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MINIMUM ABERRATION $(S^2)S^{n-k}$ DESIGNS

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Abstract: This paper extends the 4^m2^n minimum aberration designs (MA designs) of Wu and Zhang (1993) to the case of $(S^2)S^{n-k}$, where S is any prime or prime power. Some basic properties of $(S^2)S^{n-k}$ MA designs, including the relations with S^{n-k} MA designs, are discussed. The $(9)3^{n-k}$ MA designs with 27 runs and $(16)4^{n-k}$ MA designs with 64 runs are tabulated, and some $(4)2^{n-k}$ MA designs are constructed using the above relations.

Key words and phrases: Asymmetrical factorial design, fractional factorial designs, grouping scheme, minimum aberration.

1. Introduction

A factorial design in which the numbers of levels of the factors are not all equal is called an asymmetrical or mixed factorial design. In practice, many experiments require that the factors have different numbers of levels. Usually we use $S_1^{n_1}\cdots S_q^{n_q}$ to denote a design with n_1 S_1-level factors, ..., n_q S_q-level factors. Addelman (1962) established some methods for constructing 4^m2^n orthogonal arrays. By using a grouping scheme, Wu, Zhang and Wang (1992) constructed a large class of asymmetrical orthogonal arrays.

When S is a prime or prime power, the grouping scheme for constructing $(S^2)S^n$ orthogonal array is as follows. Let $OA(S^p, S^L)$ be a regular saturated orthogonal array, where $L=(S^p-1)/(S-1)$, and $x_1,\dots,x_p$ be p independent columns of the array. Then each column of the array can be represented as $\sum_{i=1}^p a_ix_i$, where a_i is an element of the finite field $GF(S)$ of S elements. For simplicity, every column can be represented by a vector $V=(a_1,\dots,a_p)$, where the first nonzero component a_i is 1. For any two independent p-vectors V_1 and V_2, let $H_{2,p}$ be the set of the $S+1$ columns of the form $c_1V_1+c_2V_2$, where

$$c_i\in GF(S),\ \text{at least one } c_i \text{ is nonzero and the first nonzero } c_i \text{ is } 1. \qquad (1)$$

We then use an S^2-level column to replace the $S+1$ S-level columns of $H_{2,p}$, and select n S-level columns from the array $OA(S^p,S^L)\backslash H_{2,p}$. Then an asymmetrical array $OA(S^p,(S^2)S^n)$ is obtained, where $n\le (S^p-1)/(S-1)-S-1$ and (S^2) denotes the S^2-level column in the design. For simplicity, we use $V_1^{c_1}V_2^{c_2}$

to denote the operation $c_1V_1 + c_2V_2$ in $GF(S)$ and we write $H_{2,p}$ as $H_{2,p} = \{V_1, V_2, V_1V_2, \ldots, V_1V_2^{S-1}\}$, where superscripts $1, \ldots, S-1 \in GF(S)$.

For any fixed n, there are many such designs. A natural question is how to find an optimal one. First, we need a criterion of goodness. Wu and Zhang (1993) described some basic ideas on asymmetrical minimum aberration factorial designs and constructed a class of minimum aberration (MA) design of the type 4^m2^n. In this paper, we extend their results to $(S^2)S^n$ designs, where S is a prime or prime power.

We only consider regular $(S^2)S^n$ designs, wherein there are an S^2-level factor and n S-level factors and, for some k, among the n S-level factors, $n-k$ of them are independent of each other and the $S+1$ S-level factors grouped as the S^2-level factor while the other k factors are defined by k independent defining relations. To emphasize the parameter k, $(S^2)S^{n-k}$ denotes this kind of designs hereafter.

In Section 2 we give some related concepts and three criteria for selecting $(S^2)S^{n-k}$ MA designs. Some properties of $(S^2)S^{n-k}$ MA designs are studied and the $(9)3^{n-k}$ MA designs with 27 runs and $(16)4^{n-k}$ MA designs with 64 runs are given in Section 3. In Section 4 we utilize the relation between $(S^2)S^{n-k}$ MA designs and S^{n-k} MA designs to give some $(4)2^{n-k}$ MA designs for certain k and n.

2. MA Criteria and Some Related Concepts

A design with n S-level factors and S^{n-k} runs is called a regular S^{n-k} design if k of its factors can be defined by k independent defining relations from its $n-k$ independent columns. For example, to construct a 3^{5-2} design, we first write a 27×3 matrix whose rows consist of all the level combinations of factors 1, 2 and 3 (0, 1 and 2 stand for the three levels of each factor). Clearly, the three factors (columns) are independent. Then we can define the factors 4 and 5 by say,

$$4 = 123, \quad 5 = 23^2, \tag{2}$$

where the levels of factors 4 and 5 are determined by addition and multiplication module 3 according to (2). The equations in (2) are called defining relations and $I = 1234^2$ and $I = 23^25^2$ are called defining contrasts. From the two defining relations we get the following:

$$I = 1234^2 = 23^25^2 = 12^24^25^2 = 13^24^25 = 1^22^23^24 = 2^235 = 1^2245 = 1^2345^2, \tag{3}$$

where $I = (0, \ldots, 0)'$ is the identity element of the group, and 1234^2, 23^25^2, $12^24^25^2$, 13^24^25 are called words. If a word contains c letters, then it is said to have wordlength c. Generally, for a regular S^{n-k} design obtained by k defining relations, I and all the words generated by the k defining relations form a group.

Since the levels in experimental designs can be treated as symbols, we have a convention: the words x and x^λ are considered to be same, where $\lambda \in GF(S)$ and $\lambda \neq 0$, and the word with the first nonzero superscript being 1 is a delegate. We use G to denote the group and call it contrast subgroup (Chen (1990)). The k independent contrasts in G are called generating elements of G. From the group theory, it follows that the number of elements in G is $|G| = S^k$ and the number of delegate words in the group is $(S^k - 1)/(S - 1)$. In the following, for convenience, we also use G to denote the set of all the different words and I in the group and still call it a contrast subgroup.

Now, let us consider a regular $(S^2)S^{n-k}$ design. In such a design, the S^2-level factor is obtained by the grouping scheme. Without loss of generality, let it be $A = (1, 2, 12, \ldots, 12^{S-1})$. Let $3, \ldots, n+2-k$ be $n-k$ independent S-level factors, and the other k S-level factors are determined by k independent defining relations, where the k relations involve the $n+2$ letters $1, \ldots, n+2$. The k independent relations also generate a contrast subgroup and the design can be completely described by this group. For example, to obtain a $(4)2^{7-3}$ design, we take $A = (1, 2, 12)$ as the 4-level factor, $3, 4, 5, 6$ as the 4 independent 2-level factors, and the other 3 factors can be obtained by 3 defining relations, for instance, $7 = 1234, 8 = 245$, and $9 = 346$. Then its contrast subgroup is $I = 12347 = 2458 = 3469 = 13578 = 12679 = 235689 = 1456789$.

Let $\mathcal{D}$ denote a regular $(S^2)S^{n-k}$ design described as above. If the S^2-level factor in $\mathcal{D}$ is replaced by the $S+1$ S-level factors of A, then $\mathcal{D}$ can be regarded as a symmetric $S^{(n+S+1)-(k+S-1)}$ design, denoted by $\mathcal{E}$. Let $G(\mathcal{D})$ denote the contrast subgroup of $\mathcal{D}$ generated by its k defining relations. We notice that for $12, 12^2, \ldots, 12^{S-1}$, every one is considered to relate one factor in $\mathcal{D}$, but in its corresponding $S^{(n+S+1)-(k+S-1)}$ design, every one of them involves two factors. Therefore, when calculating wordlength, if a word in the $(S^2)S^{n-k}$ design contains $12^t, t \in GF(S), t \neq 0$, then its wordlength in $\mathcal{D}$ is 1 less than its length in the $S^{(n+S+1)-(k+S-1)}$ design.

We partition the words in the group $G(\mathcal{D})$ into two types: type 0 and type 1. A word which does not include any 1 and 2 is of type 0, otherwise it is of type 1. Similar to symmetric designs, we define the wordlength pattern of the asymmetrical design $\mathcal{D}$ to be a nested vector $W(\mathcal{D}) = \{A_i(\mathcal{D})\}_{i\geq 3}$, where $A_i(\mathcal{D}) = (A_{i0}(\mathcal{D}), A_{i1}(\mathcal{D}))$, and $A_{ij}(\mathcal{D})$ denotes the number of words of type j with wordlength i. The wordlength pattern of the $S^{(n+S+1)-(k+S-1)}$ design $\mathcal{E}$ defined by the $k+S-1$ defining relations is denoted by $W^*(\mathcal{E})$. Similarly, we define the rth moment of $\mathcal{D}$ to be $M_r(\mathcal{D}) = \sum_j j^r A_j(\mathcal{D})$. If the two types of words are treated as equally important, we also define the combined wordlength pattern of $\mathcal{D}$ as $W^c(\mathcal{D}) = \{B_i(\mathcal{D})\}_{i\geq 3}$, where $B_i(\mathcal{D}) = A_{i0}(\mathcal{D}) + A_{i1}(\mathcal{D})$, and the combined r-moment of $\mathcal{D}$ as $M_r^c(\mathcal{D}) = \sum_j j^r B_j(\mathcal{D})$.

To find an optimal design, we first find maximum resolution designs. When two designs have the same resolution, according to the hierarchical principle of factor effects in experiments, we choose the one with least aberration. To meet different requirements, we define the following three minimum aberration criteria for selecting optimal $(S^2)S^{n-k}$ designs.

1. Minimum aberration criterion of type 0. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $(S^2)S^{n-k}$ designs and let r be the smallest i such that $A_i(\mathcal{D}_1) \neq A_i(\mathcal{D}_2)$. If $A_{r0}(\mathcal{D}_1) < A_{r0}(\mathcal{D}_2)$ or if $A_{r0}(\mathcal{D}_1) = A_{r0}(\mathcal{D}_2)$ and $A_{r1}(\mathcal{D}_1) < A_{r1}(\mathcal{D}_2)$, $\mathcal{D}_1$ is said to have less aberration of type 0 than $\mathcal{D}_2$. If there is no design which has less aberration of type 0 than $\mathcal{D}_1$, $\mathcal{D}_1$ is said to be a minimum aberration design of type 0 (MA^0 design, for short).

2. Minimum aberration criterion of type 1. Exchanging A_{r0} and A_{r1} in the above definition gives the definition of minimum aberration criterion of type 1. Also, for short, we denote the minimum aberration of type 1 by MA^1.

3. Combined minimum aberration criterion. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two designs. Suppose that r is the smallest i such that $B_i(\mathcal{D}_1) \neq B_i(\mathcal{D}_2)$. If $B_r(\mathcal{D}_1) < B_r(\mathcal{D}_2)$, $\mathcal{D}_1$ is said to have less combined aberration than $\mathcal{D}_2$. If there is no design which has less combined aberration than $\mathcal{D}_1$, $\mathcal{D}_1$ is said to be a combined minimum aberration design (MA^c design, for short).

Experimenters can choose one of the three criteria above according to practical requirements. When the S^2-level factor and all S-level factors are considered to be equally important, we can choose Criterion 3; if the S^2-level factor is regarded to be more important, we can choose Criterion 2; otherwise choose Criterion 1.

Example 2.1. Let $OA(3^4, 3^{40})$ denote a regular saturated 3-level design with independent factors 1, 2, 3 and 4. Suppose that by applying grouping schemes to $OA(3^4, 3^{40})$, we have the following three $(9)3^{4-2}$ designs: $\mathcal{D}_1 : A, B, C, D = 34^2, E = 14$, $\mathcal{D}_2 : A, B, C, D = 13, E = 24$, $\mathcal{D}_3 : A, B, C, D = 134, E = 234^2$, where $A = (1, 2, 12, 12^2), B = 3$, and $C = 4$. Then for $\mathcal{D}_1$, we have words BC^2D^2, $1CE^2$, $1BD^2E^2$, $1B^2C^2DE^2$ and $W(\mathcal{D}_1) = \{(1,1), (0,1), (0,1)\}$; for $\mathcal{D}_2$, we have words $1BD^2$, $2CE^2$, $12BCD^2E^2$, $12^2BC^2D^2E$ and $W(\mathcal{D}_2) = \{(0,2), (0,0), (0,2)\}$; for $\mathcal{D}_3$, we have words $1BCD^2$, $2BC^2E^2$, $12B^2D^2E^2$, $12^2C^2D^2E$ and $W(\mathcal{D}_3) = \{(0,0), (0,4), (0,0)\}$. Comparing $\mathcal{D}_1$ and $\mathcal{D}_2$, $\mathcal{D}_1$ is better than $\mathcal{D}_2$ with respect to Criterion 2, but $\mathcal{D}_2$ is better than $\mathcal{D}_1$ under Criteria 1 and 3. Under all three criteria $\mathcal{D}_3$ is better than $\mathcal{D}_1$ and $\mathcal{D}_2$.

For small values of parameters n and k, we can find MA designs by exhaustive computation. We obtain some $(9)3^{n-k}$ MA designs with 27 runs, $(16)4^{n-k}$ MA designs with 64 runs and we give their wordlength patterns for all the three criteria. These are tabulated in Tables 1 and 2 respectively. (For large n and k, this method is not feasible.)

3. Properties of $(S^2)S^{n-k}$ MA Designs

In this section, we study properties of $(S^2)S^{n-k}$ MA designs. We still use $A = (1, 2, 12, 12^2, \dots, 12^{S-1})$ to denote the S^2-level factor in an $(S^2)S^{n-k}$ design, and G denotes the contrast subgroup generated by its k independent defining contrasts. We only consider the case where all the $n+2$ letters appear in G, for otherwise the resolution or wordlength of the design is reduced (see Theorem 1 below). Suppose that T is a subgroup consisting of the words not including 1 and 2 and denote it by $T = I \cup F$. Then it is not difficult to verify that one of the following holds:

$$\begin{aligned} G = T \cup 12^t xQ,\ & \text{for some } t \in GF(S) \text{ and } t \neq 0, \\ & \text{where } Q = I \cup F \cup F^2 \cup \cdots \cup F^{S-1}, F^i \text{ stands for the set } iF, \\ & i \in GF(S),\ x \text{ and } F \text{ do not contain factors 1 and 2;} \end{aligned} \tag{4}$$

$$\begin{aligned} G = T \cup 1xQ \cup 2yQ \cup 12xyQ \cup \cdots \cup 12^{S-1}xy^{S-1}Q, & \\ \text{where } Q \text{ is as in (4)},\ x, y, \text{ and } F \text{ do not contain factors 1 and 2.} & \end{aligned} \tag{5}$$

In the following we give some results on $(S^2)S^{n-k}$ MA designs under Criterion 3. For simplicity, we use the notation MA design instead of MA_g design for this case. We have

Theorem 1. *Let $\mathcal{D}$ be an $(S^2)S^{n-k}$ MA design and $G(\mathcal{D})$ denote the contrast subgroup generated by its k defining relations. Then*
1. *$G(\mathcal{D})$ contains all the letters.*
2. *If $k > 1$, $G(\mathcal{D})$ satisfies* (5).
3. *If $\mathcal{D}$ satisfies* (5), *then the first moment $M_1^c(\mathcal{D})$ of the wordlength pattern of $\mathcal{D}$ is $S^{k-2}(Sn + S + 1)$.*
4. $\sum_{i \geq 3} B_i(\mathcal{D}) = (S^k - 1)/(S - 1)$.

To find $(S^2)S^{n-k}$ MA designs, we establish some ties between $(S^2)S^{n-k}$ designs and S^{n-k} designs.

Suppose that $\mathcal{D}$ is an $(S^2)S^{n-k}$ design and that its contrast subgroup $G(\mathcal{D})$ satisfies (5). By dropping letters 1 and 2 from (5) we get

$$G^- = T \cup xQ \cup yQ \cup xyQ \cup \cdots \cup xy^{S-1}Q.$$

Obviously, G^- nominally can be regarded as the contrast subgroup of an S^{n-k} design which does not contain factors 1 and 2. Denote the design by $\mathcal{D}^-$, then $G^- = G(\mathcal{D}^-)$. For such a design $\mathcal{D}^-$ we can similarly define its wordlength and wordlength pattern. Conversely, given an S^{n-k} design $\mathcal{D}^-$ without factors 1 and 2, by a selection for T, x, and y adding factors 1 and 2, we get an $(S^2)S^{n-k}$ design. For example, suppose $\mathcal{D}$ is a $(4)2^{7-3}$ design with $A = (1,2,12), 3, 4, 5, 6, 7 = 134, 8 = 12456, 9 = 345$. Then $G(\mathcal{D}) = \{I, 1347, 124568, 3459,\ 235678,\ 1579,\ 123689,\ 246789\}$, $\mathcal{D}^-$ is $3, 4, 5, 6, 7 = 34, 8 = 456, 9 = 345$, and $G(\mathcal{D}^-) = \{I, 347, 4568, 3459,\ 35678, 579, 3689, 46789\}$ with $T = \{I, 3459\}$, $x = 347$ and $y = 4568$. If we select $T = \{I, 35678\}$, $x = 347$ and $y = 579$ from $G(\mathcal{D}^-)$, we obtained another $(S^2)S^{n-k}$ design $\mathcal{D}_1$: $A = (1,2,12), 3, 4, 5, 6, 7 = 134, 8 = 1456, 9 = 12345$, which is better than $\mathcal{D}$. It can be seen that the construction of a design $\mathcal{D}$ depends on the choice of $\mathcal{D}^-$ and related Q, x and y in $G(\mathcal{D}^-)$. It is needed to note that, in a design $\mathcal{D}^-$, x and y can involve exactly two letters. In this case, the design $\mathcal{D}^-$ may have resolution II and hence may not be a main effect plan. But this does not affect Theorem 2, since the formula (A.1) in the proof of the theorem is still true.

Theorem 2. *Let $\mathcal{D}$ be an $(S^2)S^{n-k}$ design and let $G(\mathcal{D})$ satisfy* (5). *Then under Criterion* 3*:*

1. *For $k = 2$, a necessary and sufficient condition for $\mathcal{D}$ to be an $(S^2)S^{n-k}$ MA design is that $\mathcal{D}^-$ is an S^{n-k} MA design;*
2. *For $k = 3$, either $\mathcal{D}^-$ is an S^{n-k} MA design or there is an S^{n-k} MA design, say $\mathcal{D}_1^-$, such that through an appropriate selection of T, x and y, the corresponding $(S^2)S^{n-k}$ design $\mathcal{D}_1$ has the same wordlength pattern as $\mathcal{D}$, i.e., $\mathcal{D}_1$ is also an $(S^2)S^{n-k}$ MA design.*

For $k \geq 4$, we can use the following method to construct a good $(S^2)S^{n-k}$ design. Partition the wordlength pattern $W(\mathcal{D}^-)$ of $\mathcal{D}^-$ into two parts, one T, denoted by $W^1(\mathcal{D}^-) = \{B_i^1(\mathcal{D}^-)\}_{i\geq 3}$, and the other for $xQ \cup yQ \cup \cdots \cup xy^{S-1}Q$, denoted by $W^2(\mathcal{D}^-) = \{B_i^2(\mathcal{D}^-)\}_{i\geq 3}$. Therefore, the combined wordlength pattern of $\mathcal{D}$ is $W^c(\mathcal{D}) = \{B_i(\mathcal{D})\}_{i\geq 3} = \{B_i^1(\mathcal{D}^-) + B_{i-1}^2(\mathcal{D}^-)\}_{i\geq 3}$, where $B_2^2(\mathcal{D}^-) = 0$. Thus, the wordlength pattern of $\mathcal{D}$ is determined by $W^1(\mathcal{D}^-)$ and a 1-component right-shifting of $W^2(\mathcal{D}^-)$. On the other hand, for any $k > 2$, the number of words in T is $(S^{k-2}-1)/(S-1)$ and that in $xQ \cup yQ \cup \cdots \cup xy^{S-1}Q$ is $(S^k-1)/(S-1) - (S^{k-2}-1)/(S-1)$. The total number of words in $\mathcal{D}^-$ is $(S^k-1)/(S-1)$ and the number of words in the second part is $\alpha \equiv 1 - (S^{k-2}-1)/(S^k-1)$ times the total number. Obviously, the proportion α is at least $1 - 1/S^2$ for any k. For example, for $S = 2, 3, 5$, $\alpha \geq \frac{3}{4}, \frac{8}{9}, \frac{24}{25}$ respectively. Thus a 1-component right-shifting of $W^2(\mathcal{D}^-)$ contributes substantially to the wordlength pattern of $\mathcal{D}$. So, if $\mathcal{D}^-$ has small aberration then $\mathcal{D}$ also has small aberration. We call the best design obtained by this method a nearly MA design.

According to Theorem 2 and the above analysis, we can construct an (S^2) S^{n-k} MA design (at least for $k = 2, 3$) or a nearly MA design through an S^{n-k} MA design.

4. Some $(4)2^{n-k}$ MA Designs

Wu and Zhang (1993) constructed 4^m2^n MA designs for certain parameters. Using the relation between an $(S^2)S^{n-k}$ design D and its corresponding S^{n-k} design $\mathcal{D}^-$ of the last section, and the known results about 2^{n-k} MA designs (Chen (1990)), we construct more $(4)2^{n-k}$ MA designs under Criterion 3. In the following, take $A = (1, 2, 12)$ as the 4-level factor.

Case (1) $k = 2$.

Set $n - 2 = 3m + r$, $0 \le r < 3$, and let $3, \ldots, n$, $(n+1)$,$(n+2)$ denote the n 2-level factors.

For $r = 0, 1$, and 2, the defining relation of a $(4)2^{n-k}$ MA design can be taken as, respectively,

$$\begin{aligned}(n+1) &= 134\cdots(2m+1)(2m+2),\\ (n+2) &= 2(m+3)(m+4)\cdots(3m+2),\end{aligned} \tag{6}$$

$$\begin{aligned}(n+1) &= 134\cdots(2m+2)(2m+3),\\ (n+2) &= 2(m+3)(m+4)\cdots(3m+2)(3m+3),\end{aligned} \tag{7}$$

$$\begin{aligned}(n+1) &= 134\cdots(2m+2)(2m+3),\\ (n+2) &= 2(m+3)(m+4)\cdots(3m+3)(3m+4).\end{aligned} \tag{8}$$

Case (2) $k = 3$.

Set $n = 7m + r, 0 \le r < 7$, and also let $3, 4, \ldots n, (n+1), (n+2)$ denote the 2-level factors. Define $C_i = I(n-i+3)(n-i+3-7)\cdots(n-i+3-7m)$, $i = 1, \cdots, 7$, where $n - i + 3 \ge 3, n - i + 3 - 7m = r - i + 3 \ge 3$. Then we choose the defining contrasts to be $I = 1C_1C_4C_5C_6 = 2C_2C_4C_5C_7 = 12C_3C_4C_6C_7$ to obtain a $(4)2^{n-3}$ MA design.

Case (3) $k = 4$.

Set $n = 15m + r, 0 \le r < 15$. Define $C_i = I(n-i+3)(n-i+3-15)\cdots(n-i+3-15m)$, $i = 1, \ldots, 15$, where $n-i+3 \ge 3, n-i+3-15m = r-i+3 \ge 3$. Then we choose the defining contrasts

$$\begin{aligned}I &= 1C_1C_6C_7C_8C_9C_{12}C_{14}C_{15} = 2C_2C_5C_7C_8C_9C_{11}C_{13}C_{15}\\ &= 12C_3C_5C_6C_8C_{10}C_{11}C_{14}C_{15} = C_4C_5C_6C_7C_{10}C_{12}C_{13}C_{15}\end{aligned}$$

to obtain a $(4)2^{n-3}$ nearly MA design.

Example 4.1. Consider a 64-run $(4)2^{8-4}$ design. According to the above method we have $C_1 = \underline{10}, C_2 = 9, C_3 = 8, C_4 = 7, C_5 = 6$, $C_6 = 5, C_7 = 4, C_8 = 3, C_9 = C_{10} = I$ and the defining contrasts $I = 1\underline{10}345 = 29346 = 128356 = 7456$. This gives a 64-run $(4)2^{8-4}$ nearly MA design $(1, 2, 12), 3, 4, 5, 6, 456$, 12356, 2346, 1345 which has resolution IV and wordlength pattern ((2,0), (0,12), (0,0), (0,0), (1,0)).

Table 1. Minimum aberration $(9)3^{n-k}$ designs with 27 runs under the three criteria.

n	k	Design $\mathcal{D}$	$\{A_i(\mathcal{D})\}_{i\geq 3}$
2	1	$A, 3, 123$	(0,1)
3	2	$A, 3, 13, 23$	(0,3)(0,1)
4	3	$A, 3, 13, 23, 123$	(0,6)(1,4)(0,2)
5	4	$A, 3, 13, 23, 123, 12^23$	(1,10)(3,9)(0,12)(0,5)
6	5	$A, 3, 13, 23, 123, 12^23, 23^2$	(2,15)(9,18)(0,36)(2,30)(0,9)
7	6	$A, 3, 13, 23, 123, 12^23, 13^2, 23^2$	(5,21)(15,30)(9,90)(8,96)(3,69)(0,18)
8	7	$A, 3, 13, 23, 123, 13^2, 12^23, 123^2, 12^23^2$	(8,28)(30,48)(24,180)(32,256(24,276) (3,144)(0,40)

Table 2. Minimum aberration $(16)4^{n-k}$ designs with 64 runs under the three criteria.

n	k	Design $\mathcal{D}$	$\{A_i(\mathcal{D})\}_{i\geq 3}$
2	1	$A, 3, 123$	(0,1)
3	2	$A, 3, 13, 23$	(0,3)(0,2)
4	3	$A, 3, 13, 23, 123$	(0,6)(1,8)(0,6)
5	4	$A, 3, 13, 23, 123, 12^\alpha 3^\alpha$	(0,10)(4,18)(1,26)(0,26)
6	5	$A, 3, 13, 23, 123^{1+\alpha}, 12^{1+\alpha}3^\alpha, 12^\alpha 3^{1+\alpha}$	(0,15)(10,40)(5,90)(6,120)(0,55)

Remark: $GF(2^2) = \{0, 1, \alpha, 1+\alpha\}$, $A = (1, 2, 12, 12^\alpha, 12^{1+\alpha})$

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Appendix A. Proof of Theorem 1

If no word in $G(\mathcal{D})$ contains some letter j, we can obtain another design $\mathcal{D}'$ by adding the letter j to the defining relations of $\mathcal{D}$. Comparing the lengths of words in the two contrast subgroups $G(\mathcal{D}')$ and $G(\mathcal{D})$, we can see that the length of any word in $G(\mathcal{D}')$ is 1 greater than or equal to that of its corresponding word in $G(\mathcal{D})$, and there is at least one word in $G(\mathcal{D}')$ whose length is greater than the length of its corresponding word in $G(\mathcal{D})$. This means $\mathcal{D}'$ is better than $\mathcal{D}$, which contracts the assumption that $\mathcal{D}$ is an MA design, thus completing the proof of 1.

Suppose that $\mathcal{D}$ is an MA design but satisfies (4). Without loss of generality let $G(\mathcal{D}) = I \cup F \cup 12x(I \cup F \cup F^2 \cup \cdots \cup F^{S-1})$, and by the assumption $k > 1$, the subgroup $T = I \cup F$ is not trivial (i.e. it does not only contain I). Then by the properties of a group there is a z such that $F = H \cup z(I \cup H \cup \cdots \cup H^{S-1})$, where $I \cup H$ is a subgroup. Thus, $G(\mathcal{D})$ can be written as

$$\begin{aligned} G(\mathcal{D}) &= I \cup H \cup z(I \cup H \cup \cdots \cup H^{S-1}) \cup 12x(I \cup H \cup z(I \cup H \cup \\ &\quad \cdots \cup H^{S-1}) \cup \cdots \cup H^{S-1} \cup z^{S-1}(I \cup H \cup \cdots \cup H^{S-1})^{S-1}) \\ &= I \cup H \cup z(I \cup H \cup \cdots \cup H^{S-1}) \cup 12x(I \cup H \cup \cdots \cup H^{S-1} \\ &\quad \cup z(I \cup H \cup \cdots \cup H^{S-1}) \cup \cdots \cup z^{S-1}(I \cup H \cup \cdots \cup H^{S-1})) \\ &= T^* \cup zQ^* \cup 12xQ^* \cup 12xzQ^* \cup \cdots \cup 12xz^{S-1}Q^*, \end{aligned}$$

where $T^* = I \cup H$ and $Q^* = I \cup H \cup \cdots \cup H^{S-1}$.

Now we take another design $\mathcal{D}'$ which is obtained by changing z in $\mathcal{D}$ to $2z$. Then we have

$$\begin{aligned} G(\mathcal{D}') &= T^* \cup 2zQ^* \cup 12x(Q^* \cup 2zQ^* \cup \cdots \cup 2^{S-1}z^{S-1}Q^*) \\ &= T^* \cup 1xz^{S-1}Q^* \cup 2zQ^* \cup 12xQ^* \cup 12^2xzQ^* \cup \\ &\quad \cdots \cup 12^{S-1}xz^{S-2}Q^*. \end{aligned}$$

Since the length of every word of $2zQ^*$ in $G(\mathcal{D}')$ is 1 greater than that of every word of zQ^* in $G(\mathcal{D})$, and the wordlengths of the remaining words in $G(\mathcal{D}')$ and $G(\mathcal{D})$ are the same, this implies that $\mathcal{D}'$ is better than $\mathcal{D}$, also contradicting the assumption. Thus the proof of 2 is completed.

Now we prove 3. Let c be the number of words containing 12^t in G, i_0 be the number of words of length i not containing 12^t, and i_1 be the number of words of length i containing 12^t. Suppose that the combined wordlength pattern of an $(S^2)S^{n-k}$ MA design is $W^c = \{B_i\}$ and the wordlength pattern of its corresponding $S^{(n+S+1)-(k+S-1)}$ design is $W^* = \{B'_i\}$. Then we have $B_i = B_{i_0} + B_{i_1}, B'_i = B'_{i_0} + B'_{i_1}$, where $B'_{i_0} = B_{i_0}$, $B'_{i_1} = B_{(i-1)_1}$. Hence the first moment of $\mathcal{D}$ is

$$\begin{aligned} M_1^c(\mathcal{D}) &= \sum iB_i = \sum iB_{i_0} + \sum iB_{i_1} \\ &= \sum iB'_{i_0} + \sum iB'_{(i+1)_1} = \sum iB'_i - \sum iB'_{i_1} + \sum iB'_{(i+1)_1} \\ &= \sum iB'_i - \sum iB'_{i_1} + \sum (i-1)B'_{i_1} = \sum iB'_i - \sum B'_{i_1}, \end{aligned}$$

where $\sum$ is taken over $i \geq 3$. Since $G(\mathcal{D})$ has the form (5), by a simple computation we get $c = \sum B_{i_1} = \sum B'_{i_1} = S^{k-2}(S-1)$ and $\sum iB'_i = (n+2)S^{k-1}$

(Chen (1990)). Therefore we have $M_1^c(\mathcal{D}) = (n+2)S^{k-1} - S^{k-2}(S-1) = S^{k-2}(Sn+S+1)$, which is the required result.

The result of 4 is obvious. The proof of the theorem is complete.

Appendix B. Proof of Theorem 2

1. For $k=2$, the length of every word in $G(\mathcal{D})$ is equal to that of its corresponding word in $G(\mathcal{D}^-)$ plus 1. Actually, the wordlength pattern of $\mathcal{D}$ can be obtained by a 1-component right-shifting of the wordlength pattern of $\mathcal{D}^-$. This proves Part 1 of the theorem.

2. For $k=3$, by assumption, there is only one 0-type word in $G(\mathcal{D})$. We denote the length of the 0-type word by b, and denote the wordlength pattern for all 1-type words in $\mathcal{D}$ by $\{A_{i1}(\mathcal{D})\}_{i\geq 3}$. Then the wordlength pattern of $G(\mathcal{D}^-)$ corresponding to $\mathcal{D}$, denoted by $\{B_i(\mathcal{D}^-)\}_{i\geq 3}$, satisfies the conditions:

$$\begin{aligned} B_i(\mathcal{D}^-) &= A_{i+1,1}(\mathcal{D}), \quad\quad \text{if } i \neq b, \\ B_i(\mathcal{D}^-) &= A_{i+1,1}(\mathcal{D})+1, \text{ if } i = b. \end{aligned}$$

If $\mathcal{D}^-$ is not an S^{n-k} MA design, let $\mathcal{D}_1^-$ be an S^{n-k} MA design (not including factors 1 and 2). Denote the wordlength pattern of $\mathcal{D}_1^-$ by $\{B_i(\mathcal{D}_1^-)\}_{i\geq 3}$ and assume that the longest word in $G(\mathcal{D}_1^-)$ has length a. Choose a word with length a from $\mathcal{D}_1^-$ as a generator of T (see Theorem 1), properly add letters $1, 2$, or 12^t to the other words in $G(\mathcal{D}_1^-)$ and take $A = (1, 2, 12, \ldots, 12^{S-1})$ as the S^2-level factor. Then we obtain an $(S^2)S^{n-k}$ design $\mathcal{D}_1$ corresponding to $\mathcal{D}_1^-$.

There are two cases to be considered.

(i) $a \geq b$. In this case, since $\mathcal{D}_1^-$ has less aberration than $\mathcal{D}^-$, it is clear that the corresponding design $\mathcal{D}_1$ has less aberration than $\mathcal{D}$ according to Criterion 3. This contradicts the assumption that $\mathcal{D}$ is an $(S^2)S^{n-k}$ MA design. So $\mathcal{D}^-$ must be an S^{n-k} MA design.

(ii) $a < b$. Let r denote the smallest i such that $B_i(\mathcal{D}^-) \neq B_i(\mathcal{D}_1^-)$. Then $r < a$ and $B_r(\mathcal{D}^-) > B_r(\mathcal{D}_1^-)$ since $\mathcal{D}_1^-$ has less aberration than $\mathcal{D}^-$. There are two possibilities. (iia) $r < a-1$ or $B_r(\mathcal{D}^-) - B_r(\mathcal{D}_1^-) > 1$. For this case, it is easy to show that $\mathcal{D}_1$ has less aberration than $\mathcal{D}$, which contradicts the assumption. Thus $\mathcal{D}^-$ must be an S^{n-k} MA design. (iib) $r = a-1$ and $B_r(\mathcal{D}^-) - B_r(\mathcal{D}_1^-) = 1$. For this case, we first note the formulas:

$$\sum_i B_i = (S^k-1)/(S-1), \sum_i iB_i = nS^{k-1}, \tag{A.1}$$

where k is the number of defining words and n the number of letters in the design (Pless (1963)). Therefore, we have

$$B_{a-1}(\mathcal{D}_1^-) + B_a(\mathcal{D}_1^-) = \sum_{i\geq a-1} B_i(\mathcal{D}^-), \tag{A.2}$$

$$(a-1)B_{a-1}(\mathcal{D}_1^-)+aB_a(\mathcal{D}_1^-)=\sum_{i\geq a-1} iB_i(\mathcal{D}^-). \tag{A.3}$$

From $B_{a-1}(\mathcal{D}^-) > B_{a-1}(\mathcal{D}_1^-)$ and $B_b(\mathcal{D}^-) \geq 1$, it follows that $B_a(\mathcal{D}_1^-) - B_a(\mathcal{D}^-) \geq 2$. Now we show that $B_a(\mathcal{D}_1^-) - B_a(\mathcal{D}^-) = 2$. Denote $B_a(\mathcal{D}_1^-) - B_a(\mathcal{D}^-) = l$. By (A.2) we have

$$\sum_{i\geq a+1} B_i(\mathcal{D}^-) = l-1. \tag{A.4}$$

On the other hand, by (A.3) we have $la = a-1+\sum_{i\geq a+1} iB_i(\mathcal{D}^-)$. This means that if $l > 2$, $\sum_{i\geq a+1} iB_i(\mathcal{D}^-) = la-a+1 < (a+1)(l-1)$. However, by (A.4) we have $\sum_{i\geq a+1} iB_i(\mathcal{D}^-) \geq (a+1)(l-1)$, which leads to a contradiction. As a result, $B_a(\mathcal{D}_1^-) - B_a(\mathcal{D}^-)$ must be equal to 2.

Since $B_{a-1}(\mathcal{D}^-) - B_{a-1}(\mathcal{D}_1^-) = 1$ and $B_a(\mathcal{D}_1^-) - B_a(\mathcal{D}^-) = 2$, from (A.2) and (A.3) it follows that $b = a+1$, $B_b(\mathcal{D}^-) = 1$, and $B_i(\mathcal{D}^-) = 0$ for $i > a+1$. Furthermore, this implies that the design $\mathcal{D}_1$ resulting from $\mathcal{D}_1^-$ has the same wordlength pattern as that of $\mathcal{D}$, which proves Part 2 of the theorem. The proof of the theorem is complete.

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