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MINIMUM-BIAS WINDOWS FOR SPECTRAL ESTIMATION BY MEANS OF OVERLAPPED FAST FOURIER TRANSFORM PROCESSING

Albert H. Nuttall

Naval Underwater Systems Center
New London, Connecticut

11 April 1973

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36

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The time-limited nonnegative data windows that minimize the bias in auto- and cross-spectral estimation of stationary random processes by means of overlapped Fast Fourier Transform (FFT) processing are derived for a variety of constraints. We constrain the time duration of the data window, or constrain the bandwidth in various ways, such as the equivalent-noise bandwidth, the half-power bandwidth, or the root-mean-square bandwidth. In the three bandwidth-constrained cases, the window duration is adjusted to meed the constraint.

We find that the Hanning window is a reasonable compromise for achieving minimum bias, because in addition to being the optimum for one bandwidth constraint, it is very close to the optima for two other bandwidth constraints. The relative merits of the spectral characteristics of the windows are also discussed.

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ABSTRACT

The time-limited nonnegative data windows that minimize the bias in auto- and cross-spectral estimation of stationary random processes by means of overlapped Fast Fourier Transform (FFT) processing are derived for a variety of constraints. When the time duration L of the data window is constrained, the optimum data window is $(2/L)^{1/2} \cos(\pi t/L)$, $|t| \leq L/2$; when the equivalent-noise bandwidth is constrained, the optimum data window is $(8/3L)^{1/2} \cos^2(\pi t/L)$, which is the Hanning window; when the half-power bandwidth is constrained, the optimum data window is $L^{-1/2}[1.682 + 4.261 \cos(4.434\, t/L) - 4.337 \cos(3.552\, t/L)]$, which is very similar to the Hanning window; and when the root-mean-square bandwidth is constrained, the optimum data window is $4/(5L)^{1/2} \cos^3(\pi t/L)$. In the three bandwidth-constrained cases, the window duration L is adjusted to meet the constraint.

The Hanning window is a reasonable compromise for achieving minimum bias, because in addition to being the optimum for one bandwidth constraint, it is very close to the optima for two other bandwidth constraints. The relative merits of the spectral characteristics of the windows are also discussed.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

L	data window duration
t	time
E	ensemble average
f	frequency
$G(f)$	true spectrum
$\hat{G}(f)$	estimate of spectrum
$w(t)$	data window
$W(f)$	spectral window
T	available record length
$B(f)$	bias
D_1, D_2	bias constants
B_{st}	statistical bandwidth
$\phi_w(\tau)$	correlation of w(t)
B_e	equivalent-noise bandwidth
B_h	half-power bandwidth
B_r	root-mean-square bandwidth
prime	derivative
*	conjugate

MINIMUM-BIAS WINDOWS FOR SPECTRAL ESTIMATION BY MEANS OF OVERLAPPED FAST FOURIER TRANSFORM PROCESSING

INTRODUCTION

The selection of good data windows in spectral estimation of stationary random processes, to minimize leakage, is an important consideration and has received much attention [1-8]. In [6], a thorough investigation of four good data windows revealed virtually the same variance-reduction capabilities of overlapped Fast Fourier Transform (FFT) processing when the proper overlap was used for each window. The ultimate variance reduction of this direct procedure was also demonstrated to be identical to that attained by the older (indirect) analysis procedure in [4].

In this report, attention is focused on the bias in the estimation of power density spectra by means of overlapped FFT processing. Specifically, the bias is minimized by the choice of data windows that are restricted to be time-limited and nonnegative and are subject to either a time-duration constraint or a bandwidth constraint. These results complement and extend those of [8] for the indirect approach to spectral estimation.

PROBLEM DEFINITION

The overlapped FFT method for spectral estimation and the reasons for its use are documented in [6]. The mean of the spectral estimate is given by [6, eq. (5)] for auto-spectral estimation, and by [7, eq. (4A)] for cross-spectral estimation. In both cases, the mean takes the form*

$$E\ \{\hat{G}(f)\} = \int d\nu\ G(f-\nu)\ |W(\nu)|^2\ , \qquad (1)$$

where $\hat{G}(f)$ is the estimate of the true (auto or cross) spectrum $G(f)$, and

$$W(f) = \int dt\ \exp(-i2\pi ft)\ w(t), \qquad (2)$$

*Integrals without limits are over the range of nonzero integrand.

where w(t) is the data window multiplied in the time domain by the available data. It is assumed that the data window is time-limited and nonnegative:

$$w(t) \begin{Bmatrix} = 0 \text{ for } |t| > L/2 \\ \geq 0 \text{ for } |t| \leq L/2 \end{Bmatrix}, \tag{3}$$

where L is the window duration. The restriction to nonnegative data windows guarantees that the spectral window W(f) peaks at the origin. It should be noted that (1) is true with no restriction on the available record length T and with no restriction on the statistics of the random processes involved, except that the processes must be stationary; they need not be Gaussian for (1) to apply.

The desired value of (1) is the true value G(f); therefore the bias in estimation is defined as

$$B(f) = E\left\{\hat{G}(f)\right\} - G(f). \tag{4}$$

We approximate this bias by expanding $G(f-\nu)$ in (1) according to

$$G(f-\nu) \cong G(f) - \nu G'(f) + 1/2\, G''(f)\nu^2 - 1/6\, G'''(f)\nu^3 + 1/24\, G''''(f)\nu^4, \tag{5}$$

where the prime denotes a derivative. Substitution of (5) and (1) into (4) yields

$$B(f) \cong 1/2\, G''(f)\int d\nu\, \nu^2 |W(\nu)|^2 + 1/24\, G''''(f)\int d\nu\, \nu^4 |W(\nu)|^2, \tag{6}$$

where we have assumed (without loss of generality) that

$$\int d\nu\, |W(\nu)|^2 = \int dt\, w^2(t) = 1, \tag{7}$$

and that $|W(\nu)|^2$ is even about the origin; that is, w(t) is a unit-energy real waveform.

We express (6) as

$$B(f) \cong 1/2\, G''(f)\, D_1 + 1/24\, G''''(f)\, D_2 \equiv B_1(f) + B_2(f), \tag{8}$$

where window constants

$$D_1 = \int d\nu\, \nu^2\, |W(\nu)|^2,$$
$$D_2 = \int d\nu\, \nu^4\, |W(\nu)|^2, \tag{9}$$

are independent of the true spectrum G(f). To minimize the bias, we must therefore minimize D_1 and/or D_2, subject to (3) and (7). To this aim, it is useful to express (9) in terms of the time domain. There follows, by use of (2),

$$D_1 = (2\pi)^{-2} \int dt \, [w'(t)]^2 \tag{10}$$

and

$$D_2 = (2\pi)^{-4} \int dt \, [w''(t)]^2 . \tag{11}$$

Strictly, the approximation (8) to the bias is due only to local variations in true spectrum G(f) about the frequency point f of interest; equation (5) is not necessarily a good approximation for larger ν. Thus, peaks in the true spectrum that are distant from the point f under investigation are not accounted for by (5). To minimize the effects of remote spectral peaks on bias, we must also require that the spectral window W(f) decay sufficiently rapidly for large $|f|$. Thus the results of the following optimizations are not final, but must be investigated to see if they also meet the requirement of sufficiently rapid decay with frequency.

In addition to constraints (3) and (7), we shall be interested in constraining the bandwidth of the window; this is in keeping with the philosophy of requesting a specified frequency resolution for spectral estimation, and letting the window duration L and overlap be whatever is necessary to meet this requirement [6].

It should also be noted that constraints on bandwidth tend to equalize the variance-reduction capabilities of the windows. This may be seen from [6, eq. (22)], where the equivalent number of degrees of freedom is given approximately by

$$2TB_{st} \quad \text{for} \quad T \gg L , \tag{12}$$

where B_{st} is the statistical bandwidth [9, p. 278] of the window. Thus, if all windows were constrained to have the same statistical bandwidth, they would all have the same variance-reduction capabilities, and we could minimize the bias subject to this constraint. However, this constraint is not mathematically tractable.* Therefore, we resort to constraints on other, more tractable, bandwidth measures, with confidence that they too will yield comparable variance in spectral estimation (see [6, table 1]).

---

*We have not been able to express the time-domain constraint (3) directly in the frequency domain, nor have we been able to express the requirement that $\phi_w(\tau)$ be a legal correlation function directly in the time domain; see [6, eqs. (7) and (17)-(21)].

PROBLEM SOLUTION

Four different constrained problems will be addressed in this section: constrained window duration L ; constrained equivalent-noise bandwidth; constrained half-power bandwidth; and constrained root-mean-square (rms) bandwidth. The window duration L is adjusted to meet the bandwidth constraint in the latter three cases.

DURATION CONSTRAINT

Here we wish to minimize D_1 in (10), subject to constraints (3) and (7) and a fixed value of window duration L. In order that (10) be finite, w(t) must be continuous; therefore $w(\pm L/2) = 0$ from (3). When we use a calculus-of-variations approach, the optimum window $w_o(t)$ must satisfy the differential equation

$$w_o''(t) + \lambda\, w_o(t) = 0, \quad |t| < L/2, \tag{13}$$

where λ is a constant (Lagrange multiplier). The solution of (13) that satisfies the boundary conditions and (7), and has minimum D_1, is

$$w_o(t) = \left(\frac{2}{L}\right)^{1/2} \cos(\pi t/L), \quad |t| \le L/2. \tag{14}$$

The corresponding value of (10) is

$$D_1 = \frac{1}{4L^2} \; . \tag{15}$$

Several windows are compared in table 1. It is seen that the Hanning window has 33 percent greater bias, as measured by $B_1(f)$, than the optimum window, under a duration constraint.

Table 1. Window Bias Constants D_1

Data Window	$4L^2D_1$
Optimum, (14)	1
Parabola	$10/\pi^2 = 1.01$
Triangle	$12/\pi^2 = 1.22$
Hanning	$4/3 = 1.33$

The spectral window corresponding to the optimum data window, (14), is

$$W_o(f) = \frac{2}{\pi} (2L)^{1/2} \frac{\cos(\pi Lf)}{1-4L^2f^2} . \tag{16}$$

The decay for large frequencies is only as f^{-2}. The sidelobes of this and the following windows will be discussed later.

EQUIVALENT-NOISE-BANDWIDTH CONSTRAINT

The equivalent-noise bandwidth B_e of spectral window $W(f)$ is defined as

$$B_e = \frac{\int df\, |W(f)|^2}{|W(0)|^2} = \frac{1}{\left[\int dt\; w(t)\right]^2} , \tag{17}$$

where we have used (7) and (2). The quantity B_e can be interpreted physically as the bandwidth of an ideal rectangular filter that would pass the same amount of power as a filter $W(f)$, when subjected to white noise; see figure 1. The peak of $|W(f)|^2$ occurs at the origin, since data window $w(t)$ is nonnegative.

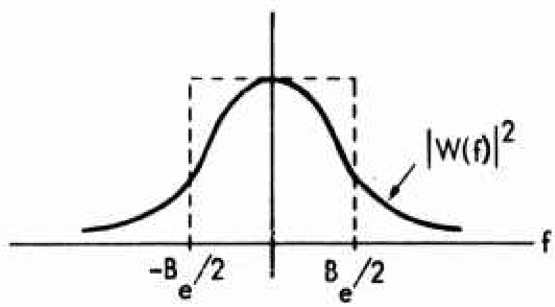

Figure 1. Equivalent-Noise-Bandwidth Interpretation

The problem here is to minimize D_1 in (10), subject to constraints (3), (7), and (17). This problem is solved in appendix A, with the result that the optimum data window is

$$w_o(t) = \left(\frac{8}{3L}\right)^{1/2} \cos^2(\pi t/L), \quad |t| \le L/2 . \tag{18}$$

This is the familiar Hanning window. The window duration L must be chosen as

$$L = \frac{3}{2}\,\frac{1}{B_e}\ , \tag{19}$$

according to (17). The minimum value of D_1 is

$$D_1 = \frac{1}{3L^2} = \frac{4}{27}\,B_e^2\ . \tag{20}$$

The specified equivalent-noise bandwidth dictates the window duration L and the minimum attainable bias constant D_1. The spectral window corresponding to (18) is

$$W_o(f) = \left(\frac{2L}{3}\right)^{1/2} \frac{\sin(\pi L f)}{\pi L f(1-L^2 f^2)}\ , \tag{21}$$

where L must be determined from (19). The decay for large frequencies varies as f^{-3}.

HALF-POWER-BANDWIDTH CONSTRAINT

The half-power bandwidth B_h of spectral window W(f) is defined as

$$\left|\frac{W(\pm B_h/2)}{W(0)}\right|^2 = \frac{1}{2}\ . \tag{22}$$

We desire to minimize D_1 in (10), subject to constraints (3), (7), and (22). Converting (22) into the time domain and restricting w(t) to be even,* constraint (22) takes a desirable integral form:

$$\int dt\ w(t)\left[\cos(\pi B_h t) - 2^{-1/2}\right] = 0\ . \tag{23}$$

The solution to this minimization problem is presented in appendix B. The optimum data window is

$$w_o(t) = L^{-1/2}\,[1.682 + 4.261\cos(4.434t/L) - 4.337\cos(3.552t/L)],\quad |t| \le L/2\ . \tag{24}$$

---

*An odd component in w(t) increases the rate of variation and therefore increases D_1.

The window duration L must be chosen as

$$L = \frac{1.411}{B_h} , \quad (25)$$

according to (23). The minimum value of D_1 is

$$D_1 = 0.1604 \, B_h^2 . \quad (26)$$

The spectral window corresponding to (24) is obtained by employing (2); this will be discussed in the next section. It is shown in appendix B that $w_o'(\pm L/2) = 0$; therefore the decay of the spectral window is according to f^{-3}.

For comparison, if the time duration of the Hanning window is adjusted to realize the specified half-power bandwidth, namely $L = 1.441/B_h$ [6, eq. (33) and table 1], it follows that $D_1 = 0.1606 \, B_h^2$. Thus the Hanning window has virtually the same bias as the optimum window under a half-power-bandwidth constraint. Further comparisons are made in the next section.

ROOT-MEAN-SQUARE-BANDWIDTH CONSTRAINT

The rms bandwidth B_r of spectral window W(f) is defined as

$$B_r^2 = \frac{\int df \, f^2 \, |W(f)|^2}{\int df \, |W(f)|^2} . \quad (27)$$

Inspection of (7) and (9) immediately reveals that

$$D_1 = B_r^2 . \quad (28)$$

Thus if the rms bandwidth is constrained, bias constant D_1 is fixed. In this case, it is reasonable to resort to minimization of the second bias constant D_2 in (9) or (11). Thus, we wish to minimize (11), subject to constraints (3), (7), and

$$\int dt \, [w'(t)]^2 = (2\pi B_r)^2 . \quad (29)$$

The solution to this problem is presented in appendix C. The optimum data window is

$$w_o(t) = \frac{4}{(5L)^{1/2}} \cos^3(\pi t/L), \quad |t| \le L/2 . \quad (30)$$

The window duration L must be chosen as

$$L = \frac{3}{2\sqrt{5}} \frac{1}{B_r} , \tag{31}$$

according to (29). The minimum value of D_2 is

$$D_2 = \frac{25}{9} B_r^4 . \tag{32}$$

The spectral window corresponding to (30) is

$$W_o(f) = \frac{16}{15\pi} (5L)^{1/2} \frac{\cos(\pi L f)}{(1-4L^2f^2)(1-4L^2f^2/9)} , \tag{33}$$

where L is determined from (31). The decay for large frequencies is according to f^{-4}.

For comparison, let the time duration of the Hanning window be adjusted to realize the specified rms bandwidth B_r. Then employing (18) in (29), we find $L = 1/(\sqrt{3}\, B_r)$, and (11) yields $D_2 = 3\, B_r^4$. Thus the Hanning window has 8 percent more bias than the optimum window under an rms bandwidth constraint, as measured by bias constant D_2.

COMPARISON OF WINDOW CHARACTERISTICS

In figure 2, one-half of the symmetric optimum data windows for the three bandwidth-constrained cases are drawn for a common time duration of $L = 1$. The equivalent-noise-bandwidth data window (Hanning) and the half-power-bandwidth data window are virtually identical and are continuous in value and derivative at 0.5. The rms-bandwidth data window is more peaked, and goes to zero in value, in derivative, and in second derivative at 0.5. Thus the last window would require greater overlap than the first two, in order to realize the same variance reduction; see [6].

In order to deduce the required overlap for the rms bandwidth data window, the quadratic and cubic data windows [6, pp. 10-18] are superposed in figure 3. Over most of the range, the quadratic and rms-bandwidth windows are very close. Near the end of the range, however, the taper of the rms-bandwidth window approaches that of the cubic; in fact, both are continuous in second derivative at 0.5. Thus, it is anticipated from earlier results [6, table 4] that slightly over 65 percent overlap would be required for the rms bandwidth data window to realize 99 percent of its maximum equivalent degrees of freedom [4, p. 22].

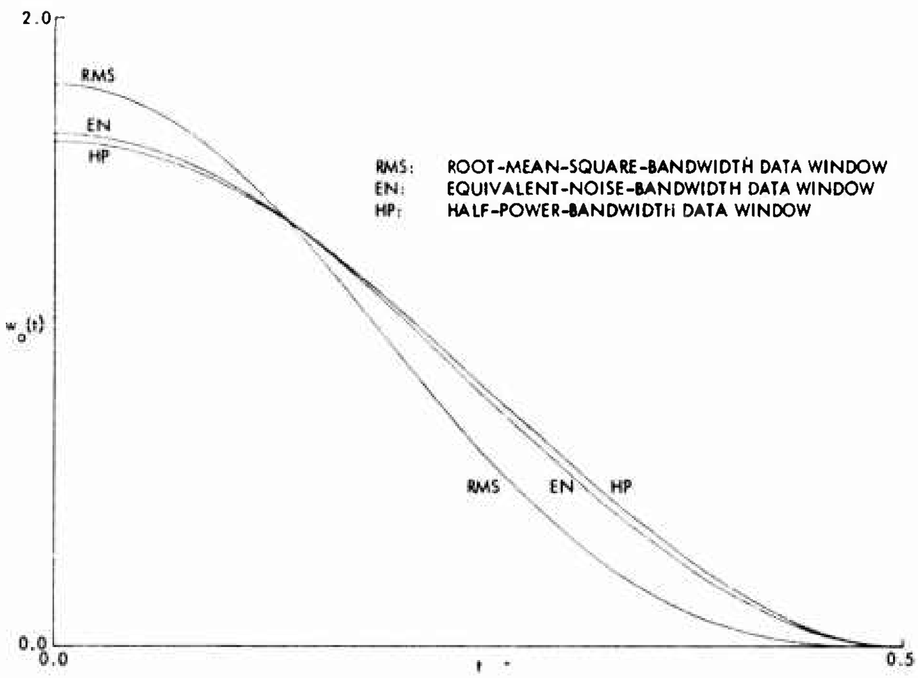

Figure 2. Three Bandwidth-Constrained Data Windows; L = 1, Unit Energy

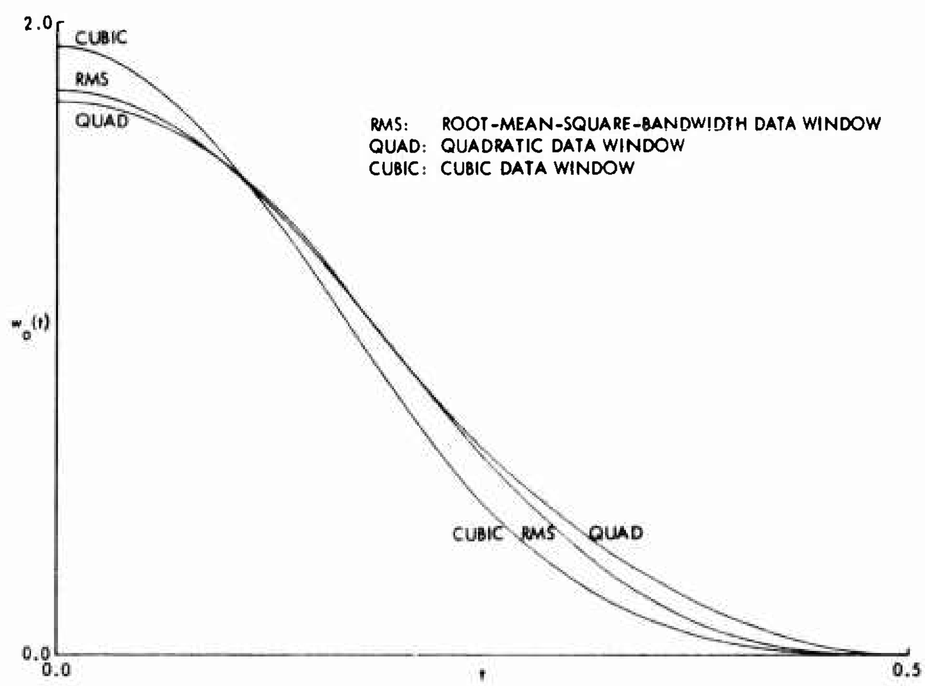

Figure 3. Comparison of RMS-Bandwidth Data Window with Quadratic and Cubic Data Windows; L = 1, Unit Energy

The spectral characteristics of the four windows derived in this report are presented in figures 4 through 7. The abscissas on every plot are in units of the half-power bandwidth; thus all the curves go through half-power (-3.01 dB) at $f/B_h = 0.5$. The duration-limited window, (14), is plotted in figure 4 and exhibits relatively slow decay with frequency, since the data window is discontinuous in derivative at its edge. The equivalent-noise-bandwidth (Hanning) and half-power-bandwidth spectral windows, plotted in figures 5 and 6, are virtually identical and have good decay with frequency, since the data windows are continuous in derivative at their edges. The spectral window for the rms-bandwidth data window is plotted in figure 7 and exhibits very rapid decay with frequency. However, as noted above, by virtue of requiring greater overlap for maximum variance reduction, this window will require somewhat greater-size FFTs than do the other windows.

CONCLUSIONS

The Hanning window is optimum under an equivalent-noise-bandwidth constraint, as far as minimization of bias constant D_1 is concerned. Furthermore, it is near the optima for two other bandwidth constraints. Its spectral decay is also sufficient for most cases that the bias is relatively unaffected by distant spectral peaks. And with 50 percent overlap, it realizes 92 percent of the maximum number of equivalent degrees of freedom [6, table 6]. Thus, the Hanning window is a reasonable compromise to utilize in spectral estimation of random stationary data.

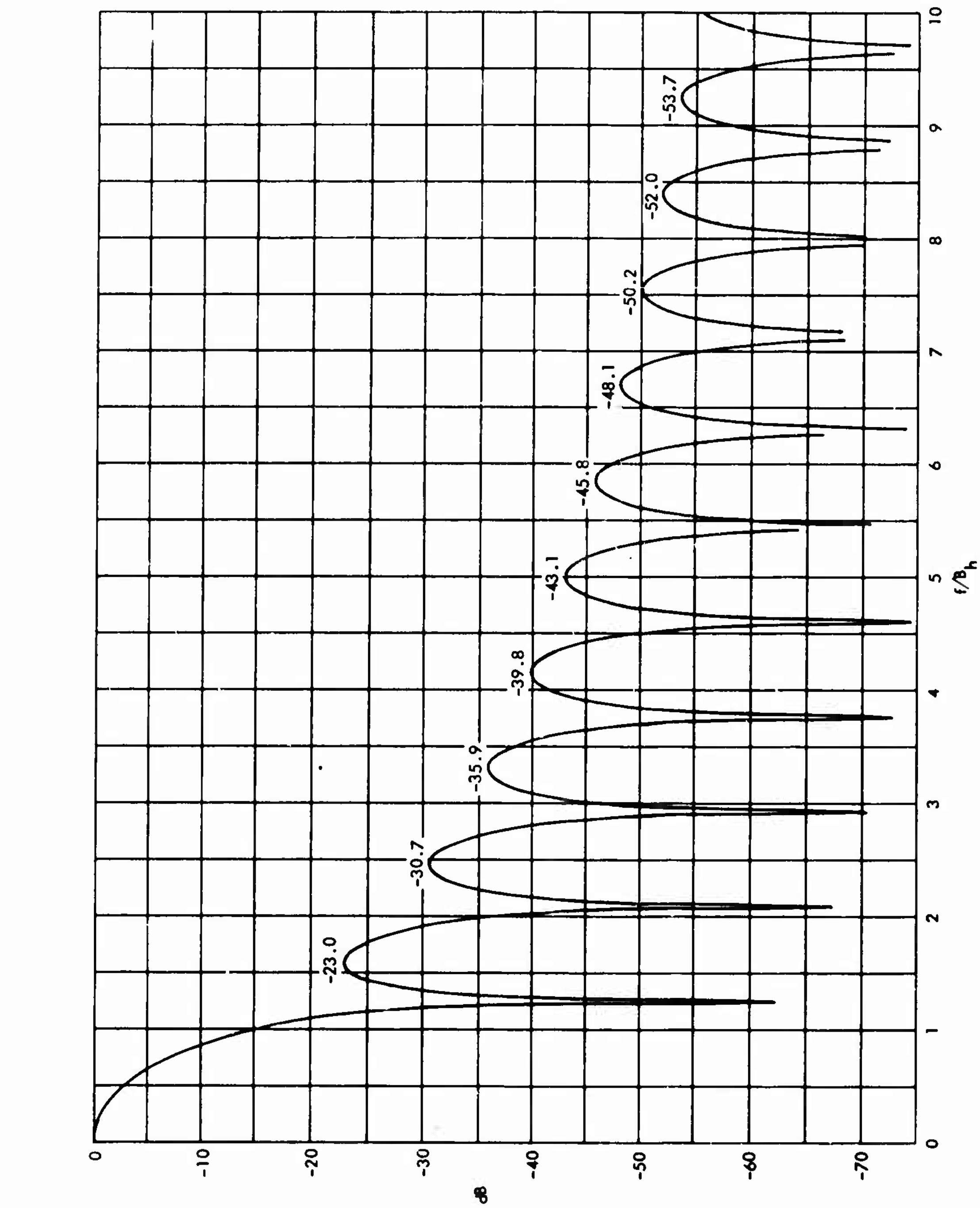

Figure 4. Spectral Window for Duration-Limited Data Window

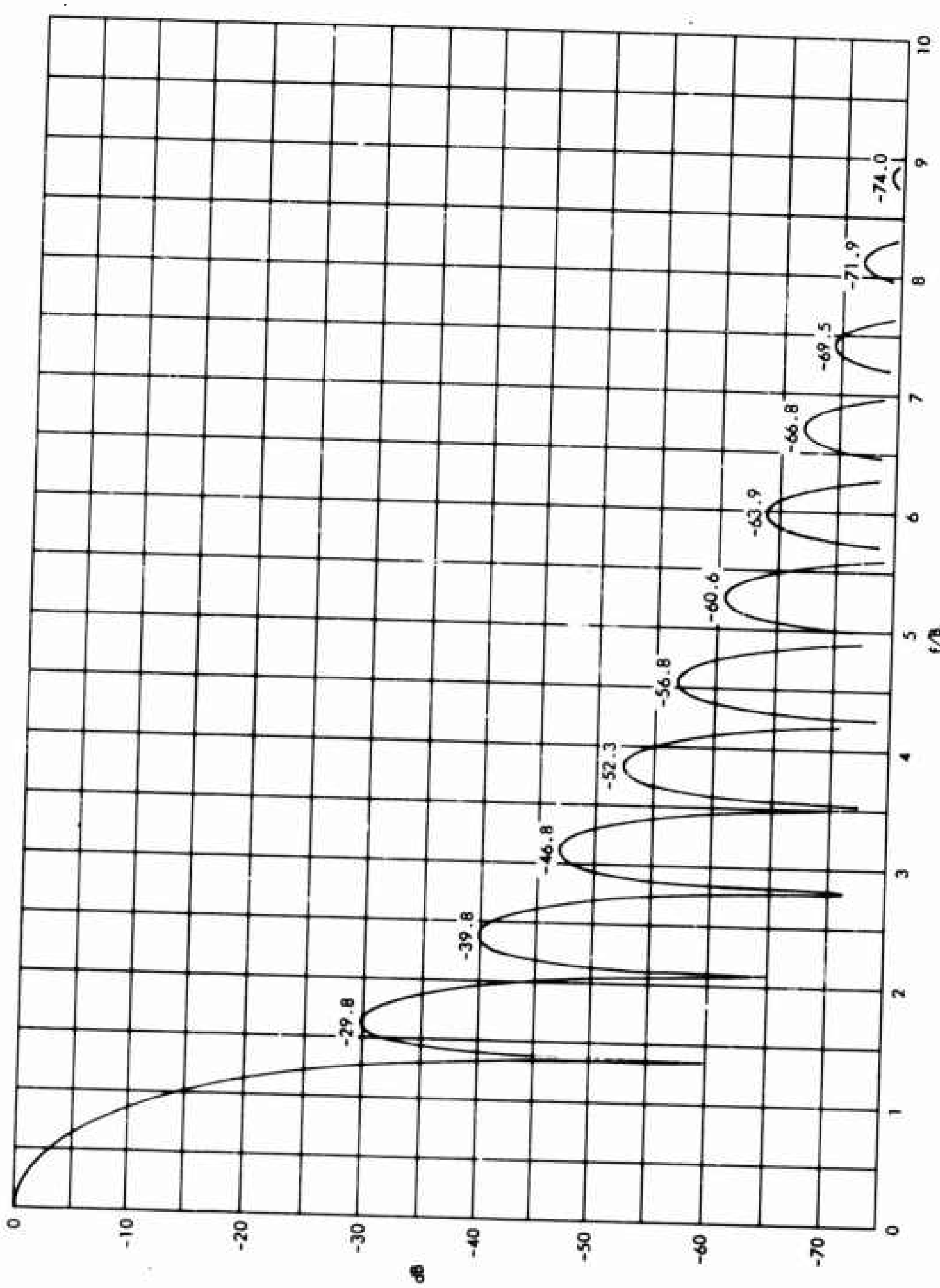

Figure 5. Spectral Window for Half-Power-Bandwidth Data Window

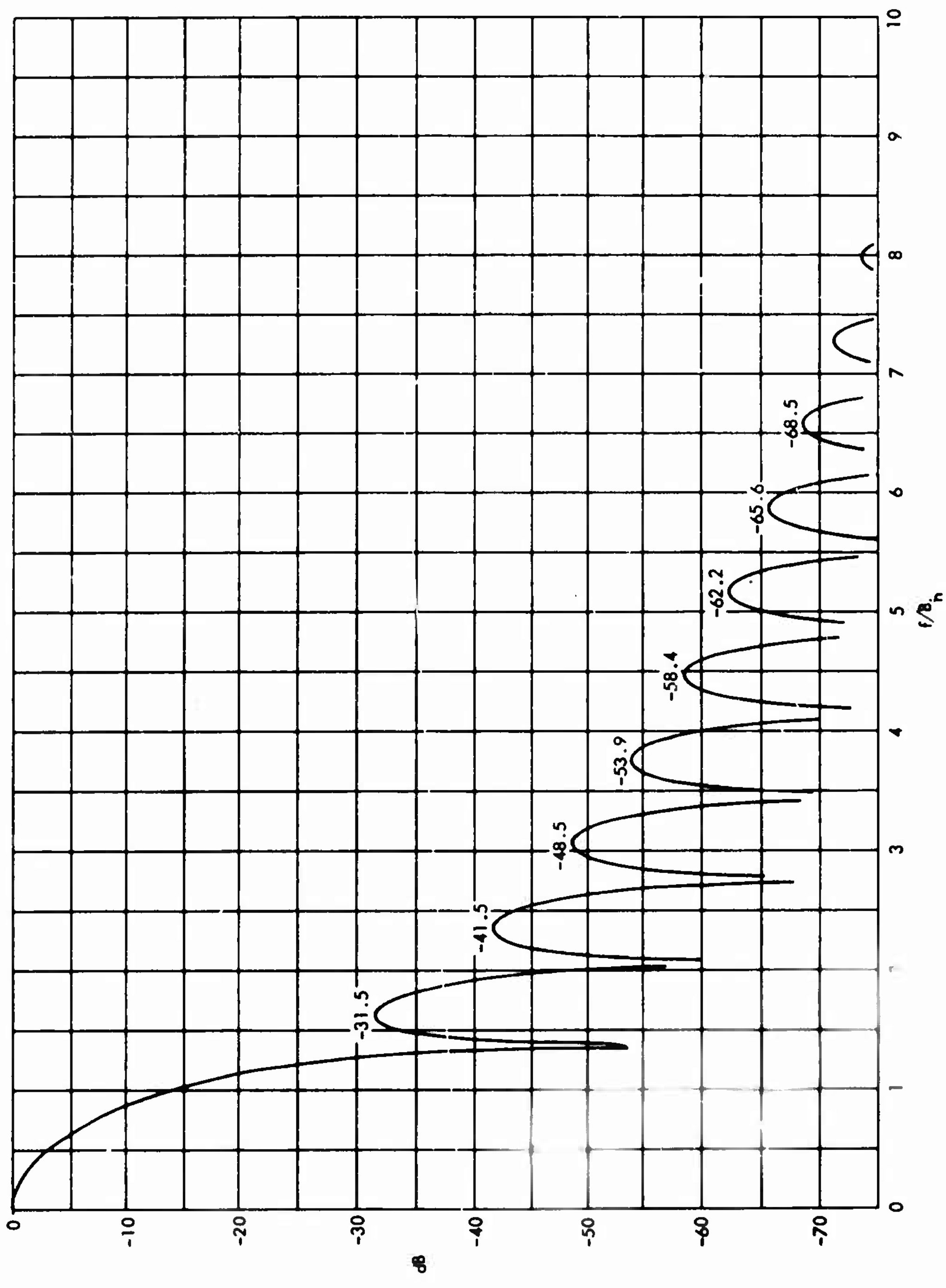

Figure 6. Spectral Window for Equivalent-Noise-Bandwidth Data Window (Hanning)

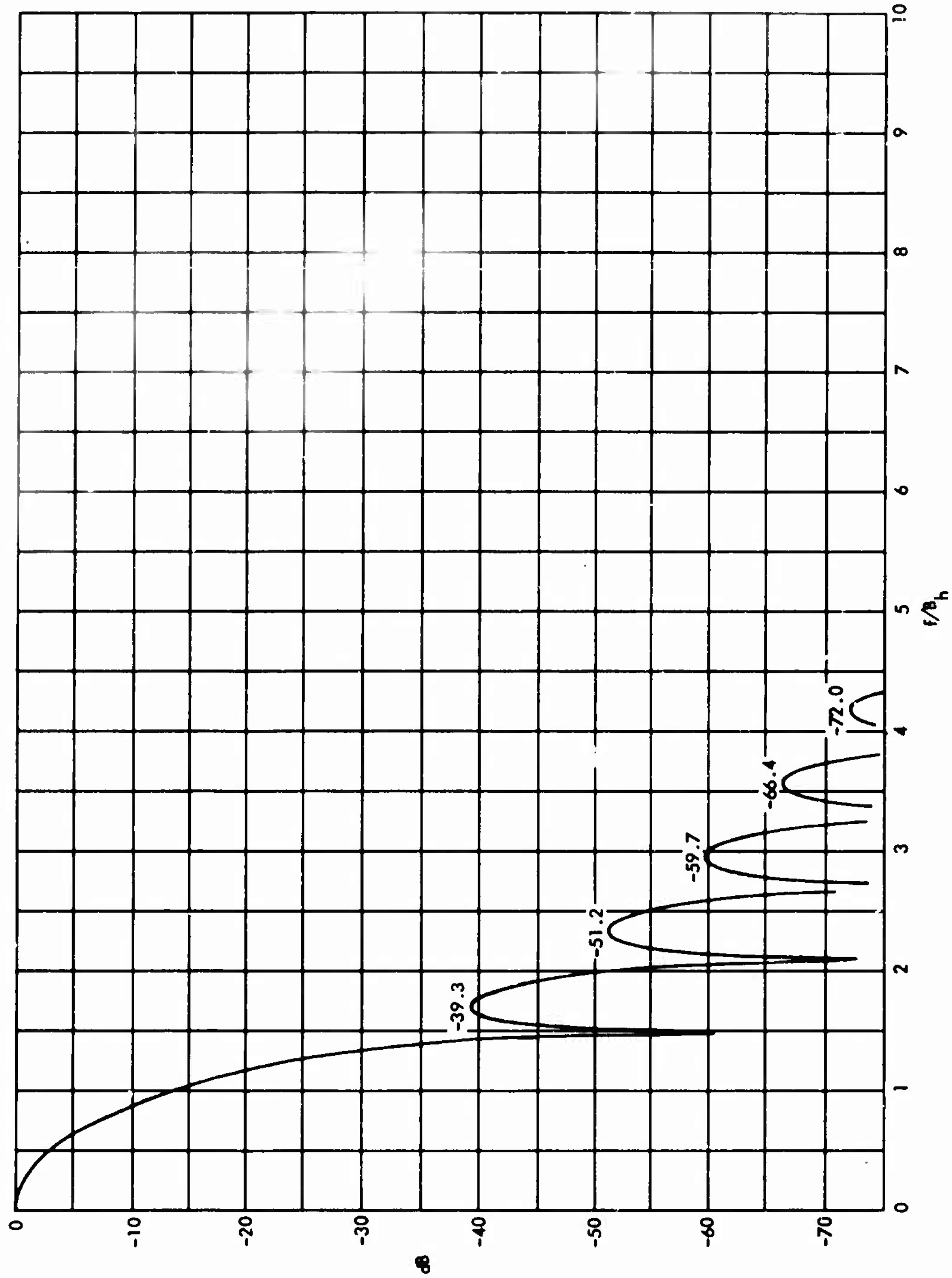

Figure 7. Spectral Window for RMS-Bandwidth Data Window

Appendix A

DERIVATION OF THE OPTIMUM DATA WINDOW FOR AN EQUIVALENT-NOISE-BANDWIDTH CONSTRAINT

We wish to minimize (10) (from the main text), subject to constraints (3), (7), and (17). Equations (7) and (17) are integral constraints and are in a convenient form for a calculus-of-variations approach. The way we handle (3) is to first ignore the nonnegative limitation; then, out of the class of allowable solutions, we restrict attention only to the nonnegative solutions and pick the best.

In order that (10) be finite, w(t) must be continuous. Using (3), we see that this means that

$$w(\pm L/2) = 0. \tag{A-1}$$

A calculus-of-variations approach tells us to minimize the quantity

$$Q = \int dt\,[w'(t)]^2 - \lambda_1 \int dt\; w^2(t) + 2\lambda_2 \int dt\; w(t), \tag{A-2}$$

where λ_1 and λ_2 are Lagrange multipliers; the resulting differential equation for the optimum window is

$$w_0''(t) + \lambda_1 w_0(t) = \lambda_2\,, \quad |t| < L/2. \tag{A-3}$$

We employed (A-1) on the allowed variations in deriving (A-3).

The general solution of (A-3) is

$$w_0(t) = \left\{ \begin{array}{c} A\cos(at) + B\sin(at) + C \\ \text{OR} \\ A\cosh(at) + B\sinh(at) + C \\ \text{OR} \\ A + Bt \end{array} \right\}, \quad |t| < L/2, \tag{A-4}$$

where a is real and positive. The third alternative in (A-4) yields the trivial solution when (A-1) is imposed.

The second alternative reduces to

$$w_0(t) = A\left[\cosh(at) - \cosh(aL/2)\right], \quad |t| \leq L/2. \tag{A-5}$$

Since A can be chosen to satisfy the energy constraint (7), and L can be chosen to satisfy the bandwidth constraint (17), there is left only the variable a in (A-5) to vary; $w_0(t)$ is certainly nonnegative by the choice of proper polarity for A. In order to find the best value of a for this second alternative of (A-4), we compute D_1 in (10) versus a and pick the minimum.

To accomplish this goal, we define

$$y_0(u) \equiv w_0\left(\frac{L}{2}u\right) \equiv A\,h(u), \quad |u| \leq 1, \tag{A-6}$$

where

$$h(u) = \cosh(\alpha u) - \cosh(\alpha), \quad |u| \leq 1 \tag{A-7}$$

and

$$\alpha = aL/2. \tag{A-8}$$

The bandwidth constraint (17) then becomes

$$A\frac{L}{2}K_0 = B_e^{-1/2}, \tag{A-9}$$

where

$$K_0 \equiv \int_{-1}^{1} du\, h(u). \tag{A-10}$$

The energy constraint (7) becomes

$$A^2 \frac{L}{2} K_2 = 1, \tag{A-11}$$

where

$$K_2 \equiv \int_{-1}^{1} du\, h^2(u). \tag{A-12}$$

The window constant (10) becomes

$$D_1 = \frac{1}{2\pi^2}\frac{A^2}{L}K_1, \tag{A-13}$$

where

$$K_1 \equiv \int_{-1}^{1} du \left[h'(u) \right]^2 . \tag{A-14}$$

If we eliminate A and L by using (A-9) and (A-11), then (A-13) becomes

$$D_1 = \frac{1}{4\pi^2} \frac{K_0^4 K_1}{K_2^3} B_e^2 . \tag{A-15}$$

The quantities necessary in (A-15) follow from (A-7), (A-10), (A-12), and (A-14):

$$\begin{aligned} K_0 &= 2\left[\cosh(\alpha) - \frac{\sinh(\alpha)}{\alpha}\right], \\ K_1 &= \alpha^2 \left[\frac{\sinh(2\alpha)}{2\alpha} - 1\right], \\ K_2 &= 2\left[1 + \frac{1}{2}\cosh(2\alpha) - \frac{3}{2}\frac{\sinh(2\alpha)}{2\alpha}\right]. \end{aligned} \tag{A-16}$$

When (A-16) is substituted in (A-15), D_1 is found to increase monotonically with increasing α ; the limit as $\alpha \to 0+$ is

$$\frac{125}{72\pi^2} B_e^2 = .1759\, B_e^2 . \tag{A-17}$$

The first alternative in (A-4), when subjected to boundary condition (A-1), breaks into two subcases. In the first, if $\sin(\alpha) = 0$, then B is arbitrary. Then $\alpha = k\pi,\ k \geq 1$. The function with smallest D_1 corresponds to $k = 1$, and yields

$$w_0(t) = A\left[\cos(2\pi t/L) + 1\right] + B \sin(2\pi t/L), \ |t| \leq L/2 . \tag{A-18}$$

However, we must have $B = 0$; otherwise $w_0(t)$ would go negative somewhere. Imposition of the energy constraint (7) and bandwidth constraint (17) yields

$$w_0(t) = \left(\frac{8}{3L}\right)^{1/2} \cos^2(\pi t/L), \ |t| \leq L/2, \tag{A-19}$$

where

$$L = \frac{3}{2} \frac{1}{B_e} . \tag{A-20}$$

The corresponding value of D_1 is

$$D_1 = \frac{1}{3L^2} = \frac{4}{27} B_e^2 , \qquad \text{(A-21)}$$

which is smaller than (A-17).

The second subcase occurs if $\sin(\alpha) \neq 0$. Then B in (A-4) must be zero, and we get

$$w_0(t) = A\left[\cos(at) - \cos(\alpha)\right], \; |t| \le L/2, \qquad \text{(A-22)}$$

where $\alpha \neq K\pi$. The comments and method immediately below (A-5) apply equally well here. Therefore we define

$$k(u) = \cos(\alpha u) - \cos(\alpha), \; |u| \le 1, \; \alpha \neq K\pi, \qquad \text{(A-23)}$$

and find

$$K_0 = 2\left[\frac{\sin(\alpha)}{\alpha} - \cos(\alpha)\right],$$

$$K_1 = \alpha^2\left[1 - \frac{\sin(2\alpha)}{2\alpha}\right], \qquad \text{(A-24)}$$

$$K_2 = 2\left[1 + \frac{1}{2}\cos(2\alpha) - \frac{3}{2}\,\frac{\sin(2\alpha)}{2\alpha}\right].$$

When (A-24) is substituted in (A-15), D_1 is found to decrease monotonically with increasing α, at least for α up to π. However, when $\alpha > \pi$, $k(u)$ becomes negative somewhere; this may be seen by noting that $k'(1) = -\alpha \sin(\alpha)$ is positive if $\alpha > \pi$. Thus the limiting member of this subclass, which is (A-19), is the optimum window.

Appendix B

DERIVATION OF THE OPTIMUM DATA WINDOW FOR A HALF-POWER-BANDWIDTH CONSTRAINT

Here we will minimize (10), subject to constraints (3), (7), and (23). In order that (10) be finite, (A-1) must again be true. A calculus-of-variations approach tells us to minimize

$$Q = \int dt\,[w'(t)]^2 - \lambda_1 \int dt\, w^2(t) + 2\lambda_2 \int dt\, w(t)\, c(t), \tag{B-1}$$

where

$$c(t) \equiv \cos(\omega t) - G, \quad \omega \equiv \pi B_h, \quad G = 2^{-1/2}, \tag{B-2}$$

and λ_1 and λ_2 are Lagrange multipliers; the resulting differential equation for the optimum window is

$$w_o''(t) + \lambda_1 w_o(t) = \lambda_2 c(t), \quad |t| < L/2. \tag{B-3}$$

If λ_1 is negative, the form for $w_o(t)$ includes sinh and cosh terms, which lead to a progressively larger value of D_1 as λ_1 becomes more negative, similar to the result of appendix A. If λ_1 is zero, the only solution to (B-3) and (A-1) is the trivial solution. If λ_1 is positive, the general solution to (B-3) is

$$w_o(t) = \begin{cases} A\cos(\lambda t) + B\sin(\lambda t) + \dfrac{\lambda_2}{\lambda^2 - \omega^2}\cos(\omega t) - \dfrac{\lambda_2}{\lambda^2} G, & \lambda \ne \omega \\ \text{OR} & \\ A\cos(\omega t) + B\sin(\omega t), & \lambda = \omega \end{cases}, \quad |t| < \frac{L}{2}. \tag{B-4}$$

We discard the odd solutions for the reason given in the footnote to (23).

If we attempt to use the second alternative in (B-4), we can eliminate A by means of energy constraint (7). However, the bandwidth constraint (23) can not be satisfied for any value of $\omega L \neq 0$. Therefore, we must discard the second alternative.

To handle the fir alternative in (B-4) conveniently, we define a function

$$y_o(u) \equiv w_o\left(\tfrac{L}{2}u\right) = A\cos(q\alpha u) + \frac{q^2 \lambda_2/\lambda^2}{q^2 - 1}\cos(\alpha u) - \frac{\lambda_2}{\lambda^2} G, \quad |u| < 1, \tag{B-5}$$

where

$$q = \frac{\lambda}{\omega} \neq 1, \quad \alpha \equiv \frac{\omega L}{2} = \frac{\pi}{2} B_h L. \tag{B-6}$$

Imposition of (A-1) forces (B-5) into the form

$$y_0(u) = C\left[\frac{\cos(q\alpha u)}{\cos(q\alpha)} - \frac{q^2\cos(\alpha u) + G(1-q^2)}{q^2\cos(\alpha) + G(1-q^2)}\right] \equiv C\,k(u), \; |u| \le 1, \tag{B-7}$$

where C is a constant. The energy constraint (7) yields

$$C = \left(\frac{2}{L K_0}\right)^{1/2}, \tag{B-8}$$

where we have used (B-5) and (B-7), and defined

$$K_0 = \int du\, k^2(u). \tag{B-9}$$

Satisfaction of bandwidth constraint (23) demands that

$$0 = \int du\left[\cos(\alpha u) - G\right] k(u), \tag{B-10}$$

where we employed (B-6) and (B-7). Substitution of the detailed form for $k(u)$, (B-7), into (B-10) yields the relation

$$\begin{aligned} &\cos(\alpha q)\left[2G S(\alpha) - 2G^2 + q^2\left\{1 + 2G^2 + S(2\alpha) - 4G S(\alpha)\right\}\right] \\ &= \left[q^2\cos(\alpha) + G(1-q^2)\right]\left[S(\alpha q + \alpha) + S(\alpha q - \alpha) - 2G S(\alpha q)\right], \end{aligned} \tag{B-11}$$

where

$$S(x) \equiv \frac{\sin(x)}{x}. \tag{B-12}$$

For a given value of q, (B-11) must be solved for the smallest value of α; G is a known specified constant. In order to find the best value of q, we compute D_1 versus q and pick the minimum, always being careful that $k(u)$ remain nonnegative for all $|u| \le 1$. The quantity

$$\begin{aligned} D_1 &= \frac{1}{4\pi^2}\int dt\left[w_0'(t)\right]^2 = \frac{1}{4\pi^2}\,\frac{2}{L}\int du\left[y_0'(u)\right]^2 \\ &= \frac{1}{\pi^2 L^2}\,\frac{K_1}{K_0} = \frac{K_1}{4\alpha^2 K_0} B_h^2, \end{aligned} \tag{B-13}$$

using (B-5), (B-7), (B-8), (B-6), and defining

$$K_1 = \int du \left[\lambda'(u) \right]^2 . \qquad \text{(B-14)}$$

The quantities K_0 and K_1 are available upon substitution of (B-7) in (B-9) and (B-14). There follows, upon use of (23),

$$K_0 = C_1^2 \left[1 + S(2\alpha q) \right] + 2C_2^2 G \left[q^2 S(\alpha) + G(1-q^2) \right]$$
$$+ C_1 C_2 \left[2G(2-q^2) S(\alpha q) + q^2 S(\alpha q + \alpha) + q^2 S(\alpha q - \alpha) \right], \qquad \text{(B-15)}$$

$$K_1 = \alpha^2 q^2 \left[C_1^2 \{ 1 - S(2\alpha q) \} + q^2 C_2^2 \{ 1 - S(2\alpha) \} \right.$$
$$\left. + 2q C_1 C_2 \{ S(\alpha q - \alpha) - S(\alpha q + \alpha) \} \right], \qquad \text{(B-16)}$$

where

$$C_1 = \left[\cos(\alpha q) \right]^{-1}, \quad C_2 = -\left[q^2 \cos(\alpha) + G(1-q^2) \right]^{-1} . \qquad \text{(B-17)}$$

The numerical approach may now be summarized as follows. A value of q is picked, and (B-11) is solved for α. Then (B-17) is computed and substituted in (B-15) and (B-16), thereby enabling evaluation of D_1 in (B-13) for that choice of q. (Up to this point, G could be any desired constant; we now restrict $G = 1/\sqrt{12}$). When this approach is tried, it is found that D_1 increases monotonically with increasing q. On the other hand, when q is made too small, $\lambda(u)$ becomes negative near $u=1$. The optimum value of q is realized when $\lambda'(1) = 0$. From (B-7) and (B-17), this requirement is

$$C_1 \sin(\alpha q) + C_2 q \sin(\alpha) = 0. \qquad \text{(B-18)}$$

The simultaneous solution of (B-11) and (B-18), with smallest α, is then given by

$$q = .8011\,1996, \quad \alpha = 2.217\,0595. \qquad \text{(B-19)}$$

The optimum value of D_1 then follows from (B-13) as

$$D_1 = .1604\,4830\, B_h^2 , \qquad \text{(B-20)}$$

and the segment length follows from (B-6) as

$$L = \frac{2\alpha}{\pi} \frac{1}{B_h} = \frac{1.411\,4239}{B_h} . \qquad \text{(B-21)}$$

Finally, the optimum window $w_0(t)$ follows from (B-5) as

$$w_0(t) = L^{-1/2} \left[1.681\,6731 + 4.261\,0617 \cos(4.434\,1191\, t/L) \right.$$
$$\left. - 4.337\,3899 \cos(3.552\,2613\, t/L) \right], \quad |t| \le L/2. \qquad \text{(B-22)}$$

Appendix C

DERIVATION OF THE OPTIMUM DATA WINDOW FOR AN RMS-BANDWIDTH CONSTRAINT

The problem here is to minimize (11), subject to constraints (3), (7), and (29). In order that (11) be finite, $w'(t)$ must be continuous. Using (3), this means that

$$w(\pm L/2) = 0, \quad w'(\pm L/2) = 0. \tag{C-1}$$

A calculus-of-variations approach tells us to minimize the quantity

$$Q = \int dt\,[w''(t)]^2 + \lambda \int dt\,[w'(t)]^2 + \mu \int dt\, w^2(t), \tag{C-2}$$

where λ and μ are Lagrange multipliers; the resulting differential equation for the optimum window is

$$w_0''''(t) - \lambda w_0''(t) + \mu w_0(t) = 0, \quad |t| < L/2. \tag{C-3}$$

We employed (C-1) on the allowed variations in deriving (C-3).

To solve (C-3), we assume a form $\exp(st)$ for $w_0(t)$. Substitution in (C-3) requires that s be chosen to satisfy

$$s^4 - \lambda s^2 + \mu = 0\,; \quad s^2 = \tfrac{1}{2}\left[\lambda \pm (\lambda^2 - 4\mu)^{1/2}\right]. \tag{C-4}$$

At this point, several alternatives are possible. The first case we pursue is a negative discriminant in (C-4). Then the four values for s in (C-4) can be expressed as

$$s = z, z^*, -z, -z^*, \tag{C-5}$$

where z is a complex constant with nonzero imaginary part. For distinct roots (i.e., z not purely real), the optimum window is

$$w_0(t) = A\exp(zt) + B\exp(z^*t) + C\exp(-zt) + D\exp(-z^*t). \tag{C-6}$$

In order that (C-1) be satisfied with a nontrivial solution, it is necessary that the determinant

$$\begin{vmatrix} E & E^* & 1/E & 1/E^* \\ 1/E & 1/E^* & E & E^* \\ zE & z^*E^* & -z/E & -z^*/E^* \\ z/E & z^*/E^* & -zE & -z^*E^* \end{vmatrix} \tag{C-7}$$

equal zero, where $E \equiv \exp(zL/2)$. This requires that

$$(z+z^*)(z-z^*)(E^4-1)(E^{*4}-1) = 0. \qquad \text{(C-8)}$$

But none of the factors in (C-8) can be zero without double roots resulting for s, in which case the form (C-6) is not appropriate. Therefore, the negative-discriminant case is self-destroying.

The second case corresponds to a zero discriminant in (C-4). Then we have three subcases:

$$s = \begin{Bmatrix} a,\ a,\ -a,\ -a \\ \text{OR} \\ ia,\ ia,\ -ia,\ -ia \\ \text{OR} \\ 0,\ 0,\ 0,\ 0 \end{Bmatrix}, \qquad a \text{ real and positive.} \qquad \text{(C-9)}$$

For subcase 1, the form for $w_o(t)$ is

$$w_o(t) = A\cosh(at) + B\sinh(at) + Ct\cosh(at) + Dt\sinh(at). \qquad \text{(C-10)}$$

Imposition of (C-1) requires that

$$\sinh(aL) = \pm aL \qquad \text{(C-11)}$$

for a nontrivial solution. But (C-11) has no solutions for positive real a .

For subcase 2 in (C-9), the form for $w_o(t)$ is

$$w_o(t) = A\cos(at) + B\sin(at) + Ct\cos(at) + Dt\sin(at). \qquad \text{(C-12)}$$

Imposition of (C-1) requires that

$$\sin(aL) = \pm aL \qquad \text{(C-13)}$$

for a nontrivial solution. Again this is disallowed.

For subcase 3 in (C-9), the form is

$$w_o(t) = A + Bt + Ct^2 + Dt^3. \qquad \text{(C-14)}$$

Imposition of (C-1) yields only the trivial solution.

The third, and last, case we must now consider is a positive discriminant in (C-4). Then we have three subcases:

$$s = \begin{Bmatrix} \pm a, \pm b \\ \pm a, \pm ib \\ \pm ia, \pm ib \end{Bmatrix}, \qquad a \text{ and } b \text{ real.} \tag{C-15}$$

For subcase 1, we have

$$w_0(t) = A\cosh(at) + B\sinh(at) + C\cosh(bt) + D\sinh(bt). \tag{C-16}$$

Imposition of (C-1) requires that

$$\alpha\tanh(\alpha) = \beta\tanh(\beta) \quad \text{or} \quad \frac{\tanh(\alpha)}{\alpha} = \frac{\tanh(\beta)}{\beta}, \tag{C-17}$$

where

$$\alpha \equiv aL/2, \quad \beta \equiv bL/2. \tag{C-18}$$

The only solutions of (C-17) are $\alpha = \beta$; that is, $a = b$. However, these have been considered already in (C-10) and found inadequate.

For subcase 2 in (C-15), we have

$$w_0(t) = A\cosh(at) + B\sinh(at) + C\cos(bt) + D\sin(bt). \tag{C-19}$$

Satisfaction of (C-1) demands that

$$-\alpha\tanh(\alpha) = \beta\tan(\beta) \quad \text{or} \quad \frac{\tanh(\alpha)}{\alpha} = \frac{\tan(\beta)}{\beta}. \tag{C-20}$$

The second alternative in (C-20) leads to odd solutions only, in (C-19), and they must be discarded because of their higher variation rate. The first alternative in (C-20) leads to

$$w_0(t) = C\left[\frac{\cosh(at)}{\cosh(\alpha)} - \frac{\cos(bt)}{\cos(\beta)}\right], \quad |t| \le \frac{L}{2}, \tag{C-21}$$

which is a legal nontrivial solution. The values of α and β are related as shown in figure C-1. To handle this alternative conveniently, we define a function

$$y_0(u) = w_0\left(\frac{L}{2}u\right) = C\left[\frac{\cosh(\alpha u)}{\cosh(\alpha)} - \frac{\cos(\beta u)}{\cos(\beta)}\right] \equiv C\,k(u), \quad |u| \le 1. \tag{C-22}$$

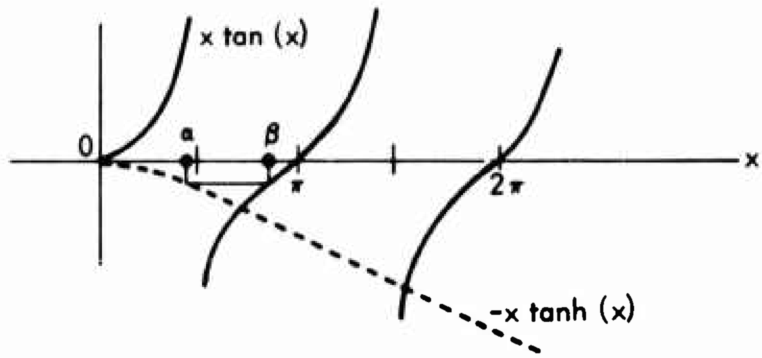

Figure C-1. Relationship of α and β in (C-20)

Then

$$y_0'(u) = \frac{L}{2} w_0'\left(\frac{L}{2}u\right), \quad y_0''(u) = \left(\frac{L}{2}\right)^2 w_0''\left(\frac{L}{2}u\right). \tag{C-23}$$

The energy constraint (7) requires that

$$C = \left(\frac{2}{LK_0}\right)^{1/2}, \tag{C-24}$$

where

$$K_j \equiv \int du\, \left[h^{(j)}(u)\right]^2. \tag{C-25}$$

The bandwidth constraint (29) requires that

$$L = \frac{1}{\pi}\left(\frac{K_1}{K_0}\right)^{1/2}\frac{1}{B_r}, \tag{C-26}$$

where we have employed (C-23), (C-24), and (C-25). Then we use (C-23) through (C-26) to determine the bias constant D_2 as

$$D_2 = \frac{1}{(2\pi)^4}\int dt\left[w_0''(t)\right]^2 = \frac{K_0 K_2}{K_1^2} B_r^4 . \tag{C-27}$$

For the current example in (C-22), we evaluate

$$K_0 = \frac{R(2\alpha)+1}{\cosh^2(\alpha)} + \frac{1+S(2\beta)}{\cos^2(\beta)},$$

$$K_1 = \alpha^2 \frac{R(2\alpha)-1}{\cosh^2(\alpha)} + \frac{4\alpha\beta}{\alpha^2+\beta^2}\left[\alpha\tan(\beta)-\beta\tanh(\alpha)\right]+\beta^2\frac{1-S(2\beta)}{\cos^2(\beta)},$$

$$K_2 = \alpha^4 \frac{R(2\alpha)+1}{\cosh^2(\alpha)} + \beta^4\frac{1+S(2\beta)}{\cos^2(\beta)}, \tag{C-28}$$

where we have employed the first alternative in (C-20), and defined

$$R(x) = \frac{\sinh(x)}{x}, \quad S(x) = \frac{\sin(x)}{x}. \tag{C-29}$$

For an arbitrary α, we solve the first alternative in (C-20) for β, and then compute D_2, by means of (C-27) and (C-28). It is found that D_2 increases monotonically with α. The minimum is realized when $\alpha = 0$, namely

$$D_2 = 3B_r^4. \tag{C-30}$$

The third subcase in (C-15) yields the form

$$w_0(t) = A\cos(at) + B\sin(at) + C\cos(bt) + D\sin(bt). \tag{C-31}$$

The boundary conditions (C-1) demand that

$$\alpha\tan(\alpha) = \beta\tan(\beta) \quad \text{or} \quad \frac{\tan(\alpha)}{\alpha} = \frac{\tan(\beta)}{\beta}. \tag{C-32}$$

However, the second alternative in (C-32) yields odd solutions and is discarded. The first alternative yields

$$w_0(t) = C\left[\frac{\cos(at)}{\cos(\alpha)} - \frac{\cos(bt)}{\cos(\beta)}\right], |t| \le \frac{L}{2}, \alpha \ne \frac{\pi}{2}. \tag{C-33}$$

The values of α and β are related as shown in figure C-2. As above,

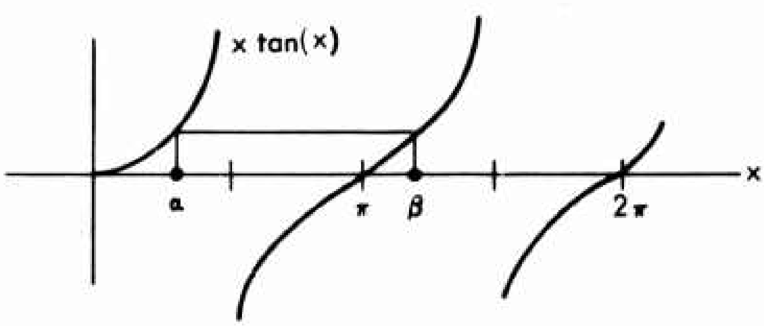

Figure C-2. Relationship of α and β in (C-32) we define a function

$$y_0(u) = W_0\left(\tfrac{1}{2}u\right) = C\,h(u) = C\left[\frac{\cos(\alpha u)}{\cos(\alpha)} - \frac{\cos(\beta u)}{\cos(\beta)}\right]. \tag{C-34}$$

We can now employ (C-25) through (C-27) immediately. We evaluate

$$K_0 = \frac{1+S(2\alpha)}{\cos^2(\alpha)} - 2\,\frac{S(\alpha+\beta)+S(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} + \frac{1+S(2\beta)}{\cos^2(\beta)},$$

$$K_1 = \alpha^2\,\frac{1-S(2\alpha)}{\cos^2(\alpha)} - 2\alpha\beta\,\frac{S(\alpha-\beta)-S(\alpha+\beta)}{\cos(\alpha)\cos(\beta)} + \beta^2\,\frac{1-S(2\beta)}{\cos^2(\beta)}, \tag{C-35}$$

$$K_2 = \alpha^4\,\frac{1+S(2\alpha)}{\cos^2(\alpha)} - 2\alpha^2\beta^2\,\frac{S(\alpha+\beta)+S(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} + \beta^4\,\frac{1+S(2\beta)}{\cos^2(\beta)}.$$

For an arbitrary $\alpha\ (\neq \pi/2)$, we solve the first alternative in (C-32) for β, and then compute D_2 using (C-27) and (C-35). It is found that D_2 decreases monotonically with increasing α, at least for α up to $\pi/2$, the limit being 25/9 at $\alpha = \pi/2 -$. However, when $\alpha > \pi/2$, $h(u)$ goes negative somewhere and is unacceptable. For $\alpha = \pi/2$, (C-32) is not an adequate form; we note instead that $\beta = 3\pi/2$ from (C-31), and then

$$h(u) = A\cos\left(\tfrac{\pi}{2}u\right) - B\cos\left(\tfrac{3\pi}{2}u\right). \tag{C-36}$$

The boundary conditions (C-1) force $A = -3B$, giving

$$h(u) = \cos^3\left(\tfrac{\pi}{2}u\right). \tag{C-37}$$

We then find

$$K_0 = \frac{5}{8},\quad K_1 = \frac{9\pi^2}{32},\quad K_2 = \frac{45\pi^4}{128}, \tag{C-38}$$

yielding

$$L = \frac{3}{2\sqrt{5}}\,\frac{1}{B_r},$$

$$D_2 = \frac{25}{9}\,B_r^4, \tag{C-39}$$

which is smaller than (C-30). Thus (C-37) is the optimum window. (Notice that $h''(1) = 0$.)

LIST OF REFERENCES

1. M. S. Bartlett, An Introduction to Stochastic Processes, Cambridge University Press, New York, 1955.
2. E. Parzen, "On Consistent Estimates of the Spectrum of a Stationary Time Series," Annals of Mathematical Statistics, vol. 28, 1957, pp. 329-348.
3. U. Grenander and M. Rosenblatt, Statistical Analysis of Stationary Time Series, John Wiley and Sons, Inc., New York, 1957.
4. R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra, Dover Publications, New York, 1959.
5. G. M. Jenkins and D. G. Watts, Spectral Analysis and Its Applications, Holden-Day Inc., San Francisco, 1968.
6. A. H. Nuttall, Spectral Estimation by Means of Overlapped Fast Fourier Transform Processing of Windowed Data, NUSC Report 4169, 13 October 1971.
7. A. H. Nuttall, "Estimation of Cross-Spectra via Overlapped FFT Processing," NUSC Technical Memorandum TC-83-72, 18 April 1972.
8. A. Papoulis, "Minimum-Bias Windows for High-Resolution Spectral Estimates," IEEE Transactions on Information Theory, vol. IT-9, no. 1, January 1973, pp. 9-12.
9. J. S. Bendat and A. G. Piersol, Random Data: Analysis and Measurement Procedures, John Wiley and Sons, Inc., New York, 1971.