Statistica Sinica **22** (2012), 869-883
doi:http://dx.doi.org/10.5705/ss.2010.040

MINIMAL SUFFICIENT CONFOUNDING INFORMATION AMONG MAIN EFFECTS AND TWO-FACTOR INTERACTIONS

Jianwei Hu[1] and Runchu Zhang[2,3]

[1]*Central China Normal University,* [2]*Nankai University and* [3]*Northeast Normal University*

Abstract: For two-level regular designs, we obtain the structures of Fisher information matrices for estimating main effects and two-factor interactions (2fi's). Based on these results, we propose the definition of minimal sufficient confounding information among main effects and 2fi's. As an application, we demonstrate that minimum aberration (MA) designs must be (M,S)-optimal designs for two-level regular designs. In addition, we show that sequentially minimizing $M_{(1,2)_1}$, $M_{(2,2)_2}$ and $M_{(2,2)_1}$, as the core of the minimum M-aberration criterion proposed by Zhu and Zeng (2005), is equivalent to sequentially minimizing word length pattern A_3 and A_4. In particular, we show that sequentially minimizing A_3 and A_4 is equivalent to sequentially maximizing the first two components of the maximum estimation capacity, $E_1(d)$ and $E_2(d)$, defined in Cheng, Steinberg, and Sun (1999).

Key words and phrases: Aliased effect-number pattern, information matrix, maximum estimation capacity, minimal sufficient confounding information, minimum aberration.

1. Introduction

The effect hierarchy principle is important in fractional factorial design. The principle states that lower-order effects are more likely to be important than higher-order ones, and effects of the same order are equally like to be important. Therefore, good designs should estimate as many lower-order effects as possible. Aimed at such a purpose, many criteria have been advised in the literature, of which the MA criterion proposed by Fries and Hunter (1980) is the most popular criterion, for it has many good properties such as model robustness. For details, we refer to Cheng, Steinberg, and Sun (1999).

For a regular 2^{n-m} fractional factorial design, say d, let $A_i(d)$ be the number of words of length i in the defining relation. Then $A(d) = (A_1(d), \ldots, A_n(d))$ is called the word length pattern of design d. For any designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If no design has less aberration than d_1, then d_1 is called the MA design.

For ease of computation, S-optimality was introduced by Shah (1960) in the context of incomplete block designs. Based on S-optimality, the (M,S) procedure proposed by Eccleston and Hedayat (1974) has been widely used and advocated in optimal design literature, see especially Shah and Sinha (1989). Cheng, Deng, and Tang (2002) and Mandal and Mukerjee (2005) also studied (M,S)-optimality in factorial designs. In fact, (M,S)-optimality can be used to quickly identify designs that might turn out to be optimal, or highly efficient, according to other meaningful criterion, such as MA, minimum moment aberration (MMA) and so on. Recently, (M,S)-optimality was once again proposed by Qu, Kushler, and Ogunyemi (2008) for selecting two-level factorial designs. Note that the two concepts of (M,S)-optimality are slight different. Shah and Sinha (1989), Cheng, Deng, and Tang (2002), and Mandal and Mukerjee (2005) considered the joint information on the main effects and two-factor interaction, while Qu, Kushler, and Ogunyemi (2008) focused on the conditional information on the two-factor interaction given main effects. Both Jacroux (2004) and Qu, Kushler, and Ogunyemi (2008) considered the connection between the (M,S) and MA criteria for two-level regular designs of resolution III or higher. They showed that, for designs of resolution IV or higher, MA designs must be (M,S)-optimal. Furthermore, Qu, Kushler, and Ogunyemi (2008) showed that all designs of resolution III up to 64 runs are also (M,S)-optimal. However, for designs of resolution III with $N(>64)$ runs, whether an MA design is (M,S)-optimal is still unresolved.

In order to give a close characterization of the aliasing patterns of a fractional factorial design, the coset pattern matrix (CPM) was defined by Zhu and Zeng (2005). Based on the CPM, the minimum M-aberration criterion was proposed, by Zhu and Zeng (2005), to rank-order designs. The minimum M-abberation criterion selects designs through sequentially minimizing the following aliasing type pattern

$$M = (M_{(1,2)_1}, M_{(2,2)_2}, M_{(2,2)_1}, M_{(1,3)_1}, M_{(2,3)_2}, M_{(2,3)_1}, \ldots).$$

They noticed that sequentially minimizing $M_{(1,2)_1}$ and $M_{(2,2)_2}$ is equivalent to sequentially maximizing the first two components of the maximum estimation capacity, $E_1(d)$ and $E_2(d)$, where $E_i(d)$ denote the number of models containing all i-factor interaction that can be estimated by d, but they did not discuss the further connection between the minimum M-aberration and MA criteria.

Recently, by introducing an aliased effect-number pattern (AENP), Zhang et al. (2008) proposed a general minimum lower-order confounding (GMC) criterion for selecting two-level regular designs. For more details of the GMC criterion, we refer to Zhang et al. (2008). Considering regular 2^{n-m} designs with n factors and $N = 2^{n-m}$ runs, they defined ${}_{i}^{\#}C_j^{(l)}$ as the number of ith-order effects aliased with l jth-order effects. They showed that the AENP can manage many other criteria. Further connotations and applications of the AENP are not known.

Throughout this paper, we only discuss the case of two-level regular designs of resolution III or higher. In Section 2, we obtain the structure of Fisher information matrices for estimating main effects and 2fi's. In Section 3, we propose the definition of minimal sufficient confounding information among main effects and 2fi's. As an application, we demonstrate that minimum aberration (MA) designs must be (M,S)-optimal designs. In Section 4, we show that the CPM is just a sufficient, but not minimal sufficient, confounding information among main effects and 2fi's. As another application of the new concept, we show that sequentially minimizing $M_{(1,2)_1}, M_{(2,2)_2}$, and $M_{(2,2)_1}$ is equivalent to sequentially minimizing word length pattern A_3 and A_4. This means that sequentially minimizing A_3 and A_4 is equivalent to sequentially maximizing the first two components of the maximum estimation capacity, $E_1(d)$ and $E_2(d)$. Thus, the essential connection between the two criteria is revealed.

2. Structures of Fisher Information Matrices for Estimating Main Effects and Two-Factor Interactions

2.1. Model

For any regular 2^{n-m} design with n factors each at two levels and $N = 2^{n-m}$ runs, we consider the scenario in which grand mean, main effects, and 2fi's are of interest and need to be estimated, three-factor and higher order interactions are negligible. To estimate the grand mean, main effects, and 2fi's, the fitted model is given by

$$Y = X\beta + \epsilon = X_0\beta_0 + X_1\beta_1 + X_2\beta_2 + \epsilon,$$

where Y denotes the vector of N observations, β_0 is the grand mean, X_0 the all $+1$ column, X_1 is the original design matrix D, β_1 is the vector of all main effects, X_2 is the collection of products of two columns from D, β_2 is the corresponding two-factor interaction effects, and ϵ is the vector of random errors, assumed to have zero mean and constant variance. Note that $X_1^T X_1 = NI$ and $X_1^T X_0 = 0$ for all designs considered.

The normal equation for estimating the grand mean, main effects and 2fi's is

$$S\beta = X^T Y,$$

where

$$S = X^T X = \begin{pmatrix} N & 0 & 0 \\ 0 & X_1^T X_1 & X_1^T X_2 \\ 0 & X_2^T X_1 & X_2^T X_2 \end{pmatrix}.$$

We say, an effect is estimable if its least squares estimate (LSE) is unique. It is easy to see that β_0 is estimable and its least squares estimate is $\hat{\beta}_0 = \sum_{i=1}^{N} Y_i/N$.

In the following two subsections, we explore the structures of Fisher information matrices for estimating main effects and two-factor interactions.

2.2. Fisher information matrix for estimating 2fi's

Consider the estimates of 2fi's. The reduced normal equation for estimating β_2 is

$$C_2\beta_2 = X_2^T Y - N^{-1} X_2^T X_1 X_1^T Y,$$

where $C_2 = X_2^T X_2 - N^{-1}(X_1^T X_2)^T (X_1^T X_2)$. Thus, β_2 is estimable if and only if C_2, the Fisher information matrix for estimating β_2, is positive definite. Since C_2 plays a key role for estimating β_2, it is important to give a clear expression for each of its elements.

Define

$$J_{stu} = J(s,t,u) = \sum_{l=1}^{N} s_l t_l u_l,$$
$$J_{stuv} = J(s,t,u,v) = \sum_{l=1}^{N} s_l t_l u_l v_l,$$

where i_l is the l-th component of column s, and so on.

We use $C_2(ij,pq)$ to denote the (ij,pq)-th element of C_2. Then

$$\begin{aligned} C_2(ij,pq) &= J_{ijpq} - N^{-1}(J_{1ij}, J_{2ij}, \ldots, J_{nij})(J_{1pq}, J_{2pq}, \ldots, J_{npq})^T \\ &= J_{ijpq} - N^{-1}\sum_{l=1}^{n} J_{lij}J_{lpq}. \end{aligned}$$

If $ij = pq$ or ij is aliased with pq, $C_2(ij,pq) = N - N^{-1}\sum_{l=1}^{n} J_{lij}^2$. In particular, if ij is aliased with one main effect, $C_2(ij,pq) = 0$, otherwise $C_2(ij,pq) = N$. If $ij \neq pq$ and ij is not aliased with pq, $C_2(ij,pq) = -N^{-1}\sum_{l=1}^{n} J_{lij}J_{lpq} = 0$ since there is no main effect aliased with both ij and pq.

Therefore, when we adjust β_2 and thus the corresponding X_2 to an appropriate order, the information matrix C_2 has a block diagonal form $C_2 = diag\{0_{t_2}, NI_{r_2}, N1_u 1_u^T, \ldots, N1_v 1_v^T\}$, where 1_l denotes a $l \times 1$ vector of 1's, and I_{r_2} denotes the identity matrix of order r_2.

From the above discussion, each of the 2fi's corresponding to I_{r_2} is neither aliased with any other 2fi nor aliased with any main effect, and thus is estimable. Each of the 2fi's corresponding to 0_{t_2} is aliased with one main effect and thus is not estimable. Each of the 2fi's corresponding to $1_l 1_l^T$ is aliased with other $l-1$ 2fi's but not aliased with any main effect, and thus is not estimable.

Clearly, there are ${}_2^{\#}C_2^{(l)}/(l+1)$ alias sets containing $l+1$ 2fi's and ${}_1^{\#}C_2^{(l+1)}$ alias sets containing $l+1$ 2fi's and one main effect. Moreover, an alias set contains at most $h = \min\{\lfloor n/2 \rfloor, 2^m\}$ 2fi's, where $\lfloor x \rfloor$ is the integer part of x. All the alias sets containing 2fi's but none of the main effects can be partitioned into h classes. The l-th class consists of the alias sets which contain $l+1$ 2fi's, $l = 0, 1, \ldots, h-1$. Let $\mathcal{C}_l$ be the l-th class. Then $|\mathcal{C}_l| = {}_2^{\#}C_2^{(l)}/(l+1) - {}_1^{\#}C_2^{(l+1)}$, where $|\cdot|$ denotes the cardinality of a set $\mathcal{C}_l$ for $l = 0, 1, \ldots, h-1$. In particular, if $l = 0$, $|\mathcal{C}_0|$ denotes the number of sets each of which contains only one 2fi but none of the main effects, i.e., the number of clear 2fi's. This means $r_2 = |\mathcal{C}_0|$.

It is also easy to see that there are $|\mathcal{C}_l|(1_{l+1}1_{l+1}^T)$. There are altogether $\binom{n}{2} - \sum_{l=0}^{h-1}(l+1)|\mathcal{C}_l|$ 2fi's, each of which is aliased with one main effect, i.e., $t_2 = \binom{n}{2} - \sum_{l=0}^{h-1}(l+1)|\mathcal{C}_l|$.

Therefore, the structure of information matrix C_2 is uniquely determined by ${}_2^{\#}C_2^{(l)}/(l+1) - {}_1^{\#}C_2^{(l+1)} (l = 0, 1, \ldots, h-1)$.

2.3. Fisher information matrix for estimating main effects

We turn to the estimates of main effects. The reduced normal equations for estimating β_1 is

$$C_1\beta_1 = X_1^T Y - (X_2^T X_2)^- X_2^T Y,$$

where $C_1 = NI_n - (X_1^T X_2)(X_2^T X_2)^-(X_1^T X_2)^T$ and $(X_2^T X_2)^-$ is the generalized inverse of $X_2^T X_2$. Thus, β_1 is estimable if and only if C_1, the Fisher information matrix for estimating β_1, is positive definite. Since C_1 plays a key role for estimating β_1, it is important to give a clear expression for each of its elements.

When we adjust β_2 and thus the corresponding X_2 to an appropriate order, $X_2^T X_2$ has a block diagonal form $X_2^T X_2 = diag\{NI_r, N1_u1_u^T, \ldots, N1_v1_v^T\}$. Since $X_2(X_2^T X_2)^- X_2^T$ is independent of the selection of $(X_2^T X_2)^-$, we can take the generalized inverse,

$$(X_2^T X_2)^- = diag\{N^{-1}I_r, N^{-1}E_u, \ldots, N^{-1}E_v\},$$

where $E_u = \{1, 0, \ldots, 0\}$ is one of the generalized inverses of $1_u1_u^T$.

We use $C_1(i,j)$ to denote the (i,j)th element of C_1. Then

$$C_1(i,j) = N\delta_{ij} - N^{-1} \begin{pmatrix} J_{js_1} \\ \vdots \\ J_{js_r} \\ J_{j(s_r+1)} \\ J_{j(s_r+2)} \\ \vdots \end{pmatrix}^T \begin{pmatrix} I_r & & & \\ & E_u & & \\ & & \ddots & \\ & & & E_v \end{pmatrix} \begin{pmatrix} J_{js_1} \\ \vdots \\ J_{js_r} \\ J_{j(s_r+1)} \\ J_{j(s_r+2)} \\ \vdots \end{pmatrix}$$
$$= N\delta_{ij} - N^{-1}(\textstyle\sum_{k=1}^{\binom{n}{2}} J_{is_k}J_{js_k} - J_{i(s_r+2)}J_{j(s_r+2)} - \cdots)$$
$$= N\delta_{ij} - N^{-1}\textstyle\sum_{s\in S_1} J_{is}J_{js},$$

where S_1: (1) contains all the 2fi's; (2) if a 2fi is aliased with other 2fi's, only one 2fi's are allowed to appear in S_1. Furthermore, δ is the Kronecker delta function defined by $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$.

If $i = j$, $C_1(i,j) = N - N^{-1}\sum_{s\in S_1} J_{is}^2$. In particular, if i is aliased with some 2fi's, then $C_1(i,j) = 0$, otherwise $C_1(i,j) = N$. If $i \neq j$, $C_1(i,j) = -N^{-1}\sum_{s\in S_1} J_{is}J_{js} = 0$ since there is no 2fi aliased with both of i and j.

Thus, the structure of information matrix C_1 has a diagonal form

$$C_1 = N\begin{pmatrix} 0_{t_1} & \\ & I_{r_1} \end{pmatrix}_{n\times n}.$$

From this, we can see that each of the main effects corresponding to I_{r_1} is not aliased with any 2fi and thus is estimable. Each of the main effects corresponding to 0_{t_1} is aliased with at least a 2fi and thus is not estimable.

Obviously, $r_1 = {}_1^{\#}C_2^{(0)}, t_1 = n - {}_1^{\#}C_2^{(0)}$. This means that the structure of information matrix C_1 is uniquely determined by ${}_1^{\#}C_2^{(0)}$. Note that C_1 and C_2 do not contain the detailed information of ${}_1^{\#}C_2^{(l)} (l = 1, \ldots, h)$.

3. Minimal Sufficient Confounding Information among Main Effects and Two-Factor Interactions

3.1. Definition

We give the definition of minimal sufficient confounding information among main effects and 2fi's.

Definition 1. If the confounding information among main effects and 2fi's is uniquely determined by information T, then T is called the sufficient confounding information among main effects and 2fi's.

Definition 2. Suppose T is sufficient confounding information among main effects and 2fi's, and for any sufficient confounding information T_1 among main effects and 2fi's, T is determined by T_1. Then T is called the minimal sufficient confounding information among main effects and 2fi's.

Results in Section 2 show that the confounding information among main effects and 2fi's is uniquely determined by ${}_1^{\#}C_2^{(l)}(l = 0, 1, \ldots, h)$ and ${}_2^{\#}C_2^{(l)}/(l + 1) - {}_1^{\#}C_2^{(l+1)}(l = 0, 1, \ldots, h - 1)$ and vice versa. Thus ${}_1^{\#}C_2^{(l)}(l = 0, 1, \ldots, h)$ and ${}_2^{\#}C_2^{(l)}/(l + 1) - {}_1^{\#}C_2^{(l+1)}(l = 0, 1, \ldots, h - 1)$ is the minimal sufficient confounding information among main effects and 2fi's.

Obviously, ${}_1^{\#}C_2^{(l)}(l = 0, 1, \ldots, h)$ and ${}_2^{\#}C_2^{(l)}/(l+1) - {}_1^{\#}C_2^{(l+1)}(l = 0, 1, \ldots, h-1)$ are uniquely determined by ${}_1^{\#}C_2^{(l)}(l = 0, 1, \ldots, h)$ and ${}_2^{\#}C_2^{(l)}(l = 0, 1, \ldots, h - 1)$ and vice versa. Thus ${}_1^{\#}C_2^{(l)}(l = 0, 1, \ldots, h)$ and ${}_2^{\#}C_2^{(l)}(l = 0, 1, \ldots, h - 1)$ is also the minimal sufficient confounding information among main effects and 2fi's.

3.2. MA criterion in view of minimal sufficient confounding information

Let A_3, as usual, denote the number of words of length 3 in the defining relation. Then A_3 reveals some confounding information among main effects and 2fi's. Also, denote by A_4 the number of words of length 4 in the defining relation. Then A_4 reveals some confounding information among 2fi's. Thus, A_3 and A_4 must be functions of the minimal sufficient confounding information among main effects and 2fi's. This can be verified by a lemma that can be deduced from Theorem 2 of Zhang et al. (2008).

Lemma 1. $A_3 = (1/3)\sum_{l=1}^{h} l\,{}_1^{\#}C_2^{(l)}$, $A_4 = (1/6)\sum_{l=1}^{h-1} l\,{}_2^{\#}C_2^{(l)}$.

Lemma 1 implies that the MA criterion loses part of confounding information among main effects and 2fi's.

3.3. Another version of minimal sufficient confounding information

The minimal sufficient confounding information among main effects and 2fi's has other versions. We show that the m_i's $(1 \le i \le g)$ defined by Cheng, Steinberg, and Sun (1999) are also the minimal sufficient confounding information among main effects and 2fi's under a certain condition.

In a 2^{n-m} design d with resolution III or higher, $2^m - 1$ of the $2^n - 1$ factorial effects appear in the defining relation. The remaining $2^n - 2^m$ effects are partitioned into $g = 2^{n-m} - 1$ alias sets each of size 2^m, n of which sets contain main effects. Let $f = g - n$, and take the f alias sets not containing main effects to be $M_1, \ldots, M_f$. Also, let the n alias sets containing main effects be $M_{f+1}, \ldots, M_g$. For $1 \le i \le g$, let $m_i(d)$ be the number of 2fi's in M_i.

Lemma 2. *For $0 \le l \le h$, we have*

$$ {}^{\#}_{1}C_2^{(l)} = \#\{i : f+1 \le i \le g, m_i = l\}, $$
$$ \frac{{}^{\#}_{2}C_2^{(l)}}{(l+1)} - {}^{\#}_{1}C_2^{(l+1)} = \#\{i : 1 \le i \le f, m_i = l+1\}, $$

where $\#$ denotes the cardinality of a set.

Under the assumption that we need not identify effects aliased with others at the same degree, the m_i's $(1 \le i \le f)$ and m_i's $(f+1 \le i \le g)$ can be arranged in nondecreasing order respectively, then obviously, ${}^{\#}_{1}C_2^{(l)} (l = 0, 1, \ldots, h)$ and ${}^{\#}_{2}C_2^{(l)}/(l+1) - {}^{\#}_{1}C_2^{(l+1)} (l = 0, 1, \ldots, h-1)$ are uniquely determined by m_i's $(1 \le i \le g)$ and vice versa. Therefore, the m_i's $(1 \le i \le g)$ are also the minimal sufficient confounding information among main effects and 2fi's under a certain condition.

By Lemmas 1 and 2, A_3 and A_4 must be functions of the m_i's $(1 \le i \le g)$.

Corollary 1. $A_3 = (1/3)\sum_{i=f+1}^{g} m_i$, $A_4 = (1/6)(\sum_{i=1}^{g} m_i^2 - \binom{n}{2})$.

By Corollary 1, and thus the minimal sufficient confounding information among main effects and 2fi's, Cheng, Steinberg, and Sun (1999) showed that the MA criterion is a good surrogate for maximum estimation capacity, a model robustness criterion. Their work can be viewed as an important application of the minimal sufficient confounding information among main effects and 2fi's.

3.4. MA designs must be (M, S)-optimal designs

There are other applications of the minimal sufficient confounding information among main effects and 2fi's. For example, by the minimal sufficient confounding information among main effects and 2fi's, the uniquely optimal confounding structure between main effects and two-factor interactions, possessed by resolution III designs with and only with minimum A_3, was found by Hu and Zhang (2011). For more applications of the minimal sufficient confounding information among main effects and 2fi's, we refer to Hu and Zhang (2009).

Denote by $f$the number of factors of the complementary design, $f = 2^{n-m} - n - 1$. For $2^{n-m-1} \le f \le 2^{n-m} - 1$, designs with A_3 being minimized must be those of resolution four or higher. Therefore, for these cases, the main effects are orthogonal to the 2fi's. Next, only $2^{\omega-1} \le f \le 2^{\omega} - 1$ and $1 \le \omega \le n-m-1$ need be considered.

Lemma 3. *For* $2^{\omega-1} \le f \le 2^{\omega} - 1, 1 \le \omega \le n-m-1$*, T is a design for which*

A_3 is minimized if and only if

$$
{}^{\#}_{1}C_2^{(k)} = \begin{cases} n-(2^\omega-1-f), & \text{if } k=\frac{1}{2}(n-f-1), \\ 2^\omega-1-f, & \text{if } k=\frac{1}{2}(n-f-1)+f-2^{\omega-1}+1, \\ 0, & \text{otherwise.} \end{cases}
$$

We turn to the connection between the (M,S) and MA criteria. First, we need the following results.

Lemma 4. *From the structure of C_2, it is easy to see that $trace(C_2) = N\sum_{l=0}^{h-1}(l+1)|\mathcal{C}_l|$, $trace(C_2^2) = N^2\sum_{l=0}^{h-1}(l+1)^2|\mathcal{C}_l|$, $trace(C_2) = N(\binom{n}{2} - 3A_3)$.*

Proof. The first two equalities are evident from the structure of C_2. We only show the third. Since there are altogether $\binom{n}{2}-\sum_{l=0}^{h-1}(l+1)|\mathcal{C}_l|$ 2fi's, each of which is aliased with one main effect, it is easy to see that $3A_3 = \binom{n}{2} - \sum_{l=0}^{h-1}(l+1)|\mathcal{C}_l|$. By the first equality, we have $trace(C_2) = N(\binom{n}{2} - 3A_3)$.

The (M,S) criterion first identifies a subclass of designs that maximize $trace(C_2)$, and then finds designs within this subclass that minimize $trace(C_2^2)$. If a design has the maximum $trace(C_2)$ and minimum $trace(C_2^2)$ within a class of designs $\mathcal{D}$, it is called an (M,S)-optimal design in $\mathcal{D}$.

Lemma 4 implies that the (M,S) criterion loses part of the confounding information among main effects and 2fi's.

Theorem 1. *For any regular design of resolution III or higher, selecting designs in a subclass of designs that maximize $trace(C_2)$, then finding designs within this subclass that minimize $trace(C_2^2)$ is equivalent to sequentially minimizing A_3 and A_4.*

Proof. By Lemma 4, $trace(C_2) = N(\binom{n}{2} - 3A_3)$. This means that maximizing $trace(C_2)$ is equivalent to minimizing A_3. By Lemmas 1 and 4, we have

$$
\begin{aligned}
trace(C_2^2) &= N^2\sum_{l=0}^{h-1}(l+1)^2|\mathcal{C}_l| \\
&= N^2\sum_{l=0}^{h-1}(l+1)^2\Big(\frac{{}^{\#}_{2}C_2^{(l)}}{(l+1)} - {}^{\#}_{1}C_2^{(l+1)}\Big) \\
&= N^2\sum_{l=0}^{h-1}(l+1){}^{\#}_{2}C_2^{(l)} - N^2\sum_{l=0}^{h-1}(l+1)^2\,{}^{\#}_{1}C_2^{(l+1)} \\
&= N^2\sum_{l=0}^{h-1} l\,{}^{\#}_{2}C_2^{(l)} + N^2\sum_{l=0}^{h-1}{}^{\#}_{2}C_2^{(l)} - N^2\sum_{l=0}^{h-1}(l+1)^2\,{}^{\#}_{1}C_2^{(l+1)} \\
&= 6N^2A_4 + N^2\binom{n}{2} - N^2\sum_{l=0}^{h-1}(l+1)^2\,{}^{\#}_{1}C_2^{(l+1)}.
\end{aligned}
$$

Under the condition that A_3 is minimized, by Lemma 3, the confounding structure between main effects and 2fi's is uniquely determined. This means that $N^2\sum_{l=0}^{h-1}(l+1)^2{}^{\#}_{1}C_2^{(l+1)}$ is a constant under the condition that A_3 is minimized. Thus, minimizing A_4 is equivalent to minimizing $trace(C_2^2)$ under the condition that A_3 is minimized. Therefore, selecting designs in a subclass of designs that maximize $trace(C_2)$, and then finding designs within this subclass that minimize $trace(C_2^2)$ is equivalent to sequentially minimizing A_3 and A_4.

By Theorem 1, it is easy to see that MA designs must be (M,S)-optimal designs.

4. Other Applications of Minimal Sufficient Confounding Information

4.1. The coset pattern matrix is sufficient confounding information

Denote by $\mathcal{F}_l$ the collection of all l-th order effects cosets and $r(\mathcal{F}_l)$ the collection of ranks of the cosets in $\mathcal{F}_l$. Suppose $i_1\cdots i_pG$ is the lth coset, that is, $r(i_1\cdots i_pG)=l$, with $0\le l\le N-1$. Let A_{lj} be the number of words of length j in $i_1\cdots i_pG$. The vector $W_l=(A_{l1},\ldots,A_{ln})$ is the coset pattern of $i_1\cdots i_pG$. The $N\times n$ matrix $A(d)=(W_0^T,\ldots,W_{N-1}^T)^T$ is called the coset pattern matrix by Zhu and Zeng (2005).

By the definitions of CPM and the aliased effect-number pattern (AENP), we get the following.

Lemma 5. *For any regular design, ${}^{\#}_{i}C_j^{(l)}$ is a function of A_{lj},*

$$
{}^{\#}_{i}C_j^{(l)}=\begin{cases}(l+1)\#\{h\in r(\bigcup_{k=0}^{x}\mathcal{F}_k):A_{hi}=l+1\}, & \text{if } i=j,\\ \sum_{\{h\in r(\bigcup_{k=0}^{x}\mathcal{F}_k):A_{hj}=l\}}A_{hi}, & \text{if } i\neq j,\end{cases}
$$

where $x=min(i,j)$.

Proof. The proof is evident. No ith-order effect is aliased with jth-order effects in the coset when $h\in r(\bigcup_{k=x+1}^{n-1}\mathcal{F}_k)$. When $i\neq j$, for any given $h\in\{h\in r(\bigcup_{k=0}^{x}\mathcal{F}_k):A_{hj}=l\}$, there are A_{hi} ith-order effects aliased with l jth-order effects. When $i=j$, for any given $h\in\{h\in r(\bigcup_{k=0}^{x}\mathcal{F}_k):A_{hi}=l+1\}$, there are $A_{hi}=l+1$ ith-order effects aliased with l ith-order effects.

In particular, when $i=1$, $A_{hi}=1$ for any regular design of resolution III or higher. Thus, we have the following corollary.

Corollary 2. *For any regular design of resolution III or higher,*

$$
{}^{\#}_{1}C_j^{(l)}=\#\{h\in r(\mathcal{F}_0\cup\mathcal{F}_1):A_{hj}=l\},\ j\ge 2.
$$

Although the minimal sufficient confounding information among main effects and 2fi's is uniquely determined by the CPM, the CPM cannot be determined by the minimal sufficient confounding information among main effects and 2fi's. This means that the CPM is just a sufficient but not minimal sufficient confounding information among main effects and 2fi's.

4.2. Relationship between minimum M-abberation and MA criteria

Based on the CPM, Zhu and Zeng (2005) proposed the minimum M-abberation criterion for regular designs of resolution at least III. Suppose e_1 and e_2 are two effects of ith-order and jth-order respectively, and aliased with each other. Then e_1 and e_2 must belong to the same coset, say $i_1 \cdots i_k G$. They have the aliasing between e_1 and e_2 as of type $(i,j)_k$, where $k \leq i$. For a given $(i,j)_k$, they took $M_{(i,j)_k}$ to be the number of pairs of aliased effects which are of the subtype $(i,j)_k$. It is easy to see that $M_{(i,j)_k}$ can be calculated from the CPM as follows,

$$M_{(i,j)_k} = \begin{cases} \sum_{h \in r(\mathcal{F}_k)} \frac{1}{2} A_{hi}(A_{hi}-1), & \text{if } i = j, \\ \sum_{h \in r(\mathcal{F}_k)} A_{hj} A_{hj}, & \text{if } i < j. \end{cases}$$

Lemma 6. *For any regular design of resolution III or higher,*

$$M_{(1,j)_1} = \sum_{l=1}^{K_j} l\, {}_1^{\#}C_j^{(l)}, M_{(i,i)_1} = \sum_{l=2}^{K_i} \binom{l}{2} {}_1^{\#}C_i^{(l)},$$
$$M_{(2,2)_2} = \sum_{l=1}^{h-1} \binom{l+1}{2} |\mathcal{C}_l|,$$

where $i \geq 2, j \geq 2$.

Proof. When $i = 1$, $A_{hi} = 1$, so $M_{(1,j)_1} = \sum_{h \in r(\mathcal{F}_1)} A_{hj}$ is just the number of jth-order effects aliased with main effects. There are ${}_1^{\#}C_j^{(l)}$ cosets with main effects as their coset leaders, each of which contains l jth-order effects. Thus, $M_{(1,j)_1} = \sum_{l=1}^{K_j} l\, {}_1^{\#}C_j^{(l)}$. $M_{(i,i)_1} = \sum_{h \in r(\mathcal{F}_1)} (1/2) A_{hi}(A_{hi}-1)$ denotes the number of pairs of aliased ith-order effects that are of type $(i,i)_1$. There are ${}_1^{\#}C_i^{(l)}$ cosets containing l ith-order effects and with main effects as their coset leaders. There are $\binom{l}{2}$ pairs of aliased ith-order effects for any coset containing l ith-order effects and with one main effect as its coset leader. Thus, $M_{(i,i)_1} = \sum_{l=2}^{K_i} \binom{l}{2} {}_1^{\#}C_i^{(l)}$. $M_{(2,2)_2} = \sum_{h \in r(\mathcal{F}_2)} (1/2) A_{hi}(A_{hi}-1)$ denotes the number of pairs of aliased 2fi's, which are of type $(2,2)_2$. There are $|\mathcal{C}_l|$ cosets containing $l+1$ ith-order effects with 2fi's as their coset leaders. There are $\binom{l+1}{2}$ pairs of aliased 2fi's for any coset containing l 2fi's and with one 2fi as its coset leader. Thus, we have $M_{(2,2)_2} = \sum_{l=1}^{h-1} \binom{l+1}{2} |\mathcal{C}_l|$.

Lemma 7. *For any regular design,*

$$\sum_{k=0}^{i} M_{(i,j)_k} = \begin{cases} \sum_{l=0}^{K_i} \frac{l}{2}\, {}_i^{\#}C_i^{(l)}, & \text{if } i = j, \\ \sum_{l=0}^{K_j} l\, {}_i^{\#}C_j^{(l)}, & \text{if } i < j. \end{cases}$$

Proof. When $i < j$, we have

$$\begin{aligned} \sum_{k=0}^{i} M_{(i,j)_k} &= \sum_{k=0}^{i} \sum_{h \in r(\mathcal{F}_k)} A_{hi} A_{hj} \\ &= \sum_{h \in r(\bigcup_{k=0}^{i} \mathcal{F}_k)} A_{hi} A_{hj} \\ &= \sum_{l=0}^{K_j} \sum_{\{h \in r(\bigcup_{k=0}^{i} \mathcal{F}_k): A_{hj} = l\}} l A_{hi} \\ &= \sum_{l=0}^{K_j} l \sum_{\{h \in r(\bigcup_{k=0}^{i} \mathcal{F}_k): A_{hj} = l\}} A_{hi} \\ &= \sum_{l=0}^{K_j} l\, {}_i^{\#}C_j^{(l)}. \end{aligned}$$

Similarly, when $i = j$, we have $\sum_{k=0}^{i} M_{(i,i)_k} = \sum_{l=0}^{K_i} \frac{l}{2}\, {}_i^{\#}C_i^{(l)}$.

Although the CPM has more detailed information than the minimal sufficient confounding information among main effects and 2fi's, the core of the minimum M-abberation criterion, the first and most important three elements of the aliasing type pattern based on the CPM, $M_{(1,2)_1}, M_{(2,2)_2}$, and $M_{(2,2)_1}$, are still functions of the minimal sufficient confounding information among main effects and 2fi's. This means that the minimum M-aberration criterion also loses part of the confounding information among main effects and 2fi's.

Now, as an application of the minimal sufficient confounding information among main effects and 2fi's, we prove a following result that reveals the close relationship between the minimum M-abberation and MA criteria.

Theorem 2. *For any regular design of resolution III or higher, sequentially minimizing $M_{(1,2)_1}, M_{(2,2)_2}$, and $M_{(2,2)_1}$ is equivalent to sequentially minimizing A_3 and A_4.*

Proof. By Lemmas 1 and 6, it is easy to see that $A_3 = (1/3)M_{(1,2)_1}$. This means that minimizing $M_{(1,2)_1}$ is equivalent to minimizing A_3. By Lemmas 1 and 6, we also have

$$\begin{aligned} A_4 &= \frac{1}{6}\sum_{l=1}^{h-1} l\,{}^{\#}_{2}C_2^{(l)} \\ &= \frac{1}{6}\sum_{l=1}^{h-1} l(l+1)\Big(\frac{{}^{\#}_{2}C_2^{(l)}}{(l+1)} - {}^{\#}_{1}C_2^{(l+1)}\Big) - \frac{1}{6}\sum_{l=1}^{h-1} l(l+1){}^{\#}_{1}C_2^{(l+1)} \\ &= \frac{1}{3}\sum_{l=1}^{h-1}\binom{l+1}{2}|\mathcal{C}_l| - \frac{1}{6}\sum_{l=1}^{h-1} l(l+1){}^{\#}_{1}C_2^{(l+1)} \\ &= \frac{1}{3}M_{(2,2)_2} - \frac{1}{6}\sum_{l=1}^{h-1} l(l+1){}^{\#}_{1}C_2^{(l+1)}. \end{aligned}$$

Under the condition that A_3 is minimized, by Lemma 3 the confounding structure between main effects and 2fi's is uniquely determined. This means that $(1/6)\sum_{l=1}^{h-1} l(l+1){}^{\#}_{1}C_2^{(l+1)}$ is a constant under the condition that A_3 is minimized. Thus, minimizing A_4 is equivalent to minimizing $M_{(2,2)_2}$ under the condition that A_3 is minimized. For the same reason, $M_{(2,2)_1}$ is also a constant under the condition that A_3 is minimized. Therefore, sequentially minimizing $M_{(1,2)_1}, M_{(2,2)_2}$, and $M_{(2,2)_1}$ is equivalent to sequentially minimizing A_3 and A_4.

4.3. MEC and MA criteria often produce quite consistent results

Theorem 3. *For any regular design of resolution III or higher, sequentially minimizing A_3 and A_4 is equivalent to sequentially maximizing $E_1(d)$ and $E_2(d)$.*

Proof. Zhu and Zeng (2005) showed that $E_1 = n(n-1)/2 - M_{(1,2)_1}$ and $E_2 = E_1(E_1+1)/2 - M_{(2,2)_2}$. By Theorem 2, it is easy to see that sequentially minimizing A_3 and A_4 is equivalent to sequentially maximizing $E_1(d)$ and $E_2(d)$.

Next, we use Theorem 3 to explain the phenomenon in Section 5 of Cheng, Steinberg, and Sun (1999): for $N = 16, 32$, the MA and MEC criteria produce quite consistent results.

On the one hand, by checking the catalog of 2^{n-m} designs in Mukerjee and Wu (2006), we know that for $N = 16, 32$, all the MA designs are uniquely determined by sequentially minimizing A_3, A_4.

On the other hand, obviously, the maximum estimation capacity criterion is a stronger condition than only sequentially maximizing E_1, E_2. As is well known, for $N = 16, 32$, a design with the maximum estimation capacity may not exist. But if it does exist, it must have minimum aberration since all the 16-run and 32-run minimum aberration designs are uniquely determined by sequentially minimizing A_3, A_4. For detailed examples, see Cheng, Steinberg, and Sun (1999) and Chen and Cheng (2004). Thus we have almost completely explained the phenomenon in Section 5 of Cheng, Steinberg, and Sun (1999). For small runs, we can illustrate this as follows.

$$\begin{aligned} MEC &\Rightarrow \textit{simutaneously maximizing } E_1 \textit{ and } E_2 \\ &\Rightarrow \textit{sequentially maximizing } E_1 \textit{ and } E_2 \\ &\Leftrightarrow \textit{sequentially minimizing } A_3 \textit{ and } A_4 \\ &\approx MA. \end{aligned}$$

Theorfore, Theorem 3 can be rephrased as follows.

Corollary 3. *For given parameters n and m, if the MEC 2^{n-m} design exists and the MA 2^{n-m} design is uniquely determined by sequentially minimizing A_3, A_4, then the MA 2^{n-m} design is just an MEC 2^{n-m} design.*

At last, we point out that, by a similar discussion, we can demonstrate that some other nice criteria, such as those in Zhang and Park (2000), Xu (2003), Ai, Li, and Zhang (2005), Fang and Qin (2005), and Xu (2006) that are equivalent to the MA criterion, also have the similar properties discussed in Theorems 1−3 and Corollary 3.

Acknowledgement

The authors would like to thank the associate editor and referees for their valuable comments and constructive suggestions for improving the paper. This work was supported by the NNSF of China grant Nos. 11171165 and 10871104.

References

Ai, M. Y., Li, P. F. and Zhang, R. C. (2005). Optimal criteria and equivalence for nonregular fractional factorial designs. *Metrika* **62**, 73-83.

Chen, H. and Cheng, C. S. (2004). Aberration, estimation capacity and estimation index aberration. *Statist. Sinica* **14**, 203-215.

Cheng, C. S., Steinberg, D. M. and Sun, D. X. (1999). Minimum aberration and model robustness for two-level factorial designs. *J. Roy. Statist. Soc. Ser. B* **61**, 85-93.

Cheng, C. S., Deng, L. Y. and Tang, B., (2002). Generalized minimum aberration and design efficiency for nonregular fractional factorial designs. *Statist. Sinica* **12**, 991-1000.

Eccleston, J. A. and Hedayat, A. S. (1974). On the theory of connected designs: characterization and optimality. *Ann. Statist.* **2**, 1238-1255.

Fang, K. T. and Qin, H. (2005). Uniformity pattern and related criteria for two-level factorials. *Sci. China A* **48**, 1-11.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22**, 601-608.

Jacroux, M (2004). A modified minimum aberration criterion for selecting regular 2^{m-k} fractional factorial designs. *J. Statist. Plann. Inference* **126**, 325-336.

Hu, J. W. and Zhang, R. C. (2009). Maximal rank minimum aberration and doubling. *Statist. Probab. Lett.* **79**, 915-919.

Hu, J. W. and Zhang, R. C. (2011). Some results on two-level regular designs with general minimum lower-order confounding. *J. Statist. Plann. Inference* **141**, 1774-1782

Mandal, A. and Mukerjee, R. (2005). Design efficiency under model uncertainty for nonregular fractions of general factorials. *Statist. Sinica* **15**, 697-707.

Mukerjee, R. and Wu, C. F. J. (2006). *A Modern Theory of Factorial Designs.* Springer, New York.

Shah, K. R. (1960). Optimality criteria for incomplete block designs. *Ann. Math. Statist.* **31**, 791-794.

Shah, K. R. and Sinha, B. K. (1989). *Lecture Notes in Statistics: Theory of Optimal Designs.* Springer, Berlin.

Qu, X. G., Kushler, R. and Ogunyemi, T. (2008). (M,S)-optimality in selecting factorial designs. *J. Statist. Plann. Inference* **138**, 65-79.

Xu, H. (2003). Minimum moment aberration for nonregular designs and supersaturated designs. *Statist. Sinica* **13**, 691-708.

Xu, H. (2006). Block regular fractional factorial designs with minimum aberration. *Ann. Statist* **34**, 2534-2553.

Zhang, R. C., Li, P., Zhao, S. L. and Ai, M. Y. (2008). A general minimum lower-order confounding criterion for two-level regular designs. *Statist. Sinica* **18**, 1689-1705.

Zhang, R. C. and Park, D. K. (2000). Optimal blocking of two-level fractional factorial designs. *J. Statist. Plann. Inference* **91**, 107-121.

Zhu, Y. and Zeng, P. (2005). On the coset pattern matrices and minimum M-aberration of 2^{n-p} designs. *Statist. Sinica* **15**, 717-730.

School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China.

E-mail: jwhu@mail.ccnu.edu.cn

KLAS and School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

LPMC and School of Mathematical Sciences, Nankai University, Tianjin 300071, China.

E-mail: zhrch@nankai.edu.cn

(Received February 2010; accepted May 2011)