

MINIMAL SURFACES IN 3-DIMENSIONAL SOLVABLE LIE GROUPS II

JUN-ICHI INOBUCHI

An integral representation formula in terms of the normal Gauss map for minimal surfaces in 3-dimensional solvable Lie groups with left invariant metric is obtained.

1. INTRODUCTION

In the previous paper [3], we obtained an integral representation formula for minimal surfaces in the 3-dimensional solvable Lie group:

$$G(\mu_1, \mu_2) = (\mathbb{R}^3(x^1, x^2, x^3), g_{(\mu_1, \mu_2)}),$$

with group structure

$$(x^1, x^2, x^3) \cdot (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (x^1 + e^{\mu_1 x^3} \tilde{x}^1, x^2 + e^{\mu_2 x^3} \tilde{x}^2, x^3 + \tilde{x}^3)$$

and metric

$$g_{(\mu_1, \mu_2)} = e^{-2\mu_1 x^3} (dx^1)^2 + e^{-2\mu_2 x^3} (dx^2)^2 + (dx^3)^2.$$

This two-parameter family of solvable Lie groups contains the following particular examples: Euclidean 3-space \mathbb{E}^3 , hyperbolic 3-space H^3 and Euclidean motion group $E(1, 1)$. Moreover, $G(0, 1)$ is isometric to the Riemannian direct product $H^2 \times \mathbb{E}^1$ of hyperbolic 2-space and the real line \mathbb{E}^1 .

In this paper, we investigate the normal Gauss maps for surfaces in $G(\mu_1, \mu_2)$ and reformulate the integral representation formula of [3] in terms of the normal Gauss map.

On the other hand, study of minimal surfaces in the reducible Riemannian symmetric space $H^2 \times \mathbb{E}^1$ has been started very recently by Rosenberg and his collaborators. See [9, 10].

In a recent paper [7], Mercuri, Montaldo and Piu obtained an integral representation formula for minimal surfaces in $H^2 \times \mathbb{E}^1$ ([7, Theorem 5.1]). Their formula coincides with our formula for $G(0, 1)$. Thus our formula is a unification of Góes–Simões–Kokubu formula [2, 5] and Mercuri–Montaldo–Piu formula [7].

Received 5th January, 2006

Partially supported by Kakenhi 16740029

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/06 \$A2.00+0.00.

2. SOLVABLE LIE GROUP

In this paper, we study the following two-parameter family of homogeneous Riemannian 3-manifolds;

$$(2.1) \quad \{(\mathbb{R}^3(x^1, x^2, x^3), g_{(\mu_1, \mu_2)}) \mid (\mu_1, \mu_2) \in \mathbb{R}^2\},$$

where the metrics $g_{(\mu_1, \mu_2)}$ are defined by

$$(2.2) \quad g_{(\mu_1, \mu_2)} := e^{-2\mu_1 x^3} (dx^1)^2 + e^{-2\mu_2 x^3} (dx^2)^2 + (dx^3)^2.$$

Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1, \mu_2)})$ is realised as the following solvable matrix Lie group:

$$G(\mu_1, \mu_2) = \left\{ \begin{pmatrix} 0 & e^{\mu_1 x^3} & 0 & x^1 \\ 0 & 0 & e^{\mu_2 x^3} & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is given explicitly by

$$(2.3) \quad \mathfrak{g}(\mu_1, \mu_2) = \left\{ \begin{pmatrix} 0 & 0 & 0 & y^3 \\ 0 & \mu_1 y^3 & 0 & y^1 \\ 0 & 0 & \mu_2 y^3 & y^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid y^1, y^2, y^3 \in \mathbb{R} \right\}.$$

Then we can take the following orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}(\mu_1, \mu_2)$:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the commutation relation of \mathfrak{g} is given by

$$[E_1, E_2] = 0, [E_2, E_3] = -\mu_2 E_2, [E_3, E_1] = \mu_1 E_1.$$

Left-translating the basis $\{E_1, E_2, E_3\}$, we obtain the following orthonormal frame field:

$$e_1 = e^{\mu_1 x^3} \frac{\partial}{\partial x^1}, e_2 = e^{\mu_2 x^3} \frac{\partial}{\partial x^2}, e_3 = \frac{\partial}{\partial x^3}.$$

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is described by

$$(2.4) \quad \begin{aligned} \nabla_{e_1} e_1 &= \mu_1 e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\mu_1 e_1, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = \mu_2 e_3, \quad \nabla_{e_2} e_3 = -\mu_2 e_2, \\ \nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0. \end{aligned}$$

EXAMPLE 2.1. (Euclidean 3-space) The Lie group $G(0, 0)$ is isomorphic and isometric to the Euclidean 3-space $\mathbb{E}^3 = (\mathbb{R}^3, +)$.

EXAMPLE 2.2. (Hyperbolic 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c, c)$ is a warped product model of the hyperbolic 3-space:

$$H^3(-c^2) = (\mathbb{R}^3(x^1, x^2, x^3), e^{-2cx^3} \{ (dx^1)^2 + (dx^2)^2 \} + (dx^3)^2).$$

This matrix group model $G(c, c)$ is used by Góes–Simões [2] and Kokubu [5].

EXAMPLE 2.3. (Riemannian product $H^2(-c^2) \times \mathbb{E}^1$) Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous space is \mathbb{R}^3 with metric:

$$(dx^1)^2 + e^{-2cx^3}(dx^2)^2 + (dx^3)^2.$$

Hence $G(0, c)$ is identified with the Riemannian direct product of the Euclidean line $\mathbb{E}^1(x^1)$ and the warped product model

$$(\mathbb{R}^2(x^2, x^3), e^{-2cx^3}(dx^2)^2 + (dx^3)^2)$$

of $H^2(-c^2)$. Thus $G(0, c)$ is identified with $\mathbb{E}^1 \times H^2(-c^2)$.

EXAMPLE 2.4. (Solvmanifold) The model space Sol of the 3-dimensional *solvegeometry* [11] is $G(1, -1)$. The Lie group $G(1, -1)$ is isomorphic to the Minkowski motion group

$$E(1, 1) := \left\{ \left(\begin{array}{ccc} e^{x^3} & 0 & x^1 \\ 0 & e^{-x^3} & x^2 \\ 0 & 0 & 1 \end{array} \right) \mid x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The full isometry group is $G(1, -1)$ itself. The homogeneous space

$$G(1, -1) = G(1, -1)/\{E\}$$

is the only proper simply connected generalised Riemannian symmetric space of dimension 3. Here E is the identity matrix.

REMARK 2.1. Let $H^2(y_1, y_2)$ be the upper half plane model of the hyperbolic 2-space of constant curvature -1 :

$$H^2(y^1, y^2) = \left(\{ (y^1, y^2) \in \mathbb{R}^2 \mid y^2 > 0 \}, \{ (dy^1)^2 + (dy^2)^2 \} / (y^2)^2 \right).$$

Consider the warped product $H^2(y^1, y^2) \times_{y^2} \mathbb{E}^1(y^3)$ with warped product metric

$$\frac{(dy^1)^2 + (dy^2)^2}{(y^2)^2} + (y^2)^2(dy^3)^2.$$

Then it is easy to verify that this warped product is isometric to $E(1, 1)$. In fact, the mapping $(y^1, y^2, y^3) := (x^1, e^{x^3}, x^2)$ is an isometry from $E(1, 1)$ onto $H^2(y^1, y^2) \times_{y^2} \mathbb{E}^1(y^3)$.

Kokubu showed that every product minimal surface in the Riemannian product $\mathbb{E}^3(y^1, y^2, y^3) = \mathbb{E}^2(y^1, y^2) \times \mathbb{E}^1(y^3)$ is minimal in the warped product $H^2(y^1, y^2) \times_{y^2} \mathbb{E}(y^3)$ (see [4, Example 3.1]).

In particular, every (totally geodesic) plane $ay^1 + by^2 + c = 0$ in the Euclidean 3-space $\mathbb{E}^3(y^1, y^2, y^3)$ is also minimal in this warped product. These planes are totally geodesic in the warped product if and only if $y^1 = \text{constant}$. Hence we notice that every plane “ $x^1 = \text{constant}$ ” in $G(1, -1)$ is a totally geodesic surface.

EXAMPLE 2.5. ($H^2 \times_{(y^2)^2} S^1$) Take $(\mu_1, \mu_2) = (-2, 1)$. Then the resulting homogeneous space is \mathbb{R}^3 with metric $e^{4x^3}(dx^1)^2 + e^{-2x^3}(dx^2)^2 + (dx^3)^2$. Under the coordinate transformation: $(y^1, y^2, y^3) := (x^2, e^{x^3}, x^1)$, this homogeneous space is represented as the warped product $H^2 \times_f \mathbb{E}$ with base

$$H^2 = \left(\{(y^1, y^2) \in \mathbb{R}^2 \mid y^2 > 0\}, \{(dy^1)^2 + (dy^2)^2\} / (y^2)^2 \right),$$

standard fibre $\mathbb{E}^1 = (\mathbb{R}(y^3), (dy^3)^2)$, and the warping function $f(y^1, y^2) = (y^2)^2$. This metric induces a Riemannian metric on the coset space $G(-2, 1)/\Gamma(-2, 1)$, where the discrete subgroup $\Gamma(-2, 1)$ is $\{(2\pi n, 0, 0) \in G(-2, 1) \mid n \in \mathbb{Z}\}$. Kokubu has shown that the catenoid in Euclidean 3-space $G(0, 0)$ is naturally regarded as a minimal surface in $G(-2, 1)/\Gamma(-2, 1)$ ([4, Example 3.3]). Note that the helicoid $z = \tan^{-1}(y/x)$ in Euclidean 3-space is naturally regarded as a “rotational” minimal surface in $\tilde{E}(2)/\Gamma$, where $\tilde{E}(2)$ is the universal covering of the Euclidean motion group $E(2)$ with flat metric and Γ is the discrete subgroup defined by $\Gamma := \{(0, 0, 2\pi n) \mid n \in \mathbb{Z}\}$. (See [3, p. 83].)

EXAMPLE 2.6. Let D be the distribution spanned by e_1 and e_2 . Since $[e_1, e_2] = 0$, this distribution is involutive. Now let M be the maximal integral surface of D through a point (x_0^1, x_0^2, x_0^3) . Then (2.4) implies that M is flat and of constant mean curvature $(\mu_1 + \mu_2)/2$. Moreover, one can check that this maximal integral surface is the plane $x^3 = x_0^3$.

- (1) If $(\mu_1, \mu_2) = (0, 0)$ then M is a totally geodesic plane.
- (2) If $\mu_1 = \mu_2 = c \neq 0$. Then M is a horosphere in the hyperbolic 3-space $H^3(-c^2)$.
- (3) If $\mu_1 = -\mu_2 \neq 0$. Then M is a non-totally geodesic minimal surface.

REMARK 2.2. Let $\text{Gr}_2(TG)$ the Grassmann bundle of 2-planes over the Lie group $G = G(\mu_1, \mu_2)$. Take a nonempty subset Σ of $\text{Gr}_2(TG)$. A surface M in G is said to be a Σ -surface if all the tangent planes of M belong to Σ . The collection of Σ -surfaces is called the Σ -geometry. In particular, if Σ is an orbit of G -action on $\text{Gr}_2(TG)$, then Σ -geometry is said to be of orbit type. Now we regard G as a homogeneous space $G/\{E\}$. Then every G -orbit in $\text{Gr}_2(TG)$ is a homogeneous subbundle of $\text{Gr}_2(TG)$. Hence the orbit space is identified with the Grassmann manifold $\text{Gr}_2(\mathfrak{g}(\mu_1, \mu_2))$. Take a unit vector W

in the Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ and denote by Π_W the linear 2-plane in $\mathfrak{g}(\mu_1, \mu_2)$ orthogonal to W . Let $\mathcal{O}(W)$ the orbit containing Π_W . Recently, Kuwabara [6] investigated $\mathcal{O}(W)$ -surfaces in $G(\mu_1, \mu_2)$ with $\mu_1 = -\mu_2 \neq 0$.

3. INTEGRAL REPRESENTATION FORMULA

Here we recall the integral representation formula obtained in the previous paper [3].

Let M be a Riemann surface and (\mathcal{D}, z) be a simply connected coordinate region. The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z} dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z},$$

with respect to the conformal structure of M . Take a triplet $\{\omega^1, \omega^2, \omega^3\}$ of (1,0)-forms which satisfies the following differential system:

$$(3.1) \quad \bar{\partial}\omega^i = \mu_i \bar{\omega}^i \wedge \omega^3, \quad i = 1, 2;$$

$$(3.2) \quad \bar{\partial}\omega^3 = \mu_1 \omega^1 \wedge \bar{\omega}^1 + \mu_2 \omega^2 \wedge \bar{\omega}^2.$$

PROPOSITION 3.1. *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to (3.1)-(3.2) on a simply connected coordinate region \mathcal{D} . Then*

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^z \operatorname{Re}(e^{\mu_1 x^3(z, \bar{z})} \cdot \omega^1, e^{\mu_2 x^3(z, \bar{z})} \cdot \omega^2, \omega^3)$$

is a harmonic map of \mathcal{D} into $G(\mu_1, \mu_2)$. Conversely, any harmonic map of \mathcal{D} into $G(\mu_1, \mu_2)$ can be represented in this form.

Equivalently, the resulting harmonic map $\varphi(z, \bar{z})$ is defined by the following data:

$$(3.3) \quad \omega^1 = e^{-\mu_1 x^3} x_z^1 dz, \quad \omega^2 = e^{-\mu_1 x^3} x_z^2 dz, \quad \omega^3 = x_z^3 dz,$$

where the coefficient functions are solutions to

$$(3.4) \quad x_{z\bar{z}}^i - \mu_i (x_z^3 x_{\bar{z}}^i + x_{\bar{z}}^3 x_z^i) = 0, \quad (i = 1, 2)$$

$$(3.5) \quad x_{z\bar{z}}^3 + \mu_1 e^{-2\mu_1 x^3} x_z^1 x_{\bar{z}}^1 + \mu_2 e^{-2\mu_2 x^3} x_z^2 x_{\bar{z}}^2 = 0.$$

COROLLARY 3.1. *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to*

$$(3.6) \quad \bar{\partial}\omega^i = \mu_i \bar{\omega}^i \wedge \omega^3, \quad i = 1, 2;$$

$$(3.7) \quad \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0$$

on a simply connected coordinate region \mathcal{D} . Then

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^z \operatorname{Re}(e^{\mu_1 x^3(z, \bar{z})} \cdot \omega^1, e^{\mu_2 x^3(z, \bar{z})} \cdot \omega^2, \omega^3)$$

is a weakly conformal harmonic map of \mathcal{D} into $G(\mu_1, \mu_2)$. Moreover $\varphi(z, \bar{z})$ is a minimal immersion if and only if

$$\omega^1 \otimes \bar{\omega}^1 + \omega^2 \otimes \bar{\omega}^2 + \omega^3 \otimes \bar{\omega}^3 \neq 0.$$

In particular for the product space $\mathbb{E}^1 \times H^2$, we have the following result.

COROLLARY 3.2. Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to

$$(3.8) \quad \bar{\partial}\omega^1 = 0, \quad \bar{\partial}\omega^2 = c\bar{\omega}^2 \wedge \omega^3;$$

$$(3.9) \quad \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0$$

on a simply connected coordinate region \mathcal{D} . Then

$$(3.10) \quad \varphi(z, \bar{z}) = 2 \int_{z_0}^z \operatorname{Re}(\omega^1, e^{cx^3(z, \bar{z})} \cdot \omega^2, \omega^3)$$

is a weakly conformal harmonic map of \mathcal{D} into the product space $G(0, c)$. Moreover $\varphi(z, \bar{z})$ is a minimal immersion if and only if

$$(3.11) \quad \omega^1 \otimes \bar{\omega}^1 + \omega^2 \otimes \bar{\omega}^2 + \omega^3 \otimes \bar{\omega}^3 \neq 0.$$

REMARK 3.1. The representation formula for minimal surfaces in $G(0, 1) = \mathbb{E}^1 \times H^2$ obtained by Mercuri–Montaldo–Piu [7, Theorem 5.1] coincides with (3.10). In [7], they assumed that the data $(\omega^1, \omega^2, \omega^3)$ satisfies (3.8), (3.9), (3.11) and the equation:

$$(3.12) \quad \bar{\partial}\omega^3 = \omega^2 \wedge \bar{\omega}^2.$$

However the equations (3.8)–(3.9) imply (3.11)–(3.12) under the assumption: there are no points on \mathcal{D} on which both ω^3 and $\bar{\partial}\omega^3$ vanish (see [5, Lemma 4.5]).

4. THE NORMAL GAUSS MAP

Let $\varphi : M \rightarrow G(\mu_1, \mu_2)$ be a conformal immersion. Take a unit normal vector field N along φ . Then, by the left translation we obtain the following smooth map:

$$\psi := \varphi^{-1} \cdot N : M \rightarrow S^2 \subset \mathfrak{g}(\mu_1, \mu_2).$$

The resulting map ψ takes value in the unit sphere in the Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$. Here, via the orthonormal basis $\{E_1, E_2, E_3\}$, we identify $\mathfrak{g}(\mu_1, \mu_2)$ with Euclidean 3-space $\mathbb{E}^3(u^1, u^2, u^3)$.

The smooth map ψ is called the *normal Gauss map* of φ .

Let $\varphi : \mathcal{D} \rightarrow G(\mu_1, \mu_2)$ be a weakly conformal harmonic map of a simply connected Riemann surface \mathcal{D} determined by the data $(\omega^1, \omega^2, \omega^3)$. Express the data as $\omega^i = \phi^i dz$. Then the induced metric I of φ is

$$I = 2 \left(\sum_{i=1}^3 |\phi^i|^2 \right) dzd\bar{z}.$$

Moreover these three coefficient functions satisfy

$$(4.1) \quad \frac{\partial \phi^3}{\partial \bar{z}} = - \sum_{i=1}^2 \mu_i |\phi^i|^2, \quad \frac{\partial \phi^i}{\partial \bar{z}} = \mu_i \bar{\phi}^i \phi^3, \quad i = 1, 2, \\ (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0.$$

The harmonic map φ is a minimal immersion if and only if

$$(4.2) \quad |\phi^1|^2 + |\phi^2|^2 + |\phi^3|^2 \neq 0.$$

Here we would like to remark that ϕ^3 is identically zero if and only if φ is a plane $x^3 = \text{constant}$. (See Example 2.6.) As we saw in Example 2.6, φ is minimal if and only if $\mu_1 + \mu_2 = 0$.

Hereafter we assume that ϕ^3 is not identically zero. Then we can introduce two mappings f and g by

$$(4.3) \quad f := \phi^1 - \sqrt{-1} \phi^2, \quad g := \frac{\phi^3}{\phi^1 - \sqrt{-1} \phi^2}.$$

By definition, f and g take values in the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using these two $\bar{\mathbb{C}}$ -valued functions, φ is rewritten as

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^z \text{Re} \left(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \right) dz.$$

The normal Gauss map is computed as

$$\psi(z, \bar{z}) = \frac{1}{1 + |g|^2} \left(2 \text{Re}(g) E_1 + 2 \text{Im}(g) E_2 + (|g|^2 - 1) E_3 \right).$$

Under the stereographic projection $\mathcal{P} : S^2 \setminus \{\infty\} \subset \mathfrak{g}(\mu_1, \mu_2) \rightarrow \mathbb{C} := \mathbb{R}E_1 + \mathbb{R}E_2$, the map ψ is identified with the $\bar{\mathbb{C}}$ -valued function g . Based on this fundamental observation, we call the function g the *normal Gauss map* of φ . The harmonicity together with the integrability (3.4)–(3.5) are equivalent to the following system for f and g :

$$(4.4) \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} |f|^2 g \{ \mu_1 (1 - \bar{g}^2) - \mu_2 (1 + \bar{g}^2) \},$$

$$(4.5) \quad \frac{\partial g}{\partial \bar{z}} = -\frac{1}{4} \{ \mu_1 (1 + g^2)(1 - \bar{g}^2) + \mu_2 (1 - g^2)(1 + \bar{g}^2) \} \bar{f}.$$

THEOREM 4.1. *Let f and g be a $\bar{\mathbb{C}}$ -valued functions which are solutions to the system: (4.4)–(4.5). Then*

$$(4.6) \quad \varphi(z, \bar{z}) = 2 \int_{z_0}^z \text{Re} \left(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \right) dz$$

is a weakly conformal harmonic map of \mathcal{D} into $G(\mu_1, \mu_2)$.

PROOF: Since the harmonicity together with integrability is equivalent to (4.4)–(4.5), Proposition 3.1 implies the result. □

EXAMPLE 4.1. For the space form $G(c, c)$ of curvature $-c^2$, (4.4)–(4.5) reduces to

$$\frac{\partial f}{\partial \bar{z}} = -c|f|^2|g|^2\bar{g}, \quad \frac{\partial g}{\partial \bar{z}} = -\frac{c}{2}\bar{f}(1 - |g|^4).$$

In particular, for Euclidean 3-space, we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = 0.$$

In the case of hyperbolic 3-space $H^3(-c^2)$, one can deduce that g is a solution to the partial differential equation:

$$(4.7) \quad \frac{\partial^2 g}{\partial z \partial \bar{z}} + \frac{2|g|^2\bar{g}}{1 - |g|^4} \frac{\partial g}{\partial z} \frac{\partial g}{\partial \bar{z}} = 0.$$

The equation (4.7) means that g is a harmonic map into the extended complex plane $\bar{\mathbb{C}}(w)$ with singular metric (so-called *Kokubu metric*) $dwd\bar{w}/(1 - |w|^4)$.

EXAMPLE 4.2. For $G(1, -1) = E(1, 1)$, (4.4)–(4.5) reduces to

$$\frac{\partial f}{\partial \bar{z}} = |f|^2g, \quad \frac{\partial g}{\partial \bar{z}} = -\frac{1}{2}(g + \bar{g})(g - \bar{g})\bar{f}.$$

EXAMPLE 4.3. For $G(0, c)$, f and g are solutions to

$$\frac{\partial f}{\partial \bar{z}} = -\frac{c}{2}|f|^2(1 + \bar{g}^2), \quad \frac{\partial g}{\partial \bar{z}} = -\frac{c}{4}(1 - g^2)(1 + \bar{g}^2)\bar{f}.$$

EXAMPLE 4.4. Assume that $\mu_1 \neq 0$. Take the following two $\bar{\mathbb{C}}$ -valued functions:

$$f = \frac{\sqrt{-1}}{\mu_1(z + \bar{z})}, \quad g = -\sqrt{-1}.$$

Then f and g are solutions to (4.4)–(4.5). By the integral representation formula, we can see that the minimal surface determined by the data (f, g) is a plane $x^2 = \text{constant}$. Note that this plane is totally geodesic in $G(1, -1)$.

EXAMPLE 4.5. Consider the product space $G(0, 1)$, and take the following two functions f and g defined on \mathbb{R}^2 .

$$\frac{\sqrt{-1}(f - 1)}{2} = \frac{\tan(2y)(\cos(2x) + \sin(2y)) + \sqrt{-1} \sin(2x)}{2 - \sin(2(x - y)) + \sin(2(x + y))},$$

$$1 - g^2 = \frac{2}{f}, \quad z = x + \sqrt{-1}y.$$

Then (f, g) is a solution to (4.4)–(4.5). Moreover it is easy to see that $\phi^1 = 1, (\phi^2)^2 + (\phi^3)^2 = -1$. One can check that the minimal surface determined by the data (f, g) is the *minimal helicoid* in the sense of Nelli and Rosenberg [8] (See also [7, Example 5.2]).

REMARK 4.1. In [7], the following two auxiliary functions were introduced.

$$\mathbf{G}^2 = \frac{f}{2}, \quad \mathbf{H} = g \cdot \mathbf{G}.$$

Then we have

$$\phi^1 = \mathbf{G}^2 - \mathbf{H}^2, \quad \phi^2 = \sqrt{-1}(\mathbf{G}^2 + \mathbf{H}^2), \quad \phi^3 = 2\mathbf{GH}.$$

These functions are solutions to the system:

$$\begin{aligned} \mathbf{G}_z &= \frac{\mathbf{H}}{2} \{ \mu_1(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) - \mu_2(\mathbf{G}^2 + \mathbf{H}^2) \}, \\ \mathbf{H}_{\bar{z}} &= -\frac{1}{2\mathbf{G}} \{ \mu_1(\mathbf{G}^2 + \mathbf{H}^2)(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) + \mu_2(\mathbf{G}^2 - \mathbf{H}^2)(\overline{\mathbf{G}}^2 + \overline{\mathbf{H}}^2) \} \\ &\quad + \frac{\mathbf{H}^2}{2\mathbf{G}} \{ \mu_1(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) - \mu_2(\mathbf{G}^2 + \mathbf{H}^2) \}. \end{aligned}$$

The integral representation formula is rewritten as

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^z \operatorname{Re}(e^{\mu_1 x^3}(\mathbf{G}^2 - \mathbf{H}^2), \sqrt{-1}e^{\mu_2 x^3}(\mathbf{G}^2 + \mathbf{H}^2), 2\mathbf{GH}) dz.$$

In [1], Berdinskiĭ and Taĭmanov obtained a Weierstrass type representaion for minimal surfaces in Sol in terms of spinors and Dirac operators.

REFERENCES

- [1] D.A. Berdinskiĭ and I.A. Taĭmanov, 'Surfaces in three-dimensional Lie groups', (in Russian), *Sibirsk. Mat. Zh.* **46** (2005), 1248–1264. English Translation: *Siberian Math. J.* **46** (2005), 1005–1019.
- [2] C.C. Góes and P.A.Q. Simões, 'The generalized Gauss map of minimal surfaces in H^3 and H^4 ', *Bol. Soc. Brasil Mat.* **18** (1987), 35–47.
- [3] J. Inoguchi, 'Minimal surfaces in 3-dimensional solvable Lie groups', *Chinese Ann. Math. Ser. B* **24** (2003), 73–84.
- [4] M. Kokubu, 'On minimal submanifolds in product manifolds with a certain Riemannian metric', *Tsukuba J. Math.* **20** (1996), 191–198.
- [5] M. Kokubu, 'Weierstrass representation for minimal surfaces in hyperbolic space', *Tôhoku Math. J.* **49** (1997), 367–377.
- [6] K. Kuwabara, 'Grassmann geometry on the groups of rigid motions on the Euclidean and Minkowski planes', *Tsukuba J. Math.* (to appear).
- [7] F. Mercuri, S. Montaldo and P. Piu, 'A Weierstrass representation formula for minimal surfaces in \mathbb{H}_3 and $\mathbb{H}^2 \times \mathbb{R}$ ', *Acta Math. Sin.* (to appear).
- [8] B. Nelli and H. Rosenberg, 'Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ ', *Bull. Brazil Math. Soc. (N.S.)* **33** (2002), 263–292.
- [9] R. Pedrosa and M. Ritoré, 'Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems', *Indiana Univ. Math. J.* **48** (1999), 1357–1349.

- [10] H. Rosenberg, 'Minimal surfaces in $M^2 \times \mathbf{R}$ ', *Illinois J. Math.* **46** (2002), 1177–1195.
- [11] W.M. Thurston, *Three-dimensional geometry and topology I*, Princeton Math. Series. **35** (Princeton University Press, Princeton N.J., 1997).

Department of Mathematics Education
Faculty of Education
Utsunomiya University
Utsunomiya, 321-8505
Japan
e-mail: inoguchi@cc.utsunomiya-u.ac.jp