

MINIMAX MULTI-DISTRICT APPORTIONMENTS

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THE PROBLEM

GOVERNABILITY

DEMOCRACY



**MAJORITY
SYSTEMS**

**PROPORTIONAL
SYSTEMS**



ROUNDINGS

ROUNDINGS

	Votes	Seats
A	50	5
B	30	3
C	20	2
Totals	100	10

$$s_i = v_i \cdot S / V$$

BUT...

	Votes	Seats
A	50	2.5
B	30	1.5
C	20	1.0
Totals	100	5

CRITERIA

- Equal votes \rightarrow Equal seats
- Monotonicity
(*more votes \rightarrow not less seats*)
- Symmetry
- Hare (*roundings*)
- Super-additivity
- Majority (*power indices*)

Monotonicity

	Vote s	Hare quotas	Seats				
			α	β	γ	δ	ϵ
A	50	2.5					
B	30	1.5					
C	20	1.0					
tot	100	5.0					

Monotonicity

	Vote s	Hare quotas	Seats				
			α	β	γ	δ	ϵ
A	50	2.5	5	4	3	3	2
B	30	1.5	0	1	2	1	2
C	20	1.0	0	0	0	1	1
tot	100	5.0	5	5	5	5	5

Monotonicity

	Vote s	Hare quotas	Seats				
			α	β	γ	δ	ϵ
A	50	2.5	5	4	3	3	2
B	30	1.5	0	1	2	1	2
C	20	1.0	0	0	0	1	1
tot	100	5.0	5	5	5	5	5

↑ ↑
Hare

Majorities

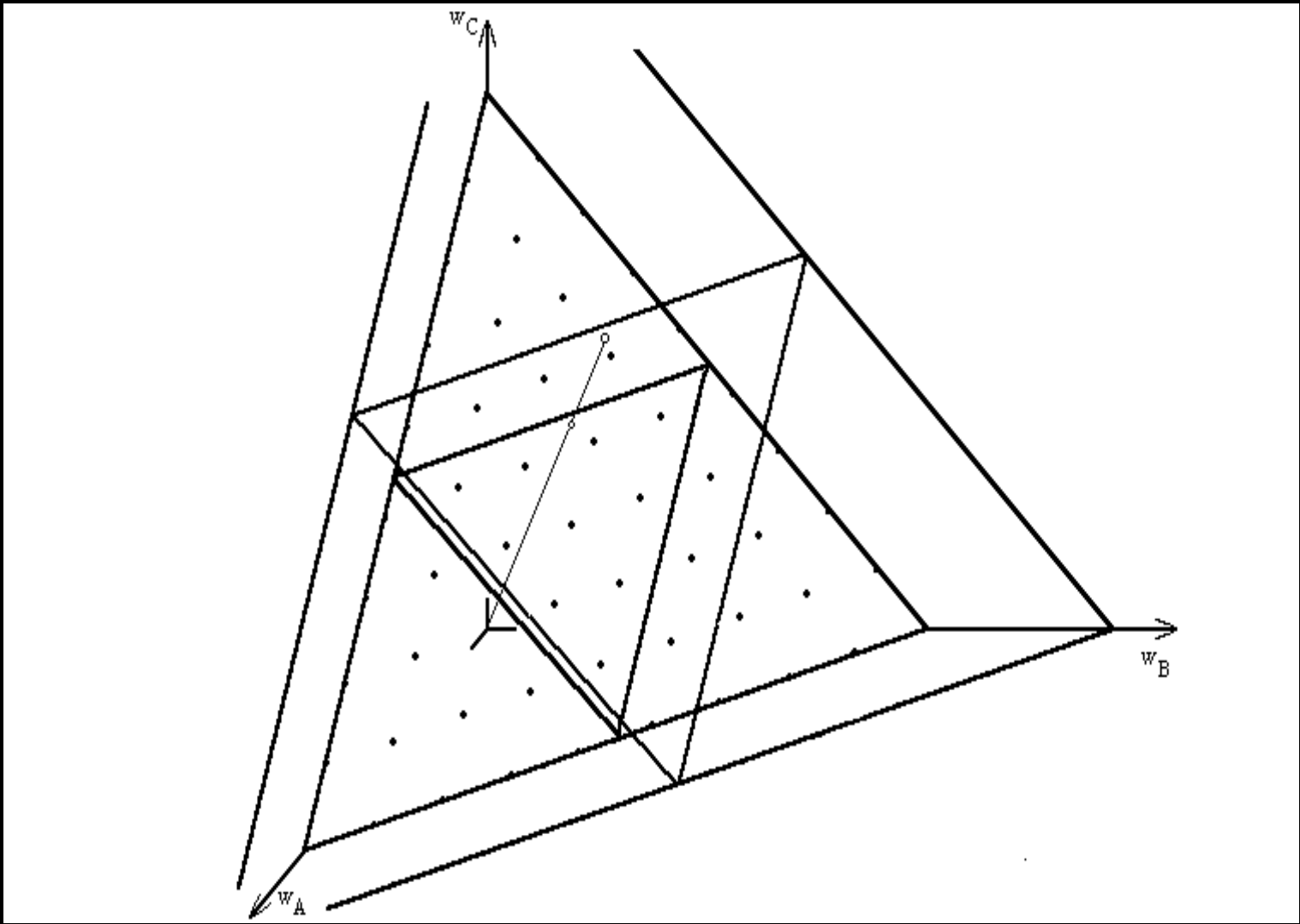
	Votes	Crucialities	N.	Prop.
A	50			
B	30			
C	20			
tot	100			

Majorities

	Votes	Crucialities	N.	Prop.
A	50	(AB)(AC)(ABC)		
B	30	(AB)		
C	20	(AC)		
tot	100			

Majorities

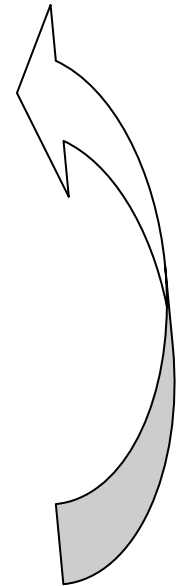
	Votes	Crucialities	N.	Prop.
A	50	(AB)(AC)(ABC)	3	3/5
B	30	(AB)	1	1/5
C	20	(AC)	1	1/5
tot	100		5	1



CONTRADICTIONS

	Votes	Seats
A	49	0
B	49	0
C	2	1
tot	100	1

Equal votes → Equal seats



Two parties, one seat

	Votes	Seats
A	50	
B	50	
tot	100	1

- Equal votes \rightarrow Equal seats
- Symmetry
- Majority

IN GENERAL

Hare Maximum & Superadditivity

$\neg \downarrow$

Symmetry & Monotonicity

ROUNDING METHODS

- PROPORTIONAL SYSTEM

(Hamilton)

- GREATEST DIVISORS

(D'Hondt)

- QUOTA GREAT. DIVISORS

(Balinski & Young)

- JUMP GREAT. DIVISORS

(St. Lague, Niemeyer, ...)

-MINIMAX

(Gambarelli)

**Gambarelli, G. (1999) "Minimax Apportionments",
Group Decision and Negotiation, 8, 6, 441-461.**

**Gambarelli, G. and A. Palestini (2007) "Minimax
Multi-District Apportionments" *Homo
economicus*, Vol. 23, 335-356.**

PROPORTIONAL SYSTEM

	Votes	Hare quotas	Seats	
A	50	2.5		
B	30	1.5		
C	20	1.0		
tot	100	5	5	5

exogenous methods
(oldness of candidates, draw, ...)

GREATEST DIVISORS

VOTES	:1	:2	:3	...
50	<u>50</u>	<u>25</u>	<u>16,[̄]6</u>	...
30	<u>30</u>	15	10,0	...
20	<u>20</u>	10	<u>6,[̄]6</u>	...

	Votes	Seats
A	50	3
B	30	1
C	20	1
tot	100	5

QUOTA GREATEST DIVISORS

respect of Hare maximum

JUMP GREATEST DIVISORS

divisions by 1, 3, 5 .. or other jumps

MINIMAX

Previously

- a method was invented
- then it was applied
- then it caused complaint about broken criteria

Now

- the priority order of criteria is established
- the set of apportionments respecting the first criterion is generated
- such set is successively restricted by applying subsequent criteria.

(now computers exist...)

Ordering of criteria

1) First criterion:

- Monotonicity
- Hare minimum
- Hare maximum

THEOREM

*non-empty survived
apportionments*

2) Minimization of the maximum damage votes-seats,
in terms of

- percentage
- power indices

Damages in terms of Percentage

δ		Votes %	$\delta\%$	Damages
3	A	50	60	-10
1	B	30	20	10
1	C	20	20	0

Ordered Damages: (10, 0, -10)

ε		Votes %	$\varepsilon\%$	Damages
2	A	50	40	10
2	B	30	40	-10
1	C	20	20	0

Ordered Damages: (10, 0, -10)

Damages in terms of Power Indices

	Votes	Votes p.i.	δ	δ p.i.	Damages
A	50	60	3	100	-40
B	30	20	1	0	20
C	20	20	1	0	20

Ordered Damages: (20, 20, -40)

V		Votes p.i.	
50	A	$3/5 =$	60
30	B	$1/5 =$	20
20	C	$1/5 =$	20

	Votes	Votes p.i.	ϵ	ϵ p.i.	Damages
A	50	60	2	$100/3$	$80/3$
B	30	20	2	$100/3$	$-40/3$
C	20	20	1	$100/3$	$-40/3$

Ordered Damages: (26. $\bar{6}$, -13. $\bar{3}$, -13. $\bar{3}$)

THEN
the solution is δ

THE ADVANTAGES

To the party (or to the coalition) having relative majority

- 1) the uninominal voting system
- 2) the majority prize.

To the remaining average and large parties

- 1) thresholds
- 2) greatest divisors.

To the smaller parties with peculiar linguistic or ethnical characteristics

- 1) the respect for such minorities.

To the remaining small parties

- 1) quota and jump greatest divisors
- 2) the proportional voting system.

**No party
gets any advantage
from the Minimax Method**

because this method:

- 1) respects all the principal equity criteria
- 2) minimizes distortions as far as possible.

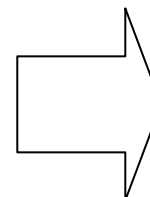
MULTI-DISTRICT SYSTEMS

VOTES	A	B	C	Totals
District .I	50	60	10	120
District II	10	10	60	80
National Totals	60	70	70	200

Local Hare Quotas	A	B	C	Totals
I	2,500	3,000	0,500	6
II	0,625	0,625	3,750	5
Totals of Local H.Q.	3,125	3,625	4,250	11

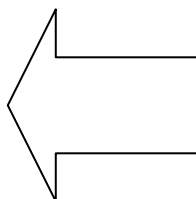
CLASSICAL ROUNDINGS

Hamilton (1)	A	B	C	Totals
I	3	3	0	6
II	1	0	4	5
Totals	4	3	4	11



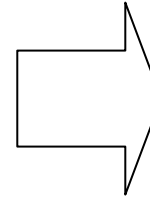
Breaks:
-Symmetry totals
-Monotonicity totals
-Power indices in I

Breaks:
-Symmetry totals
-Hare max totals



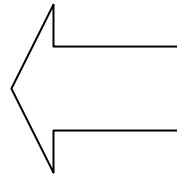
Hamilton (2)	A	B	C	Totals
I	2	3	1	6
II	1	0	4	5
Totals	3	3	5	11

Hamilton (3)	A	B	C	Totals
I	3	3	0	6
II	0	1	4	5
Totals	3	4	4	11

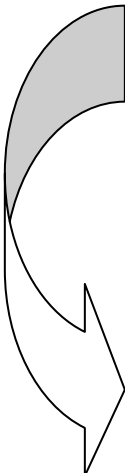


Breaks:
- Power index in I

Breaks:
-Symmetry totals
-Hare max totals



Hamilton (4)	A	B	C	Totals
I	2	3	1	6
II	0	1	4	5
Totals	2	4	5	11



Hondt	A	B	C	Total s
I	3	3	0	6
II	0	0	5	5
Totals	3	3	5	11

Breaks:

- the Power index local criterion in I,
- the Hare maximum local criterion in II
- both Hare maximum and symmetry criteria of totals.

Bal.&Young (1)	A	B	C	Totals
I	3	3	0	6
II	0	1	4	5
Totals	3	4	4	11

Bal.&Young (2)	A	B	C	Totals
I	3	3	0	6
II	1	0	4	5
Totals	4	3	4	11

Breaks:

- the local criterion of power indices (I district)
in both configurations.

MINIMAX

- Begin from an order of criteria, for instance:
 - Monotonicity and Hare at national level
 - Monotonicity and Hare at local level
 - Damage minimization at national level
 - Damage minimization at local level
 - Power index damage minimization at national level
 - Power index damage minimization at local levelor any other sequence.
- Build the set of solutions verifying the first criterion
- Restrict it step by step

Minimax	A	B	C	Totals
I	2	3	1	6
II	1	1	3	5
Totals	3	4	4	11

The minimax apportionment respects both at a local and a national level:

- **Symmetry**
- **Monotonicity**
- **Hare minimum**
- **Hare maximum**
- **Equal seats for equal votes**
- **Power Indices**

Theorem

For every multi-district apportionment a not-empty solution exists which respects, at national level,

- Monotonicity
- Hare minimum
- Hare maximum



Algorithm

The Italian Parliament: how it is

List	Chamber		Senate	
	% seats	Seats	% seats	Seats
CENTRE-RIGHT				
Il Popolo della Libertà	43,81	276	46,67	147
Lega Nord	9,52	60	7,94	25
Movimento per l'autonomia	1,27	8	0,63	2
<i>Totals centre-right coalition</i>	54,60	344	55,24	174
CENTRE-LEFT				
Partito democratico	34,44	217	37,46	118
Italia dei valori	4,60	29	4,44	14
<i>Totals centre-left coalition</i>	39,05	246	41,90	132
			0,00	
NON UNITED PARTIES			0,00	
Unione di centro	5,71	36	0,95	3
SVP	0,32	2	0,63	2
Movimento associativo italiani all'estero	0,16	1	0,32	1
Autonomie liberté democratie	0,16	1	0,00	0
Vallée d'Aoste	0,00	0	0,32	1
SVP-Insieme per le autonomie	0,00	0	0,63	2
<i>Totals non united parties</i>	6,35	40	2,86	9
Totals	100,00	630	100,00	315

Minimax - Same conditions (premia – thresholds)

HOW IT WOULD BE

List	Chamber		Senate	
	% seats	Seats	% seats	Seats
CENTRE-RIGHT				
Il Popolo della Libertà	43,17	272	42,54	134
Lega Nord	9,52	60	7,94	25
Movimento per l'autonomia	1,27	8	1,27	4
<i>Totals centre-right coalition</i>	53,97	340	51,75	163
CENTRE-LEFT				
Partito democratico	34,76	219	36,51%	115
Italia dei valori	4,60	29	4,76%	15
<i>Totals centre-left coalition</i>	39,37	248	41,27%	130
NON UNITED PARTIES				
Unione di centro	6,03	38	6,35	20
SVP	0,48	3	0,32	1
Movimento associativo italiani all'estero	0,16	1	0,00	0
Autonomie liberté démocratie	0,00	0	0,32	1
Vallée d'Aoste	6,67	42	6,98	22
Totals	100,00	630	100,00	315

Minimax - Same conditions (premia – thresholds)

THE DIFFERENCES

List	Δ seats	
	Chamber	Senate
CENTRE-RIGHT		
Il Popolo della Libertà	-4	-13
Lega Nord	0	0
Movimento per l'autonomia	0	+2
<i>Totals centre-right coalition</i>	-4	-11
CENTRE-LEFT		
Partito democratico	+2	-3
Italia dei valori	0	+1
<i>Totals centre-left coalition</i>	+2	-2
NON UNITED PARTIES		
Unione di centro	+2	+17
SVP	+1	-1
Movimento associativo italiani all'estero	0	-1
Autonomie liberté démocratie	-1	0
Vallée d'Aoste	0	-1
SVP-Insieme per le autonomie	0	-1
<i>Totals non united parties</i>	+2	+13
Totals	0	0

Minimax without premia – thresholds

HOW IT WOULD BE

List	Chamber		Senate	
	% seats	Seats	% seats	Seats
EXTREME LEFT				
Partito comunista dei lavoratori	0,63	4	0,63	2
Liga veneta repubblica	0,16	1	0,32	1
<i>Totals extreme left</i>	0,79	5	0,95	3
LEFT				
La Sinistra-L'Arcobaleno	3,02	19	3,17	10
Partito socialista	0,95	6	0,95	3
Sinistra critica	0,48	3	0,32	1
<i>Totals left</i>	4,44	28	4,44	14
CENTRE-LEFT				
Partito democratico	33,17	209	33,02	104
Italia dei valori	4,44	28	4,13	13
SVP	0,48	3	0,32	1
Per il bene comune	0,32	2	0,32	1
Unione democratica per i consumatori	0,32	2	0,32	1
Autonomie liberté démocratie	0,16	1	0,00	0
Vallée d'Aoste	0,00	0	0,32	1
SVP-Insieme per le autonomie	0,00	0	0,32	1
<i>Totals centre-left</i>	38,89	245	38,73	122

CENTRE				
Unione di centro	5,71	36	5,71	18
Movimento associativo italiani all'estero	0,16	1	0,32	1
Associazione italiani in Sud America	0,16	1	0,32	1
<i>Totals centre</i>	6,03	38	6,35	20
CENTRE-RIGHT				
Il Popolo della Libertà	36,98	233	37,78	119
Lega Nord	8,10	51	7,62	24
Movimento per l'autonomia	1,11	7	0,95	3
Ass. difesa della vita Aborto? No, grazie	0,32	2	0,00	0
Partito liberale italiano	0,32	2	0,00	0
Lega lombarda	0,00	0	0,32	1
<i>Totals centre-right</i>	46,83	295	46,67	147
RIGHT				
La Destra-fiamma tricolore	2,38	15	2,22	7
Lista dei grilli parlanti	0,16	1	0,32	1
Die freiheitlichen	0,16	1	0,00	0
<i>Totals right</i>	2,70	17	2,54	8
EXTREME-RIGHT				
Forza nuova	0,32	2	0,32	1
<i>Totals extreme right</i>	0,32	2	0,32	1
Totals	100,00	630	100,00	315

Minimax without premia – thresholds

How it would be - SUMMARY

	Chamber		Senate	
	% seats	n. seats	% seats	n. seats
Extreme Left	0,79	5	0,95	3
Left	4,44	28	4,44	14
Centre-Left	38,89	245	38,73	122
Centre	6,03	38	6,35	20
Centre-Right	46,83	295	46,67	147
Right	2,70	17	2,54	8
Extreme Right	0,32	2	0,32	1
Totals	100,00	630	100,00	315

Minimax without premia – thresholds

THE DIFFERENCES

List	Δ seats	
	Chamber	Senate
EXTREME LEFT		
Partito comunista dei lavoratori	+4	+2
Liga veneta repubblica	+1	+1
<i>Totals extreme left</i>	+5	+3
LEFT		
La Sinistra-L'Arcobaleno	+19	+10
Partito socialista	+6	+3
Sinistra critica	+3	+1
<i>Totals left</i>	+28	+14
CENTRE-LEFT		
Partito democratico	-8	-14
Italia dei valori	-1	-1
SVP	+1	-1
Per il bene comune	+2	+1
Unione democratica per i consumatori	+2	+1
Autonomie liberté démocratie	0	0
Vallée d'Aoste	0	0
SVP-Insieme per le autonomie	0	-1
<i>Totals centre-left</i>	-4	-15

CENTRE		
Unione di centro	0	+15
Movimento associativo italiani all'estero	0	0
Associazione italiani in Sud America	+1	+1
<i>Totals centre</i>	+1	+16
CENTRE-RIGHT		
Il Popolo della Libertà	-43	-28
Lega Nord	-9	-1
Movimento per l'autonomia	-1	+1
Ass. difesa della vita Aborto? No, grazie	+2	0
Partito liberale italiano	+2	0
Lega lombarda	0	+1
<i>Totals centre-right</i>	-49	-27
RIGHT		
La Destra-fiamma tricolore	+15	+7
Lista dei grilli parlanti	+1	+1
Die freiheitlichen	+1	0
<i>Totals right</i>	+17	+8
EXTREME-RIGHT		
Forza nuova	+2	+1
<i>Totals extreme-right</i>	+2	+1
TOTALS	0	0

Minimax without premia – thresholds

The Differences - SUMMARY

	Δ seats	
	Chamber	Senate
Extreme Left	+5	+3
Left	+28	+14
Centre-Left	-4	-15
Centre	+1	+16
Centre-Right	-49	-27
Right	+17	+8
Extreme Right	+2	+1
Totals	0	0

THE ADVANTAGES

- The choice of an order of criteria affects the kind of solution.
- It is very important to decide whether to privilege the damage minimization globally or locally.
Globally, the advantage goes to average and large parties powerful on the whole national territory.
Locally, it goes to the minorities concentrated in particular areas.
- All the previous considerations on pros and cons at one-district level keep valuable for multi-district systems.

**No party
gets any advantage
from the Minimax Method**

since this technique respects all the fundamental equity criteria and minimizes distortions as much as possible.



Probably

it will never be adopted



Open problems:

Gambarelli, G. (1999) "Minimax Apportionments", *Group Decision and Negotiation*, 8, 6, 441-461.

Bertini, C., Gambarelli, G. and I. Stach (2005) "Apportionment Strategies for the European Parliament via GDP - Applications of Minimax Apportionments outside Italy" *Homo Oeconomicus* (G. Gambarelli and M. Holler, eds), 22(4), 589-604.

Gambarelli, G. and A. Palestini (2007) "Minimax Multi-District Apportionments" *Power Measures IV - a Special Issue of Homo Oeconomicus* (G. Gambarelli, ed.), Vol. 24, n. 3/4, 335-356.

MY WARMEST THANKS TO...



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