Minimizing the Expected Rank with Full Information

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Abstract: The full-information secretary problem in which the objective is to minimize the expected rank is seen to have a value smaller than 7/3 for all n (the number of options). This can be achieved by a simple memoryless threshold rule. The asymptotically optimal value for the class of such rules is about 2.3266. For a large finite number of options, the optimal stopping rule depends on the whole sequence of observations and seems to be intractable. This raises the question whether the influence of the history of all observations may asymptotically fade. We have not solved this problem, but we show that the values for finite n are nondecreasing in n and exhibit a sequence of lower bounds that converges to the asymptotic value which is not smaller than 1.908.

§**1. Introduction and Summary.** We consider the full-information secretary problem with the objective of minimizing the expected rank. Thus, we have a stopping rule problem with observations, $X_1, \cdots, X_n$, known to be independent, identically and uniformly distributed on the interval $[0,1]$. The payoff for selecting the kth observation, X_k, is equal to R_k, the rank of X_k among $X_1, \cdots, X_n$,

$$R_k = R_k(n) = \sum_{i=1}^{n} \mathrm{I}(X_i \leq X_k), \tag{1.1}$$

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where $\mathrm{I}(A)$ denotes the indicator function of the event A. Note that we define the smaller observations to be the better ones (i.e. having smaller ranks), which is here more convienent. The problem is to find a stopping rule, N^*, adapted to the sequence $X_1, \cdots, X_n$, that minimizes the expected rank,

$$\mathrm{E}\, R_{N^*} = \inf_N \mathrm{E}\, R_N = V_n. \tag{1.2}$$

We refer to this problem as Robbins' problem, as it was posed as an unsolved problem by Robbins at the AMS/IMS/SIAM Conference on Sequential Search and Selection in Real Time in Amherst, June 1990. In the "classical" secretary problem, solved by Lindley (1961), one is only allowed to use rules that depend on the relative ranks of the observations (called the no-information problem) and the objective is to maximize the probability of selecting the observation of absolute rank 1 (the best-choice problem). The full-information best-choice problem was solved by Gilbert and Mosteller (1966) and the no-information expected-rank problem was solved by Chow et al. (1964). Thus the solution to the full-information expected-rank problem would complete a two by two factorial design of secretary problems. The main motivation to study this problem is however that it touches questions of general interest. Given that a finite problem requires history-dependent optimal stopping rules, what structure of the problem would imply an "asymptotic irrelevance" of the history.

MOSER'S PROBLEM.

A closely related problem, due to Moser (1956), is to choose a stopping rule, $\hat{N}$, adapted to $X_1, \cdots, X_n$, to minimize the expectation of X_N,

$$\mathrm{E}\, X_{\hat{N}} = \inf_N \mathrm{E}\, X_N. \tag{1.3}$$

Moser shows that the optimal stopping rule for this problem is to stop after observing X_k if $X_k \le a_{n-k}$, where $a_0 = 1$, and inductively for $j \ge 1$, $a_{j+1} = a_j - \frac{1}{2}a_j^2$, and moreover, the optimal return is a_n. Thus, the optimal rule is

$$\hat{N} = \min\{k \ge 1 : X_k \le a_{n-k}\}, \tag{1.4}$$

and its value is $\mathrm{E}\,X_{\hat{N}} = a_n$. Moser also shows that

$$a_n \simeq \frac{2}{n + \log(n) + O(1)}. \tag{1.5}$$

Gilbert and Mosteller (1966) refine the $O(1)$ to $1.767\ldots + o(1)$.

RELATIONSHIP BETWEEN MOSER'S PROBLEM AND ROBBINS' PROBLEM

There is a strong relation between the ranks, R_k, and the values, X_k. The distribution of X_k given $R_k = r$ is the beta distribution with parameters r and $n-r+1$, so that

$$\mathrm{E}(X_k | R_k = r) = \frac{r}{n+1}. \tag{1.6}$$

Similarly, the other regression function is

$$\mathrm{E}(R_k | X_k = x) = 1 + (n-1)x. \tag{1.7}$$

Moreover, the correlation between X_k and R_k tends to one as $n \to \infty$: $\mathrm{E}\,X_k = 1/2$, $\mathrm{Var}\,X_k = 1/12$, $\mathrm{E}\,R_k = (n+1)/2$, $\mathrm{Var}\,R_k = (n+1)(n-1)/12$, and $\mathrm{Cov}(X_k, R_k) = (n-1)/12$, so that

$$\mathrm{Corr}(X_k, R_k) = \sqrt{\frac{n-1}{n+1}} \to 1. \tag{1.8}$$

The regressions of X_k on R_k, and of R_k on X_k, are both linear and the correlation tends to one.

Thus, it would seem that if we replace the payoff, X_k, by $R_k/(n+1)$ in Moser's problem, we have not changed the problem much if n is large. That is, the problem of minimizing $\mathrm{E}\,X_N$ and of minimizing $\mathrm{E}\,R_N/(n+1)$ should be asymptotically equivalent. Since the former has $\hat{N}$ as the optimal rule and value approximately $2/n$, we expect that the problem of minimizing $\mathrm{E}\,R_N$ should have $\hat{N}$ as an asymptotically optimal rule and a value asymptotically equal to 2.

Surprisingly, this is not true. In Section 2, we investigate the class of memoryless rules. These are rules for which the decision to stop at stage k depends only on X_k and not on the values of any previously observed X_i. We restrict attention to memoryless rules of the form,

$$N = N(\mathbf{p}) = \min\{k \geq 1 : X_k \leq p_k\}, \tag{1.9}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ is a given sequence of numbers in the interval $[0,1]$ with $p_n = 1$ (to guarantee $N \leq n$). We derive a formula for the expected rank of the object chosen by such a rule, and we apply the formula to the rule,

$$\tilde{N} = \min\{k \geq 1 : X_k \leq \frac{2}{n-k+2}\}, \tag{1.10}$$

which is asymptotically optimal for Moser's problem, and show that it has asymptotic value $7/3$. This formula is also useful for finding the optimal rule of the form (1.9). In Section 3, we report briefly on the results of a numerical investigation of the optimal value and the optimal rule within this class.

In Section 4, we look at lower bounds for the value, V_n, of Robbins' problem. We identify a sequence of problems, indexed by m, whose values, $V_n^{(m)}$, are always less than V_n. We define the asymptotic values as $V^{(m)} = \lim_{n\to\infty} V_n^{(m)}$ and $V = \lim_{n\to\infty} V_n$ and show that these asymptotic values exist and that $V^{(m)} \to V$ as $m \to \infty$. Then we investigate numerically the asymptotic values of these problems for small m.

§2. Upper Bounds on the Value of Robbins' Problem.

Theorem 1. *If $N(\mathbf{p})$ is the rule (1.9) with $p_n = 1$, then, denoting $q_k = 1 - p_k$ and $(x)^+ = \max(0, x)$,*

$$\mathrm{E}\, R_{N(\mathbf{p})} = 1 + \frac{1}{2}\sum_{k=1}^{n}\left(\prod_{i=1}^{k-1} q_i\right)\left[(n-k)p_k^2 + \sum_{j=1}^{k-1}\frac{((p_k - p_j)^+)^2}{q_j}\right]. \tag{2.1}$$

Proof. We first find the distribution of the stopping time of the rule $N = N(\mathbf{p})$, given by (1.9).

$$\mathrm{P}(N = k) = \left(\prod_{i=1}^{k-1} q_i\right) p_k \qquad \text{for} \quad k = 1, \ldots, n. \tag{2.2}$$

To compute the expected rank using N,

$$\mathrm{E}\, R_N = \sum_{k=1}^{n} \mathrm{E}(R_k | N = k)\mathrm{P}(N = k), \tag{2.3}$$

we need to compute

$$\mathrm{E}(R_k|N = k) = 1 + \sum_{j=1}^{k-1} \mathrm{P}(X_j < X_k | N = k) + \sum_{j=k+1}^{n} \mathrm{P}(X_j < X_k | N = k). \tag{2.4}$$

For $j > k$,

$$\mathrm{P}(X_j < X_k | N = k) = \int_0^{p_k} \frac{x}{p_k}\, dx = \frac{p_k}{2}, \tag{2.5}$$

while for $j < k$, this quantity is zero if $p_j \geq p_k$, and

$$\mathrm{P}(X_j < X_k | N = k) = \int_{p_j}^{p_k} \frac{x - p_j}{(1 - p_j)p_k}\, dx = \frac{(p_k - p_j)^2}{2(1 - p_j)p_k} \tag{2.6}$$

if $p_j < p_k$. Combining this into the formulas above, we find

$$\mathrm{E}(R_k|N = k) = 1 + \frac{(n-k)p_k}{2} + \sum_{j=1}^{k-1} \frac{((p_k - p_j)^+)^2}{2q_j p_k}, \tag{2.7}$$

and substituting this and (2.2) into (2.3) completes the proof. ■

As an example of the use of this formula, let us evaluate $\mathrm{E}\, R_{\tilde{N}}$ where $\tilde{N}$ is the rule given in (1.10). The distribution of $\tilde{N}$ telescopes from (2.2) into the simple form,

$$\mathrm{P}(\tilde{N} = k) = \frac{2(n - k + 1)}{n(n + 1)}. \tag{2.8}$$

The conditional expected rank simplifies to

$$\mathrm{E}(R_k|\tilde{N} = k) = 1 + \frac{n - k}{n - k + 2} + \sum_{j=1}^{k-1} \frac{(k - j)^2}{(n - k + 2)(n - j + 2)(n - j)}, \tag{2.9}$$

so that the expected rank becomes

$$\mathrm{E}\, R_{\tilde{N}} = 1 + \sum_{k=1}^{n} \frac{2(n - k + 1)(n - k)}{n(n + 1)(n - k + 2)} + \sum_{k=1}^{n} \frac{2(n - k + 1)}{n(n + 1)} \sum_{j=1}^{k-1} \frac{(k - j)^2}{(n - k + 2)(n - j + 2)(n - j)} \tag{2.10}$$

The first summation above is the contribution to the expected rank, $\mathrm{E}\,R_{\tilde{N}}$, given by those observations that come after $\tilde{N}$, and the second summation is the contribution of those observations that come before $\tilde{N}$.

The first summation above is a Riemann approximation for large n to the integral,

$$\int_0^1 2(1-x)\,dx = 1. \tag{2.11}$$

Thus, the contribution of those observations that appear after $\tilde{N}$ already push the expected rank up to 2 as $n \to \infty$. The second summation above is similarly a Riemann approximation to the double integral,

$$\int_0^1 2(1-x)\int_0^x \frac{(x-y)^2}{(1-x)(1-y)^2}\,dy\,dx = \frac{1}{3}. \tag{2.12}$$

Thus, the expected rank using $\tilde{N}$ converges to $7/3$ as $n \to \infty$.

We may use (2.10) to find simple upper bounds for $\mathrm{E}\,R_{\tilde{N}}$, and hence for $\mathrm{E}\,R_{N^*}$ of Robbins' problem. Let V_n denote the minimum expected rank, (1.2), for for finite n and let $V = \lim_{n\to\infty} V_n$.

Corollary 1. *For all finite n,*

$$V_n \le 1 + \frac{4(n-1)}{3(n+1)}, \tag{2.13}$$

and hence $V \le 7/3$.

Proof. The first summation in (2.10) is

$$\frac{2}{n(n+1)}\sum_{k=1}^{n}\frac{(n-k+1)(n-k)}{n-k+2} \le \frac{2}{n(n+1)}\sum_{k=1}^{n}(n-k) = \frac{n-1}{n+1}. \tag{2.14}$$

and the second summation is

$$\begin{aligned}\frac{2}{n(n+1)}\sum_{k=1}^{n}\frac{n-k+1}{n-k+2}\sum_{j=1}^{k-1}\frac{(k-j)^2}{(n-j+2)(n-j)} &\le \frac{2}{n(n+1)}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\frac{(k-j)^2}{(n-j+2)(n-j)}\\ &= \frac{2}{n(n+1)}\sum_{j=1}^{n-1}\frac{(n-j+1)(2n-2j+1)}{6(n-j+2)}\\ &\le \frac{2}{3n(n+1)}\sum_{j=1}^{n-1}(n-j) = \frac{n-1}{3(n+1)}.\end{aligned} \tag{2.15}$$

Hence, we have the bound, valid for all finite n, $\mathrm{E}\, R_{\tilde N} \le 1 + 4(n-1)/(3(n+1))$, which implies (2.13). ■

The minimum expected rank for the no-information problem (where rules must be based only on the relative ranks of the observations), was found by Chow et al (1964) to be $3.8695\ldots$. By allowing the rule to depend on the actual values of the X_j, we can thus reduce the expected rank by at least $1.5362\ldots$.

Assaf and Samuel-Cahn (1991) study memoryless rules of the form

$$N_c = \min\{k \ge 1 : X_k \le c/(n-k+c)\}. \tag{2.16}$$

and find that the limiting value exists for all $c > 1$ and is minimized by $c = 1.9469\cdots$ giving a value of $2.3318\cdots$ for the expected rank.

§**3. Numerical Investigation of Memoryless Rules.** Formula (2.1) is quite useful for numerical evaluation of the optimal rule within the class of memoryless rules of the form (1.9) with nondecreasing p_k. For computational purposes, it is important to notice that it is multiquadratic; that is, it is quadratic in each variable with the others held fixed. This is easily observed if we write it as a function of the q_k in the following form.

$$\begin{aligned} \mathrm{E}\, R_{N(\mathbf{p})} = 1 &+ \frac{1}{2}\sum_{k=1}^{n}\left(\prod_{i=1}^{k-1} q_i\right)\left[(n-k)(1-q_k)^2 + \sum_{j=1}^{k-1} q_j - 2(k-1)q_k\right] \\ &+ \frac{1}{2}\sum_{k=1}^{n} q_k^2 \sum_{j=1}^{k-1}\left(\prod_{\substack{i=1\\ i\ne j}}^{k-1} q_i\right). \end{aligned} \tag{3.1}$$

It may also be noticed that the coefficient of each q_k^2 is positive so that (3.1) is convex in each variable with the others held fixed. For a given $\mathbf{p}$, $\mathrm{E}\, R_{N(\mathbf{p})}$ can be made smaller by moving any p_k toward its argminimum of (3.1) with the other p_i, $i \ne k$, held fixed.

We report briefly on the results of a numerical investigation of the optimal value and the optimal rule within the class of memoryless rules. The optimal value $\mathbf{p}^{opt}$ of $\mathbf{p}$ may be found by successive approximation as follows. An initial estimate of $\mathbf{p}$ is chosen, (Moser's rule is a reasonable choice), and an improved choice of $\mathbf{p}$ is found by successively

Table 1. Values of Certain Rules.

n	$\mathrm{E}\,R_{\tilde{N}}$	$\mathrm{E}\,R_{\hat{N}}$	$\mathrm{E}\,R_{N(\mathbf{p}^{opt})}$
5	1.644629	1.589185	1.586131
10	1.875204	1.815253	1.810876
20	2.044858	1.994183	1.989000
50	2.185534	2.154156	2.148230
100	2.246813	2.227616	2.221327
200	2.283601	2.272676	2.266170
400	2.305174	2.299233	2.292611
800	2.317578	2.314449	2.307768

evaluating the partial derivatives with respect to the q_m for $m = 1, \ldots, n-1$, solving the resulting linear equation for q_m and moving q_m to this root, or at least as far as possible without violating monotonicity. This is repeated until no significant change is observed. In Table 1, we compare the returns of the rules $\hat{N}$, $\tilde{N}$, and $N(\mathbf{p}^{opt})$. The value of the optimal rule within the class (1.9) seems to be approaching again a value close to $7/3$. In Table 2, selected values of the optimal rule within the class (1.9) are displayed. The rule $N(\mathbf{p}^{opt})$ seems to be approaching Moser's rule, $\hat{N}$, as $n \to \infty$.

Extrapolation techniques allow us to estimate the limit of the sequence $\mathrm{E}\,R_{N(\mathbf{p}^{opt})}$ to be $2.32659\cdots$. Assaf and Samuel-Cahn (1991) succeed in giving an explicit rule with the value 2.3267 already very close to this. In the terminology of our paper their rule

Table 2. Certain Values of $\mathbf{p}^{opt}$.

n	p_{n-1}	p_{n-2}	p_{n-3}	p_1	Moser
5	.535825	.391809	.302076	.238507	.258270
10	.528306	.397852	.320840	.134065	.150178
20	.517963	.391693	.319302	.073283	.083608
50	.508435	.383362	.312630	.031612	.036522
100	.504479	.379518	.309072	.016392	.018972
200	.502314	.377351	.306989	.008373	.009704
Moser	.500000	.375000	.304688		

corresponds to (2.16) with c being replaced by a polynomial in the (limiting) argument $z=(n-k)/n$. (See also Kennedy and Kertz (1990) Section 2.)

§**4. Lower Bounds on the Value.** We have been unsuccessful at finding any upper bounds on the values V_n and V for Robbins' problem other than those given by $\mathrm{E}\, R_{N(\mathbf{p}^{opt})}$. In this section, we consider a sequence of lower bounds to these values, denoted by $V_n^{(m)}$, and show that $V^{(m)}=\lim_{n\to\infty} V_n^{(m)}$ converges to V as $m\to\infty$.

There is a simple argument that gives a lower bound to V of about $1.42\ldots$ as follows. Suppose we modify the problem in favor of the decision maker by changing the payoff for stopping at k to be 1 if $R_k=1$ and 2 if $R_k>1$. The problem then becomes one of finding a stopping rule N to minimize $\mathrm{P}(R_N=1)+2\mathrm{P}(R_N>1)=2-\mathrm{P}(R_N=1)$, or, equivalently, to maximize $\mathrm{P}(R_N=1)$. This is just the full-information best-choice problem solved by Gilbert and Mosteller (1966) and seen to have an asymptotic value of $\theta=.580164\ldots$. Using the same rule for the modified problem, we have an expected payoff of $2-\theta=1.419836\ldots$. Since this is the best that can be done in the modified problem, in the original problem we cannot do better either; that is,

$$V\geq 1.419836\ldots. \tag{4.1}$$

We extend this idea as follows. To make it harder for the decision maker, we count ranks 1 to m at their value and any higher rank as $m+1$. This gives an increasing sequence of modified payoffs indexed by m, converging monotonically to the actual rank, $R_k(n)$, as $m\to n$. Specifically, the payoff for stopping at k is

$$R_k(n,m)=\min\{m+1,\sum_{i=1}^{n}\mathrm{I}(X_i\leq X_k)\}. \tag{4.2}$$

The decision maker would now only stop at a relative rank less than or equal to m, except at the very last stage when he must stop.

Let $V_n^{(m)}=\inf_N \mathrm{E}\, R_N(n,m)$ denote the value of this problem. We first show that $V_n^{(m)}$ and V_n are monotone in n.

Theorem 2. *The sequences $V_n^{(m)}$ and V_n are monotone nondecreasing in n.*

Proof. We give the proof for V_n, the proof for $V_n^{(m)}$ being similar. The method we use compares V_n with how well we can do knowing $X_{(n)}$, the largest order statistic of $X_1, \ldots, X_n$ from the start. For this purpose, let $\mathcal{F}_k$ denote the σ-field generated by $X_1, \ldots, X_k$, and let $\mathcal{F}_k^{(n)}$ denote the σ-field generated by $X_{(n)}, X_1, \ldots, X_k$ for $k = 0, 1, \ldots, n$, so that both $\{\mathcal{F}_k\}$ and $\{\mathcal{F}_k^{(n)}\}$ are increasing in k, and $\mathcal{F}_k \subset \mathcal{F}_k^{(n)}$. Let $\mathcal{C}_n$ denote the class of stopping rules, $N \leq n$, adapted to $\{\mathcal{F}_k\}$, and let $\mathcal{C}'_n$ denote the larger class of stopping rules stopping adapted to $\{\mathcal{F}_k^{(n)}\}$. Then,

$$
\begin{aligned}
V_n &= \inf_{N \in \mathcal{C}_n} \mathrm{E}\, R_N(n) \\
&\geq \inf_{N \in \mathcal{C}'_n} \mathrm{E}\, R_N(n) \\
&= \inf_{N \in \mathcal{C}'_n} \mathrm{E}\{\mathrm{E}\{R_N^{(n)} | \mathcal{F}_0^{(n)}\}\}
\end{aligned} \tag{4.3}
$$

In the problem of minimizing $\mathrm{E}\{R_N^{(n)} | \mathcal{F}_0^{(n)}\}$ using $N \in \mathcal{C}'_n$, we are given the value of $X_{(n)}$, so the optimal rule will never stop on it. Moreover, conditioned on $X_{(n)}$, the other X_j are i.i.d. from a uniform distribution on $(0, X_{(n)})$. This is the original problem with sample size reduced to $n-1$ and distribution changed to the uniform on $(0, X_{(n)})$. It has value V_{n-1} for all $X_{(n)}$, and optimal strategy that may be obtained by scaling the optimal strategy for $X_{(n)} = 1$, i.e. replacing X_k by $X_k / X_{(n)}$. Thus, a measurable rule, $N^* \in \mathcal{C}'_n$ may be constructed that achieves the minimum value of $V_{n-1} = \mathrm{E}\{R_{N^*}^{(n)} | \mathcal{F}_0^{(n)}\}$ for all $X_{(n)}$. Substitution of this value into (4.3) completes the proof. ∎

From this theorem, we have that $V^{(m)} = \lim_{n\to\infty} V_n^{(m)}$ exists for all m and that $V = \lim_{n\to\infty} V_n$ exists.

Theorem 3. *$V^{(m)} \to V$ as $m \to \infty$.*

Proof. $V_n^{(m)}$ is monotone nondecreasing in m for each n since $R_k(n, m)$ is a.s. nondecreasing in m for each n and k. Then, since $V_n^{(n)} = V_n$,

$$
V^{(m)} = \lim_{n\to\infty} V_n^{(m)} \leq \lim_{n\to\infty} V_n^{(n)} = \lim_{n\to\infty} V_n = V
$$

This shows that $\limsup_{m\to\infty} V^{(m)} \le V$. From Theorem 2, $V_n^{(m)}$ is monotone nondecreasing in n for each m, so that

$$V^{(m)} \ge V_m^{(m)} = V_m \to V \quad \text{as} \quad m \to \infty$$

so that $\liminf_{m\to\infty} V^{(m)} \ge V$. ■

We describe briefly the computational problems involved in evaluating $V_n^{(m)}$. First we make a simple modification that improves the lower bound. We count against the decision maker all better observations that arrive after he has made his choice. This amounts to replacing the payoff (4.2) by

$$Y_k = 1 + \min\{m, \sum_{i=1}^{k-1} \mathrm{I}(X_i < X_k)\} + (n-k)X_k. \tag{4.4}$$

We have replaced the future payoff by its expectation given $X_1, \ldots, X_k$ without loss of generality as far as the decision maker is concerned. (The prophet would object though.) Let $W_n^{(m)}$ denote the minimum expected return for this problem, and let $W^{(m)}$ denote $\lim_{n\to\infty} W_n^{(m)}$. We have automatically $V_n^{(m)} < W_n^{(m)}$. When $m = 1$, this change only increases the lower bound from $V^{(1)} = 1.420$ in (4.1) to $W^{(1)} = 1.462$. With this modification too, the decision maker stops only at a relative rank less than or equal to m.

The value function for this problem depends only on the smallest m observations seen so far, and on k, the number of observations to go. Let $V_k(y_1, \ldots, y_m)$ denote the minimum expected payoff, **minus 1,** using (4.4) when there are k observations to go and m smallest order statistics of the X_j seen so far are $0 < y_1 \le \ldots \le y_m \le 1$. It satisfies the equation

$$\begin{aligned} V_{k+1}(y_1, \ldots, y_m) = \sum_{j=0}^{m-1} \int_{y_j}^{y_{j+1}} & \min\{kx + j, V_k(y_1, \ldots, y_j, x, y_{j+1}, \ldots, y_{m-1})\}\, dx \\ & + (1 - y_m) V_k(y_1, \ldots, y_m), \end{aligned} \tag{4.5}$$

with boundary condition $V_0 \equiv m$ or $V_1(y_1, \ldots, y_m) = m - y_1 - \cdots - y_m$, where y_0 denotes 0. At the initial stages of the problem when there are not yet m observations, we obtain the value by replacing the unobserved order statistics by 1. Thus we seek $W_n^{(m)} = V_n(1,\ldots,1)$ **+ 1**.

Although (4.5) represents a considerable simplification over the original problem, the computational aspects involved with this approach are still severe. Both the time and storage requirements for the computations increase exponentially in m. Table 3 contains the results of some calculations that were carried out for $m = 1$ through $m = 5$. We are indebted to Janis Hardwick and Nicholas Schork of the University of Michigan for the computations for $m = 4$ and $m = 5$. The results for the column of $W^{(m)}$ are extrapolated from the table.

This pushes the lower bound up from 1.462 when $m = 1$, to 1.658 for $m = 2$, to 1.782 for $m = 3$, to 1.860 at $m = 4$ and to 1.908 at $m = 5$.

REMARKS: What does the optimal limiting strategy look like? Is it a memoryless threshold rule? We do not know. For fixed n, the instantaneous reward by selecting a point is a function of both its relative rank and its magnitude. All other information is redundant. But the future reward under optimal behavior depends also on the pattern of the previously observed points especially those close to zero. For example, suppose $n = 100$, and we have just observed $X_{20} = .025$ which is of relative rank 2. If the best observation is has value .024, we would be more inclined to continue than if it were .011, say. Similarly, if the next best three observations were .026, .028, and .029, we would be more inclined to stop than

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Table 3. Certain Values of $W_n^{(m)}$.

m	$n=2$	3	5	9	17	$W^{(m)}$
1	1.2500	1.3242	1.3803	1.4171	1.4385	1.462
2	-	1.3915	1.4991	1.5689	1.6092	1.658
3	-	-	1.5509	1.6490	1.7070	1.782
4	-	-	1.5710	1.6956	1.7668	1.860
5	-	-	-	1.7260	1.8090	1.908
V_n	1.2500	1.3915	1.5710			

if the next best observation were .101, say. The question is whether and how the relevant information can be packed into tractable limiting threshold functions. Being able to do so would then open the way to the extremal type theorems for threshold-stopped random variables derived by Kennedy and Kertz (1991, Section 3). Unfortunately, exchangeability does not seem to be enough to reveal the shape of the optimal threshold function.

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