Minimization of the norm, the norm of the inverse and the condition number of a matrix by completion

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Abstract

We study the problem of minimizing the norm, the norm of the inverse and the condition number with respect to the spectral norm, when a submatrix of a matrix can be chosen arbitrarily. For the norm minimization problem we give a different proof than that given by Davis/Kahan/Weinberger. This new approach can then also be used to characterize the completions that minimize the norm of the inverse. For the problem of optimizing the condition number we give a partial result.
Keywords: condition number, norm of a matrix, matrix completion, dilation theory, robust regularization of descriptor systems

1 Introduction

We study the following optimization problem: Given integers $n, m, N > n, m$ and matrices $A \in \mathbf{C}^{n,m}, B \in \mathbf{C}^{n,N-m}, C \in \mathbf{C}^{N-n,m}$, find $X \in \mathbf{C}^{N-n,N-m}$ such that the matrix

$$W(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix} \tag{1.1}$$

satisfies

$$\begin{aligned} \operatorname{cond}(W(X)) &= \min_{Z\in \mathbf{C}^{N-n,N-m}}\{\|W(Z)\|\|W(Z)^{-1}\|\} \\ &= \min_{Z\in \mathbf{C}^{N-n,N-m}}\{\operatorname{cond}(W(Z))\}. \end{aligned} \tag{1.2}$$

Throughout the paper $\| \quad \|$ will denote the spectral norm and N the order of $W(X)$. In order to study the solution of this problem, we study the following two related problems:

Find $X \in \mathbf{C}^{N-n,N-m}$ such that $W(X)$ as in (1.1) satisfies

$$\|W(X)\| = \min_{Z\in \mathbf{C}^{N-n,N-m}} \|W(Z)\|, \tag{1.3}$$

and find $X \in \mathbf{C}^{N-n,N-m}$ such that $W(X)$ as in (1.1) satisfies

$$\|W(X)^{-1}\| = \min_{Z\in \mathbf{C}^{N-n,N-m}} \|W(Z)^{-1}\|. \tag{1.4}$$

Problem (1.3) is well known in dilation theory and was solved by Davis/Kahan/Weinberger, see [3] and the references therein. It has many applications in perturbation theory for eigenvalues, e.g. Parlett [9] and numerical quadrature, e.g. Davis/Kahan/Weinberger [3].

Problems (1.2) and (1.4) are not as well studied but they have applications in the construction of numerically stable parallel methods for block structured linear systems, e.g. Mehrmann [7] and in robust control, e.g. Bunse-Gerstner/Mehrmann/Nichols [1, 2]. Consider for example the descriptor control problem:

$$E\dot{x} = Fx + Gu, \; y = Hx \tag{1.5}$$

with $E, F \in \mathbf{C}^{n_1,n_1}, G \in \mathbf{C}^{n_1,m_1}, H \in \mathbf{C}^{p_1,n_1}$. Here x is the state, u the input and y the output of the system and E is assumed to be singular. Without loss of generality let us assume that

$$E = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix},\; F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},\; G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},\; H = \begin{bmatrix} H_1 & H_2 \end{bmatrix},$$

with Σ square, nonsingular and diagonal. This can easily be achieved via a singular value decomposition of E. If F_{22} is nonsingular, then problem (1.5) can be reduced to a lower order ordinary control system, by eliminating the second block row. For details see [1, 2].

Using a linear feedback $u = My$ in (1.5), we can modify the properties of the matrix F. In particular under some regularizability assumptions, (see [1, 2]), we can choose the feedback matrix M to make $F_{22} + G_2MH_2$ nonsingular. If we have done so, we can transform the system to a reduced order ordinary control system. To do this in practice, however, we need that the matrix F_{22} is well-conditioned with respect to inversion. Thus, it is obvious that we should choose the feedback M such that $F_{22} + G_2MH_2$ is well-conditioned. Using singular value decompositions of G_2, H_2 the problem of choosing the feedback that minimizes the condition number of $F_{22} + G_2MH_2$ is easily transformed to problem (1.2).

In [3] explicit solutions for problem (1.3) are given. Here we will give explicit solutions for problem (1.4). In fact, problems (1.3) and (1.4) are equivalent to Riccati inequalities which have explicit solutions. Using this characterization, we give elementary representations of solutions to problems (1.3) and (1.4). We discuss the two problems separately in the Hermitian case, i.e. $A = A^H$, $C = B^H$ and $X = X^H$ in Section 2 and in the non-Hermitian case in Section 3.

For problem (1.2) we give bounds for the solutions in terms of Riccati inequalities for solutions of (1.3) and (1.4) in Section 4. In general the solution of (1.2) is an open problem, except for the case that either n or m is 0. In this case particular solutions were given in [1]. A complete characterization of the solutions for this case is also given in Section 4.

We use the following notation:

By $\mathbf{C}^{n,m}$ we denote the set of complex $n \times m$ matrices. For $A \in \mathbf{C}^{n,m}$ we denote by A^H the conjugate transpose of A. For Hermitian matrices $A, B \in \mathbf{C}^{n,n}$ we write $A < B$ ($A \leq B$) if $B - A$ is positive definite (positive semidefinite). For a matrix $A \in \mathbf{C}^{n,m}$, we denote by $0 \leq \sigma_p(A) \leq \sigma_{p-1}(A) \leq \ldots \leq \sigma_1(A)$ the singular values of A, where $p = \min\{m, n\}$.

We make use of the Sherman-Morrison-Woodbury formula in the form

$$(A + UV^H)^{-1} = A^{-1} - A^{-1}U(I + V^HA^{-1}U)^{-1}V^HA^{-1}, \tag{1.6}$$

e.g. [5] and we also use the congruence, e.g. [8]

$$\begin{bmatrix} A & B \\ B^H & D \end{bmatrix} \cong \begin{bmatrix} A & 0 \\ 0 & D - B^HA^{-1}B \end{bmatrix} \cong \begin{bmatrix} A - BD^{-1}B^H & 0 \\ 0 & D \end{bmatrix} \tag{1.7}$$

with the Schur complements

$$D - B^HA^{-1}B$$

and

$$A - BD^{-1}B^H,$$

provided that the inverses of A and D exist.

2 Minimizing the norm and the norm of the inverse. The Hermitian case.

In this section we discuss the Hermitian case. We discuss the minimization of the norm and the norm of the inverse jointly and show that both results can be obtained in a similar fashion. To do this we give a proof for the minimization of the norm different from that given in [3].

Let

$$W(X) = \begin{bmatrix} A & B \\ B^H & X \end{bmatrix} \in \mathbf{C}^{N,N}, \tag{2.1}$$

where A is Hermitian and $X \in \mathbf{C}^{m,m}$ is required to be Hermitian, too. The main idea for the solution of problems (1.3) and (1.4) is to consider the limiting case of the following two problems.

For $\sigma_1([A, B]) < \beta$ find an X such that

$$\|W(X)\| \leq \beta, \tag{2.2}$$

and for $0 < \alpha < \sigma_n([A, B])$, find an X such that

$$\|W(X)^{-1}\| \leq \frac{1}{\alpha}. \tag{2.3}$$

Here we assume that $0 < \sigma_n([A, B])$ in order to guarantee that $W(X)$ is invertible for some X. We call Hermitian matrices X satisfying (2.2) or (2.3) solutions of (2.2) or (2.3), respectively.

The theory that we develop depends strongly on the following Lemma, which shows that both problems (2.2) and (2.3) are equivalent to Riccati inequalities. Introduce the matrix function

$$R(X, \gamma) := \\ (B^H B + X^2 - \gamma^2 I) - (B^H A + X B^H)(A^2 + B B^H - \gamma^2 I)^{-1}(AB + BX). \tag{2.4}$$

Lemma 1 *Let $W(X)$ be as in (2.1).*
(i) If $\sigma_1([A, B]) < \beta$, then problem (2.2) and the Riccati inequality

$$R(X, \beta) \leq 0 \tag{2.5}$$

have the same set of solutions.
(ii) Let $0 < \sigma_n([A, B])$. If $0 < \alpha < \sigma_n([A, B])$, then problem (2.3) and the Riccati inequality

$$R(X, \alpha) \geq 0 \tag{2.6}$$

have the same set of solutions.

Proof. (i) To prove that (2.2) and (2.5) have a common set of solutions note that if $\|W(X)\| \leq \beta$ has a solution X, then

$$W(X)^2 \leq \beta^2 I$$

or equivalently

$$\begin{bmatrix} A^2 + BB^H & AB + BX \\ B^H A + X B^H & B^H B + X^2 \end{bmatrix} - \beta^2 I \leq 0.$$

We have $A^2+BB^H-\beta^2 I < 0$, since $\sigma_1([A,B]) < \beta$. Since positive definiteness is preserved when taking Schur complements, see (1.7), it follows that

$$R(X,\beta) = (B^H B+X^2-\beta^2 I)-(B^H A+XB^H)(A^2+BB^H-\beta^2 I)^{-1}(AB+BX) \leq 0.$$

Therefore, X is a solution of $R(X,\beta) \leq 0$. Observe that every step of above proof is reversible, so (2.2) and (2.5) have a common set of solutions.

In the same way it follows that (2.3) and (2.6) have a common set of solutions, since $\alpha^2 I \leq W(X)^2$ and from $0 < \alpha < \sigma_n([A,B])$ it follows that $A^2 + BB^H - \alpha^2 > 0$. ■

Lemma 1 shows that problems (2.2) and (2.3) are equivalent to the Riccati inequalities (2.5) and (2.6) which are well studied, since they are central problems in control theory, e.g. [11].

To proceed, we consider a general Riccati equation of the form

$$XRX - XV - V^H X - Q = 0, \tag{2.7}$$

where R and Q are Hermitian matrices with R invertible and V is a square matrix.

Several papers are devoted to give explicit Hermitian solutions of (2.7) under the condition that $RV^H = VR$ and some further control theoretic conditions, e.g. [6, 10]. The main idea is to rewrite (2.7) as

$$(X - V^H R^{-1})R(X - R^{-1}V) = Q + V^H R^{-1} V. \tag{2.8}$$

Here we are not interested in these problems of control theory, but this approach of solving Riccati equations plays a central role in solving our problems.

We rewrite $R(X,\beta) = 0$ with $R(X,\beta)$ as in (2.5) in the form of (2.7) and obtain

$$R(X,\beta) = XR_\beta X - XV_\beta - V_\beta^H X - Q_\beta = 0, \tag{2.9}$$

where

$$\begin{aligned} R_\beta &= I - B^H(A^2 + BB^H - \beta^2 I)^{-1}B, \\ V_\beta &= B^H(A^2 + BB^H - \beta^2 I)^{-1}AB, \\ Q_\beta &= B^H A(A^2 + BB^H - \beta^2 I)^{-1}AB + \beta^2 I - B^H B. \end{aligned} \tag{2.10}$$

Now we have that $R_\beta V_\beta^H = V_\beta R_\beta$. Before verifying this fact, we note that

$$R_\beta^{-1} = I + B^H(A^2 - \beta^2 I)^{-1}B, \tag{2.11}$$

which is a direct consequence of the Sherman-Morrison-Woodbury formula (1.6) and the fact that $A^2 - \beta^2 I$ is invertible. Thus, we have that

$$\begin{aligned} &R_\beta^{-1}V_\beta = B^H(A^2 + BB^H - \beta^2 I)^{-1}AB+ \\ &B^H(A^2 - \beta^2 I)^{-1}BB^H(A^2 + BB^H - \beta^2 I)^{-1}AB \\ &= B^H(I + (A^2 - \beta^2 I)^{-1}BB^H)(A^2 + BB^H - \beta^2 I)^{-1}AB \\ &= B^H(A^2 - \beta^2 I)^{-1}AB. \end{aligned} \tag{2.12}$$

This shows $V_\beta^H R_\beta^{-1} = R_\beta^{-1} V_\beta$ and $R_\beta V_\beta^H = V_\beta R_\beta$ from which we obtain that the Riccati equation $R(X, \beta) = 0$ has solutions, and also obtain general solutions of $R(X, \beta) \leq 0$.

Replacing β by α in (2.9) we obtain formulas analogous to (2.9)-(2.12) for $R(X, \alpha), R_\alpha, V_\alpha, Q_\alpha, R_\alpha^{-1}$.

Before we establish the structure of solutions for $R(X, \beta) \leq 0$ and $R(X, \alpha) \geq 0$, let us discuss the inertias of the Hermitian matrices R_β^{-1} and R_α^{-1}. This is done in the following Lemma.

Lemma 2 *(i) Let $\sigma_1([A, B]) < \beta$. Then R_β^{-1} given by (2.11) is positive definite.*

(ii) Let $0 < \alpha < \sigma_n([A, B])$. Suppose that α is not a singular value of A, and that the inertia of $A^2 - \alpha^2 I$ corresponding to the numbers of positive, negative and zero eigenvalues of $A^2 - \alpha^2 I$ is $(p, n - p, 0)$. Then, the inertia of R_α^{-1} is $(N + p - 2n, n - p, 0)$.

Proof. Taking Schur complements in

$$\begin{bmatrix} \beta^2 I_n - A^2 & B \\ B^H & I_{N-n} \end{bmatrix}$$

we obtain as in (1.7) the following congruences

$$\begin{bmatrix} \beta^2 I_n - A^2 - BB^H & 0 \\ 0 & I_{N-n} \end{bmatrix} \cong \begin{bmatrix} \beta^2 I_n - A^2 & B \\ B^H & I_{N-n} \end{bmatrix}$$
$$\cong \begin{bmatrix} \beta^2 I_n - A^2 & 0 \\ 0 & R_\beta^{-1} \end{bmatrix}.$$

The left matrix is positive definite by assumption. Thus, we have R_β^{-1} is positive definite.

Similarly, taking Schur complements in

$$\begin{bmatrix} -(A^2 - \alpha^2 I_n) & B \\ B^H & I_{N-n} \end{bmatrix}$$

we obtain the congruences

$$\begin{bmatrix} -(A^2 + BB^H - \alpha^2 I_n) & 0 \\ 0 & I_{N-n} \end{bmatrix}$$
$$\cong \begin{bmatrix} -(A^2 - \alpha^2 I_n) & B \\ B^H & I_{N-n} \end{bmatrix} \cong \begin{bmatrix} -(A^2 - \alpha^2 I_n) & 0 \\ 0 & R_\alpha^{-1} \end{bmatrix}.$$

Since $\alpha < \sigma_n([A, B])$, it follows that the left matrix has the inertia $(N-n, n, 0)$ and the inertia of the right matrix is the sum of $(n - p, p, 0)$ and the inertia of R_α^{-1}. Thus, the inertia of R_α^{-1} is $(N+p-2n, n-p, 0)$. ■

It is immediate from the fact that R_β^{-1} is positive definite and R_α^{-1} is indefinite that solving $R(X, \beta) \leq 0$ is easier than solving $R(X, \alpha) \geq 0$, though both problems are solvable. In fact $\sigma_1([A,B]) < \beta$ implies $\sigma_1(A) < \beta$. However, when $0 < \alpha < \sigma_n([A,B])$, $(A-\alpha I)^{-1}$ may have several poles in this interval. This fact makes numerical calculations in the latter case more complicated than in the definite case.

Theorem 1 *Let $W(X)$ be as in (2.1).*
(i) Let $\sigma_1([A,B]) < \beta$. Then,

$$\|W(X)\| \le \beta$$

always has a solution. In fact it has the two extremal solutions

$$\begin{array}{l} X_1 = \beta I + B^H (A-\beta I)^{-1} B, \\ X_2 = -\beta I + B^H (A+\beta I)^{-1} B, \end{array} \tag{2.13}$$

and the set of solutions is the 'interval' of matrices given by

$$X_2 \le X \le X_1. \tag{2.14}$$

(ii) Let $0 < \sigma_n([A,B])$. If $0 < \alpha < \sigma_n([A,B])$ and α is not a singular value of A, then

$$\|W(X)^{-1}\| \le \frac{1}{\alpha}$$

always has a solution. In fact it has the two solutions

$$\begin{array}{l} Y_1 = \alpha I + B^H (A-\alpha I)^{-1} B, \\ Y_2 = -\alpha I + B^H (A+\alpha I)^{-1} B, \end{array} \tag{2.15}$$

and general solutions are of the form

$$X = Y_1 + Y \tag{2.16}$$

or

$$X = Y_2 - Y, \tag{2.17}$$

where Y satisfies

$$Y \ge Y\left(\frac{B^H (A^2 + BB^H - \alpha^2 I)^{-1} B - I}{2\alpha}\right) Y. \tag{2.18}$$

Proof. According to Lemma 1, we only need to solve $R(X,\beta) \le 0$ to solve (2.2) and $R(X,\alpha) \ge 0$ to solve (2.3).

(i) We begin with the Riccati inequality based on the equation (2.9). With R_β, V_β and Q_β as in (2.10), we have to solve

$$R(X,\beta) = (X - V_\beta^H R_\beta^{-1}) R_\beta (X - R_\beta^{-1} V_\beta) - (Q_\beta + V_\beta^H R_\beta^{-1} V_\beta) \le 0. \tag{2.19}$$

From (2.12) and the Sherman-Morrison-Woodbury formula (1.6)

$$A(A^2-\beta^2 I)^{-1}A - I = \beta^2(A^2-\beta^2 I)^{-1},$$

we obtain

$$\begin{aligned}
&Q_\beta + V_\beta^H R_\beta^{-1} V_\beta = B^H A(A^2+BB^H-\beta^2 I)^{-1}AB \\
&+\beta^2 I - B^H B + B^H A(A^2+BB^H-\beta^2)^{-1}BB^H(A^2-\beta^2)^{-1}AB \\
&= B^H A(A^2-\beta^2)^{-1}AB - B^H B + \beta^2 I \\
&= \beta^2(B^H(A^2-\beta^2 I)^{-1}B + I) = \beta^2 R_\beta^{-1}.
\end{aligned}$$

So the above Riccati inequality (2.19) is equivalent to

$$ZR_\beta Z \leq R_\beta^{-1}, \tag{2.20}$$

where $Z = (X - V_\beta^H R_\beta^{-1})/\beta$. Two particular solutions of (2.20) are $Z_1 = R_\beta^{-1}$ and $Z_2 = -R_\beta^{-1}$. This implies that we have the two particular solutions

$$X_1 = V_\beta^H R_\beta^{-1} + \beta Z_1 = \beta I + B^H(A-\beta I)^{-1}B \tag{2.21}$$

and

$$X_2 = V_\beta^H R_\beta^{-1} + \beta Z_2 = -\beta I + B^H(A+\beta I)^{-1}B \tag{2.22}$$

for (2.5).

As R_β^{-1} is positive definite, the set of solutions of (2.20) is the 'interval'

$$Z_2 \leq Z \leq Z_1$$

implying that the set of solutions of (2.5) is given by

$$X_2 \leq X \leq X_1.$$

(ii) Following the proof of (i), we can verify in the same way that Y_1 and Y_2 defined by (2.15) are solutions of $R(X,\alpha)=0$.

However, to get the general structure of solutions of $R(X,\alpha)\geq 0$, the procedure of (i) is not satisfactory as (2.20) with indefinite R_α^{-1} is hard to solve. Thus, we prefer the form $X = Y_1 + Y$ or $X = Y_2 - Y$. Recall that by (2.11) and (2.12) with β replaced by α we have

$$\begin{aligned}
&R(Y_1+Y,\alpha) = R(Y_1,\alpha) + Y^2 + Y_1 Y + YY_1 - \\
&YB^H(A-\alpha I)^{-1}B - B^H(A-\alpha I)^{-1}BY - YB^H(A^2+BB^H-\alpha^2 I)^{-1}BY \\
&= Y^2 + 2\alpha Y - YB^H(A^2+BB^H-\alpha^2 I)^{-1}BY.
\end{aligned}$$

Thus, $R(Y_1+Y,\alpha)\geq 0$ if and only if Y satisfies

$$Y \geq Y\left(\frac{B^H(A^2+BB^H-\alpha^2 I)^{-1}B - I}{2\alpha}\right)Y. \tag{2.23}$$

Similarly we can prove that $X = Y_2 - Y$ with Y satisfying (2.18) is also a solution of $R(X,\alpha)\geq 0$. ■

Remark 1 For the two particular solutions X_1 and X_2 of $R(X, \beta) \le 0$, we always have

$$X_1 - X_2 = 2\beta R_\beta^{-1} \ge 0$$

and the "central" solution of $R(X, \beta) \le 0$ is

$$\frac{X_1 + X_2}{2} = -B^H(\beta^2 I - A^2)AB. \tag{2.24}$$

However, for $R(X, \alpha) \ge 0$, the two special solutions Y_1 and Y_2 do not have a natural order, as

$$Y_1 - Y_2 = 2\alpha R_\alpha^{-1} \tag{2.25}$$

is indefinite. Only in the case that $0 < \alpha < \sigma_n(A)$, we have that R_α is positive definite by Lemma 2 and

$$Y_2 \le Y_1$$

holds.

Remark 2 If Y as in (2.23) is chosen to be nonsingular, then (2.18) implies that

$$Y^{-1} \ge \frac{B^H(A^2 + BB^H - \alpha^2 I)^{-1}B - I}{2\alpha} = -\frac{R_\alpha}{2\alpha}.$$

In particular when $\alpha < \sigma_n(A)$, R_α is positive definite and $Y_2 \le Y_1$, and thus any positive definite matrix Y will satisfy (2.18).

Thus, in this case we have that the solutions of (2.2) lie inside of $X_2 \le X \le X_1$, and those of (2.3) lie outside of $Y_2 \le Y \le Y_1$.

Remark 3 Note that the solutions X_1 and X_2 of (2.2) depend on the inverses of $(A - \beta I)$ and $(A + \beta I)$, or more precisely on $B^H(A - \beta I)^{-1}B$ and $B^H(A + \beta I)^{-1}B$.

Taking the limit $\beta \to \sigma_1([A, B])$ with $\beta > \sigma_1([A, B])$, it is known that the limits of $B^H(A - \beta I)^{-1}B$ and $B^H(A + \beta I)^{-1}B$ always exist even in the case that $\sigma_1([A, B]) = \sigma_1(A)$ and that they always give a minimizing solution, e.g. [9]. Thus

$$\|W(X)\| = \sigma_1([A, B])$$

always has a finite solution.

However, when we take the limit $\alpha \to \sigma_n([A, B])$ with $\alpha < \sigma_n([A, B])$, the problem

$$\|W(X)^{-1}\| = \frac{1}{\sigma_n([A, B])}$$

may not have a finite solution.

An example is given by

$$W(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & x \end{bmatrix}.$$

The minimal singular value of $[A, b]$ is 1. Both 1 and -1 are eigenvalues of A and neither of the limits in (2.15) for $\alpha \to 1$ exists. Since the minimal singular value of $W(x)$ is obviously in the interval $[0, 1]$ it follows that $\|W(x)^{-1}\|$ has 1 as a minimum, which is only obtained as a limit for $x \to \infty$.

Theorem 2 *Let* $W(X) = \begin{bmatrix} A & B \\ B^H & X \end{bmatrix}$ *with* A *Hermitian.*
(i) Then

$$\inf_X \|W(X)\| = \sigma_1([A, B]).$$

Moreover, there exists an Hermitian matrix X *such that*

$$\|W(X)\| = \sigma_1([A, B]).$$

(ii) Let $0 < \sigma_n([A, B])$. *Then,*

$$\inf_X \|W(X)^{-1}\| = \frac{1}{\sigma_n([A, B])}.$$

Moreover, if $\sigma_n([A, B])$ *is not an eigenvalue of both* A *and* $-A$*, then there exists an Hermitian matrix* X *such that*

$$\|W(X)^{-1}\| = \frac{1}{\sigma_n([A, B])}.$$

Proof. (i) Since for any X we have $\|W(X)\| \geq \sigma_1([A, B])$, it follows by Theorem 1 that

$$\inf_X \|W(X)\| = \sigma_1([A, B]).$$

As already pointed out in the limit discussion of Remark 3, there always exists an Hermitian matrix X such that $\|W(X)\| = \sigma_1([A, B])$.

(ii) It follows from the Cauchy Interlacing Theorem, (e.g. [5]), that for any Hermitian matrix X we have

$$\sigma_N(W(X)) \leq \sigma_n([A, B]).$$

Consequently we have

$$\inf_X \|W(X)^{-1}\| \geq \frac{1}{\sigma_n([A, B])}.$$

It follows by Theorem 1(ii) that

$$\inf_X \|W(X)^{-1}\| = \frac{1}{\sigma_n([A, B])}.$$

Under the condition that $\sigma_n([A, B])$ is not an eigenvalue of both A and $-A$, the discussion in Remark 3 shows that there exists an Hermitian matrix X such that

$$\|W(X)^{-1}\| = \frac{1}{\sigma_n([A, B])}.$$

Thus, we have finished the proof. ■

Remark 4 The minimizing solutions of (1.3) in the Hermitian case satisfy

$$X_2 \le X \le X_1,$$

where X_1 and X_2 are given by (2.13) with $\beta = \sigma_1([A, B])$.

The minimizing solutions of (1.4) in the Hermitian case are

$$X = Y_1 + Y, \quad or \quad X = Y_2 - Y,$$

where Y_1 , Y_2 and Y are given by (2.15) and (2.18) with $\alpha = \sigma_n([A, B])$.

3 Minimizing the norm and the norm of the inverse. The non-Hermitian case.

In this section we discuss the minimization of the norm and the norm of the inverse of

$$W(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix}, \tag{3.1}$$

where $A \in \mathbf{C}^{n,m}, B \in \mathbf{C}^{n,N-m}, C \in \mathbf{C}^{N-n,m}$, and $X \in \mathbf{C}^{N-n,N-m}$.

In the following let α_0, β_0 be defined as

$$\beta_0 = \max\{\sigma_1([A, B]), \sigma_1(\begin{bmatrix} A \\ C \end{bmatrix})\}, \alpha_0 = \min\{\sigma_n([A, B]), \sigma_m(\begin{bmatrix} A \\ C \end{bmatrix})\}. \tag{3.2}$$

We furthermore assume that $0 < \alpha_0$ in order to guarantee that the inverse of $W(X)$ exists for some X.

The results of Section 2 are naturally extended to the non-Hermitian case. As in the previous section we solve the problems

$$\|W(X)\| \le \beta, \tag{3.3}$$

for $\beta_0 < \beta$ and

$$\|W(X)^{-1}\| \le \frac{1}{\alpha}, \tag{3.4}$$

for $0 < \alpha < \alpha_0$, and then take limits for $\alpha \to \alpha_0$ and $\beta \to \beta_0$

The following two lemmata are direct analogues to Lemma 1 and Lemma 2, and are stated without proofs.

Lemma 3 *Let $W(X)$ be as in (3.1).*
(i) If $\beta_0 < \beta$, then problem (3.3) and the Riccati inequality

$$\begin{aligned} R(X, \beta) = (CC^H + XX^H - \beta^2 I) - \\ (CA^H + XB^H)(AA^H + BB^H - \beta^2 I)^{-1}(AC^H + BX^H) \le 0. \end{aligned} \tag{3.5}$$

have the same set of solutions.

(ii) Let $0 < \alpha_0$. If $0 < \alpha < \alpha_0$, then problem (3.4) and the Riccati inequality

$$\begin{aligned} R(X,\alpha) = (CC^H + XX^H - \alpha^2 I) - \\ (CA^H + XB^H)(AA^H + BB^H - \alpha^2 I)^{-1}(AC^H + BX^H) \geq 0. \end{aligned} \tag{3.6}$$

have the same set of solutions.

Proof. Analogous to the proof of Lemma 1. ■

If we rewrite $R(X,\beta) = 0$ in the form

$$R(X,\beta) = XR_{\beta,B}X^H - XV_\beta - V_\beta^H X^H - Q_\beta = 0, \tag{3.7}$$

where

$$\begin{aligned} R_{\beta,B} &= I - B^H(AA^H + BB^H - \beta^2 I)^{-1}B \\ R_{\beta,C} &= I - C(A^H A + C^H C - \beta^2 I)^{-1}C^H \\ V_\beta &= B^H(AA^H + BB^H - \beta^2 I)^{-1}AC^H \\ Q_\beta &= CA^H(AA^H + BB^H - \beta^2 I)^{-1}AC^H + \beta^2 I - CC^H \end{aligned} \tag{3.8}$$

then the following formulas are obtained parallel to those in Section 2.

$$R_{\beta,B}^{-1} = I + B^H(AA^H - \beta^2 I)^{-1}B, \tag{3.9}$$

$$R_{\beta,B}^{-1}V_\beta = B^H(AA^H - \beta^2 I)^{-1}AC^H \tag{3.10}$$

and

$$Q_\beta + V_\beta^H R_{\beta,B}^{-1} V_\beta = \beta^2 R_{\beta,C}^{-1} \tag{3.11}$$

where

$$R_{\beta,C}^{-1} = I + C(A^H A - \beta^2 I)^{-1}C^H. \tag{3.12}$$

Replacing β by α in (3.8), we obtain analogous formulas to (3.9)-(3.12) from

$$R(X,\alpha) = XR_{\alpha,B}X - XV_\alpha - V_\alpha^H X - Q_\alpha = 0.$$

For the inertias of $R_{\beta,B}$, $R_{\beta,C}$, $R_{\alpha,B}$ and $R_{\alpha,C}$, we have the following Lemma.

Lemma 4 *Let $\beta_0 < \beta$. Then, both $R_{\beta,B}^{-1}$ and $R_{\beta,C}^{-1}$ given by (3.9) and (3.12) are positive definite.*
If $0 < \alpha < \alpha_0$ and α is not a singular value of A and if the inertia of $AA^H - \alpha^2 I$ is $(p, n-p, 0)$, then the inertia of $R_{\alpha,B}^{-1}$ is $(N+p-n-m, n-p, 0)$, and that of $R_{\alpha,C}^{-1}$ is $(N+p-m-n, m-p, 0)$.

Proof. Analogous to the proof of Lemma 2. ■

We now prove the main theorem of this section.

Theorem 3 *Let $W(X)$ be as in (3.1).*
(i) Let $\beta_0 < \beta$ with β_0 as in (3.2). Then the inequality

$$\|W(X)\| \leq \beta \tag{3.13}$$

has a solution. All matrices X solving inequality (3.13) are given by

$$X = CA^H(AA^H - \beta^2 I)^{-1}B + \beta R_{\beta,C}^{-1/2} K R_{\beta,B}^{-1/2}, \tag{3.14}$$

where K is any matrix such that $\|K\| \leq 1$.
(ii) Suppose that $0 < \alpha_0$. Let $0 < \alpha < \alpha_0$ and assume that α is not a singular value of A. Then the inequality

$$\|W(X)^{-1}\| \leq \frac{1}{\alpha} \tag{3.15}$$

has a solution. All matrices X solving inequality (3.15) are given by

$$X = CA^H(AA^H - \alpha^2 I)^{-1}B + \alpha Y, \tag{3.16}$$

where Y satisfies the inequality

$$Y R_{\alpha,B} Y^H \geq R_{\alpha,C}^{-1}. \tag{3.17}$$

Proof. (i) By formulas (3.8), (3.9) and (3.11) we obtain

$$R(X,\beta) = (X - V_\beta^H R_{\beta,B}^{-1}) R_{\beta,B} (X^H - R_{\beta,B}^{-1} V_\beta) - (Q_\beta + V_\beta^H R_{\beta,B}^{-1} V_\beta),$$

where

$$(Q_\beta + V_\beta^H R_{\beta,B}^{-1} V_\beta) = \beta^2 R_{\beta,C}^{-1}.$$

Consider the inequality

$$Z R_{\beta,B} Z^H \leq R_{\beta,C}^{-1}, \tag{3.18}$$

where $R_{\beta,B}$ and $R_{\beta,C}^{-1}$ are given in (3.8) and (3.12) and $Z = (X - V_\beta^H R_{\beta,B}^{-1})/\beta$.

Both $R_{\beta,B}$ and $R_{\beta,C}$ are positive definite by Lemma 4. Thus, it follows that Z satisfying (3.18) has the form

$$Z = R_{\beta,C}^{-1/2} K R_{\beta,B}^{-1/2},$$

where $\|K\| \leq 1$. So the solution X is given by (3.14).

(ii) As in case (i), $R(X,\alpha) \geq 0$ is equivalent to the inequality

$$Y R_{\alpha,B} Y^H \geq R_{\alpha,C}^{-1}, \tag{3.19}$$

where $Y = (X - V_\alpha^H R_{\alpha,B}^{-1})/\alpha$.

By Lemma 4, the inertia of $R_{\alpha,B}$ is $(N+p-m-n, n-p, 0)$ and that of $R_{\alpha,C}$ is $(N+p-m-n, m-p, 0)$ with p being the number of positive eigenvalues of $AA^H - \alpha^2 I$. Let Q_1 and Q_2 be transformations such that

$$Q_1^H R_{\alpha,B} Q_1 = \begin{bmatrix} I_{N+p-m-n} & \\ & -I_{n-p} \end{bmatrix}$$

and

$$Q_2^H R_{\alpha,C} Q_2 = \begin{bmatrix} I_{N+p-m-n} & \\ & -I_{m-p} \end{bmatrix},$$

then (3.19) is equivalent to

$$\tilde{Y} \begin{bmatrix} I_{N+p-m-n} & \\ & -I_{n-p} \end{bmatrix} \tilde{Y}^H \geq \begin{bmatrix} I_{N+p-m-n} & \\ & -I_{m-p} \end{bmatrix}, \tag{3.20}$$

where $\tilde{Y} = Q_2^{-H} Y Q_1^H$. Inequality (3.20) is always solvable. In fact when $n \leq m$, one possible solution is

$$\tilde{Y} = \begin{bmatrix} I_{N+p-m-n} & 0 \\ 0 & -I_{n-p} \\ 0 & 0 \end{bmatrix},$$

where $\tilde{Y}$ is completed by a 0 block to be an $(N-n) \times (N-m)$ matrix, and when $n \geq m$, a possible solution is

$$\tilde{Y} = \begin{bmatrix} I_{N+p-m-n} & 0 & 0 \\ 0 & -I_{n-p} & 0 \end{bmatrix}.$$

■

Remark 5 The general solution (3.14) of (3.3) does not depend on the inverses of $(AA^H + BB^H - \beta^2 I)$ and $(A^H A + C^H C - \beta^2 I)$ respectively, but instead on the matrices $B^H (AA^H - \beta^2 I)^{-1} B$ and $C(A^H A - \beta^2 I)^{-1} C^H$, which both have limits when $\beta \to \beta_0$, e.g. [9].

However, the existence of the limiting solutions of (3.4) when $\alpha \to \alpha_0$ requires as an extra condition that α_0 is not a singular value of A.

A counterexample is given below, to show that when α_0 is a singular value of A, then problem (1.4) may not have finite solutions. Let

$$W(x) = \begin{bmatrix} -1 & 0 \\ 1 & x \end{bmatrix}$$

with $\alpha_0 = \sigma_1(A) = 1$. Now

$$W(x)^{-1} = \begin{bmatrix} -1 & 0 \\ 1/x & 1/x \end{bmatrix}.$$

Only when we take the limit $x \to \infty$, we obtain $\|W(x)^{-1}\| \to 1$, the minimum of $\|W(x)^{-1}\|$. So problem (1.4) has no finite solution.

Now let us come back to problems (1.3) and (1.4). The following theorem characterizes the solutions for these problems.

Theorem 4 *Let $W(X)$ be as in (3.1).*
(i) Then

$$\inf_X \|W(X)\| = \beta_0.$$

Furthermore, there exists a matrix X such that

$$\|W(X)\| = \beta_0.$$

(ii) Let $0 < \alpha_0$. Then

$$\inf_X \|W(X)^{-1}\| = \frac{1}{\alpha_0}.$$

Furthermore, if we assume that α_0 is not a singular value of A, then there exists a matrix X such that

$$\|W(X)^{-1}\| = \frac{1}{\alpha_0}.$$

Proof. For any appropriately sized matrix X the Cauchy interlacing theorem, (e.g. [5]), implies the inequalities

$$\|W(X)\| \geq \beta_0$$

and

$$\|W(X)^{-1}\| \geq \frac{1}{\alpha_0}$$

Thus, Theorem 3 and Remark 5 yield the conclusions of Theorem 4. ■

Remark 6 It should be noted that Theorems 3 and 4 are also valid if $n = 0$ or $m = 0$. The results are immediately modified by removing all terms involving A, C or A, B, respectively, from the formulas.

4 Minimizing the condition number.

In this section we will discuss problem (1.2) of finding a matrix X such that the condition number of $W(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix}$ is minimal. **In general this is still an open problem.**

Since the condition number of $W(X)$ depends continuously on X, and since any elements of X going to infinity will cause $\|W(X)\|$ to go to infinity and hence also cond$(W(X))$ to go to infinity, the problem (1.2) always has a finite minimizing solution.

From Theorem 4, a lower bound of $\|W(X)\|\|W(X)^{-1}\|$ is given by β_0/α_0. Thus, it follows that

$$\min_X \operatorname{cond}(W(X)) \geq \frac{\beta_0}{\alpha_0}. \tag{4.1}$$

In order to find X such that

$$\|W(X)\|\|W(X)^{-1}\| = \frac{\beta_0}{\alpha_0},$$

we have to find a common solution for both Riccati inequalities $R(X, \beta_0) \le 0$ and $R(X, \alpha_0) \ge 0$. Unfortunately such solutions exist only in special cases.

One such special case is $m = 0$, i.e. the submatrices A, C of $W(X)$ are void. In this case we will give the complete set of solutions. This result generalizes a result of Bunse-Gerstner/Mehrmann/Nichols [2], where partial solutions for this problem were obtained.

Let $W(X) = \begin{bmatrix} B \\ X \end{bmatrix}$, with $B \in \mathbf{C}^{n,N}, X \in \mathbf{C}^{N-n,N}$ and let

$$B = U \begin{bmatrix} \beta_0 I_{n_1} & 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 & 0 \\ 0 & 0 & \alpha_0 I_{n_3} & 0 \end{bmatrix} V^H \tag{4.2}$$

be the singular value decomposition of B with U, V unitary and Σ_2 diagonal having no diagonal element equal to α_0 or β_0. Let $Y = XV = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix}$ be partitioned analogously, with block sizes n_1, n_2, n_3, n_4, each of which may be 0.

Due to the special form of the matrix $W(X)$, the Riccati inequalities (3.5) and (3.6) simplify to

$$X(I - B^H(BB^H - \beta^2 I)^{-1}B)X^H - \beta^2 I \le 0 \tag{4.3}$$

and

$$X(I - B^H(BB^H - \alpha^2 I)^{-1}B)X^H - \alpha^2 I \ge 0. \tag{4.4}$$

A simple calculation shows that (4.3) is equivalent to

$$Y \begin{bmatrix} -\frac{\beta^2}{\beta_0^2-\beta^2} I_{n_1} & 0 & 0 & 0 \\ 0 & -\beta^2(\Sigma_2^2 - \beta^2 I_{n_2})^{-1} & 0 & 0 \\ 0 & 0 & -\frac{\beta^2}{\alpha_0^2-\beta^2} I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{bmatrix} Y^H \le \beta^2 I_{N-n} \tag{4.5}$$

and (4.4) is equivalent to

$$Y \begin{bmatrix} -\frac{\alpha^2}{\beta_0^2-\alpha^2} I_{n_1} & 0 & 0 & 0 \\ 0 & -\alpha^2(\Sigma_2^2 - \alpha^2 I_{n_2})^{-1} & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2}{\alpha_0^2-\alpha^2} I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{bmatrix} Y^H \ge \alpha^2 I_{N-n} \tag{4.6}$$

Taking limits $\alpha \to \alpha_0$ and $\beta \to \beta_0$ we obtain from (4.5) that Y_1 has to be the zero matrix and from (4.6) that Y_3 has to be the zero matrix.

We combine these considerations in the following proposition:

Proposition 1 *Let* $W(X) = \begin{bmatrix} B \\ X \end{bmatrix}$, *with* $B \in \mathbf{C}^{n,N}, X \in \mathbf{C}^{N-n,N}$ *and let* B *be factorized as in (4.2). Then the set of solutions of problem (1.1) is given by all matrices* $Y = \begin{bmatrix} 0 & Y_2 & 0 & Y_4 \end{bmatrix}$ *with* Y_2, Y_4 *satisfying the inequality*

$$\begin{aligned} &\alpha_0^2(I_{N-n} + Y_2(\Sigma_2^2 - \alpha_0^2 I_{n_2})^{-1} Y_2^H) \le Y_4 Y_4^H \\ &\quad \le \beta_0^2(I_{N-n} + Y_2(\Sigma_2^2 - \beta_0^2 I_{n_2})^{-1} Y_2^H). \end{aligned} \tag{4.7}$$

A necessary and sufficient condition for the existence of a matrix Y_4 satisfying (4.7) is that

$$\alpha_0^2(I_{N-n} + Y_2(\Sigma_2^2 - \alpha_0^2 I_{n_2})^{-1} Y_2^H) \le \beta_0^2(I_{N-n} + Y_2(\Sigma_2^2 - \beta_0^2 I_{n_2})^{-1} Y_2^H) \tag{4.8}$$

and

$$0 \le I_{N-n} + Y_2(\Sigma_2^2 - \beta_0^2 I_{n_2})^{-1} Y_2^H. \tag{4.9}$$

Inequality (4.8) is equivalent to

$$Y_2 \Sigma_2^2 (\Sigma_2^2 - \alpha_0^2)^{-1} (\beta_0^2 - \Sigma_2^2)^{-1} Y_2^H \le I_{n_2}, \tag{4.10}$$

i.e.

$$\| Y_2 \Sigma_2 (\Sigma_2^2 - \alpha_0^2)^{-1/2} (\beta_0^2 - \Sigma_2^2)^{-1/2} \| \le 1$$

and (4.9) is equivalent to

$$\| Y_2 (\beta_0^2 - \Sigma_2^2)^{-1/2} \| \le 1.$$

Clearly for $Y_2 = 0$ and $Y_4 = \delta I_{N-n}$ with $\alpha_0 \le \delta \le \beta_0$ we obtain a solution. This solution was obtained already in [2] via a different approach, but there are obviously more solutions to (4.7). For $n = 0, m > 0$ we get an analogous result by transposition.

To characterize the complete set of minimizing solutions for (1.2) for $m > 0, n > 0$ is still an open problem. If we are just interested in one solution, then since problem (1.2) is an optimization problem with $(N-n)(N-m)$ free variables, we could use numerical optimization to find the solution. In the case that X is a scalar one can apply the standard scalar Newton method, since in this case the derivative of the condition number can be explicitely computed via formulas given in [4]. Using the solution of problem (1.3) or (1.4) that gives the smaller condition number as starting point for Newton's method, the convergence is usually very fast. Consider the following numerical example:

Example 1 Let

$$A = \text{diag}(-0.5, 1, 2, 3), c = [1, 1, 1, 1], b = c^T.$$

The minimizing solutions of (1.3) and (1.4) are $x_1 = x_2 = -2.2980$ and $y_1 = y_2 = 4.8405$, respectively. We obtain

$$\min_x \|W(x)\| = 3.2480, \ \min_x \|W(x)^{-1}\| = 1.3708$$

and a lower bound for the condition number is

$$\min_x \operatorname{cond}(W(x)) \geq \beta_0/\alpha_0 = 4.4523.$$

The minimizing point obtained by Newton's method is at $x^* = 3.8885$ with $\min_x \operatorname{cond}(W(x)) = 7.7633$.

5 Conclusion

We have given explicit solutions for the problem of minimizing the norm and the norm of the inverse of a matrix, for which a submatrix can be assigned arbitrarily. The solution sets are characterized completely and from these results lower bounds for the optimal solution for the problem of minimizing the condition numbers are obtained. In the special case that $n = 0$ or $m = 0$ a complete characterization is given for this problem.

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