ITOH, T.
KŌDAI MATH. SEM. REP
25 (1973), 202–214

MINIMAL SURFACES IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

BY TAKEHIRO ITOH

For surfaces in a 4-dimensional Riemannian manifold of constant curvature, the author [3] proved the following

THEOREM. *Let M be a 2-dimensional connected compact Riemannian manifold which is minimally immersed in a unit sphere of dimension 4. If the normal scalar curvature K_N is non-zero constant, then M may be regarded as a Veronese surface.*

In this paper, he generalizes the above theorem and proves the following

THEOREM. *Let M be a 2-dimensional connected compact Riemannian manifold which is minimally immersed in a $(2+\nu)$-dimensional unit sphere $S^{2+\nu}$. If the normal scalar curvature K_N is non-zero constant and the square of the second curvature k_2 is less then $K_N/4$, then M is a generalized Veronese surface.*

By a generalized Veronese surface we mean a surface defined by Ōtsuki [6].

§ 1. Preliminaries.

Let $\bar{M}$ be a $(2+\nu)$-dimensional Riemannian manifold of constant curvature $\bar{c}$ and M be a 2-dimensional Riemannian manifold immersed isometrically in $\bar{M}$ by the immersion x: $M \to \bar{M}$. $F(\bar{M})$ and $F(M)$ denote the orthonormal frame bundles over $\bar{M}$ and M respectively. Let B be the set of all elements $b=(p, e_1, e_2, e_3, \cdots, e_{2+\nu})$ such that $(p, e_1, e_2) \in F(M)$ and $(p, e_1, e_2, e_3, \cdots, e_{2+\nu}) \in F(\bar{M})$ identifying $p \in M$ with $x(p)$ and e_i with $dx(e_i)$, $i=1, 2$. Then B is naturally considered as a smooth submanifold of $F(\bar{M})$. Let $\bar{\omega}_A$, $\bar{\omega}_{AB}=-\bar{\omega}_{BA}$, $A, B=1, 2, 3, \cdots, 2+\nu$, be the basic and connection forms of $\bar{M}$ on $F(\bar{M})$ which satisfy the structure equations:

$$(1.1)\qquad \begin{aligned} d\bar{\omega}_A &= \sum_B \bar{\omega}_{AB} \wedge \bar{\omega}_B, \\ d\bar{\omega}_{AB} &= \sum_C \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} - \bar{c}\,\bar{\omega}_A \wedge \bar{\omega}_B. \end{aligned}$$

Received May 18, 1972.

In this paper, we use the following convention on the range of indices:

$$1\leqq A, B, C, \cdots \leqq 2+\nu, \qquad 1\leqq i, j, \cdots \leqq 2, \qquad 3\leqq \alpha, \beta, \gamma, \cdots \leqq 2+\nu.$$

Deleting the bars of $\bar{\omega}_A, \bar{\omega}_{AB}$ on B, as is well known, we have

$$\omega_\alpha=0, \tag{1.2}$$

$$\omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j, \qquad A_{\alpha ij}=A_{\alpha ji}, \tag{1.3}$$

$$d\omega_i=\omega_{ij}\wedge\omega_j, \qquad i\neq j, \tag{1.4}$$

$$d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{ki}-\Omega_{ij}, \qquad \Omega_{ij}=\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{1.5}$$

$$R_{ijkl}=\bar{c}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_\alpha (A_{\alpha ik}A_{\alpha jl}-A_{\alpha il}A_{\alpha jk}), \tag{1.6}$$

$$d\omega_{\alpha\beta}=\sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\Omega_{\alpha\beta}, \qquad \Omega_{\alpha\beta}=\frac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i\wedge\omega_j, \tag{1.7}$$

$$R_{\alpha\beta ij}=\sum_k (A_{\alpha ik}A_{\beta jk}-A_{\alpha jk}A_{\beta ik}). \tag{1.8}$$

M is said to be *minimal* if its mean curvature vector $(1/2)\sum_{\alpha,i} A_{\alpha ii}e_\alpha$ vanishes identically, i.e., if trace $A_\alpha=0$ for all α, $A_\alpha=(A_{\alpha ij})$. We say the dimension of the linear space of all second fundamental forms corresponding to normal vectors at $p\in M$ with vanishing trace the *minimal index* at p and denote it by m-index$_pM$. We have easily

$$m\text{-index}_pM\leqq 2 \qquad \text{at each point} \quad p\in M. \tag{1.9}$$

We denote the square of the norm of the system of all 2nd fundamental forms by

$$S=\frac{1}{2}\sum_{\alpha\, i,j} A_{\alpha ij}A_{\alpha ij}=\sum_\alpha \|A_\alpha\|^2, \tag{1.10}$$

where for symmetric matrices A, B we define the inner product of A and B by

$$\langle A, B\rangle=\frac{1}{2}\text{trace } AB.$$

We define the *normal scalar curvature* K_N of M in $\bar{M}$ as follows:

$$K_N=\sum_{i<j,\, \alpha<\beta} R_{\alpha\beta ij}R_{\alpha\beta ij}=\sum_{i<j,\, \alpha<\beta}\Big(\sum_k (A_{\alpha ik}A_{\beta jk}-A_{\alpha jk}A_{\beta ik})\Big)^2. \tag{1.11}$$

Now, we assume that M is minimal in $\bar{M}$ and K_N is non-zero non M. Then we have

(1.12) $$m\text{-index}_p M=2 \quad \text{at each point} \quad p \text{ on } M.$$

Hence, as stated in [5], we can decompose the normal space N_p at $p\in M$ as follows:

(1.13) $$N_p=N'_p+O_p, \qquad N'_p\perp O_p, \qquad O_p=\phi_b^{-1}(0),$$
$$\dim N'_p=2,$$

where ϕ_b is a linear mapping from N_p into the set of all symmetric matrices of order 2 defined by $\phi_b(\sum_\alpha v_\alpha e_\alpha)=\sum_\alpha v_\alpha A_\alpha$. This decomposition does not depend on the choice of a frame b over p and is smooth. Let B_0 be the set of all $b\in B$ such that $e_3, e_4\in N'_p$. Then B_0 is a smooth submanifold of B. On B_0, we have

(1.14) $$\omega_{i\beta}=0, \quad \text{i.e.,} \quad A_\beta=0 \qquad \text{for} \quad \beta>4.$$

Therefore we have

(1.15) $$K_N=R_{3412}^2=\Big(\sum_k (A_{31k}A_{42k}-A_{32k}A_{41k})\Big)^2.$$

As a special case of [6], we can verify the following

LEMMA 1. *On B_0, for a fixed $\beta>4$, we have $\omega_{3\beta}\equiv\omega_{4\beta}\equiv 0$ (mod ω_1, ω_2) and $\omega_{3\beta}=\omega_{4\beta}=0$ or else $\omega_{3\beta}\wedge\omega_{4\beta}\neq 0$.*

Now, by virtue of Lemma 1, we can define two linear mappings φ_{11} and φ_{12} from M_p into O_p corresponding to the normal vector e_3 and e_4 as follows: for any $X\in M_p$,

(1.16) $$\varphi_{11}(X)=\sum_{\beta>4}||A_3||\cdot\omega_{3\beta}(X)e_\beta, \qquad \varphi_{12}(X)=\sum_{\beta>4}||A_4||\cdot\omega_{4\beta}(X)e_\beta.$$

As stated in [5], these two linear mappings have the same image of the tangent unit sphere $S_p^1=\{X\in M_p\colon ||X||=1\}$ and $\varphi_{11}(X)$ and $\varphi_{12}(X)$ are conjugate to each other with respect to the image when it is an ellipse. We define the second curvature $k_2(p)$ of M at p by

(1.17) $$k_2(p)=\max_{X\in S_p^1}||\varphi_{11}(X)||=\max_{X\in S_p^1}||\varphi_{12}(X)||.$$

It is clear that $k_2(p)$ is continuous on M.

§ 2. Minimal surfaces with non-zero constant K_N.

In this section we assume that *M is connected, compact and minimal in $\bar{M}$, K_N is non-zero constant and $4k_2^2<K_N$ at each point of M.*

LEMMA 2. *We have identically*

(2.1) $$S^2=K_N \qquad \textit{on} \quad M.$$

Proof. From (1.10), (1.14) and (1.15), we have

$$S^2-K_N=\{(A_{311}-A_{412})^2+(A_{312}+A_{411})^2\}\{(A_{311}+A_{412})^2+(A_{312}-A_{411})^2\}\geqq 0. \tag{2.2}$$

Hence, if it is not identically $S^2=K_N$, the function S^2-K_N takes its positive maximum at some point $p_0\in M$, because M is compact. Let U be a neighborhood of p_0 in which we can choose $b\in B_0$ such that

$$A_3=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix},\quad A_4=\begin{pmatrix}0 & \mu\\ \mu & 0\end{pmatrix},\quad \lambda^2>\mu^2>0, \tag{2.3}$$

where λ and μ are differentiable functions on U. Then we have

$$K_N=4\lambda^2\mu^2,\qquad S=\lambda^2+\mu^2.$$

From (1.1) and (2.3) we have

$$\begin{aligned} &d\lambda\wedge\omega_1+(2\lambda\omega_{12}-\mu\omega_{34})\wedge\omega_2=0,\\ &d\lambda\wedge\omega_2-(2\lambda\omega_{12}-\mu\omega_{34})\wedge\omega_1=0;\end{aligned} \tag{2.4}$$

$$\begin{aligned} &d\mu\wedge\omega_1+(2\mu\omega_{12}-\lambda\omega_{34})\wedge\omega_2=0,\\ &d\mu\wedge\omega_2-(2\mu\omega_{12}-\lambda\omega_{34})\wedge\omega_1=0.\end{aligned} \tag{2.5}$$

Since $K_N=4\lambda^2\mu^2$ is constant, from (2.4) and (2.5) we have

$$4\lambda\mu\omega_{12}=(\lambda^2+\mu^2)\omega_{34},$$

and hence

$$K_N\omega_{12}=\lambda\mu S\omega_{34}. \tag{2.6}$$

Differentiating both sides of (2.6), we get

$$\begin{aligned} K_N d\omega_{12}&=\lambda\mu dS\wedge\omega_{34}+\lambda\mu S d\omega_{34}\\ &=\lambda\mu dS\wedge\omega_{34}-2\lambda^2\mu^2 S\omega_1\wedge\omega_2-\lambda\mu S\sum_{\beta>4}\omega_{3\beta}\wedge\omega_{4\beta}.\end{aligned} \tag{2.7}$$

On the other hand, since $\omega_{i\beta}=0$ $(\beta>4)$, $i=1,2$, we have

$$d\omega_{i\beta}=\omega_{i3}\wedge\omega_{3\beta}+\omega_{i4}\wedge\omega_{4\beta}=0,$$

which reduce to

$$\lambda\omega_{3\beta}\wedge\omega_1+\mu\omega_{4\beta}\wedge\omega_2=0,$$

$$\lambda\omega_{3\beta}\wedge\omega_2-\mu\omega_{4\beta}\wedge\omega_1=0.$$

By Cartan's Lemma, we may put

$$\lambda\omega_{3\beta}=f_\beta\omega_1+g_\beta\omega_2,$$

$$\mu\omega_{4\beta}=g_\beta\omega_1-f_\beta\omega_2,$$

and define two normal vectors $F_1=\sum_{4<\alpha} f_\alpha e_\alpha$ and $G_1=\sum_{4<\alpha} g_\alpha e_\alpha$. Then we have

$$\lambda\mu\sum_{\beta>4}\omega_{3\beta}\wedge\omega_{4\beta}=-(||F_1||^2+||G_1||^2)\omega_1\wedge\omega_2. \tag{2.8}$$

By means of Cartan's Lemma, from (2.4) and (2.5) we have

$$2\lambda\omega_{12}-\mu\omega_{34}=\lambda_2\omega_1-\lambda_1\omega_2,$$

$$2\mu\omega_{12}-\lambda\omega_{34}=\mu_2\omega_1-\mu_1\omega_2,$$

putting $d\lambda=\lambda_1\omega_1+\lambda_2\omega_2$ and $d\mu=\mu_1\omega_1+\mu_2\omega_2$. Thus we get

$$(\lambda^2-\mu^2)\omega_{34}=(\lambda_2\mu-\lambda\mu_2)\omega_1-(\lambda_1\mu-\lambda\mu_1)\omega_2.$$

Since $\lambda\mu$=constant and hence $\lambda\mu_i+\lambda_i\mu=0$, we have

$$\omega_{34}=\frac{2\mu}{\lambda^2-\mu^2}(\lambda_2\omega_1-\lambda_1\omega_2)$$

and

$$\lambda\mu dS=2\mu(\lambda^2-\mu^2)d\lambda=2\mu(\lambda^2-\mu^2)(\lambda_1\omega_1+\lambda_2\omega_2).$$

Hence we have

$$\lambda\mu dS\wedge\omega_{34}=-4\mu^2||\nabla\lambda||^2\omega_1\wedge\omega_2, \tag{2.9}$$

where $\nabla\lambda$ is the gradient vector of λ. From (2.7), (2.8) and (2.9), we have

$$KK_N=4\mu^2||\nabla\lambda||^2+\frac{K_N S}{2}-S(||F_1||^2+||G_1||^2), \tag{2.10}$$

where K is the Gaussian curvature of M. Since $K_N>4k_2^2\geqq 0$, $2k_2^2\geqq||F_1||^2+||G_1||^2$, and $S>0$, we have

$$KK_N\geqq 4\mu^2||\nabla\lambda||^2+\frac{1}{2}K_N S-2Sk_2^2>0,$$

so that we get

$$K>0 \quad \text{on} \quad U. \tag{2.11}$$

From (2.4) and (2.5), we have

$$d(\lambda^2-\mu^2)\wedge\omega_1+4(\lambda^2-\mu^2)d\omega_1=0,$$

$$d(\lambda^2-\mu^2)\wedge\omega_2+4(\lambda^2-\mu^2)d\omega_2=0,$$

which imply that there exists a neighborhood V of p_0 where we have isothermal coordinates (u, v) such that

$$ds^2=E\{du^2+dv^2\}, \qquad \omega_1=\sqrt{E}\,du, \qquad \omega_2=\sqrt{E}\,dv, \qquad \sqrt{\lambda^2-\mu^2}\,E=1,$$

where $E=E(u, v)$ is a positive function on V. With respect to these isothermal coordinates, K is given by $K=-(1/2E)\,\Delta\log E$. Since $\sqrt{\lambda^2-\mu^2}\,E=1$ and $(\lambda^2-\mu^2)^2=S^2-K_N$, we obtain

$$K=\frac{\sqrt{\lambda^2-\mu^2}}{8}\,\Delta\log(S^2-K_N), \tag{2.12}$$

which, together with (2.11), implies

$$\Delta\log(S^2-K_N)>0 \qquad \text{on} \quad V.$$

Thus $\log(S^2-K_N)$ is a subharmonic function on V and takes its maximum at p_0 by our assumption, so that $\log(S^2-K_N)$ must be constant. Then, (2.12) implies $K=0$, which contradicts $K>0$. Q. E. D.

By Lemma 2, for a frame $b\in B_0$ we have $\|A_3\|=\|A_4\|$ and $\langle A_3, A_4\rangle=0$. Therefore, on a neighborhood $U(p)$ of p of M we can choose a frame field $b\in B_0$ such that

$$A_3=\begin{pmatrix} k_1 & 0 \\ 0 & -k_1 \end{pmatrix}, \qquad A_4=\begin{pmatrix} 0 & k_1 \\ k_1 & 0 \end{pmatrix}, \tag{2.13}$$

where k_1 is non-zero constant on M. It follows from (2.13) that

$$\omega_{34}=2\omega_{12} \tag{2.14}$$

and we may put

$$k_1\omega_{3\beta}=f_\beta\omega_1+g_\beta\omega_2,$$

$$k_1\omega_{4\beta}=g_\beta\omega_1-f_\beta\omega_2, \qquad 4<\beta,$$

as in the proof of Lemma 2. Then, (2.14) implies

$$\|F_1\|^2+\|G_1\|^2=2k_1^2(k_1^2-K),$$

where $F_1=\sum_{4<\beta}f_\beta e_\beta$ and $G_1=\sum_{4<\beta}g_\beta e_\beta$.

Since $\|F_1\|^2+\|G_1\|^2\leqq 2k_2^2<K_N/2=2k_1^4$ and $K=\bar{c}-S=\bar{c}-2k_1^2$, we see

$$K=\text{positive constant on } M. \tag{2.15}$$

Then, we have the following

LEMMA 3. *The image of S_p^1 under φ_{11} (or φ_{12}) is a circle with constant radius $k_2=k_1\sqrt{k_1^2-K}$, where the circle is a point if $k_2=0$ on M.*

Proof. Putting

$$l_2=\operatorname*{Min}_{X\in S_p^1}\|\varphi_{11}(X)\|=\operatorname*{Min}_{X\in S_p^1}\|\varphi_{12}(X)\|,$$

we can see

$$(k_2^2-l_2^2)^2=(\|F_1\|^2-\|G_1\|^2)^2+4\langle F_1,G_1\rangle^2\geqq 0,$$

so $(k_2^2-l_2^2)^2$ is a differentiable function on M, because $\{p,F_1,G_1\}$ obey an analogous rule to the rotation of the 2-frame $\{p,e_3,e_4\}$. Hence, if $k_2=l_2$ does not hold identically on M, then $(k_2^2-l_2^2)^2$ takes its positive maximum at some point p_1 on M. Let U_1 be a neighborhood of p_1 on which $k_2>l_2$ and we can choose isothermal coordinate (u,v) and a frame $b\in B_0$ satisfying (2.13) and

$$ds^2=E\{du^2+dv^2\},\qquad \omega_1=\sqrt{E}\,du,\qquad \omega_2=\sqrt{E}\,dv, \tag{2.16}$$

where $E=E(u,v)$ is a positive function on U_1. Since $\|F_1\|^2+\|G_1\|^2>0$ on U_1, we may assume $F_1\neq 0$ on a small neighborhood V_1 of p_1 in U_1. Then we can choose a frame field $b\in B_0$ satisfying (2.13), (2.16) and

$$F_1=fe_5,\qquad G_1=\sum_{4<\alpha}g_\alpha e_\alpha,$$

where f is a non-zero differentiable function and g_α are differentiable functions on V_1. Then we have

$$k_1\omega_{35}=f\omega_1+g_5\omega_2,\qquad k_1\omega_{3\beta}=g_\beta\omega_2,$$

$$k_1\omega_{45}=g_5\omega_1-f\omega_2,\qquad k_1\omega_{4\beta}=g_\beta\omega_1,\qquad 5<\beta.$$

Using these equations and $\omega_{34}=2\omega_{12}$, from the structure equations we obtain

$$df\wedge\omega_1+dg_5\wedge\omega_2+3fd\omega_1+3g_5d\omega_2=\omega_2\wedge\Big(\sum_{5<\beta}g_\beta\omega_{\beta 5}\Big),$$

$$dg_5\wedge\omega_1-df\wedge\omega_2+3g_5d\omega_1-3fd\omega_2=\omega_1\wedge\Big(\sum_{5<\beta}g_\beta\omega_{\beta 5}\Big),$$

$$dg_\beta\wedge\omega_2+3g_\beta d\omega_2=-f\omega_1\wedge\omega_{\beta 5}-g_5\omega_2\wedge\omega_{\beta 5}+\omega_2\wedge\Big(\sum_\gamma g_\gamma\omega_{\gamma\beta}\Big),$$

$$dg_\beta\wedge\omega_1+3g_\beta d\omega_1=-g_5\omega_1\wedge\omega_{\beta 5}+f\omega_2\wedge\omega_{\beta 5}+\omega_1\wedge\Big(\sum_\gamma g_\gamma\omega_{\gamma\beta}\Big),$$

which imply that the complex valued function $E^3(\|G_1\|^2-\|F_1\|^2)+2iE^3\langle F_1,G_1\rangle$ is holomorphic in $z=u+iv$, so that

$$\begin{aligned}-6\Delta\log E&=\Delta\log\{(\|F_1\|^2-\|G_1\|^2)^2+4\langle F_1,G_1\rangle^2\}\\&=\Delta\log(k_2^2-l_2^2)^2\qquad\text{on}\quad V_1.\end{aligned} \tag{2.17}$$

Since K is given by $K=-(1/2E)\Delta\log E$ and is positive from (2.15), (2.17) implies that $\log(k_2^2-l_2^2)^2$ is a subharmonic function on V_1. Since $(k_2^2-l_2^2)^2$ takes its positive maximum at p_1 in V_1, $\log(k_2^2-l_2^2)^2$ must be constant so that (2.7) implies $K=0$, which contradicts $K>0$. Thus, $k_2=l_2$ holds at every point on M. Furthermore, we have $||F_1||=||G_1||$ and $\langle F_1, G_1\rangle=0$ for any frame $b\in B_0$ satisfying (2.13). Since $2k_2^2=||F_1||^2+||G_1||^2=2k_1^2(k_1^2-K)$ is constant on M, $k_2=l_2=k_1\sqrt{k_1^2-K}$ is constant on M. Q. E. D.

By Lemma 3, if $k_2=0$ on M, then the geodesic codimension of M in $\bar{M}$ is 2, so that M is a Veronese surface (see [3]). If $k_2\neq 0$ on M, then by Lemma 2 and 3, on a neighborhood $U(p)$ of a p of M we can choose a frame field $b\in B_0$ satisfying (2.13) and

$$(2.18)\qquad \begin{aligned} &k_1\omega_{35}=k_2\omega_1=k_1\omega_{46}\\ &k_1\omega_{36}=k_2\omega_2=-k_1\omega_{45}, \qquad \omega_{3\beta}=\omega_{4\beta}=0, \qquad 6<\beta, \end{aligned}$$

where k_2 is non-zero constant on M. Then, from (2.14) and (2.18) we obtain

$$(2.19)\qquad \omega_{56}=3\omega_{12}$$

and we may put

$$\begin{aligned} &k_2\omega_{5\beta}=f_\beta\omega_1+g_\beta\omega_2\\ &k_2\omega_{6\beta}=g_\beta\omega_1-f_\beta\omega_2, \qquad 6<\beta, \end{aligned}$$

where f_β and g_β are differentiable functions on $U(p)$. We consider two linear mappings φ_{21} and φ_{22} from M_p into N_p as follows

$$\begin{aligned} &\varphi_{21}(X)=\sum_\beta k_2\omega_{5\beta}(X)e_\beta=\omega_1(X)F_2+\omega_2(X)G_2,\\ &\varphi_{22}(X)=\sum_\beta k_2\omega_{6\beta}(X)e_\beta=\omega_1(X)G_2-\omega_2(X)F_2, \end{aligned}$$

where X is a tangent vector to M and $F_2=\sum_{6<\beta}f_\beta e_\beta$ and $G_2=\sum_{6<\beta}g_\beta e_\beta$ are normal vector fields on $U(p)$. Using (2.19) and the structure equations, we obtain

$$||F_2||^2+||G_2||^2=k_2^2\left(\frac{2k_2^2}{k_1^2}-3K\right)=\text{constant on } M.$$

In the same manner as the proof of Lemma 3, we can prove the following

LEMMA 4. *If $k_2=constant\neq 0$ on M, the image of S_p^1 under φ_{21} (or φ_{22}) is a circle with constant radius $k_2\sqrt{k_2^2/k_1^2-3K/2}$, where the circle is a point if $2k_2^2=3k_1^2K$ on M.*

If $2k_2^2=3k_1^2K$ on M, then the geodesic codimension of M is 4, because $\omega_{i\beta}=0$ $(4<\beta)$, $\omega_{3\gamma_1}=\omega_{4\gamma_1}=0$ $(6<\gamma_1)$ and $\omega_{5\gamma_2}=\omega_{6\gamma_2}=0$ $(8<\gamma_2)$. Henceforth, we may consider the case $2k_2^2\neq 3k_1^2K$ on M. Then, by Lemmas 2, 3 and 4, on a neighborhood of a point p on M we can choose a frame field $b\in B_0$ satisfying (2.13), (2.18) and the following conditions:

$$k_2\omega_{57}=k_3\omega_1=k_2\omega_{68},$$

$$k_2\omega_{58}=k_3\omega_2=-k_2\omega_{67}, \qquad \omega_{5\gamma}=\omega_{6\gamma}=0, \qquad 8<\gamma,$$

where k_3 is a non-zero constant on M. From the above equations we get

$$\omega_{78}=4\omega_{12}. \tag{2.20}$$

We use the following convention about indices:

$$I_0=\{1, 2\}, \qquad I_t=\{2t+1, 2t+2\}, \qquad t=1, 2, \cdots, m,$$

and if we write $\alpha_1, \alpha_2 \in I_t$, then $\alpha_1<\alpha_2$.

Now we shall prove the following

THEOREM 1. *Let M be a 2-dimensional connected compact Riemannian manifold which is isometrically and minimally immersed in a Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. If the normal scalar curvature K_N is non-zero constant on M and the square of the second curvature k_2 is less than $K_N/4$, then the geodesic codimension of M in $\bar{M}$ is even $2m$ (m is a positive integer), and we can choose a frame $b\in B_0$ such that*

$$\begin{aligned}
&k_{t-1}\omega_{\alpha_1\beta_1}=k_t\omega_1=k_{t-1}\omega_{\alpha_2\beta_2}, \qquad \omega_{\alpha_1\gamma}=0,\\
&k_{t-1}\omega_{\alpha_1\beta_2}=k_t\omega_2=-k_{t-1}\omega_{\alpha_2\beta_1}, \qquad \omega_{\alpha_2\gamma}=0,\\
&\alpha_1, \alpha_2\in I_{t-1}, \qquad \beta_1, \beta_2\in I_t, \qquad 2t+2<\gamma,\\
&t=1, 2, \cdots, m,
\end{aligned} \tag{2.21}$$

where $k_0=1$ and k_t ($2\leqq t\leqq m$) are non-zero constant on M. Furthermore, we obtain

$$\omega_{\alpha_1\alpha_2}=(t+1)\omega_{12}, \qquad \alpha_1, \alpha_2\in I_t \qquad (t=1, 2, \cdots, m), \tag{2.22}$$

$$(t+1)K=\frac{2k_t^2}{k_{t-1}^2}-\frac{2k_{t+1}^2}{k_t^2} \qquad (t=1, \cdots, m-1), \tag{2.23}$$

$$(m+1)K=\frac{2k_m^2}{k_{m-1}^2}. \tag{2.24}$$

Proof. By induction with respect to t, we shall prove the theorem. For $t=1$, 2 and 3, we proved our assertions by Lemmas 2, 3 and 4 respectively. Hence, we suppose that our (2.21), (2.22) and (2.23) hold for all $t\leqq t_0$. In this case, we shall prove that our assertion holds for t_0+1. Then, since $\omega_{\alpha_1\gamma}=\omega_{\alpha_2\gamma}=0$, $\alpha_1, \alpha_2\in I_{t_0-1}$, $2t_0+2<\gamma$, we have

$$k_{t_0}\omega_{\beta_1\gamma}\wedge\omega_1+k_{t_0}\omega_{\beta_2\gamma}\wedge\omega_2=0,$$

$$k_{t_0}\omega_{\beta_1\gamma}\wedge\omega_2-k_{t_0}\omega_{\beta_2\gamma}\wedge\omega_1=0, \qquad \beta_1, \beta_2\in I_{t_0},$$

which, together with Cartan's lemma, imply that we may put

$$k_{t_0}\omega_{\beta_1\gamma}=f_\gamma\omega_1+g_\gamma\omega_2,$$

$$k_{t_0}\omega_{\beta_2\gamma}=g_\gamma\omega_1-f_\gamma\omega_2, \qquad 2t_0+2<\gamma,$$

and define two normal vector fields $F_{t_0}=\sum_\gamma f_\gamma e_\gamma$ and $G_{t_0}=\sum_\gamma g_\gamma e_\gamma$. We consider two linear mappings $\varphi_{t_0 1}$ and $\varphi_{t_0 2}$ from M_p into N_p as follows:

$$\varphi_{t_0 1}(X)=\sum_\gamma k_{t_0}\omega_{\beta_1\gamma}(X)e_\gamma=\omega_1(X)F_{t_0}+\omega_2(X)G_{t_0},$$

$$\varphi_{t_0 2}(X)=\sum_\gamma k_{t_0}\omega_{\beta_2\gamma}(X)e_\gamma=\omega_1(X)G_{t_0}-\omega_2(X)F_{t_0},$$

where X is a tangent vector to M. Putting

$$k_{t_0+1}=\operatorname*{Max}_{X\in S_p^1}||\varphi_{t_0 1}(X)||=\operatorname*{Max}_{X\in S_p^1}||\varphi_{t_0 2}(X)|| \quad \text{and} \quad l_{t_0+1}=\operatorname*{Min}_{X\in S_p^1}||\varphi_{t_0 1}(X)||=\operatorname*{Min}_{X\in S_p^1}||\varphi_{t_0 2}(X)||,$$

we can see

$$(k_{t_0+1}^2-l_{t_0+1}^2)^2=(||F_{t_0}||^2-||G_{t_0}||^2)^2+4\langle F_{t_0}, G_{t_0}\rangle^2,$$

so $(k_{t_0+1}^2-l_{t_0+1}^2)^2$ is a differentiable function on M, because $\{p, F_{t_0}, G_{t_0}\}$ obey an analogous rule to the rotation of the 2-frame $\{p, e_{\beta_1}, e_{\beta_2}\}$. Hence, similarly to the proof of Lemma 3, we can see that $k_{t_0+1}=l_{t_0+1}$ holds everywhere on M. On the other hand, since $\omega_{\beta_1\beta_2}=(t_0+1)\omega_{12}$, we get

$$2k_{t_0+1}^2=||F_{t_0}||^2+||G_{t_0}||^2=k_{t_0}^2\left(\frac{2k_{t_0}^2}{k_{t_0-1}^2}-(t_0+1)K\right)=\text{constant on } M.$$

If $2k_{t_0}^2=(t+1)k_{t_0-1}^2K$ on M, we can see that the geodesic codimension of M is $2t_0$ and (2.24) holds. Therefore, we consider the case $2k_{t_0}^2\neq(t_0+1)k_{t_0-1}^2K$ on M. Then, by the above argument, we can choose a frame field $b\in B_0$ satisfying (2.21) for all $t\leqq t_0$ and

$$k_{t_0}\omega_{\beta_1\gamma_1}=k_{t_0+1}\omega_1=k_{t_0}\omega_{\beta_2\gamma_2}, \qquad \omega_{\beta_1\gamma}=0,$$

$$k_{t_0}\omega_{\beta_1\gamma_2}=k_{t_0+1}\omega_2=-k_{t_0}\omega_{\beta_2\gamma_1}, \qquad \omega_{\beta_2\gamma}=0,$$

$$\gamma_1, \gamma_2\in I_{t_0+1}, \qquad 2t_0+4<\gamma,$$

where k_{t_0+1} is non-zero constant on M, which imply that (2.21), (2.22) and (2.23) hold for t_0+1. Thus, it is clear that the geodesic codimension of M in $\bar{M}$ is even $2m$. Then, since we have $\omega_{2m+1\,2m+2}=(m+1)\omega_{12}$ and $\omega_{2m+1\,\gamma}=\omega_{2m\,\gamma}=0$ $(2m+2<\gamma)$, we obtain (2.24). Q. E. D.

§3. The proof of the main theorem.

By an analogous computation to the one in §4 in [7], from Theorem 1 we obtain the following

THEOREM 2. *Let M be a 2-dimensional connected compact Riemannian manifold which is isometrically and minimally immersed in a Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. If the normal scalar curvature K_N is non-zero constant on M and if the square of the second curvature k_2 is less than $K_N/4$, then the geodesic codimension is even $2m$ and the Gaussian curvature K is positive constant, and supposing $K=1$, there exist m constants $b_t=(m-t+1)(m+t+2)/4$ $1\leqq t\leqq m$, and m complex normal vector fields $\xi_1, \cdots, \xi_m$ such that*

$$\text{(I)}\qquad \begin{aligned} &\xi_t\cdot\xi_s=\xi_t\cdot\bar{\xi}_s=0, \qquad t\neq s,\\ &\xi_t\cdot\xi_t=0, \qquad \xi_t\cdot\bar{\xi}_t=2, \qquad t=2, 3, \cdots, m \end{aligned}$$

and

$$\text{(II)}\qquad \begin{aligned} dx &= \frac{1}{h}(\bar{\xi}_0 dz+\xi_0 d\bar{z}), \qquad \xi_0=e_1+ie_2,\\ \bar{D}\xi_0 &= \frac{1}{h}\xi_0(\bar{z}dz-zd\bar{z}) + \frac{2\sqrt{b_1}}{h}\xi_1 d\bar{z}, \qquad h=1+z\bar{z},\\ \bar{D}\xi_1 &= -\frac{2\sqrt{b_1}}{h}\xi_0 dz+\frac{2}{h}\xi_1(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_2}}{h}\xi_2 d\bar{z},\\ &\cdots\cdots\cdots\cdots\cdots,\\ \bar{D}\xi_t &= -\frac{2\sqrt{b_t}}{h}\xi_{t-1}dz+\frac{t+1}{h}\xi_t(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}d\bar{z},\\ &\cdots\cdots\cdots\cdots\cdots,\\ \bar{D}\xi_m &= -\frac{2\sqrt{b_m}}{h}\xi_{m-1}dz+\frac{m+1}{h}\xi_m(\bar{z}dz-zd\bar{z}), \end{aligned}$$

where z is an isothermal complex coordinate of M and $\bar{D}$ denotes the covariant differentiation of $\bar{M}$.

In Theorem 2 we may consider $\bar{M}^{2+2m}=\bar{M}=S^{2+2m}(R)$, where $S^{2+2m}(R)$ denotes the $(2+2m)$-sphere of radius R:

$$\frac{1}{R^2}=\bar{c}=\frac{(m+1)(m+2)}{2}.$$

We regard as $S^{2+2m}(R)\subset E^{3+2m}$ and put

$$\frac{x}{R}=e_{3+2m}. \tag{3.1}$$

By (3.1) we have

$$dx=Rde_{3+2m}=\frac{1}{h}(\bar{\xi}_0 dz+\xi_0 d\bar{z}). \tag{3.2}$$

From (II) in Theorem 2 and the above relation, we have easily

$$
\begin{aligned}
d\xi_0 &= \frac{1}{h}\xi_0(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{b_1}}{h}\xi_1 dz - \frac{2}{Rh}e_{3+2m}dz,\\
d\xi_1 &= \frac{2\sqrt{b_1}}{h}\xi_0 dz + \frac{2}{h}\xi_1(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{b_2}}{h}\xi_2 d\bar{z},\\
&\cdots\cdots\cdots\cdots\cdots,\\
d\xi_t &= -\frac{2\sqrt{b_t}}{h}\xi_{t-1}dz + \frac{t+1}{h}\xi_t(\bar{z}dz - zd\bar{z}) + \frac{2\sqrt{b_{t+1}}}{h}\xi_{t+1}d\bar{z},\\
&\cdots\cdots\cdots\cdots\cdots,\\
d\xi_m &= -\frac{2\sqrt{b_m}}{h}\xi_{m-1}dz + \frac{m+1}{h}\xi_m(\bar{z}dz - zd\bar{z}),
\end{aligned}
\tag{3.3}
$$

where d denotes the ordinary differential operator in E^{3+2m}. Since equations (3.3) are the same ones as (II) in Theorem 2 in Ōtsuki [7] when we put formally $P = -(1/R)e_{3+2m}$ in the case $\bar{M}^{n+2m} = E^{n+2m}$, M is congruent to the surface given by

$$
\begin{aligned}
x = &\frac{\sqrt{m!}}{(m+2)\sqrt{(2m+2)(2m+1)\cdots(m+3)}\,(1+z\bar{z})^{m+1}}\\
&\times\left[\sum_{j=0}^{m}(-1)^{j+1}\left\{\sum_{s=0}^{j}(-1)^s\binom{2m+2-j}{m+1-s}\binom{j}{s}(z\bar{z})^s\right\}(\bar{z}^{m+1-j}A_j + z^{m+1-j}\bar{A}_j)\right.\\
&\left.+(-1)^m\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s}^2(z\bar{z})^s A_{m+1}\right],
\end{aligned}
\tag{3.4}
$$

where $A_0, A_1, \cdots, A_{m+1}$ are constant complex vectors in C^{m+2} such that

$$
\begin{aligned}
&A_t\cdot A_t = 0, \qquad t = 0, 1, \cdots, m, \qquad A_{m+1} = \bar{A}_{m+1},\\
&A_t\cdot A_s = A_t\cdot\bar{A}_s = 0, \qquad t \neq s, \qquad t, s = 0, 1, \cdots, m+1,\\
&A_0\cdot\bar{A}_0 = 2, \qquad A_t\cdot\bar{A}_t = 2\binom{2m+2}{t}, \qquad t = 1, 2, \cdots, m+1.
\end{aligned}
\tag{3.5}
$$

Thus we have proved that M may be regarded as a generalized Veronese surface of index m defined by Ōtsuki [7].

REFERENCES

[1] CHERN, S. S., M. DO CARMO, AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length. Functional analysis and related fields, Proc. Conf. in Honor of Marshall Stone, Springer, Berlin (1970).

[2] ITOH, T., Minimal surfaces with M-index 2, T_1-index 2 and T_2-index 2. Kōdai Math. Sem. Rep. **24** (1972), 1–16.

[3] ———, Minimal surfaces in 4-dimensional Riemannian manifolds of constant curvature. Kōdai Math. Sem. Rep. **24** (1972), 451–458.

[4] LITTLE, J. A., On singularities of submanifolds of higher dimensional Euclidean spaces. Ann. Mat. Pura Appl. **83** (1969), 261-336.
[5] ŌTSUKI, T., A Theory of Riemannian submanifolds. Kōdai Math. Sem. Rep. **20** (1968), 282-295.
[6] ———, Minimal submanifolds with M-index 2. J. Diff. Geometry **6** (1971), 193-211.
[7] ———, Minimal submanifolds with M-index 2 and generalized Veronese surfaces. J. Math. Soc. Japan **24** (1972), 89-122.

DEPARTMENT OF THE FOUNDATIONS OF MATHEMATICAL SCIENCES,
TOKYO UNIVERSITY OF EDUCATION.