Minimum classical bit for remote preparation and measurement of a qubit

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We show that a qubit chosen from equatorial or polar great circles on a Bloch sphere can be remotely prepared with one cbit from Alice to Bob if they share one ebit of entanglement. Also we show that any single-particle measurement on an arbitrary qubit can be remotely simulated with one ebit of shared entanglement and communication of one cbit.

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The state of a quantum system contains a large amount of information that cannot be accessed by an observer. How well one can extract and utilize the largely inaccessible quantum information is the subject of quantum information theory. One of the surprising discoveries in this area is the *teleportation* of an *unknown* quantum state by Bennett *et al.* [1] from one place to another without ever physically sending the particle. A qubit, for example, can be sent from Alice to Bob provided they share an Einstein-Podolsky-Rosen (EPR) pair and Alice carries out a Bell-state measurement on the qubit and one-half of the EPR pair, and sends two bits of classical information to Bob, who in turn can perform a unitary operation on his particle to get the original state. The quantum teleportation of a photon has been demonstrated experimentally by Bouwmeester *et al.*, [2] and Boschi *et al.*, [3]. The continuous version of quantum teleportation has also been verified by Furusawa *et al.* [4]. Although a qubit contains a double infinity of bits of information [5] (corresponding to two real numbers), only two classical bits (cbits) are necessary to transmit a qubit in the teleportation process. This raises the question, is it really the minimum number of cbits needed to transmit a qubit? What about the rest of the infinity of this number of bits? It has been suggested that the remaining bits flow across the entanglement channel [5]. Is it that two cbits are required just to preserve the causality (the peaceful co-existence of quantum theory and relativity) or is it the essence of an *unknown* qubit (without which the qubit cannot be reconstructed, the particle is just being in a random mixture at Bob's place)?

Recently several philosophical implications of quantum teleportation and its experimental verification have been brought out by Vaidman [6]. Though quantum teleportation requires a quantum channel that is an entangled pair, doubts have been raised whether teleportation is really a nonlocal phenomenon [7]. Hardy [8] has argued that one can construct a local theory where cloning of a state is not possible but teleportation is. Interestingly, the old issue of mimicking quantum theory by a local hidden variable (LHV) theory has been revived by Brassard *et al.* [9] and Steiner [10], who show that nonlocal correlations of quantum theory can be simulated by local hidden variable theory with classical communication. A natural question then is, if classical communication can help in mimicking nonlocal correlation, can one teleport a quantum state with extra number of cbits. This has been answered by Cerf *et al.*, [11] who have proved that one can construct a classical teleportation scheme of a *known* state from Alice to Bob with the help of 2.19 cbits (on an average) provided they have initially shared local hidden variables. This is an interesting result. They compare the cbits required in classical teleportation to cbits required in quantum teleportation and argue that only 0.19 bit more is required when one uses local hidden variables. But if it is compared with our scheme, it requires 1.19 cbits more.

In this Brief Report we show that there is a simple scheme for remote preparation of a particular ensemble of qubit and remote measurement of an arbitrary qubit *known* to Alice but *unknown* to Bob. This requires only one cbit to be transmitted from Alice to Bob. Unlike the teleportation of an *unknown* qubit, here, we do not require a Bell-state measurement. Only a single-particle von Neumann measurement is necessary. The qubit that is intended to be transmitted does not play any direct role in the measurement process except for the fact that its state is known to Alice. A qubit chosen from equatorial or polar great circles on a Bloch sphere can be remotely prepared with one cbit from Alice to Bob if they share one ebit of entanglement. Further we show that any single-particle measurement on an arbitrary qubit can be remotely simulated with one ebit and communication of one cbit. This also shows that the classical teleportation envisaged by Cerf *et al.* [11] actually requires 1.19 bits more than that of a situation where one uses entangled pairs rather than local hidden variables. Since they think of transmitting a *known* qubit, and in teleportation one sends an *unknown* qubit, one should not compare the classical information cost in the above situation.

Let us consider a pure input state $|\Psi\rangle \in \mathcal{H}=C^2$, which is the state of a qubit. An arbitrary qubit can be represented as

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{1}$$

where we can choose α to be real and β to be a complex number, in general [up to U(1) equivalance classes of states]. This qubit can be represented by a point on a sphere S^2 [which is the projective Hilbert space $\mathcal{P}=CP(1)$ for any two-state system] with the help of two real parameters θ and ϕ, where $\alpha=\cos(\theta/2)$ and $\beta=\sin(\theta/2)\exp(i\phi)$. Now Alice wants to transmit the above qubit to Bob. She can either

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physically send the particle (which is not interesting) or she needs to send a doubly infinity of bits of information across a classical channel to Bob. However, as we will show, there is a very simple procedure to send the information content of particular ensemble of qubit without ever sending it or without ever sending an infinity of bits of information. Just one cbit is required to send the information content of a qubit provided Alice and Bob share one half of the particles from an EPR source. The EPR state of the particles 1 and 2 is given by

$$|\Psi^-\rangle_{12}=\frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2-|1\rangle_1|0\rangle_2). \tag{2}$$

Suppose Alice is in possession of 1 and Bob is in possession of 2. The qubit $|\Psi\rangle$ is *known* to Alice and *unknown* to Bob. Since Alice knows the state she can chose to measure the particle 1 in any basis she wants. Alice carries out measurement on particle 1 by projecting onto the ''qubit basis'' $\{|\Psi\rangle,|\Psi_\perp\rangle\}$, where the ''qubit basis'' is related to the old basis $\{|0\rangle,|1\rangle\}$ in the following manner

$$\begin{aligned}|0\rangle_1&=\alpha|\Psi\rangle_1-\beta|\Psi_\perp\rangle_1,\\ |1\rangle_1&=\beta^*|\Psi\rangle_1+\alpha|\Psi_\perp\rangle_1.\end{aligned} \tag{3}$$

By this change of basis the normalization and orthogonality relation between basis vectors are preserved. Now writing the entangled state $|\Psi^-\rangle_{12}$ in the ''qubit basis'' $\{|\Psi\rangle_1,|\Psi_\perp\rangle_1\}$ gives us

$$|\Psi^-\rangle_{12}=\frac{1}{\sqrt{2}}[|\Psi\rangle_1|\Psi_\perp\rangle_2-|\Psi_\perp\rangle_1|\Psi\rangle_2], \tag{4}$$

which is also a consequence of invariance of $|\Psi^-\rangle_{12}$ under the $U_1\otimes U_2$ operation. The total state after a single-particle von Neumann measurement (if the outcome of Alice is $|\Psi_\perp\rangle_1$) is given by $|\Psi_\perp\rangle_1\otimes|\Psi\rangle_2$.

When she sends her measurement result (one bit of classical information) to Bob, then particle 2 can been found in the original state $(\alpha|0\rangle_2+\beta|1\rangle_2)$, which is nothing but the remote prepartion of a *known* qubit. But this is not a successful remote state preparation because Bob will succeed only half of the time in getting the original qubit. If the outcome of Alice's measurement result is $|\Psi\rangle_1$, then the classical communication from Alice would tell Bob that he has obtained a state that is $|\Psi_\perp\rangle=(\alpha|1\rangle_2-\beta^*|0\rangle_2)$. This is a complement qubit that is orthogonal to the original one. The resulting state (if the outcome is $|\Psi\rangle_1$) is given by $|\Psi\rangle_1\otimes|\Psi_\perp\rangle_2$.

There is nothing special about sharing an EPR singlet state. In fact, Alice and Bob can share any other maximally entangled state from the basis $\{|\Psi^+\rangle_{12},|\Phi^\pm\rangle_{12}\}$. These can be expressed in terms of the qubit basis as

$$|\Psi^+\rangle_{12}=-\frac{1}{\sqrt{2}}[|\Psi\rangle_1(\sigma_z)|\Psi_\perp\rangle_2+|\Psi_\perp\rangle_1(\sigma_z)|\Psi\rangle_2],$$

$$|\Phi^+\rangle_{12}=\frac{1}{\sqrt{2}}[|\Psi\rangle_1(i\sigma_y)|\Psi_\perp\rangle_2+|\Psi_\perp\rangle_1(i\sigma_y)|\Psi\rangle_2],$$

$$|\Phi^-\rangle_{12}=\frac{1}{\sqrt{2}}[|\Psi\rangle_1(\sigma_x)|\Psi_\perp\rangle_2+|\Psi_\perp\rangle_1(\sigma_x)|\Psi\rangle_2], \tag{5}$$

where σ_x, σ_y, and σ_z are the Pauli matrices. When Alice and Bob share $|\Psi^+\rangle_{12}$, $|\Phi^+\rangle_{12}$, and $|\Phi^-\rangle_{12}$, then the resulting states after a single-particle von Neumann measurement and classical communication are given by $|\Psi_\perp\rangle_1\otimes(\sigma_z)|\Psi\rangle_2$, $|\Psi_\perp\rangle_1\otimes(i\sigma_y)|\Psi\rangle_2$, and $|\Psi_\perp\rangle_1\otimes(\sigma_x)|\Psi\rangle_2$, respectively.

In general, if Alice finds $|\Psi_\perp\rangle_1$ in a single-particle measurement, then one cbit from Alice to Bob will result in a qubit or a qubit up to a rotation operator at Bob's place. If Alice finds $|\Psi\rangle_1$, then sending of one cbit will yield an exact a complement qubit or a complement-qubit state up to a rotation operator. The overall rotation operators that Bob has to apply to get a qubit depends on the type of entangled state they have shared initially. In the following we discuss successful remote state preparation of a special ensemble of qubits when Alice and Bob share an EPR singlet. If Alice chooses to prepare a real qubit, i.e., $|\Psi\rangle=\cos(\theta/2)|0\rangle+\sin(\theta/2)|1\rangle$, which means on the projective Hilbert space S^2 the point lies on the polar line, then the azimuthal angle ϕ is zero. In this case Bob just has to perform a rotation (i.e., apply $\sigma_x\sigma_y$) or do nothing after receiving the classical information from Alice. Alternatively, Alice could wish to prepare a qubit chosen from equatorial line on Bloch sphere such as $|\Psi\rangle=(1/\sqrt{2})(|0\rangle+e^{i\phi}|1\rangle)$ with $\theta=\pi/2$. In this case when Bob gets $|\Psi_\perp\rangle_2=(1/\sqrt{2})(|0\rangle-e^{i\phi}|1\rangle)$ then he can still get $|\Psi\rangle$ by applying σ_z. Therefore, when the measurement outcome is $|\Psi\rangle_1$ or $|\Psi_\perp\rangle_1$ (in the both cases) Bob's particle is prepared in the (un)known state. Thus for any real qubit our simple scheme remotely prepares a known state with certainty. Since a real qubit requires a single infinity of bits of information (as one real number θ or ϕ is necessary) to be sent across a classical channel, use of shared entanglement reduces it to sending just one cbit across a classical channel and this can be done with certainty. For an arbitrary but *known* qubit this protocol is able to transmit half of the time. This is because Bob cannot convert the orthogonal-complement qubit (which he gets half of the time) since it is *unknown* to him. We know that an arbitrary unknown state cannot be complemented [12–14]. Though we can design a NOT gate which can take $|0\rangle\to|1\rangle$ and $|1\rangle\to|0\rangle$, there is no universal NOT gate that can take an unknown qubit $|\Psi\rangle\to|\Psi_\perp\rangle$ as it involves an *antiunitary* operation. Thus, a double infinity of bits of information cannot be passed all the time with the use of entanglement by sending just one classical bit.

This shows that to remotely prepare a *known* qubit chosen from a special ensemble one need not do a Bell-state measurement and send two cbits. Only single-particle measurement and one cbit is necessary from Alice to Bob, provided they share an entangled state. In ''classical teleportation'' of a qubit it is aimed to simulate any possible measurement on the qubit sent to Bob (unknown to him) [11]. One may tend to think that since, in our scheme, we can remotely prepare an arbitrary *known* state one-half of the time, Bob might not be able to simulate the measurement statistics all the time (as Bob cannot get a *unknown* qubit from the complement qubit). However, there is no problem for Bob to simulate the measurement statistics on the complement qubit (also called time-reversed qubit). This is because the quantum-mechanical probabilities and transition probabilities are invariant under unitary and antiunitary operations (thanks to Wigner's theorem). This says that for any two nonorthogonal rays $|\langle\Psi|\Phi\rangle|^2=|\langle\Psi'|\Phi'\rangle|^2$, where $|\Psi'\rangle,|\Phi'\rangle$ are related to $|\Psi\rangle,|\Phi\rangle$ either by unitary or antiunitary transformations. For example, if Bob wants to measure an observable $(\mathbf{b}\cdot\boldsymbol{\sigma})$, then the probability of measurement outcome in the state $\rho=|\Psi\rangle\langle\Psi|=\frac{1}{2}(1+\mathbf{n}\cdot\boldsymbol{\sigma})$ is given by

$$P_{\pm}(\rho)=\mathrm{tr}[P_{\pm}(\mathbf{b})\rho]=\frac{1}{2}(1\pm\mathbf{b}\cdot\mathbf{n}), \tag{6}$$

where the projection operator $P_{\pm}(\mathbf{b})=\frac{1}{2}(1\pm\mathbf{b}\cdot\boldsymbol{\sigma})$. But suppose Bob gets $\rho_{\perp}=|\Psi_{\perp}\rangle\langle\Psi_{\perp}|=\frac{1}{2}(1-\mathbf{n}\cdot\boldsymbol{\sigma})$. In this case the measurement gives a result

$$P_{\pm}(\rho_{\perp})=\mathrm{tr}[P_{\pm}(\mathbf{b})\rho_{\perp}]=\frac{1}{2}(1\mp\mathbf{b}\cdot\mathbf{n}), \tag{7}$$

which is different than Eq. (6). However, Bob can always chose his apparatus (by reversing the direction of **b**) such that he can make $P_{\pm}(\rho)=P_{\pm}(\rho_{\perp})$. Note that Bob cannot reverse the direction of **n** but can in principle reverse the direction of **b**. So even if Bob cannot get a qubit from a complement qubit (half of the time) still he can get the same measurement outcomes from it. Therefore, Bob can simulate with 100% efficiency the statistics of his measurements on a qubit *known* to Alice but *unknown* to him, provided they share an EPR pair and communicate one cbit. Whether one can always remotely simulate all the measurement results for a higher dimensional quantum systems is still an open question.

Our observation also shows that the extra cbits required in a hidden variable scenario is 1.19 and not just 0.19 bits as mentioned in Ref. [11]. So to fill the gap between LHV and quantum theory, 1.19 cbits are necessary (for lower dimensional Hilbert spaces). It should be remarked that the 2.19 cbit needed in classical teleportation protocol [11] is not optimal. If a better protocol exists, it will bring down the cbit cost. We can formally say that any LHV model that simulates teleportation of a *known* qubit without entanglement will require at least one cbit (because no LHV can beat the use of entanglement) to be transmitted from Alice to Bob. This shows that one cbit is sufficient for classical teleportation. That this is also necessary can be seen easily: Alice can use the classical teleportation scheme to transmit one classical bit. Therefore, any LHV scheme that realizes the classical teleportation, needs to use one cbit, otherwise we would have sent one classical bit with less than one classical bit of information.

The entanglement channel is a *passive* communication channel, which on its own cannot be used for communication purposes. Supplemented with cbits it becomes *active*, so we can regard cbits as the *essence of the entanglement channel*. Thus we can say that the minimum cbits required to remotely prepare a *real known* qubit is one cbit (using shared entanglement), where as to transmit an *unknown* qubit one needs two cbits (as in teleportation protocol). The scenario presented here is also useful in the context of ''assisted cloning'' and ''orthogonal complementing'' of unknown states [15]. Recently, the classical communication cost of remote state preparation and distributed quantum information has been studied by Lo [16]. In an important paper Bennett *et al.*, [17] have shown that asymptotically one needs one cbit per qubit for remote state preparation of any qubit and have studied the remote preparation of entangled states.

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