

Minimizing topological entropy for maps of the circle

LOUIS BLOCK, ETHAN M. COVEN† AND ZBIGNIEW NITECKI

From the University of Florida, Gainesville, Fla., Wesleyan University, Middletown, Conn., and Tufts University, Medford, Mass.

(Received 10 December 1980)

Abstract. For each $n \geq 2$, we find the minimum value of the topological entropies of all continuous self-maps of the circle having a fixed point and a point of least period n , and we exhibit a map with this minimal entropy.

1. Introduction

This paper is concerned with the following problem.

For each $n \geq 2$, find a continuous map f_n of the circle to itself, having a fixed point and a point of least period n , minimal in the sense that $\text{ent}(f_n) \leq \text{ent}(f)$ for every continuous map f of the circle having a fixed point and a point of least period n . Determine $\text{ent}(f_n)$.

Here $\text{ent}(\cdot)$ denotes topological entropy.

The solution to the analogous entropy-minimizing problem for maps of the interval was discovered in the course of investigations having to do with Šarkovskii's theorem. Recall the Šarkovskii ordering \triangleleft of the positive integers

$$3 \triangleleft 5 \triangleleft \cdots \triangleleft 2 \cdot 3 \triangleleft 2 \cdot 5 \triangleleft \cdots \triangleleft 2^2 \cdot 3 \triangleleft 2^2 \cdot 5 \triangleleft \cdots \triangleleft 2^2 \triangleleft 2^1 \triangleleft 2^0.$$

Šarkovskii's theorem [4], [5], [3] states that, if a continuous map of a compact interval to itself (or to the reals) has a point of least period n , then it has a point of least period m for every $m \triangleright n$.

P. Štefan [5] has described a set of constructions which, for each $n \geq 2$, yields a map g_n of the interval having a point of least period n but no point of least period m for any $m \triangleleft n$. It turns out [5], [3] that the maps g_n are the solution to the entropy-minimizing problem for maps of the interval: $\text{ent}(g_n) \leq \text{ent}(g)$ for every continuous map g of the interval having a point of least period n . (If g maps a compact interval I into the reals, then $\text{ent}(g)$ is defined to be $\text{ent}(g|I')$, where $I' = \bigcap_{j \geq 0} g^{-j}(I)$. Note that I' need not be an interval.)

The topological entropy of g_n is given by the formula

$$\text{ent}(g_n) = \log \sigma_n,$$

† Address for correspondence: Ethan M. Coven, Department of Mathematics, Wesleyan University, Middletown, Conn. 06457, USA.

where σ_n is defined as follows. For $k \geq 3$, let $L_k(x) = x^k - 2x^{k-2} - 1$ and let λ_k denote the largest root of L_k . For $n = 2^s k$ where k is odd, let $\sigma_n = 1$ if $k = 1$ and $\sigma_n = (\lambda_k)^{2^{-s}}$ if $k \geq 3$.

The result corresponding to Šarkovskii's theorem for maps of the circle is the following [2]: if a continuous map of the circle to itself has a fixed point and a point of least period n , then either (a) it has a point of least period m for every $m \triangleright n$ or (b) it has a point of least period m for every $m > n$.

The examples g_n of Štefan can be extended to maps of the circle without changing the set of least periods or the topological entropy. L. Block [1] has constructed maps f_n of the circle having a fixed point and a point of least period n , but no point of least period m for any m , $1 < m < n$. The topological entropy of f_n is given by the formula

$$\text{ent}(f_n) = \log \mu_n,$$

where μ_n is the largest root of $M_n(x) = x^{n+1} - x^n - x - 1$.

THEOREM A. *If a continuous map of the circle has a fixed point and a point of least period $n \geq 2$, then $\text{ent}(f) \geq \min\{\log \mu_n, \log \sigma_n\}$.*

This result was proved in [3] except for certain maps of degree -1 . See § 2 for details.

In light of the constructions described above, in order to solve the problem stated at the beginning of this paper we need only determine which is smaller, μ_n or σ_n .

THEOREM B. *Let $2 \leq n = 2^s k$ where k is odd.*

- (1) *If $s = 0$, then $\mu_n < \sigma_n$ except that $\mu_3 = \sigma_3$.*
- (2) *If $1 \leq s \leq 6$, then $\sigma_n < \mu_n$ when $k \leq 2s + 3$ and $\mu_n < \sigma_n$ when $k \geq 2s + 5$.*
- (3) *If $s \geq 7$, then $\sigma_n < \mu_n$ when $k \leq 2s + 5$ and $\mu_n < \sigma_n$ when $k \geq 2s + 7$.*

2. Proof of theorem A

We prove theorem A by restricting our attention to maps of the circle of a fixed degree, denoted by $\deg(\cdot)$, and using the following result.

THEOREM [3]. *Let f be a continuous map of the circle having a fixed point and a point of least period $n \geq 2$.*

- (a) *If $|\deg(f)| \geq 2$, then $\text{ent}(f) \geq \log |\deg(f)|$.*
- (b) *If $\deg(f) = 0$, then $\text{ent}(f) \geq \log \sigma_n$.*
- (c) *If $\deg(f) = 1$, then $\text{ent}(f) \geq \min\{\log \mu_n, \log \sigma_n\}$.*
- (d) *If $\deg(f) = -1$ and n is odd, then $\text{ent}(f) \geq \log \sigma_n$.*

It is an elementary fact (see § 3) that $\mu_n, \sigma_n < 2$. Therefore to prove theorem A it suffices to show that (d) holds when n is even.

LEMMA 1. *Let f be a continuous map of the circle having a fixed point and a point of least period $n \geq 3$. If f has no point of least period $n + 1$, then $\text{ent}(f) \geq \log \sigma_n$.*

Proof. By theorem A₁ of [2], the hypotheses of theorem A₂ of [2] are satisfied. Then, as in the proof of theorem A₂, there is a proper closed subinterval K of the circle, containing the orbit of a point of least period n , a homeomorphism h from

K onto a compact subinterval I of the reals, and a continuous map g from I into the reals such that for all $x \in K$, $f(x) \in K$ if and only if $g(h(x)) \in I$, and in this case $h(f(x)) = g(h(x))$. In particular, g has a point of least period n .

Let $K' = \bigcap_{j \geq 0} f^{-j}(K)$ and $I' = \bigcap_{j \geq 0} g^{-j}(I)$. Then $f|K'$ and $g|I'$ are topologically conjugate (via the appropriate restriction of h) and by definition, $\text{ent}(g) = \text{ent}(g|I')$. Then

$$\text{ent}(f) \geq \text{ent}(f|K') = \text{ent}(g|I') = \text{ent}(g) \geq \log \sigma_n,$$

the last inequality by Štefan's results. \square

We now complete the proof of theorem A. Suppose f is a map of the circle of degree -1 having a fixed point and a point of least period n , where n is even. We may assume that $n > 2$ for, if $n = 2$, then $\sigma_n = 1$ and there is nothing to prove. If f has no point of least period $n + 1$, then by lemma 1, $\text{ent}(f) \geq \log \sigma_n$. If f has a point of least period $n + 1$, then since $n + 1$ is odd, $\text{ent}(f) \geq \log \sigma_{n+1}$. But it is another elementary fact (see § 3) that $\sigma_m > \sqrt{2}$ if m is odd, and $\sigma_m < \sqrt{2}$ if m is even. Thus $\log \sigma_{n+1} > \log \sigma_n$. \square

3. Proof of theorem B

We begin by listing some elementary facts about the polynomials

$$M_n(x) = x^{n+1} - x^n - x - 1 \quad (n \geq 2)$$

and

$$L_k(x) = x^k - 2x^{k-2} - 1 \quad (k \geq 3).$$

M_n is increasing on $(1, \infty)$. Since $M_n(1) < 0$ and $M_n(2) > 0$, M_n has a unique root μ_n in $(1, \infty)$ and

$$1 < \mu_n < 2. \quad (1)$$

$$(\mu_n)^n = \frac{\mu_n + 1}{\mu_n - 1}. \quad (2)$$

L'_k has exactly one root in $(0, \infty)$ and L'_k changes from negative to positive at this root. Since $L_k(0) < 0$, $L_k(\sqrt{2}) < 0$ and $L_k(2) > 0$, L_k has a unique root λ_k in $(0, \infty)$ and

$$\sqrt{2} < \lambda_k < 2. \quad (3)$$

Recall that for $n = 2^s k$ where k is odd, $\sigma_n = 1$ if $k = 1$ and $\sigma_n = (\lambda_k)^{2^{-s}}$ if $k \geq 3$.

$$\sigma_n > \sqrt{2} \quad \text{if } n \text{ is odd} \quad \text{and} \quad \sigma_n < \sqrt{2} \quad \text{if } n \text{ is even.} \quad (4)$$

LEMMA 2. Let $n = 2^s k$ where k is odd. If $k \geq 2s + 7$, then $\mu_n < \sigma_n$.

Proof. Let $q = 2^{-(s+1)}$. By (3), $\sigma_n > 2^q$. On the other hand,

$$M_n(2^q) > 0 \quad \text{if} \quad 2^{k/2} > \frac{2^q + 1}{2^q - 1}.$$

But

$$\frac{2^x + 1}{2^x - 1} < \frac{4}{x} \quad \text{whenever} \quad 0 < x \leq 1.$$

(To see this, look at $F(x) = (4-x)2^x - x - 4$; $F(0) = 0$ and $F'(x) > 0$ if $0 \leq x \leq 1$.) Thus

$$\frac{2^q + 1}{2^q - 1} < 2^{s+3},$$

and hence $M_n(2^q) > 0$ if $k > 2(s+3)$. Therefore

$$\mu_{2^s k} < 2^q < \sigma_{2^s k}$$

for all odd $k \geq 2s+7$. □

We shall find it desirable to use the polynomials

$$T_s(x) = x^{2^{s+1}} - x - 1 \quad (s \geq 0).$$

T_s is increasing on $(1, \infty)$. Since $T_s(1) < 0$ and $T_s(2) > 0$, T_s has a unique root τ_s in $(1, \infty)$ and

$$(\tau_s)^{2^{s+1}} = \tau_s + 1. \quad (5)$$

LEMMA 3. Let $n = 2^s k$ where $k \geq 3$ is odd. Then $\sigma_n - \mu_n$ has the same sign (positive, negative, zero) as $(\tau_s + 1)^{k-2}(\tau_s - 1)^2 - 1$.

Proof. Suppose $\sigma_n - \mu_n > 0$. Let $r = 2^s$. Then $\lambda_k > (\mu_n)^r$ and hence $L_k((\mu_n)^r) < 0$. Writing μ in place of μ_n and using (2), we have

$$L_k(\mu^r) = \frac{2}{\mu^{2r}(\mu-1)} T_s(\mu).$$

Hence $T_s(\mu) < 0$ and so $\mu < \tau_s$. Writing τ in place of τ_s , using (2) and (5) and the fact that $G(x) = ((x+1)/(x-1))^2$ is decreasing on $(1, \infty)$, we have

$$\left(\frac{\tau+1}{\tau-1}\right)^2 < \left(\frac{\mu+1}{\mu-1}\right)^2 = \mu^{2n} < \tau^{2n} = (\tau+1)^k.$$

Thus $(\tau_s + 1)^{k-2}(\tau_s - 1)^2 > 1$.

The same argument goes through with all the inequalities reversed or all replaced by equalities. □

An immediate consequence of lemma 3 is

LEMMA 4. Let $k \geq 3$ be odd. If $\mu_{2^s k} < \sigma_{2^s k}$, then $\mu_{2^s l} < \sigma_{2^s l}$ for all odd $l > k$.

LEMMA 5. If $\sigma_{2^s(2s+5)} < \mu_{2^s(2s+5)}$, then $\sigma_{2^t(2t+5)} < \mu_{2^t(2t+5)}$ for all $t > s$.

Proof.† It suffices to show that the result holds for $t = s+1$. Let $\alpha = \tau_s$ and $\beta = \tau_{s+1}$. Using (5), we have that $T_{s+1}(\alpha) = \alpha(\alpha+1) > 0$, and so $\beta < \alpha$. Using (5) again,

$$(\beta^{2^{s+1}})^2 = \beta^{2^{s+2}} = \beta + 1 < \alpha + 1 = \alpha^{2^{s+1}}.$$

Thus $\beta^2 < \alpha$ and hence $(\beta^2 - 1)^2 < (\alpha - 1)^2$. Then

$$(\beta + 1)^{2^{s+1}+3}(\beta - 1)^2 = (\beta + 1)^{2^{s+3}}(\beta^2 - 1)^2 < (\alpha + 1)^{2^{s+3}}(\alpha - 1)^2 < 1,$$

the last inequality by lemma 3. By lemma 3 again, $\sigma_{2^{s+1}(2(s+1)+5)} < \mu_{2^{s+1}(2(s+1)+5)}$. □

We now complete the proof of theorem B.

Since $M_3(x) = (x^2 - x - 1)(x^2 + 1)$ and $L_3(x) = (x^2 - x - 1)(x + 1)$, $\mu_3 = \lambda_3 = \sigma_3$. Since $L_5(1.5) < 0 < M_5(1.5)$, $\mu_5 < 1.5 < \lambda_5 = \sigma_5$. Then (1) follows from lemma 2.

† This proof is due to M. L. Ginsberg (personal communication).

It can be checked (using, for example, a hand-held calculator) that if $1 \leq s \leq 6$, then

$$\sigma 2^s(2s+3) < \mu 2^s(2s+3)$$

and

$$\mu 2^s(2s+5) < \sigma 2^s(2s+5),$$

and if $s = 7$, then

$$\sigma 2^s(2s+5) < \mu 2^s(2s+5).$$

Then (2) follows from lemma 4 and (3) follows from lemmas 2, 4 and 5. \square

REFERENCES

- [1] L. Block. Periodic orbits of continuous maps of the circle. *Trans. Amer. Math. Soc.* **260** (1980), 555–562.
- [2] L. Block. Periods of periodic points of maps of the circle which have a fixed point. *Proc. Amer. Math. Soc.* **82** (1981), 481–486.
- [3] L. Block, J. Guckenheimer, M. Misiurewicz, & L.-S. Young. Periodic points and topological entropy of one dimensional maps. In *Global Theory of Dynamical Systems*, Proceedings, pp. 18–34. Northwestern, 1979. Lecture Notes in Math. no. 819. Springer: Berlin, 1980.
- [4] A. N. Šarkovskii. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.* **16** (1964) 61–71. (Russian, English summary.)
- [5] P. Štefan. A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line. *Comm. Math. Phys.* **54** (1977), 237–248.