Minimizing Effective Resistance of a Graph

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***Abstract*— The effective resistance between two nodes of a weighted graph is the electrical resistance seen between the nodes of the corresponding resistor network with branch conductances given by the edge weights. The effective resistance comes up in many applications and fields in addition to electrical network analysis, including, for example, Markov chains and continuous-time averaging networks. In this paper we study the problem of allocating edge weights on a given graph in order to minimize the total effective resistance, *i.e.*, the sum of the effective resistances between all pairs of nodes. We show that this is a convex optimization problem which can be solved efficiently. We show that optimal allocation of the edge weights can reduce the total effective resistance of the graph (compared to uniform weights) by a factor that grows unboundedly with the size of the graph.**

I. Introduction

Let $\mathcal{N}$ be a network with n nodes and m edges, *i.e.*, an undirected graph (V, E) with $|V| = n$, $|E| = m$, and nonnegative weights on the edges. We call the weight on edge l its *conductance*, and denote it by g_l. The *effective resistance* between a pair of nodes i and j, denoted R_{ij}, is the electrical resistance measured across nodes i and j, when the network represents an electrical circuit with each edge (or branch, in the terminology of electrical circuits) a resistor with (electrical) conductance g_l. In other words, R_{ij} is the potential difference that appears across terminals i and j when a unit current source is applied between them. We will give a formal, precise definition of effective resistance later; for now we simply note that it is a measure of how 'close' the nodes i and j are: R_{ij} is small when there are many paths between nodes i and j with high conductance edges, and R_{ij} is large when there are few paths, with lower conductance, between nodes i and j. Indeed, the resistance R_{ij} is sometimes referred to as the *resistance distance* between nodes i and j.

We define the *total effective resistance*, $R_{\rm tot}$, as the sum of the effective resistance between all distinct pairs of nodes,

$$R_{\rm tot} = \frac{1}{2}\sum_{i,j=1}^{n} R_{ij} = \sum_{i<j} R_{ij}. \tag{1}$$

The total effective resistance is evidently a quantitative scalar measure of how well 'connected' the network is, or how 'large' the network is, in terms of resistance distance. The total effective resistance comes up in a number of contexts. In an electrical network, $R_{\rm tot}$ is related to the average power dissipation of the circuit, with a random current excitation. The total effective resistance arises in Markov chains as well: $R_{\rm tot}$ is, up to a scale factor, the average commute time (or average hitting time) of a Markov chain on the graph, with weights given by the edge conductances g_l. In this context, a network with small total effective resistance corresponds to a Markov chain with small hitting or commute times between nodes, and a large total effective resistance corresponds to a Markov chain with large hitting or commute times between at least some pairs of nodes. The total effective resistance comes up in a number of other applications as well, including averaging networks, experiment design, and Euclidean distance embeddings.

In this paper we address the problem of allocating a fixed total conductance among the edges so as to minimize the total effective resistance of the graph. We can assume without loss of generality that the total conductance to be allocated is one, so we have the optimization problem

$$\begin{array}{ll} \text{minimize} & R_{\rm tot} \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0. \end{array} \tag{2}$$

Here, the optimization variable is $g \in \mathbf{R}^m$, the vector of edge conductances, and the problem data is the graph (topology) (V, E). The symbol $\mathbf{1}$ denotes the vector with all entries one, and the inequality symbol $\geq$ between vectors means componentwise inequality. We refer to the problem (2) as the *effective resistance minimization problem* (ERMP).

We will give several interpretations of this problem. In the context of electrical networks, the ERMP corresponds to allocating conductance to the branches of a circuit so as to achieve low resistance between the nodes; in a Markov chain context, the ERMP is the problem of selecting the weights on the edges to minimize the average commute (or hitting) time between nodes. When R_{ij} are interpreted as distances, the ERMP is the problem of allocating conductance to a graph to make the graph 'small', in the sense of average distance between nodes. We denote the optimal solution of the ERMP (which we will show always exists, and is unique) as $g^\star$, and the corresponding optimal total effective resistance as $R^\star_{\rm tot}$.

In this paper, we will show that the problem (2) is a *convex optimization problem*, which can be formulated as a semidefinite program (SDP) [BV04]. This has several implications, practical and theoretical. One practical consequence is that we can solve the ERMP efficiently. On the theoretical side, convexity of the ERMP allows us to form necessary and sufficient optimality conditions, and an associated dual problem (with zero duality gap). Feasible points in the dual problem gives us lower bounds on $R^\star_{\rm tot}$; in fact, we obtain a lower bound on $R^\star_{\rm tot}$ given any feasible allocation of conductances. This gives us an

easily computable upper bound on the suboptimality, *i.e.*, a duality gap, given a conductance allocation g. We use this duality gap in a simple interior point algorithm for solving the ERMP. We show that for the barbell graph, the ratio of $R^\star_{\rm tot}$ to $R_{\rm tot}$ obtained with uniform edge weights converges to zero as the size of the graph increases. Thus, the total effective resistance of a graph, with optimized edge weights, can be unboundedly better (*i.e.*, smaller) than the total effective resistance of a graph with uniform allocation of weights to the edges.

This paper is organized as follows. In §II, we give a formal definition of the effective resistance, derive a number of formulas and expressions for R_{ij}, $R_{\rm tot}$, and the first and second derivatives of $R_{\rm tot}$, and establish several important properties, such as convexity of R_{ij} and $R_{\rm tot}$ as a function of the edge conductances. In §III, we give several interpretations of R_{ij}, $R_{\rm tot}$, and the ERMP. We study the ERMP in §IV, giving the SDP formulation, (necessary and sufficient) optimality conditions, two dual problems, and a simple but effective custom interior point method for solving it. In §V, we analytically solve the ERMP for the barbell. We describe some extensions in §VII.

A. Related problems

The ERMP is related to several other convex optimization problems that involve choice of some weights on the edges of a graph. One such problem (already mentioned above) is to assign nonnegative weights, that add to one, to the edges of a graph so as to maximize the second smallest eigenvalue of the Laplacian:

$$\begin{array}{ll} \text{maximize} & \lambda_2(L) \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0. \end{array} \tag{3}$$

Here L denotes the Laplacian of the weighted graph. This problem has been studied in different contexts. The eigenvalue $\lambda_2(L)$ is related to the *mixing rate* of the Markov process with edge transition rates given by the edge weights. In [SBXD], the weights g are optimized to obtain the fastest mixing Markov process on the given graph. The problem (3) has also been studied in the context of algebraic connectivity [Fie73]. The algebraic connectivity is the second smallest eigenvalue of the Laplacian matrix L of a graph (with unit edge weights), and is a measure of how well connected the graph is. Fiedler defines the absolute algebraic connectivity of a graph as the maximum value of $\lambda_2(L)$ over all nonnegative edge weights that add up to m, *i.e.*, m times the optimal value of (3). The problem of finding the absolute algebraic connectivity of a graph is discussed in [Fie90], [Fie93], and an analytical solution is presented for tree graphs.

Other convex problems involving edge weights on graphs include the problem of finding the fastest mixing Markov chain on a given graph [BDX04], [GHW05], [SBXD], the problem of finding the edge weights (which can be negative) that give the fastest convergence in an averaging network [XB04], and the problem of finding edge weights that give the smallest least mean-square (LMS) consensus error [XBK05]. Convex optimization can also be used to obtain bounds on various quantities over a family of graphs; see [GB]. For an overview of such problems, see [Boy06].

In [BVGY01], Boyd et al consider the sizing of the wires in the power supply network of an integrated circuit, with unknown load currents modeled stochastically. This turns out to be closely related to our ERMP, with the wire segment widths proportional to the edge weights.

Some papers on various aspects of resistance distance include [Kle02], [XG03], [Bap99], [KR93].

II. The effective resistance

A. Definition

Suppose edge l connects nodes i and j. We define $a_l \in \mathbf{R}^n$ as $(a_l)_i = 1$, $(a_l)_j = -1$, and all other entries 0. The *conductance matrix* (or *weighted Laplacian*) of the network is defined as

$$G = \sum_{l=1}^m g_l a_l a_l^T = A\,\mathbf{diag}(g) A^T,$$

where $\mathbf{diag}(g) \in \mathbf{R}^m$ is the diagonal matrix formed from g, and $A \in \mathbf{R}^{n\times m}$ is the *incidence* matrix of the graph:

$$A = [a_1 \ \cdots \ a_m].$$

Since $g_l \geq 0$, G is positive semidefinite, which we write as $G \succeq 0$. (The symbol $\succeq$ denotes denotes matrix inequality, between symmetric matrices.) The matrix G satisfies $G\mathbf{1} = 0$, since $a_l^T \mathbf{1} = 0$ for each edge l. Thus, G has smallest eigenvalue 0, corresponding to the eigenvector $\mathbf{1}$.

Throughout this paper we make the following assumption about the edge weights:

The subgraph of edges with positive edge weights is connected. (4)

(If this is not the case, the effective resistance between any pair of nodes not connected by a path of edges with positive conductance is infinite, and many of our formulas are no longer valid.)

With this assumption, all other eigenvalues of G are positive. We denote the eigenvalues of G as

$$0 < \lambda_2 \leq \cdots \leq \lambda_n.$$

The nullspace of G is one-dimensional, the line along $\mathbf{1}$; its range has co-dimension one, and is given by $\mathbf{1}^\perp$ (*i.e.*, all vectors v with $\mathbf{1}^T v = 0$).

Let $G^{(k)}$ be the submatrix obtained by deleting the kth row and column of G. Our assumption (4) implies that each $G^{(k)}$ is nonsingular (see, *e.g.*, [DK69]). We will refer to $G^{(k)}$ as the *reduced conductance matrix* (obtained by *grounding* node k).

Now we can define the effective resistance R_{ij} between a pair of nodes i and j. Let v be a solution to the equation

$$Gv = e_i - e_j,$$

where e_i denotes the ith unit vector, with 1 in the ith position, and 0 elsewhere. This equation has a solution since $e_i - e_j$ is in the range of G. We define R_{ij} as

$$R_{ij} = v_i - v_j.$$

This is well defined; *all* solutions of $Gv = e_i - e_j$ give the same value of $v_i - v_j$. (This follows since the difference of any two solutions has the form $\alpha\mathbf{1}$, for some $\alpha \in \mathbf{R}$.) We define the *effective resistance matrix* $R \in \mathbf{R}^{n\times n}$ as the matrix with i, j entry R_{ij}. The effective resistance matrix is evidently symmetric, and has diagonal entries zero, since $R_{ii} = 0$.

B. Effective resistance in an electrical network

The term *effective resistance* (as well as several other terms used here) comes from electrical network analysis. We consider an electrical network, with conductance g_l on branch (or edge) l. Let $v \in \mathbf{R}^n$ denote the vector of node potentials, and suppose a current J_i is injected into node i. The sum of the currents injected into the network must be zero, in order for Kirchhoff's current law to hold, *i.e.*, we must have $\mathbf{1}^T J = 0$. The injected currents and node potentials are related by $Gv = J$. There are many solutions of this equation, but all differ by a constant vector. Thus, the potential difference between a pair of nodes is always well defined.

One way to fix the node potentials is to assign a potential zero to some node, say the kth node. This corresponds to grounding the kth node. When this is done, the circuit equations are given by $G^{(k)}v^{(k)} = J^{(k)}$, where $G^{(k)}$ is the reduced conductance matrix, $v^{(k)}$ is the reduced potential vector, obtained by deleting the kth entry of v (which is zero), and $J^{(k)}$ is the reduced current vector, obtained by deleting the kth entry of J. In this formulation, $J^{(k)}$ has no restrictions; alternatively, we can say that J_k is implicitly defined as $J_k = -\mathbf{1}^T J^{(k)}$. From our assumption (4), $G^{(k)}$ is nonsingular, so there is a unique reduced potential vector $v^{(k)}$ for any vector of injected currents $J^{(k)}$.

Now consider the specific case when the external current is $J = e_i - e_j$, which corresponds to a one ampere current source connected from node j to node i. Any solution v of $Gv = e_i - e_j$ is a valid vector of node potentials; all of these differ by a constant. The difference $v_i - v_j$ is the same for all valid node potentials, and is the voltage developed across terminals i and j. This voltage is R_{ij}, the effective resistance between nodes i and j. (The effective resistance between two nodes of a circuit is defined as the ratio of voltage across the nodes to the current flow injected into them.)

The effective resistance R_{ij} is the total power dissipated in the resistor network when $J = e_i - e_j$, *i.e.*, a one ampere current source is applied between nodes i and j. This can be shown directly, or by a power conservation argument. The voltage developed across nodes i and j is R_{ij} (by definition), so the power supplied by the current source, which is current times voltage, is R_{ij}. The power supplied by the external current source must equal the total power dissipated in the resistors of the network, so the latter is also R_{ij}.

C. Some formulas for effective resistance

In this section we derive several formulas for the effective resistance between a pair of nodes and the total effective resistance of a general graph. Our first expressions involve the reduced conductance matrix, which we write here as $\tilde{G}$ (since the particular node that is grounded will not matter). We form the reduced conductance matrix $\tilde{G}$ by removing, say, the kth row and column of G. Let $\tilde{v}$, $\tilde{e}_i$, and $\tilde{e}_j$ be, respectively, the vectors v, e_i and e_j, each with the kth component removed. If $Gv = e_i - e_j$, then we have $\tilde{G}\tilde{v} = \tilde{e}_i - \tilde{e}_j$. This equation has a unique solution, $\tilde{v} = \tilde{G}^{-1}(\tilde{e}_i - \tilde{e}_j)$. The effective resistance between nodes i and j is given by $v_i - v_j = \tilde{v}_i - \tilde{v}_j$, *i.e.*,

$$R_{ij} = (\tilde{e}_i - \tilde{e}_j)^T \tilde{G}^{-1}(\tilde{e}_i - \tilde{e}_j). \tag{5}$$

(This is independent of the choice of node grounded, *i.e.*, which row and column is removed.) When neither i nor j is k, the node that is grounded, we can write (5) as

$$R_{ij} = (\tilde{G}^{-1})_{ii} + (\tilde{G}^{-1})_{jj} - 2(\tilde{G}^{-1})_{ij}.$$

If j is k, the node that is grounded, then $\tilde{e}_j = 0$, so (5) becomes

$$R_{ij} = (\tilde{G}^{-1})_{ii}.$$

We can also write the effective resistance R_{ij} in terms of the pseudo-inverse $G^\dagger$ of G. We have

$$G^\dagger G = I - \mathbf{1}\mathbf{1}^T/n,$$

which is the projection matrix onto the range of G. (Here we use the simpler notation $\mathbf{1}\mathbf{1}^T/n$ to mean $(1/n)\mathbf{1}\mathbf{1}^T$.) Using this it can verified that

$$G^\dagger = (G + \mathbf{1}\mathbf{1}^T/n)^{-1} - \mathbf{1}\mathbf{1}^T/n. \tag{6}$$

The following formula gives R_{ij} in terms of $G^\dagger$ (see, *e.g.*, [KR93]):

$$\begin{aligned} R_{ij} &= (e_i - e_j)^T G^\dagger (e_i - e_j) && (7)\\ &= (G^\dagger)_{ii} + (G^\dagger)_{jj} - 2(G^\dagger)_{ij}. && (8) \end{aligned}$$

To see this, multiply $Gv = e_i - e_j$ on the left by $G^\dagger$ to get $(I - \mathbf{1}\mathbf{1}^T/n)v = G^\dagger(e_i - e_j)$, so

$$(e_i - e_j)^T G^\dagger (e_i - e_j) = (e_i - e_j)^T v = v_i - v_j$$

(since $e_i - e_j \perp \mathbf{1}$). From (6), we get another formula for the effective resistance,

$$R_{ij} = (e_i - e_j)^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1}(e_i - e_j). \tag{9}$$

We can derive several formulas for the effective resistance matrix R, using (5) and (7). From (7), we see that

$$R = \mathbf{1}\,\mathbf{diag}(G^\dagger)^T + \mathbf{diag}(G^\dagger)\mathbf{1}^T - 2G^\dagger, \tag{10}$$

where $\mathbf{diag}(G^\dagger) \in \mathbf{R}^n$ is the vector consisting of the diagonal entries of $G^\dagger$.

Using (6), this can be rewritten as

$$\begin{aligned} R &= \mathbf{1}\,\mathbf{diag}((G+\mathbf{1}\mathbf{1}^T/n)^{-1})^T \\ &+ \mathbf{diag}((G+\mathbf{1}\mathbf{1}^T/n)^{-1})\mathbf{1}^T - 2(G+\mathbf{1}\mathbf{1}^T/n)^{-1}. \end{aligned} \tag{11}$$

We can also derive a matrix expression for R in terms of the reduced conductance matrix $\tilde{G}$. Suppose $\tilde{G}$ is formed by removing the kth row and column from G. Form a matrix $H \in \mathbf{R}^{n\times n}$ from $\tilde{G}^{-1}$ by adding a kth row and column, with all entries zero. Then, using (5), R can be written as

$$R = \mathbf{1}\,\mathbf{diag}(H)^T + \mathbf{diag}(H)\mathbf{1}^T - 2H. \tag{12}$$

D. Some formulas for total effective resistance

In this section we give several general formulas for the total effective resistance,

$$R_{\rm tot} = \sum_{i<j} R_{ij} = (1/2)\mathbf{1}^T R\mathbf{1}.$$

From (10) we get

$$\begin{aligned} R_{\rm tot} &= (1/2)\mathbf{1}^T\mathbf{1}\,\mathbf{diag}(G^\dagger)^T\mathbf{1} + \\ &\quad (1/2)\mathbf{1}^T\,\mathbf{diag}(G^\dagger)\mathbf{1}^T\mathbf{1} - \mathbf{1}^T G^\dagger\mathbf{1} \qquad (13)\\ &= n\,\mathbf{Tr}\,G^\dagger \qquad (14)\\ &= n\,\mathbf{Tr}(G+\mathbf{1}\mathbf{1}^T/n)^{-1} - n, \qquad (15) \end{aligned}$$

using $G^\dagger\mathbf{1} = 0$ to get the second line, and (6) to get the third line. ($\mathbf{Tr}\,Z$ denotes the trace of a square matrix Z.)

We can use (14) to get a formula for $R_{\rm tot}$ in terms of the eigenvalues of G. The eigenvalues of $G^\dagger$ are $1/\lambda_i$, for $i = 2,\ldots,n$, and 0. So we can rewrite (14) as

$$R_{\rm tot} = n\sum_{i=2}^{n}\frac{1}{\lambda_i}. \tag{16}$$

This expression for the total effective resistance can be found in [AF03, §3.4].

The total effective resistance can also be expressed in terms of the reduced conductance matrix $\tilde{G}$. Multiplying (12) on the left and right by $\mathbf{1}^T$ and $\mathbf{1}$ and dividing by 2, we have

$$R_{\rm tot} = n\,\mathbf{Tr}\,\tilde{G}^{-1} - \mathbf{1}^T\tilde{G}^{-1}\mathbf{1} = n\,\mathbf{Tr}(I-\mathbf{1}\mathbf{1}^T/n)\tilde{G}^{-1}. \tag{17}$$

(Note that $\tilde{G}\in\mathbf{R}^{(n-1)\times(n-1)}$, so the vectors denoted $\mathbf{1}$ in this formula have dimension $n-1$.)

The total effective resistance can also be written in terms of an integral:

$$R_{\rm tot} = n\,\mathbf{Tr}\int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n)\,dt. \tag{18}$$

This can be seen as follows. Let the eigenvectors of G be $n^{-1/2}\mathbf{1}, v_2, \ldots, v_n$, corresponding to the eigenvalues $\lambda_1 = 0 < \lambda_2 \le \cdots \le \lambda_n$. The matrix e^{-tG} has the same eigenvectors, with corresponding eigenvalues 1, and $e^{-\lambda_i t}$, for $i = 2,\ldots,n$. Therefore we have

$$\begin{aligned} &n\,\mathbf{Tr}\int_0^\infty \left(e^{-tG} - \mathbf{1}\mathbf{1}^T/n\right)\,dt \\ &= n\,\mathbf{Tr}\int_0^\infty \sum_{i=2}^{n} e^{-\lambda_i t} v_i v_i^T\,dt \\ &= n\sum_{i=2}^{n}\int_0^\infty e^{-\lambda_i t}\,dt \\ &= n\sum_{i=2}^{n}\frac{1}{\lambda_i}, \end{aligned}$$

using $\mathbf{Tr}\,v_i v_i^T = \|v_i\|^2 = 1$ to get the second line.

E. Basic properties

The effective resistance R_{ij}, and the total effective resistance $R_{\rm tot}$, are rational functions of g. This can be seen from (5), since the inverse of a matrix is a rational function of the matrix, and R_{ij} is a linear function of $\tilde{G}^{-1}$. They are also homogeneous with degree -1: if $\hat{g} = cg$, where $c > 0$, then $\hat{R}_{ij} = R_{ij}/c$, and $\hat{R}_{\rm tot} = R_{\rm tot}/c$.

The effective resistance R_{ij}, with $i \ne j$, is always positive: the matrix $\tilde{G}^{-1}$ is positive definite (since $\tilde{G} \succ 0$), so from (5), $R_{ij} > 0$ when $i \ne j$. Nonnegativity of R_{ij} can also be seen by noting that $\tilde{G}$ is an M-matrix. The inverse of an M-matrix is elementwise nonnegative [HJ91], and since R_{ij} is the (i,i)th entry of $(G^{(j)})^{-1}$, it is nonnegative as well.

The effective resistance also satisfies the triangle inequality (see, *e.g.*, [KR93]):

$$R_{ik} \le R_{ij} + R_{jk}. \tag{19}$$

Therefore, the effective resistance defines a metric on the graph, called the *resistance distance* [KR93].

The effective resistance R_{ij} is a monotone decreasing function of g, *i.e.*, if $g \le \hat{g}$, then $R_{ij} \ge \hat{R}_{ij}$. To show this, suppose $0 \le g \le \hat{g}$, and let G and $\hat{G}$ denote the associated conductance matrices. Evidently we have $G + \mathbf{1}\mathbf{1}^T/n \preceq \hat{G} + \mathbf{1}\mathbf{1}^T/n$, so $(G+\mathbf{1}\mathbf{1}^T/n)^{-1} \succeq (\hat{G}+\mathbf{1}\mathbf{1}^T/n)^{-1}$. From (9),

$$\begin{aligned} R_{ij} &= (e_i-e_j)^T(G+\mathbf{1}\mathbf{1}^T/n)^{-1}(e_i-e_j) \\ &\ge (e_i-e_j)^T(\hat{G}+\mathbf{1}\mathbf{1}^T/n)^{-1}(e_i-e_j) \\ &= \hat{R}_{ij}. \end{aligned}$$

F. Convexity of effective resistance

The effective resistance R_{ij} is a convex function of g: for $g, \hat{g} \ge 0$ (both satisfying the basic assumption (4)), and any $\theta \in [0,1]$, we have

$$R_{ij}(\theta g + (1-\theta)\hat{g}) \le \theta R_{ij}(g) + (1-\theta)R_{ij}(\hat{g}).$$

To show this, we first observe that $f(X) = c^T Y^{-1} c$, where $Y = Y^T \in \mathbf{R}^{n\times n}$ and $c \in \mathbf{R}^n$, is a convex function of Y, for $Y \succ 0$ (see, *e.g.*, [BV04, §3.1.7]). Since $G + \mathbf{1}\mathbf{1}^T/n$ is an affine function of g, R_{ij} is a convex function of g.

It follows that $R_{\rm tot}$ is also convex, since it is a sum of convex functions.

The total effective resistance is, in fact, a *strictly* convex function of g: for $g, \hat{g} \geq 0$ (both satisfying the basic assumption (4)), with $g \neq \hat{g}$, and any $\theta \in (0,1)$, we have

$$R_{\rm tot}(\theta g + (1-\theta)\hat{g}) < \theta R_{\rm tot}(g) + (1-\theta)R_{\rm tot}(\hat{g}).$$

To establish this, we first show that $\mathbf{Tr}\, X^{-1}$ is a strictly convex function of X, for X symmetric and positive definite. Its second order Taylor approximation is

$$\begin{aligned}\mathbf{Tr}(X+\Delta)^{-1} \approx \mathbf{Tr}\, X^{-1} - \mathbf{Tr}\, X^{-1}\Delta X^{-1} \\ + \mathbf{Tr}\, X^{-1}\Delta X^{-1}\Delta X^{-1}.\end{aligned}$$

The second order term can be expressed as

$$\mathbf{Tr}\, X^{-1}\Delta X^{-1}\Delta X^{-1} = \|X^{-1}\Delta X^{-1/2}\|_F^2,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. This second order term vanishes only if $\Delta = 0$ (since X^{-1} and $X^{-1/2}$ are both invertible), *i.e.*, it is a positive definite quadratic function of Δ. This shows that $\mathbf{Tr}\, X^{-1}$ is a strictly convex function of $X = X^T \succ 0$. Since the affine mapping from g to $G + \mathbf{1}\mathbf{1}^T/n$ is one-to-one, we conclude that

$$R_{\rm tot} = n\,\mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n$$

is a strictly convex function of g.

G. Gradient and Hessian

In this section we work out some formulas for the gradient and Hessian of $R_{\rm tot}$ with respect to g. (A similar approach can be used to find the derivatives of R_{ij} with respect to g, but we will not need these in the sequel.) We will use the following fact. Suppose the invertible symmetric matrix $X(t)$ is a differentiable function of the parameter $t \in \mathbf{R}$. Then we have ([BV04, §A.4.1])

$$\frac{\partial X^{-1}}{\partial t} = -X^{-1}\frac{\partial X}{\partial t}X^{-1}.$$

Using this formula, and $R_{\rm tot} = n\,\mathbf{Tr}(G+\mathbf{1}\mathbf{1}^T/n)^{-1} - n$, we have

$$\begin{aligned}\frac{\partial R_{\rm tot}}{\partial g_l} &= -n\,\mathbf{Tr}(G+\mathbf{1}\mathbf{1}^T/n)^{-1}\frac{\partial G}{\partial g_l}(G+\mathbf{1}\mathbf{1}^T/n)^{-1} \\ &= -n\,\mathbf{Tr}(G+\mathbf{1}\mathbf{1}^T/n)^{-1}a_l a_l^T(G+\mathbf{1}\mathbf{1}^T/n)^{-1} \\ &= -n\|(G+\mathbf{1}\mathbf{1}^T/n)^{-1}a_l\|^2. \qquad (20)\end{aligned}$$

We can express the gradient as

$$\nabla R_{\rm tot} = -n\,\mathbf{diag}(A^T(G+\mathbf{1}\mathbf{1}^T/n)^{-2}A).$$

The gradient can also be expressed in terms of a reduced conductance matrix:

$$\nabla R_{\rm tot} = -n\,\mathbf{diag}(\tilde{A}^T\tilde{G}^{-1}(I-\mathbf{1}\mathbf{1}^T/n)\tilde{G}^{-1}\tilde{A}).$$

For future reference, we note the formula

$$\nabla R_{\rm tot}^T g = -R_{\rm tot}, \qquad (21)$$

which holds since $R_{\rm tot}$ is a homogeneous function of g of degree -1. It is easily verified by taking the derivative with respect to α of $R_{\rm tot}(\alpha g) = R_{\rm tot}(g)/\alpha$, evaluated at $\alpha = 1$.

We now derive the second derivative or Hessian matrix of $R_{\rm tot}$. From (20), we have

$$\begin{aligned}\frac{\partial^2}{\partial g_l \partial g_k}R_{\rm tot} &= -n\frac{\partial}{\partial g_k}\|(G+\mathbf{1}\mathbf{1}^T/n)^{-1}a_l\|^2 \\ &= 2n a_l^T(G+\mathbf{1}\mathbf{1}^T/n)^{-2}a_k a_k^T \\ &\quad \cdot(G+\mathbf{1}\mathbf{1}^T/n)^{-1}a_l. \qquad (22)\end{aligned}$$

We can express the Hessian of $R_{\rm tot}$ as

$$\begin{aligned}\nabla^2 R_{\rm tot} &= 2n\left(A^T(G+\mathbf{1}\mathbf{1}^T/n)^{-2}A\right) \\ &\quad \circ \left(A^T(G+\mathbf{1}\mathbf{1}^T/n)^{-1}A\right),\end{aligned}$$

where $\circ$ denotes the Hadamard (elementwise) product. A similar expression can be derived using reduced matrices:

$$\nabla^2 R_{\rm tot} = 2n(\tilde{A}^T\tilde{G}^{-1}(I-\mathbf{1}\mathbf{1}^T/n)\tilde{G}^{-1}\tilde{A}) \circ (\tilde{A}^T\tilde{G}^{-1}\tilde{A}). \qquad (23)$$

III. Interpretations

A. Average commute time

The effective resistance between a pair of nodes i and j is related to the commute time between i and j for the Markov chain defined by the conductances g [CRR+89]. Let $\mathcal{M}$ be a Markov chain on the graph $\mathcal{N}$, with transition probabilities determined by the conductances:

$$P_{ij} = \frac{g_{ij}}{\sum_{l\sim(i,k)} g_l},$$

where g_{ij} is the conductance across edge (i,j), and $l \sim (i,k)$ means that edge l lies between nodes i and k. This Markov chain is reversible, with stationary distribution

$$\pi_i = \frac{\sum_{l\sim(i,j)} g_l}{\sum_{l=1}^m g_l}.$$

The *hitting time* H_{ij} is the expected time taken by the random walk to reach node j for the first time starting from node i. The *commute time* C_{ij} is the expected time the random walk takes to return to node i for the first time after starting from i and passing through node j. The following well known result relates commute times and effective resistance (see, for example, [AF03, §3.3]):

$$C_{ij} = (\mathbf{1}^T g)R_{ij}.$$

That is, the effective resistance between i and j is proportional to the commute time between i and j.

Therefore, the total effective resistance is proportional to $\overline{C}$, the commute time averaged over all pairs of nodes:

$$\overline{C} = \frac{2(\mathbf{1}^T g)}{n(n-1)}R_{\rm tot}.$$

Since $C_{ij} = H_{ij} + H_{ji}$, $R_{\rm tot}$ is also proportional to $\overline{H}$, the hitting time averaged over all pairs of nodes:

$$\overline{H} = \frac{\mathbf{1}^T g}{n(n-1)}R_{\rm tot}.$$

In the context of Markov chains, the ERMP (2) is the problem of choosing edge weights on a graph so as to minimize its average commute time or hitting time.

B. Power dissipation in a resistor network

The total effective resistance is related to the average power dissipated in a resistor network with random injected currents. Suppose a random current $J \in \mathbf{R}^n$ is injected into the network. The current must satisfy $\mathbf{1}^T J = 0$, since the total current entering the network must be zero. We assume that

$$\mathbf{E}\, J = 0, \qquad \mathbf{E}\, JJ^T = I - \mathbf{1}\mathbf{1}^T/n.$$

Roughly speaking, this means J is a random current vector, with covariance matrix I on $\mathbf{1}^\perp$.

The power dissipated in the resistor network with injected current vector J is $J^T G^\dagger J$. The expected dissipated power is

$$\mathbf{E}\, J^T G^\dagger J = \mathbf{Tr}\, G^\dagger\, \mathbf{E}\, JJ^T = \mathbf{Tr}\, G^\dagger = \frac{1}{n} R_{\rm tot},$$

where the second equality follows from $G^\dagger \mathbf{1} = 0$.

Thus, the total effective resistance is proportional to the average power dissipated in the network when the injected current is random, with mean 0 and covariance $I - \mathbf{1}\mathbf{1}^T/n$. A network with small $R_{\rm tot}$ is one which dissipates little power, under random current excitation; large $R_{\rm tot}$ means the average power dissipation is large.

The ERMP (2) is the problem of allocating unit conductance among the branches of a resistor network, so as to minimize the average power dissipated under random current excitation. (See, *e.g.*, [BVGY01].)

We can also give an interpretation of the gradient $\nabla R_{\rm tot}$ in the context of a resistor network. With random current excitation J, with $\mathbf{E}\, J = 0$, $\mathbf{E}\, JJ^T = I - \mathbf{1}\mathbf{1}^T/n$, the partial derivative $\partial R_{\rm tot}/\partial g_l$ is proportional to the mean square voltage across edge l. This can be seen from (20) as follows. The voltage v_l across edge l, with current excitation J is $a_l^T G^\dagger J = a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} J$, since $a_l^T \mathbf{1} = 0$. The expected value of the squared voltage is

$$\begin{aligned} & \mathbf{E}(a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} J)^2 \\ = \;& a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1}\, \mathbf{E}\, JJ^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l \\ = \;& a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-1} (I - \mathbf{1}\mathbf{1}^T/n)(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l \\ = \;& a_l^T (G + \mathbf{1}\mathbf{1}^T/n)^{-2} a_l, \end{aligned}$$

where the last equality follows since $(G + \mathbf{1}\mathbf{1}^T/n)^{-1}\mathbf{1} = \mathbf{1}$, and $\mathbf{1}^T a_l = 0$. Comparing this with (20), we see that

$$\frac{\partial R_{\rm tot}}{\partial g_l} = -n\, \mathbf{E}\, v_l^2. \tag{24}$$

The gradient $\nabla R_{\rm tot}$ is equal to $-n$ times the vector of mean square voltage appearing across the edges.

C. Elmore delay in an RC circuit

We consider again a resistor network, with branch (electrical) conductances given by g_l. To this network we add a separate ground node, and a unit capacitance between every other node and the ground node. The vector of node voltages (with respect to the ground node) in this RC (resistor-capacitor) circuit evolves according to $\dot v = -Gv$. This has solution $v(t) = e^{-tG} v(0)$. Since e^{-tG} has largest eigenvalue 1, associated with the eigenvector $\mathbf{1}$, with other eigenvalues $e^{-\lambda_i t}$, for $i = 2, \ldots, n$, we see that $v(t)$ converges to the vector $\mathbf{1}\mathbf{1}^T v(0)/n$. In other words, the voltage (or equivalently, charge) equilibrates itself across the nodes in the circuit.

Suppose we start with the initial voltage $v(0) = e_k$, *i.e.*, one volt on node k, with zero voltage on all other nodes. It can be shown that the voltage at node k monotonically decreases to the average value, $1/n$. The *Elmore delay* at node k is defined as

$$T_k = \int_0^\infty (v_k(t) - 1/n)\, dt$$

(see, for example, [Elm48], [WH04]). The Elmore delay T_k gives a measure of the speed at which charge starting at node k equilibrates.

The average Elmore delay, over all nodes, is

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^n T_k &= \frac{1}{n}\sum_{k=1}^n e_k^T \int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n) e_k\, dt \\ &= \frac{1}{n}\, \mathbf{Tr} \left(\int_0^\infty (e^{-tG} - \mathbf{1}\mathbf{1}^T/n)\, dt \right) \\ &= \frac{1}{n^2} R_{\rm tot}, \end{aligned}$$

where the last equality follows from (18). The total effective resistance of the network is thus equal to the sum of the Elmore delay to each node in the RC circuit.

The ERMP (2) is the problem of allocating a total conductance of one to the resistor branches of an RC circuit, so as to minimize the average Elmore delay of the nodes.

D. Total time constant of an averaging network

We can interpret $R_{\rm tot}$ in terms of the time constants in an averaging network. We consider the dynamical system $\dot x = -Gx$, where G is the conductance matrix. This system carries out (asymptotic) averaging: e^{-tG} is a doubly stochastic matrix, which converges to $\mathbf{1}\mathbf{1}^T/n$ as $t \to \infty$, so $x(t) = e^{-tG} x(0)$ converges to $\mathbf{1}\mathbf{1}^T x(0)/n$.

The eigenvalues $\lambda_2, \ldots, \lambda_n$ of G determine the rate at which the averaging takes place. The eigenvectors $v_2, \ldots, v_n$ are the modes of the system, and the associated time constants are given by

$$T_k = \frac{1}{\lambda_k}.$$

(This gives the time for mode k to decay by a factor e.) Therefore, $R_{\rm tot} = n \sum_{k=2}^n T_k$ is proportional to the sum of the time constants of the averaging system.

E. A-optimal experiment design

The ERMP can be interpreted as a certain type of optimal experiment design problem. The goal is to estimate a parameter vector $x \in \mathbf{R}^n$ from noisy linear measurements

$$y_i = v_i^T x + w_i, \quad i = 1, \ldots, K,$$

where each v_i can be any of the vectors $a_1, \ldots, a_m$, and w_i are independent random (noise) variables with zero mean and unit variance. Thus, each measurement consists of measuring a difference between two components of x, corresponding to some edge of our graph, with some additive noise. With these measurements of differences of components, we can only estimate x up to some additive constant; the parameter x and $x + \alpha\mathbf{1}$, for any $\alpha \in \mathbf{R}$, produce exactly the same measurements. We will therefore assume that the parameter to be estimated satisfies $\mathbf{1}^T x = 0$.

Now suppose a total of k_l measurements are made using a_l, for $l = 1, \ldots, m$, so we have $\sum_{l=1}^m k_l = K$. The minimum variance unbiased estimate of x, given the measurements, is

$$\begin{aligned}\hat{x} &= \left(\sum_{i=1}^K v_i v_i^T\right)^\dagger \left(\sum_{i=1}^K v_i^T y_i\right) \\ &= \left(\sum_{l=1}^m k_l a_l a_l^T\right)^\dagger \left(\sum_{l=1}^m k_l a_l^T y\right).\end{aligned}$$

The associated estimation error $e = \hat{x} - x$ has zero mean and covariance matrix

$$\Sigma_{\rm err} = \left(\sum_{l=1}^m k_l a_l a_l^T\right)^\dagger.$$

(There is no estimation error in the direction $\mathbf{1}$, since we have assumed that $\mathbf{1}^T x = 0$, and we always have $\mathbf{1}^T \hat{x} = 0$.) The goal of experiment design is to choose the integers $k_1, \ldots, k_l$, subject to $\sum_l k_l = K$, to make the estimation error covariance matrix $\Sigma_{\rm err}$ small. There are several ways to define 'small', which yield different experiment design problems. In A-optimal experiment design, the objective is the trace of $\Sigma_{\rm err}$. This is proportional to the sum of the squares of the semi-axis lengths of the confidence ellipsoid associated with the estimate $\hat{x}$.

We now change variables to $\theta_l = k_l/K$, which is the fraction of the total number of experiments (*i.e.*, K) that are carried out using $v = a_l$. The variables θ_l are nonnegative and add to one, and must be integer multiples of $1/K$. If K is large, we can ignore the last requirement, and take the variables θ_l to be real. This yields the (relaxed) A-optimal experiment design problem [Puk93]:

$$\begin{array}{ll}\text{minimize} & (1/K)\,\mathbf{Tr}(\sum_{l=1}^m \theta_l a_l a_l^T)^\dagger \\ \text{subject to} & \sum_{l=1}^m \theta_l = 1, \quad \theta_l \geq 0.\end{array}$$

This is a convex optimization problem, with variable $\theta \in \mathbf{R}^m$. (See [BV04, §7.5] and its references for more on experiment design problems.)

Identifying θ_l with g_l, we see that the A-optimal experiment design problem above is the same as our ERMP (up to scale factor in the objective). Thus, we can interpret the ERMP as follows. We have real numbers, $x_1, \ldots, x_n$ at the nodes of our graph, which have zero sum. Each edge in our graph corresponds to a possible measurement we can make, which gives the difference in its adjacent node values, plus a noise. We are going to make a large number of these measurements, in order to estimate x. The problem is to choose the fraction of the experiments that should be devoted to each edge measurement. Using the trace of the error covariance matrix as our measure of estimation quality, the optimal fractions are exactly the optimal conductances in the ERMP.

IV. Minimizing total effective resistance

In this section we study the ERMP,

$$\begin{array}{ll}\text{minimize} & R_{\rm tot} \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0,\end{array} \tag{25}$$

in detail. This is a convex optimization problem, since the objective is a convex function of g, and the constraint functions are linear. The problem is clearly feasible, since $g = (1/m)\mathbf{1}$, the uniform allocation of conductance to edges, is feasible. Since the objective function is strictly convex, the solution to (25) is unique. We denote the unique optimal point as $g^\star$, and the associated value of the objective as $R_{\rm tot}^\star$.

A. SDP formulation

The ERMP (25) can be formulated as a *semidefinite program* (SDP),

$$\begin{array}{ll}\text{minimize} & n\,\mathbf{Tr}\,Y \\ \text{subject to} & \mathbf{1}^T g = 1, \quad g \geq 0, \\ & \left[\begin{array}{cc} G + \mathbf{1}\mathbf{1}^T/n & I \\ I & Y \end{array}\right] \succeq 0,\end{array} \tag{26}$$

where $G = \sum_{l=1}^m g_l a_l a_l^T$. The variables are the conductances $g \in \mathbf{R}^m$, and the slack symmetric matrix $Y \in \mathbf{R}^{n\times n}$. To see the equivalence, we note that whenever $G + \mathbf{1}\mathbf{1}^T/n \succ 0$ (which is guaranteed whenever the basic assumption (4) holds),

$$\left[\begin{array}{cc} G + \mathbf{1}\mathbf{1}^T/n & I \\ I & Y \end{array}\right] \succeq 0 \iff Y \succeq (G + \mathbf{1}\mathbf{1}^T/n)^{-1}.$$

To minimize the SDP objective $n\,\mathbf{Tr}\,Y$, subject to this constraint, with G fixed, we simply take $Y = (G + \mathbf{1}\mathbf{1}^T/n)^{-1}$, so the objective of the SDP becomes $R_{\rm tot} + n$.

B. Optimality conditions

The optimal conductance $g^\star$ satisfies

$$\mathbf{1}^T g = 1, \quad g \geq 0, \quad \nabla R_{\rm tot} + R_{\rm tot}\mathbf{1} \geq 0, \tag{27}$$

where $R_{\rm tot}$ is the total effective resistance with g. Conversely, if g is any vector of conductances that satisfies (27), then it is optimal, *i.e.*, $g = g^\star$. The first two conditions in (27) require that g be feasible.

These optimality conditions can be derived as follows. Since the ERMP is a convex problem with differentiable objective, a necessary and sufficient condition for optimality of a feasible g is

$$\nabla R_{\rm tot}^T(\hat{g} - g) \geq 0 \quad \text{for all } \hat{g} \text{ with } \mathbf{1}^T\hat{g} = 1,\ \hat{g} \geq 0$$

(see, *e.g.*, [BV04, §4.2.3]). This is the same as

$$\nabla R_{\rm tot}^T(e_l - g) \geq 0, \quad l = 1, \ldots, m.$$

Since $R_{\rm tot}$ is a homogeneous function of g of degree -1, we have $\nabla R_{\rm tot}^T g = -R_{\rm tot}$ (see (21)), so the condition above can be written as

$$\frac{\partial R_{\rm tot}}{\partial g_l} + R_{\rm tot} \geq 0, \quad l = 1, \ldots, m, \tag{28}$$

which is precisely the third condition in (27).

From the optimality conditions (27) we can derive a complementary slackness condition:

$$g_l \left(\frac{\partial R_{\rm tot}}{\partial g_l} + R_{\rm tot} \right) = 0, \qquad l = 1, \ldots, m. \tag{29}$$

This means that for each edge, we have either $g_l = 0$ or $\partial R_{\rm tot}/\partial g_l + R_{\rm tot} = 0$. To establish the complementarity condition, we note that

$$g^T(\nabla R_{\rm tot} + R_{\rm tot}\mathbf{1}) = 0,$$

since $g^T \nabla R_{\rm tot} = -R_{\rm tot}$ and $g^T R_{\rm tot}\mathbf{1} = R_{\rm tot}$. If g satisfies (27), then this states that the inner product of two nonnegative vectors, g and $\nabla R_{\rm tot} + R_{\rm tot}\mathbf{1}$, is zero; it follows that the products of the corresponding entries are zero. This is exactly the complementarity condition above.

We can give the optimality conditions a simple interpretation in the context of a circuit driven by a random current, as described in §III-B. We suppose the circuit is driven by a random current excitation J with zero mean and covariance $\mathbf{E}\, JJ^T = I - \mathbf{1}\mathbf{1}^T/n$. By (24), we have $\partial R_{\rm tot}/\partial g_l = -n\,\mathbf{E}\, v_l^2$, where v_l is the (random) voltage appearing across edge l. The optimality condition is that g is feasible, and we have

$$\mathbf{E}\, v_l^2 \leq (1/n) R_{\rm tot}, \qquad l = 1, \ldots, m.$$

Thus, the conductances are optimal when the mean square voltage across each edge is less than or equal to $(1/n)R_{\rm tot}$. Using the complementarity condition (29), we can be a bit more specific: each edge that has positive conductance allocated to it must have a mean square voltage equal to $(1/n)R_{\rm tot}$; any edge with zero conductance must have a mean square voltage no more than $(1/n)R_{\rm tot}$.

C. The dual problem

In this section we derive the Lagrange dual problem for the ERMP (25), as well as some interesting variations on it. We start by writing the ERMP as

$$\begin{array}{ll} \text{minimize} & n\,\mathbf{Tr}\, X^{-1} - n \\ \text{subject to} & X = \sum_{l=1}^m g_l a_l a_l^T + \mathbf{1}\mathbf{1}^T/n, \\ & \mathbf{1}^T g = 1, \quad g \geq 0, \end{array} \tag{30}$$

with variables $g \in \mathbf{R}^m$, and $X = X^T \in \mathbf{R}^{n\times n}$. Associating dual variables $Z = Z^T \in \mathbf{R}^{n\times n}$, $\nu \in \mathbf{R}$ with the equality constraints, and $\lambda \in \mathbf{R}^m$ with the nonnegativity constraint $g \geq 0$, the Lagrangian is

$$\begin{aligned} &L(X, g, Z, \nu, \lambda) \\ &\quad = \; n\,\mathbf{Tr}\, X^{-1} - n + \nu(\mathbf{1}^T g - 1) - \lambda^T g \\ &\qquad + \mathbf{Tr}\, Z\left(X - \sum_{l=1}^m g_l a_l a_l^T - \mathbf{1}\mathbf{1}^T/n \right). \end{aligned}$$

The dual function is

$$\begin{aligned} &h(Z, \nu, \lambda) = \inf_{X \succeq 0,\; g} L(X, g, Z, \nu, \lambda) \\ &\quad = \; \inf_{X \succeq 0} \mathbf{Tr}(nX^{-1} + ZX) + \\ &\qquad \inf_g \left(\sum_{l=1}^m g_l(-a_l^T Z a_l + \nu - \lambda_l) \right) - n \\ &\qquad -\nu - (1/n)\mathbf{1}^T Z \mathbf{1}^T \\ &\quad = \; \begin{cases} -\nu - (1/n)\mathbf{1}^T Z \mathbf{1} + 2\,\mathbf{Tr}(nZ)^{1/2} - n \\ \quad \text{if } -a_l^T Z a_l + \nu = \lambda_l,\; l = 1, \ldots, m,\; Z \preceq 0; \\ \infty \;\; \text{otherwise.} \end{cases} \end{aligned}$$

To justify the last line, we note that $\mathbf{Tr}(nX^{-1} + ZX)$ is unbounded below, as a function of X, unless $Z \succeq 0$; when $Z \succ 0$, the unique X that minimizes it is $X = (Z/n)^{-1/2}$, so it has the value

$$\begin{aligned} \mathbf{Tr}(nX^{-1} + ZX) &= \mathbf{Tr}(n(Z/n)^{1/2} + Z(Z/n)^{-1/2}) \\ &= 2\,\mathbf{Tr}(nZ)^{1/2}. \end{aligned}$$

When Z is positive semidefinite, but not positive definite, we get the same minimal value, but it is not achieved by any X. (This calculation is equivalent to working out the conjugate of the function $\mathbf{Tr}\, U^{-1}$, for $U \succ 0$, which is $-2\,\mathbf{Tr}(-V)^{1/2}$, with domain $V \preceq 0$; see, *e.g.*, [BV04, Ex.3.37].)

The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & h(Z, \nu, \lambda) \\ \text{subject to} & \lambda \geq 0. \end{array}$$

Using the explicit formula for g derived above, and eliminating λ, which serves as a slack variable, we obtain the dual problem

$$\begin{array}{ll} \text{maximize} & -\nu - (1/n)\mathbf{1}^T Z \mathbf{1} + 2\,\mathbf{Tr}(nZ)^{1/2} - n \\ \text{subject to} & a_l^T Z a_l \leq \nu, \quad l = 1, \ldots, m, \\ & Z \succeq 0. \end{array} \tag{31}$$

This problem is another convex optimization problem, with variables $Z = Z^T \in \mathbf{R}^{n\times n}$ and $\nu \in \mathbf{R}$. The scalar variable ν could be eliminated, since its optimal value is evidently $\nu = \max_l a_l^T Z a_l$.

Since the ERMP is convex, has only linear equality and inequality constraints, and Slater's condition is satisfied (for example by $g = (1/m)\mathbf{1}$), we know that the optimal duality gap for the ERMP (30) and the dual problem (31) is zero. In other words, the optimal value of the dual (31) is equal to $R_{\rm tot}^\star$, the optimal value of the ERMP. In fact,

we can be very explicit: if $X^\star$ is the optimal solution of the primal ERMP (30), then

$$Z^\star = n(X^\star)^{-2}, \qquad \nu^\star = \max_l a_l^T Z^\star a_l$$

are optimal for the dual ERMP (31). Conversely, if $Z^\star$ is optimal for the dual ERMP (31), then $X^\star = (Z^\star/n)^{-1/2}$ is the optimal point for the primal ERMP (30).

We can use the dual problem (31) to derive a useful bound on the suboptimality of any feasible conductance vector g, by constructing a dual feasible point from g. With $G = A\,\mathbf{diag}(g)A^T$, we define

$$Z = n(G + \mathbf{1}\mathbf{1}^T/n)^{-2},$$

with $\nu = \max_l a_l^T Z^\star a_l$. The pair (Z, ν) is evidently feasible for the dual problem, so its dual objective value gives a lower bound $\underline{R}$ on $R^\star_{\rm tot}$:

$$\begin{aligned} R^\star_{\rm tot} &\geq -\nu - (1/n)\mathbf{1}^T Z \mathbf{1} + 2\,\mathbf{Tr}(nZ)^{1/2} - n \\ &= -\max_l n\|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 \\ &\quad + 2n\,\mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - 2n \\ &= \underline{R}, \end{aligned}$$

where we use $(G + \mathbf{1}\mathbf{1}^T/n)^{-1}\mathbf{1} = \mathbf{1}$ in the second line.

Let η denote the difference between this lower bound $\underline{R}$ and the value of $R_{\rm tot}$ achieved by the conductance vector g. This is a *duality gap* associated with g, *i.e.*, an upper bound on the suboptimality of g. Using $R_{\rm tot} = n\,\mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} - n$, we can express this duality gap as

$$\begin{aligned} \eta &= R_{\rm tot} - \underline{R} \\ &= -n\,\mathbf{Tr}(G + \mathbf{1}\mathbf{1}^T/n)^{-1} + n \\ &\quad + \max_l n\|(G + \mathbf{1}\mathbf{1}^T/n)^{-1} a_l\|^2 \\ &= -R_{\rm tot} + \max_l \left(-\frac{\partial R_{\rm tot}}{\partial g_l}\right) \\ &= -\min_l \left(\frac{\partial R_{\rm tot}}{\partial g_l} + R_{\rm tot}\right). \end{aligned}$$

In summary, we have the following inequality: given any feasible g, its associated total effective resistance $R_{\rm tot}$ satisfies

$$R_{\rm tot} - R^\star_{\rm tot} \leq -\min_l \left(\frac{\partial R_{\rm tot}}{\partial g_l} + R_{\rm tot}\right). \qquad (32)$$

When $g = g^\star$, the righthand side is zero, by our optimality condition (27). This shows that the duality gap converges to zero as g converges to $g^\star$.

A second formulation of the dual, which leads to a tighter duality gap, can be found in a longer version of this paper [GBS05].

D. An interior-point algorithm

The ERMP can be solved numerically using several methods, for example via the SDP formulation (26), using a standard solver such as SeDuMi [Stu99] or DSDP [BY04], or by implementing a standard barrier method [BV04, §11.3], using the gradient and Hessian formulas given in §II-G. In this section we describe a simple custom interior-point algorithm for the ERMP, that uses the duality gap $\hat\eta$ derived in §IV-C. This interior-point method is substantially faster than an SDP formulation, or a more generic method.

The logarithmic barrier for the nonnegativity constraint $g \geq 0$ is

$$\Phi(g) = -\sum_{l=1}^m \log g_l,$$

defined for $g > 0$. In a primal interior-point method, we minimize $tR_{\rm tot} + \Phi$, subject to $\mathbf{1}^T g = 1$, using Newton's method, where $t > 0$ is a parameter; the solution of this subproblem is guaranteed to be at most m/t suboptimal. Our formula (32) gives us a nice bound on suboptimality,

$$\eta = -\left(\min_l \frac{\partial R_{\rm tot}}{\partial g_l} + R_{\rm tot}\right),$$

given any feasible g. We can turn this around, and use this bound to update the parameter t in each step of an interior-point method, by taking $t = \beta m/\eta$, where β is some constant. (If $\eta = 0$, we can stop because g is optimal.) This yields the following algorithm.

Given relative tolerance $\epsilon \in (0, 1)$, $\beta \geq 1$.

Set $g := (1/m)\mathbf{1}$.

while $\hat\eta > \epsilon R_{\rm tot}$ **repeat**

1. Set $t = \beta m/\eta$.
2. Compute Newton step δg for $tR_{\rm tot} + \Phi$ by solving, with $H = t\nabla^2 R_{\rm tot} + \nabla^2\Phi$, and $f = t\nabla R_{\rm tot} + \nabla\Phi$,
$$\begin{bmatrix} H & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \delta g \\ \nu \end{bmatrix} = -\begin{bmatrix} f \\ 0 \end{bmatrix}.$$
3. Find step length s by backtracking line search ([BV04, §9.2]).
4. Set $g := g + s\delta g$.

When the algorithm exits, we have $R_{\rm tot} - R^\star_{\rm tot} \leq \eta \leq \epsilon R_{\rm tot}$, which implies that

$$\frac{R_{\rm tot} - R^\star_{\rm tot}}{R^\star_{\rm tot}} \leq \frac{\epsilon}{1-\epsilon}.$$

Thus, the algorithm computes a conductance vector guaranteed to be no more than $\epsilon/(1-\epsilon)$ suboptimal.

Using $\beta = 1$ and relative tolerance $\epsilon = 0.001$, we found the algorithm to be very effective, never requiring more than 20 or so steps to converge for the many graphs we tried. The main computational effort is in computing the Newton step (*i.e.*, step 2), which requires $O(m^3)$ arithmetic operations, if no structure in the equations is exploited. For graphs with no more than around $m = 2000$ edges, the algorithm is quite fast.

V. The barbell graph

Consider the barbell graph $K_n - K_n$ on $2n$ nodes, which consists of two fully connected components of size n, joined by a single edge between nodes n and $n+1$, shown in figure 1. For this graph, we will show that the ratio

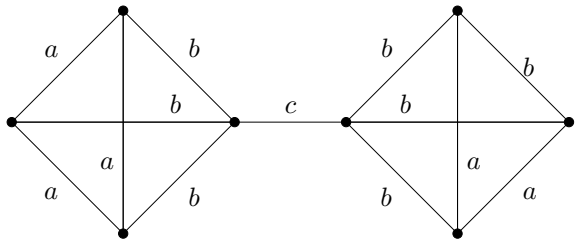

Fig. 1. A barbell graph on 8 nodes.

of $R_{\rm tot}$ with uniform weights to $R^\star_{\rm tot}$ grows unboundedly with n.

Using symmetry and convexity of the ERMP, it can be shown that the optimal $g^\star$ has exactly three distinct weights: the weights on edges neither of whose endpoints is n or $n+1$, a; the weights on edges with exactly one endpoint n or $n+1$, b; and the weight on the edge between n and $n+1$, c. The conductance matrix G for these weights is

$$G = \begin{bmatrix} \alpha I - a\mathbf{1}\mathbf{1}^T & -b\mathbf{1} & & \\ -b\mathbf{1} & \gamma & -c & \\ & -c & \gamma & -b\mathbf{1} \\ & & -b\mathbf{1} & \alpha I - a\mathbf{1}\mathbf{1}^T \end{bmatrix},$$

where $\mathbf{1} \in \mathbf{R}^{n-1}$, $\alpha = a(n-1)+b$, and $\gamma = (n-1)b+2c$.

The $2n$ eigenvalues of the conductance matrix with these weights are shown in the longer version of this paper [GBS05] to be

$$\begin{array}{ll} 0, & a(n-1)+b \ \text{ with multiplicity } 2n-4, \\ nb, & (1/2)(nb+2c \pm \left((2c+nb)^2 - 8bc\right))^{1/2}. \end{array}$$

Therefore, we have reduced the ERMP problem (which has $m = n(n-1)+1$ variables) to the problem

$$\begin{array}{ll} \text{minimize} & (2n-4)/(a(n-1)+b) + 1/nb \\ & + (nb+2c)/2bc \\ \text{subject to} & (n-1)(n-2)a + 2(n-1)b + c \leq 1, \\ & a, b, c \geq 0, \end{array} \tag{33}$$

which has three variables: a, b, and c.

This problem has an analytical solution: the optimal weights are

$$a^\star = \frac{\mu}{n-1}\left(\sqrt{2} - \frac{\sqrt{n+1}}{n}\right), \qquad b^\star = \mu\frac{\sqrt{n+1}}{n},$$
$$c^\star = \mu\sqrt{\frac{n}{2}},$$

where μ is a normalizing constant,

$$1/\mu = (n-2)\left(\sqrt{2} - \frac{\sqrt{n+1}}{n}\right) + 2(n-1)\frac{\sqrt{n+1}}{n} + \sqrt{\frac{n}{2}}.$$

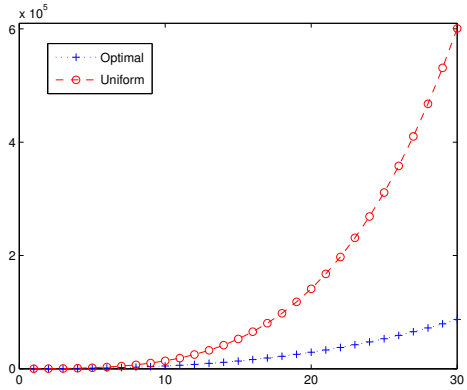

Fig. 2. The optimal value $R^\star_{\rm tot}$, and $R_{\rm tot}$ with uniform weights, as a function of n, for the barbell graph.

Clearly, $R^\star_{\rm tot}$ scales as n^3 for the barbell graph $K_n - K_n$. For the same graph, the total effective resistance obtained with uniform weights $g_l = 1/m$ scales as n^4. We conclude that the suboptimality of the uniform weights grows unboundedly with n, *i.e.*, the optimal conductances are unboundedly better than uniform conductances. In figure 2, the optimal $R^\star_{\rm tot}$ is plotted as a function of n for the barbell graph, along with the total effective resistance with uniform weights.

VI. Examples

In this section we show some examples of optimal conductance allocations on graphs. In each example, we draw the edges with width and color saturation proportional to the optimal edge conductance.

Our first two examples are a path on 11 nodes, and a tree on 25 nodes, shown in figure 3. As expected, the conductance is larger on edges with more paths passing through them than edges near the leaves, which have fewer paths passing through them.

Our next example is an 8×8 mesh, shown in figure 4. We plot the optimal conductances for the 8×8 mesh, and for a graph that is formed by removing some edges from the mesh.

In figure 5, we plot the optimal conductances for a barbell. Finally, in figure 6, we plot the optimal conductances for a randomly generated graph with 25 nodes and 88 edges. Here too we see large conductance allocated to edges across sparse cuts.

VII. Extensions

We conclude by listing some variations on the ERMP that are convex optimization problems, and can be handled using similar methods. Since each R_{ij} is a convex function of the conductances, we can minimize any nondecreasing convex function of the R_{ij} (which is convex). Some interesting (convex) objectives, in addition to $R_{\rm tot}$, include the following.

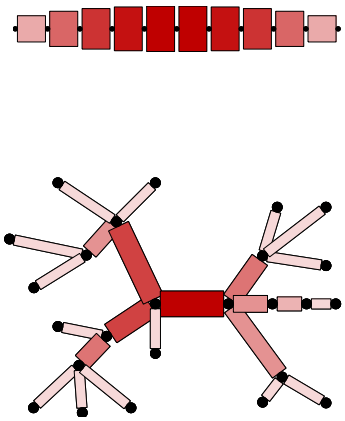

Fig. 3. Left: Optimal conductance allocation on a path on 11 nodes. Right: Optimal conductance allocation on a tree with 25 nodes.

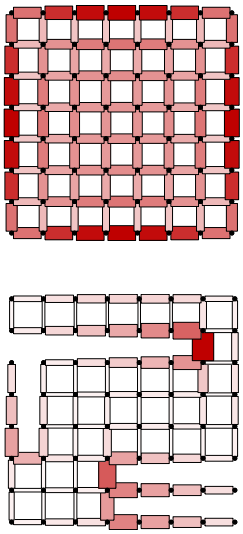

Fig. 4. Left: Optimal conductance allocation on an 8×8 mesh. Right: Optimal allocation for a modified 8×8 mesh.

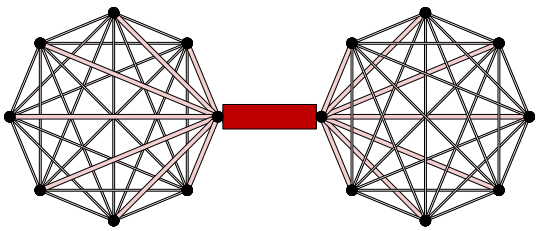

Fig. 5. Optimal conductance allocation on barbell graph with 16 nodes.

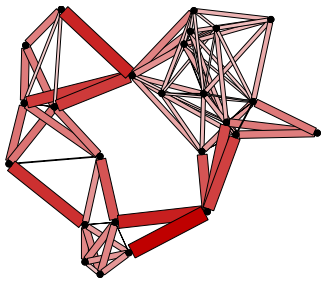

Fig. 6. Optimal conductance allocation on randomly generated graph with 25 nodes and 88 edges.

- *Minimizing effective resistance between a specific pair of nodes.* We allocate conductance to minimize the effective resistance between a specific pair of nodes, i and j. This problem has the following simple solution: Allocate the conductance equally to the edges lying on a shortest path between i and j. This conductance allocation violates our assumption (4); the effective resistance between any pair of nodes not on the path is undefined.
- *Minimizing sum of effective resistances to a specific node.* This problem can be formulated as an SDP, and solved by modifications to the methods discussed in this paper. Examples show that the optimal allocation of weights need not be a tree.
- *Minimizing maximum effective resistance.* The maximum effective resistance over all pairs of nodes, $\max_{i,j} R_{ij}$, is the pointwise maximum of the convex functions R_{ij}, and is therefore also a convex function of the conductances g [BV04, §3.2.3]. The problem of minimizing the maximum effective resistance is a convex optimization problem, can be formulated as an SDP, and solved using standard interior-point methods.

We mention one more extension: minimizing R_{tot} without the nonnegativity constraints on the edge conductances:

$$\begin{array}{ll} \text{minimize} & R_{\mathrm{tot}} \\ \text{subject to} & \mathbf{1}^T g = 1. \end{array} \tag{34}$$

In this problem the conductances can be negative, but we restrict the domain of the objective R_{tot} to $\{g \mid G + \mathbf{1}\mathbf{1}^T/n \succ 0\}$. This problem can be solved using Newton's method, using the derivatives found in §II-G. The optimality conditions for this problem are simply that $\mathbf{1}^T g = 1$ (feasibility), and that all components of the gradient, ∇R_{tot}, are equal, specifically, $\nabla R_{\mathrm{tot}}(g) = -R_{\mathrm{tot}}\mathbf{1}$.

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REFERENCES

[AF03] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs*. 2003. Book in preparation. Available at `stat-www.berkeley.edu/users/aldous/RWG/book`.

[Bap99] R. Bapat. Resistance distance in graphs. *Mathematics Student*, 68:87–98, 1999.

[BDX04] S. Boyd, P. Diaconis, and L. Xiao. Fastest mixing Markov chain on a graph. *SIAM Review, problems and techniques section*, 46(4):667–689, 2004.

[Boy06] S. Boyd. Convex optimization of graph laplacian eigenvalues. volume 3, 2006. Available at `www/~boyd/cvx_opt_graph_lapl_eigs`.

[BV04] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004. Available at `www/~boyd/cvxbook`.

[BVGY01] S. Boyd, L. Vandenberghe, A. El Gamal, and S. Yun. Design of robust global power and ground networks. In *Proc. ACM/SIGDA Int. Symposium on Physical Design (ISPD)*, pages 60–65. ACM, April 2001.

[BY04] S. Benson and Y. Ye. DSDP5 user guide — software for semidefinite programming. Technical Report ANL/MCS-TM-277, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, July 2004. Available online at: `www.mcs.anl.gov/~benson/dsdp`.

[CRR+89] A. Chandra, P. Raghavan, W. Ruzzo, R. Smolensky, and P. Tiwari. The electrical resistance of a graph captures its commute and cover times. In *Proceedings of the 21st Annual Symposium on Foundations of Computer Science*. ACM, 1989.

[DK69] C. Desoer and E. Kuh. *Basic Circuit Theory*. McGraw-Hill, Inc., 1969.

[Elm48] W. Elmore. The transient response of damped linear networks. *Journal of Applied Physics*, 19:55–63, January 1948.

[Fie73] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematics Journal*, 23:298–305, 1973.

[Fie90] M. Fiedler. Absolute algebraic connectivity of trees. *Linear and Multilinear Algebra*, 26:85–106, 1990.

[Fie93] M. Fiedler. Some minimax problems for graphs. *Discrete Mathematics*, 121:65–74, 1993.

[GB] A. Ghosh and S. Boyd. Upper bounds on algebraic connectivity via convex optimization. To appear in *Linear Algebra Appl.*

[GBS05] A. Ghosh, S. Boyd, and A. Saberi. Optimizing effective resistance of a graph. Submitted to *SIAM Review, Problems and Techniques*, 2005. Available at `www/~boyd/eff_res`.

[GHW05] F. Göring, C. Helmberg, and M. Wappler. Embedded in the shadow of the separator. Preprint 2005-12, Fakultät für Mathematik, Technische Universität Chemnitz, Chemnitz, Germany, September 2005.

[HJ91] R. Horn and C. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, UK, 1991.

[Kle02] D. Klein. Resistance-distance sum rules. *Croatia Chem. Acta*, 73:957–967, 2002.

[KR93] D. Klein and M. Randic. Resistance distance. *Journal of Mathematical Chemistry*, 12:81–85, 1993.

[Puk93] F. Pukelsheim. *Optimal Design of Experiments*. Wiley and Sons, 1993.

[SBXD] J. Sun, S. Boyd, L. Xiao, and P. Diaconis. The fastest mixing Markov process on a graph and a connection to a maximum variance unfolding problem.

[Stu99] J. Sturm. Using SeDuMi, a Matlab toolbox for optimization over symmetric cones, 1999. `sedumi.mcmaster.ca`.

[WH04] N. Weste and D. Harris. *CMOS VLSI Design*. Addison Wiley, 2004.

[XB04] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. In *Systems and Control Letters*, volume 53, pages 65–78, 2004.

[XBK05] L. Xiao, S. Boyd, and S.-J. Kim. Distributed average consensus with least-mean-square deviation. Submitted to *Journal of Parallel and Distributed Computing*, May 2005. Available at `www/~boyd/lms_consensus`.

[XG03] W. Xiao and I. Gutman. Resistance distance and Laplacian spectrum. *Theor. Chem. Acc.*, 110:284–289, 2003.