $\mathscr{C}$ associated to $\mathscr{M}$ with the following property:

Given a local chart of $\mathscr{F}$, (U, α), where $\alpha : U \to \mathbf{C}$ is a distinguished map, there is a measure ν_U on $\alpha(U) =$ plaque space, such that for any differential form ω with support in U

$$\langle \omega, \mathscr{C} \rangle = \int \left(\int_\pi \omega \right) d\nu_U(\pi), \tag{5}$$

where π denotes a plaque of (U, α).

We define the *asymptotic Chern class of* $\mathscr{F}$ as follows: Let $(U_j, \alpha_j)_{j=1}^{N}$ be a covering of $\mathscr{M}$ by local charts of $\mathscr{F}$ and (U_j, φ_j) a partition of unity subordinated to the covering (U_j). Then

$$c(\mathscr{M}) = \sum_{j=1}^{N} \int_{\alpha_j(U_j)} \left(\int_{\pi} \varphi_j(x)\, \Theta_x(x, \alpha_j) \right) d\nu_j(\pi), \tag{6}$$

where $\nu_j = \nu_{U_j}$ and Θ is the curvature form associated to a connection ∇ on the normal bundle $N(\mathscr{M})$ to the foliation restricted to $\mathscr{M}$.

It can be verified that $c(\mathscr{M})$ is independent from all the choices. Now, given any section X of $N(\mathscr{M})$ such that on each leaf its zeros form a locally finite set in the topology of the leaf, we can define an index $\mathrm{Ind}(X, \mathscr{M})$ as follows. As before, cover $\mathscr{M}$ with a finite number of charts of $\mathscr{F}$, (U_j, α_j), and suppose that: (a) the foliated part of ∂U_j does not meet $\mathscr{M}$ and the vertical part of ∂U_j does not meet $\mathrm{Zer}(X, \mathscr{M})$, the set of zeros of X. (b) $\varphi_j = 1$ on $\mathrm{Zer}(X, \mathscr{M})$ for all $j = 1, \ldots, N$. Then

$$\mathrm{Ind}(X, \mathscr{M}) = \sum_{j=1}^{N} \int_{\alpha_j(U_j)} n(X, \varphi_j, \pi)\, d\nu_j(\pi),$$

where
$$n(X, \varphi_j, \pi) = \sum_{x \in \mathrm{Zer}(X, \pi)} \varphi_j(x)\, \mathrm{Ind}(X, x).$$

Now, we prove that

$$c(\mathscr{M}) = \mathrm{Ind}(X, \mathscr{M}).$$

In fact, let $N_\theta = N(M)\backslash$zero section and $p : N_\theta \to \mathscr{M}$ the fiber projection restricted to N_θ. Then there exists a form ψ on N_θ such that i) $p^*(\Theta) = d\psi$ and ii) if $i_x : N_{\theta_x} \to N$ is the injection of the fiber N_{θ_x} then $i_x^*(\psi)$ is a generator of $H^1(N_{\theta_x})$. So if $D \subset \pi_x$ is a small disc contained in the plaque of x centered at x, then we have $\mathrm{Ind}(X, x) = \int_{\partial D} X^* \psi$. Therefore,

$$\begin{aligned}
\mathrm{Ind}(X, \mathscr{M}) &= \sum_{j=1}^{N} \left(\int_{\alpha_j(U_j)} \varphi_j(X^* \psi) \right) d\nu_j(\pi) \\
&= \sum_{j=1}^{N} \int_{\alpha_j(U_j)} \left[\int_{\pi_x} d\varphi_j \wedge X^* \psi + \int_{\pi_x} \varphi\, dX^* \psi \right] d\nu_j(\pi) \\
&= \sum_{j=1}^{N} \int_{\alpha_j(U_j)} \left(\int_{\pi_x} d\varphi_j \wedge X^* \psi \right) d\nu_j(\pi) + \sum_{j=1}^{N} \int_{\alpha_j(U_j)} \varphi\Theta\, d\nu_j(\pi) \\
&= \langle \Sigma\, d\varphi_j \wedge X^* \psi, \mathscr{C} \rangle + c(\mathscr{M}) = c(\mathscr{M}) \quad \text{since } \Sigma\, d\varphi_j = 0.
\end{aligned}$$

Now, there are sections of $N(\mathscr{M})$ admitting zeros. To see this it is enough to take a linea vector field on $\mathbf{CP}(2)$ such that it is tangent to $\mathscr{F}$ at some regular point. So $c(\mathscr{M}) \neq 0$. Moreover, since $\mathscr{M}$ is embedded in a foliation, the connection ∇ in $N(\mathscr{M})$ is flat, thus $c(\mathscr{M}) = 0$ and this is a contradiction.

3. Hyperbolicity in minimal sets

We proceed to discuss now the construction of metrics which induce negative Gaussian curvature on the leaves of a foliation of $\mathbf{C}P(2)$.

Let $\mathscr{F}$ be a holomorphic foliation on $\mathbf{C}P(2)$ of degree $n \geqslant 2$. As we have seen, $\mathscr{F}$ can be expressed in the affine coordinate system $(x, y) = (x : y : 1)$ by a differential equation of the form $\mathrm{P}\, dy - \mathrm{Q}\, dx = 0$, where $\mathrm{P} = p + xg$ and $\mathrm{Q} = q + yg$, p and q polynomials of degree $\leqslant n$ and g a homogeneous polynomial of degree n. The leaves of $\mathscr{F} \mid \mathbf{C}^2$ are defined locally as the solutions of the complex differential equation

$$(1') \qquad \dot{x} = \mathrm{P}(x, y), \qquad \dot{y} = \mathrm{Q}(x, y).$$

If we set $\mathrm{R}(x, y) = y\mathrm{P}(x, y) - x\mathrm{Q}(x, y) = yp(x, y) - xq(x, y)$, then R has degree $\leqslant n + 1$, and the leaves of $\mathscr{F}$ in the affine coordinate system $(v, u) = (1 : v : u)$, are the solutions of the complex differential equation

$$(3') \qquad \dot{u} = \widetilde{\mathrm{P}}(u, v), \qquad \dot{v} = \widetilde{\mathrm{Q}}(u, v),$$

where $\widetilde{\mathrm{P}}(u, v) = u^{n+1}\, \mathrm{P}(1/u, v/u)$ and $\widetilde{\mathrm{Q}}(u, v) = u^{n+1}\, \mathrm{R}(1/u, v/u)$.

Analogously, the leaves of $\mathscr{F}$ in the coordinate system $(z, w) = (z : 1 : w)$ are the solutions of the differential equation

$$(7) \qquad \dot{z} = \hat{\mathrm{P}}(z, w), \qquad \dot{w} = \hat{\mathrm{Q}}(z, w),$$

where $\hat{\mathrm{P}}(z, w) = -\, w^{n+1}\, \mathrm{R}(z/w, 1/w)$ and $\hat{\mathrm{Q}}(z, w) = w^{n+1}\, \mathrm{Q}(z/w, 1/w)$.

It follows that the set of singularities of $\mathscr{F}$ is given by

$$(8) \qquad \operatorname{sing}(\mathscr{F}) = \overline{\{\mathrm{P} = 0\}} \cap \overline{\{\mathrm{Q} = 0\}} \cap \overline{\{\mathrm{R} = 0\}}.$$

Now consider the Hermitian metric in $\mathbf{C}^2 - \operatorname{sing}(\mathscr{F})$ whose associated quadratic form is

$$(9) \qquad \mu = (1 + |x|^2 + |y|^2)^{n-1} \cdot \frac{|dx|^2 + |dy|^2 + |x\, dy - y\, dx|^2}{|\mathrm{P}(x, y)|^2 + |\mathrm{Q}(x, y)|^2 + |\mathrm{R}(x, y)|^2},$$

where $\mathbf{C}^2 = \{(x : y : 1) \in \mathbf{C}P(2);\, x, y \in \mathbf{C}\}$. It is not difficult to see that the expression of μ in the coordinate system $(1 : v : u)$ is

$$(10) \qquad \mu = (1 + |u|^2 + |v|^2)^{n-1} \cdot \frac{|du|^2 + |dv|^2 + |u\, dv - v\, du|^2}{|\widetilde{\mathrm{P}}(u, v)|^2 + |\widetilde{\mathrm{Q}}(u, v)|^2 + |\widetilde{\mathrm{R}}(u, v)|^2}.$$

Its expression in the coordinate system $(z : 1 : w)$ is analogous. Therefore μ extends to $\mathbf{C}P(2) - \operatorname{sing}(\mathscr{F})$ and defines a Riemannian metric in this set. Let us compute the curvature of μ restricted to a leaf of $\mathscr{F}$.

The leaf of $\mathscr{F}$ through $p_0 = (x_0, y_0) \in \mathbf{C}^2$ can be parametrized locally by $p(\mathrm{T}) = (x(\mathrm{T}), y(\mathrm{T}))$, where $p(\mathrm{T})$, $|\mathrm{T}| < \varepsilon$, is the solution of (1′) such that $p(0) = p_0$. The quadratic form induced in L by this parametrization is given by

$$\mu^* = (1 + |x(\mathrm{T})|^2 + |y(\mathrm{T})|^2)^{n-1}\, |d\mathrm{T}|^2 = \psi(\mathrm{T})\, |d\mathrm{T}|^2$$

because $x'(T) = P(x(T), y(T))$ and $y'(T) = Q(x(T), y(T))$. On the other hand, the Gaussian curvature of L at the point $p(T)$ is given by

$$K(T) = -\frac{2}{\psi(T)} \frac{\partial}{\partial T} \frac{\partial}{\partial \overline{T}} \log \psi(T).$$

A direct computation shows that

$$\begin{aligned}\frac{\partial}{\partial T} \frac{\partial}{\partial \overline{T}} \log \psi(T) &= (n-1) \frac{\partial}{\partial T} \frac{\partial}{\partial \overline{T}} \log(1 + |x(T)|^2 + |y(T)|^2) \\ &= (n-1) \frac{|x'(T)|^2 + |y'(T)|^2 + |xy'(T) - yx'(T)|^2}{(1 + |x(T)|^2 + |y(T)|^2)^2} \\ &= \frac{(n-1) F(x(T), y(T))}{(1 + |x(T)|^2 + |y(T)|^2)^2},\end{aligned}$$

where $$F(x,y) = |P(x,y)|^2 + |Q(x,y)|^2 + |R(x,y)|^2.$$

This implies that $K(T) = k(x(T), y(T))$, where

$$k(x,y) = -2(n-1) F(x,y)/(1 + |x|^2 + |y|^2)^{n+1}.$$

Since $F > 0$ in $\mathbf{C}^2 - \operatorname{sing}(\mathscr{F})$, it follows that the curvature of the leaves of $\mathscr{F}$ in the metric μ is < 0. It is not difficult to see that the function k extends to $\mathbf{C}P(2) - \operatorname{sing}(\mathscr{F})$, and the curvature of the leaf L through $p_0 \in \mathbf{C}P(2) - \operatorname{sing}(\mathscr{F})$ at p_0 is $k(p_0)$. From this argument we obtain the following result:

Theorem **3.** — *There exists a Hermitian metric in* $\mathbf{C}P(2) - \operatorname{sing}(\mathscr{F})$ *which induces negative Gaussian curvature on the leaves of* $\mathscr{F}$. *In particular, if* $\mathscr{F}$ *has a nontrivial minimal set* $\mathscr{M}$, *then any leaf of* $\mathscr{M}$ *is hyperbolic (in the sense that it is covered by the unit disc of* $\mathbf{C}$*).*

Proof. — Let μ be the Riemannian metric given by (10). As we have seen, there exists a function $k : \mathbf{C}P(2) - \operatorname{sing}(\mathscr{F}) \to (-\infty, 0)$ such that the curvature of the leaf of $\mathscr{F}$ through p is $k(p)$. Suppose that $\mathscr{F}$ has a nontrivial minimal set $\mathscr{M}$. Then $\mathscr{M}$ is a compact subset of $\mathbf{C}P(2) - \operatorname{sing}(\mathscr{F})$. It follows that there are numbers $0 < B < C$ such that $-C < k(p) < -B$ for all $p \in \mathscr{M}$. This implies that the universal covering of any leaf $L \subset \mathscr{M}$ is the unit disc ([4]).

4. Consequences of hyperbolicity

In the last section we proved that the leaves of the minimal set $\mathscr{M}$ of a foliation $\mathscr{F}$ have strict negative curvature induced by a suitable Riemannian metric in $\mathbf{C}P(2)\backslash\operatorname{sing}(\mathscr{F})$. We are going to explore now some quantitative consequences of this fact.

The first basic tool is the following (see [4]):

Theorem (Ahlfors). — Let L *be a one-dimensional Kaehler manifold with metric* μ_L *whose Gaussian curvature is bounded above by a negative constant* — B. *Then every holomorphic map* $\varphi : \mathbf{D} \to L$ *satisfies*

$$\varphi^* \mu_L \leqslant \frac{\mu_D}{B}$$

where μ_D *is the Poincaré metric of the unit disc* $\mathbf{D} \subset \mathbf{C}$.

This result implies the following in our context. Let L be a leaf contained in $\mathscr{M}$ and $\varphi : \mathbf{D} \to L$ a uniformization for L. Set $\mathring{L} = L \cap \mathbf{C}^2$; then $\mathring{L}$ is locally parametrized by the complex time $T \in \mathbf{C}$ in such a way that

$$\frac{dx}{dT} = P(x,y), \qquad \frac{dy}{dT} = Q(x,y).$$

Since

$$\mu_L = (1 + |x|^2 + |y|^2)^{n-1} \frac{|dx|^2 + |dy|^2 + |x\,dy - y\,dx|^2}{|P(x,y)|^2 + |Q(x,y)|^2 + |R(x,y)|^2}$$

we find

$$\begin{aligned}\mu_L &= (1 + |x|^2 + |y|^2)^{n-1} \frac{(|P|^2 + |Q|^2 + |xQ - yP|^2)\,|dT|^2}{|P|^2 + |Q|^2 + |xQ - yP|^2} \\ &= (1 + |x|^2 + |y|^2)^{n-1}\,|dT|^2.\end{aligned}$$

Therefore

$$|dT|^2 = \frac{\mu_L}{(1 + |x|^2 + |y|^2)^{n-1}}.$$

If we consider now $T = T(z)$, where $(x,y) = \varphi(z)$, we find

$$|T'(z)|^2\,|dz|^2 \leqslant \varphi^* \mu_L \leqslant \frac{|dz|^2}{B(1 - |z|^2)^2},$$

so that

$$|T'(z)| \leqslant \frac{1}{\sqrt{B}(1 - |z|^2)}.$$

We observe that $S(z) = \varphi^*\left(P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}\right)$ is a meromorphic vector field in $\mathbf{D}$ with poles at the point $\varphi^{-1}(L \backslash \mathring{L})$ of order $n - 1 \in \mathbf{N}$, if the tangency order of $\mathscr{F}$ with the line $\mathbf{CP}(2) \backslash \mathbf{C}^2$ is $n \in \mathbf{N}$, *and with no zeros*. It follows that

$$\hat{T}(z) = \int_0^z \frac{dz}{S}$$

is well-defined. We have reached the following conclusion:

Theorem **4.** — *The complex time* $T \in \mathbf{C}$ *which parametrizes a leaf* L *of* $\mathcal{M}$ *is a multivalued Bloch function, i.e.:*

1) there exists $\hat{T} : \mathbf{D} \to \mathbf{C}$, *a holomorphic function, such that* $\hat{T} = T \circ \varphi$ *satisfies*

$$\frac{dx}{dT} = P(x,y), \qquad \frac{dy}{dT} = Q(x,y);$$

2) $\sup_{z \in \mathbf{D}} |\hat{T}'(z)| (1 - |z|^2) < \infty$. *In fact, the upper bound can be taken as the lowest value of the curvature of* L *(in absolute value).*

The same inequality as in Ahlfors' Theorem leads to another interesting conclusion

Theorem **5.** — *The family of uniformizations of the leaves in the minimal set is a normal family (according to the topology of uniform convergence in compact subsets).*

Proof. — Let $\varphi : \mathbf{D} \to L$ be a uniformization for the leaf L. Given $z_1, z_2 \in \mathbf{D}$ we have

$$\begin{aligned} d_L(\varphi(z_1), \varphi(z_2)) &\leqslant \int_{\varphi(c)} \sqrt{\mu_L} = \int_c \sqrt{\varphi^* \mu_L} \\ &\leqslant \int_c \sqrt{B^{-1} \mu_D} = B^{-1/2} d_D(z_1, z_2), \end{aligned}$$

where $c \subset \mathbf{D}$ is the geodesic from $z_1 \in \mathbf{D}$ to $z_2 \in \mathbf{D}$, and d_L, d_D denote the distances in L and $\mathbf{D}$ defined by μ_L and μ_D.

It follows that the family of uniformizations is equicontinuous.

A natural question now is: what about the limit functions of this family? Are they also uniformizations? The answer is positive, but first we have to state a sort of converse to Ahlfors' Theorem.

Theorem (Bland and Kalka, see [1]*).* — *Let* η *be a conformal complete metric in the unit disc* $\mathbf{D}$ *such that* $-C \leqslant k \leqslant -B < 0$, *where* k *is the Gaussian curvature of* η. *Then*

$$\frac{\mu_D}{C} \leqslant \eta.$$

Let us apply this result to a uniformization $\varphi : \mathbf{D} \to L \subset \mathcal{M}$: since the Gauss curvature k of L satisfies $-C \leqslant k \leqslant -B < 0$, we find

$$\frac{\mu_D}{C} \leqslant \varphi^*(\mu_L).$$

The uniformization φ is then a local quasi-isometry, since we have simultaneously

$$\frac{\mu_D}{C} \leqslant \varphi^*(\mu_L) \leqslant \frac{\mu_D}{B}.$$

Theorem **6.** — *The family of uniformizations of the leaves of $\mathcal{M}$ is a compact family.*

Proof. — This is quite simple at this point. Given a sequence $\varphi_n : \mathbf{D} \to \mathbf{CP}(2)$ in the family, we may assume from Theorem 5 that $\varphi_n \to \varphi$ uniformly in compact subsets of $\mathbf{D}$. If $p = \lim_{n\to\infty} \varphi_n(0)$, it is clear that φ maps $\mathbf{D}$ into the leaf of $\mathcal{M}$ through $p \in \mathbf{CP}(2)$. From

$$\frac{\mu_D}{C} \leqslant \varphi_n^*(\mu_L) \leqslant \frac{\mu_D}{B}$$

we get

$$\frac{\mu_D}{C} \leqslant \varphi^*(\mu_L) \leqslant \frac{\mu_D}{B}.$$

Therefore φ is a local quasi-isometry from $\mathbf{D}$ into the leaf, so it is a covering map.

To close this section, we make a last remark. Let $\varphi : \mathbf{D} \to L$ be a uniformization of $L \subset \mathcal{M}$ and suppose that the fundamental polygon in $\mathbf{D}$ has a parabolic vertex $a \in \partial\,\mathbf{D}$. This means that an angle A of vertex $a \in \partial\,\mathbf{D}$ inside the fundamental polygon has bounded hyperbolic area, so that the area according to the metric $\varphi^*(\mu_L)$ is also finite. Thus, $\varphi(A)$ has finite μ_L-area. Since $\mathcal{M}$ is a minimal set, we find that $\overline{\varphi(A)} = \mathcal{M}$. Therefore, if $p \in \varphi(A)$ and $R_0 \ni p$ is a small region contained in $\varphi(A)$, we get, from the local trivialization theorem for foliations, that R_0 is accumulated by regions $R_n \subset \varphi(A)$ contained in different plaques. All these regions have comparable μ_L-areas, that is

$$\beta \operatorname{area}(R_0) \leqslant \operatorname{area}(R_n) \leqslant \alpha \operatorname{area}(R_0)$$

for constants $\alpha > 0$, $\beta > 0$. Therefore $\varphi(A)$ has infinite μ_L-area, a contradiction. We have proven the following

Theorem **7.** — *The leaves of the minimal set have no parabolic ends.*

M. Gromov pointed out to us that the curvature of the normal bundle to the foliation $\mathcal{F}$ is positive. Thus Omori's Maximum Principle [9] may be useful in proving the existence of leaves with nontrivial holonomy.

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I.M.P.A.
Estrada Dona Castorina 110
Jardim Botânico
22460, Rio de Janeiro, R.J., Brasil

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