2004

# Minimum energy as the general form of critical flow and maximum flow efficiency and for explaining variations in river channel pattern

He Qing Huang
*University of Oxford*

Howard H. Chang
*San Diego State University*

Gerald Nanson
*University of Wollongong*, gnanson@uow.edu.au

# Minimum energy as the general form of critical flow and maximum flow efficiency and for explaining variations in river channel pattern

## Abstract

Although the Bélanger-Böss theorem of critical flow has been widely applied in open channel hydraulics, it was derived from the laws governing ideal frictionless flow. This study explores a more general expression of this theorem and examines its applicability to flow with friction and sediment transport. It demonstrates that the theorem can be more generally presented as the principle of minimum energy (PME), with maximum efficiency of energy use and minimum friction or minimum energy dissipation as its equivalents. Critical flow depth under frictionless conditions, the best hydraulic section where friction is introduced, and the most efficient alluvial channel geometry where both friction and sediment transport apply are all shown to be the products of PME. Because PME in liquids characterizes the stationary state of motion in solid materials, flow tends to rapidly expend excess energy when more than minimally demanded energy is available. This leads to the formation of relatively stable but dynamic energy-consuming meandering and braided channel planforms and explains the existence of various extremal hypotheses.

## Keywords

channel, variations, efficiency, maximum, minimum, flow, form, general, river, explaining, pattern, critical, energy, GeoQuest

## Disciplines

Medicine and Health Sciences | Social and Behavioral Sciences

## Publication Details

Huang, H., Chang, H. H. & Nanson, G. C. (2004). Minimum energy as the general form of critical flow and maximum flow efficiency and for explaining variations in river channel pattern. Water Resources Research, 40 (4), 1-13.

# Minimum energy as the general form of critical flow and maximum flow efficiency and for explaining variations in river channel pattern

He Qing Huang
School of Geography and the Environment, University of Oxford, Oxford, UK

Howard H. Chang
Department of Civil and Environmental Engineering, San Diego State University, San Diego, California, USA

Gerald C. Nanson
School of Geosciences, University of Wollongong, Wollongong, New South Wales, Australia

Received 30 July 2003; revised 14 January 2004; accepted 13 February 2004; published 15 April 2004.

[1] Although the Bélanger-Böss theorem of critical flow has been widely applied in open channel hydraulics, it was derived from the laws governing ideal frictionless flow. This study explores a more general expression of this theorem and examines its applicability to flow with friction and sediment transport. It demonstrates that the theorem can be more generally presented as the principle of minimum energy (PME), with maximum efficiency of energy use and minimum friction or minimum energy dissipation as its equivalents. Critical flow depth under frictionless conditions, the best hydraulic section where friction is introduced, and the most efficient alluvial channel geometry where both friction and sediment transport apply are all shown to be the products of PME. Because PME in liquids characterizes the stationary state of motion in solid materials, flow tends to rapidly expend excess energy when more than minimally demanded energy is available. This leads to the formation of relatively stable but dynamic energy-consuming meandering and braided channel planforms and explains the existence of various extremal hypotheses. *INDEX TERMS:* 1815 Hydrology: Erosion and sedimentation; 1824 Hydrology: Geomorphology (1625); 1625 Global Change: Geomorphology and weathering (1824, 1886); 8125 Tectonophysics: Evolution of the Earth; *KEYWORDS:* critical flow, extremal hypothesis, least action principle, minimum energy, most efficient alluvial channel geometry, regime theory

**Citation:** Huang, H. Q., H. H. Chang, and G. C. Nanson (2004), Minimum energy as the general form of critical flow and maximum flow efficiency and for explaining variations in river channel pattern, *Water Resour. Res.*, *40*, W04502, doi:10.1029/2003WR002539.

## 1. Introduction

[2] The motion of fluids through space under the action of external forces is generally described by the laws of conservation of energy and matter (flow continuity). These two laws can be satisfied with many possibilities, but among which there exists a unique solution. This implies another inherent law of motion. Flow in an open channel is a typical example of such conservation. Although there are many cross sections that fulfill the two laws, a unique solution appears when energy reaches a permissible minimum.

[3] This principle of minimum specific energy for a given discharge was first recognized by P. Böss in 1919. As identified by Bëlanger in 1849 [*Lamb*, 1945; *Jaeger*, 1955], a different form of stating the same principle is to express it as the maximum discharge for a given amount of energy. Although this so-called Bëlanger-Böss theorem of critical flow has been widely applied in open channel hydraulics, it is definable only in terms of the laws governing ideal frictionless flow. In order to deal with more complicated hydrodynamic problems, such as flow with friction in a deformable channel, numerous attempts have been made to link this theorem with a more general principle established in hydrodynamics, notably the Helmholtz and Korteweg's minimum energy dissipation theorems [*Lamb*, 1945] and the Boussinesq's stability principle [*Jaeger*, 1955]. A number of extremal hypotheses have also been proposed, including maximum energy loss [*Jefferson*, 1902; *Schoklitsch*, 1937; *Inglis*, 1947], minimum variance and least work [*Leopold and Langbein*, 1962], maximum sediment transporting capacity [*Pickup*, 1976; *Kirkby*, 1977; *White et al.*, 1982; *Bettess and White*, 1987], minimum energy dissipation rate [*Yang*, 1971, 1987; *Yang and Song*, 1979; *Song and Yang*, 1980, 1982; *Yang et al.*, 1981], minimum stream power [*Chang*, 1979a, 1979b; 1980a, 1980b; 1985a, 1985b, 1988; *Millar and Quick*, 1993, 1998; *Millar*, 2000], maximum friction factor [*Davies and Sutherland*, 1980, 1983; *Lamberti*, 1988; *Phillips*, 1991], maximum flow resistance [*Abrahams et al.*, 1995], maximum rate of energy dissipation [*Huang*, 1983, 1988], minimum Froude number [*Jia*, 1990; *Yalin and Silva*, 1999, 2000], and critical flow constraint [*Grant*, 1997; *Tinkler*, 1997a, 1997b]. However, the Helm-


0043-1397/04/2003WR002539

holtz and Korteweg's minimum dissipation theorems were derived under the condition that the inertia terms (i.e., kinetic energy) in the Navier-Stokes equations can be ignored. This assumption has made the theorems inapplicable, at least in theory, directly to open-channel flow [*Chen*, 1980]. The physics behind Boussinesq's stability principle, on the other hand, has not been elucidated clearly, and there are conflicting opinions over its general applicability [*Jaeger*, 1955; *Chow*, 1959]. Among the extremal hypotheses, two of them, maximum sediment transporting capacity and minimum stream power, have been shown to be the essential expressions of a principle inherent in basic flow relationships, that of maximum flow efficiency (MFE) [*Huang and Nanson*, 2000, 2001, 2002; *Huang et al.*, 2002]. Although MFE provides a convincing explanation as to why rivers exhibit a consistent channel geometry in very different geographic regions, it is derived from laws governing flow in a straight single-thread alluvial channel system, thus applicable only to straight rivers. Most rivers tend to adopt sinuous or braided channel patterns which are inefficient and therefore high-energy consuming systems [e.g., *Richards*, 1982; *Knighton*, 1998].

[4] By examining the way in which the Bëlanger-Böss theorem of critical flow has gained wide application in open channel hydraulics, the purpose of this study is to address the issue of flow in other than straight channels, for the theorem also illustrates the condition of MFE [*Chow*, 1959; *Henderson*, 1966]. By pursing a clear definition of the stationary state for the motion of liquids in an open channel system, this study shows that the Bëlanger-Böss theorem of critical flow can be generalized into the less specific principle of minimum energy (PME). We demonstrate that PME is equivalent to the conditions of maximum efficiency of energy use and minimum force, while it characterizes for liquids what in solid materials is termed the stationary state of motion. In flow with friction and sediment transport, PME determines the "best hydraulic section" [*Chow*, 1959] and the most efficient alluvial channel geometry [*Huang and Nanson*, 2000, 2002; *Huang et al.*, 2002]. Most importantly, this study indicates that in a situation where there is excess energy available, PME can be satisfied only when the *excess* energy is maximally dissipated in a fixed boundary channel. In alluvial channels with mobile boundaries, the excess energy will lead flow to erode the channel bed and banks, resulting in flow-resisting bedforms and in-channel planform adjustment and the formation of relatively stable (dynamical equilibrium) meandering and braided patterns that are able to consume the excess energy. With the recognition of the general form of PME, this study also gives a clear illustration of the applicable conditions for various extremal hypotheses.

## 2. Role of Stationary State in Dynamic Systems

[5] Any system in motion must possess a certain amount of energy, which normally consists of several types that can be converted from one to another. As a result, if only the law of conservation of energy is applied, there is no unique solution to the form of the motion. This can be seen more clearly from an essential case where total available energy $E$ consists of only two parts; the energy stored in a system above a position of ultimate rest, or potential energy $E_p$, and the energy of motion, or kinetic energy $E_k$. That is:

$$E = E_p + E_k \tag{1}$$

[6] For the conservation of energy in such a dynamic system, i.e., $\Delta E = 0$, the following condition needs to be satisfied:

$$\Delta E_p = -\Delta E_k \tag{2}$$

The meaning of equation (2) is that in a conservative system, for a given amount of energy $E$, potential energy always needs to be converted into kinetic energy, or vice versa. Thus a pendulum can swing to the two extremities of its arc at which points it has its least kinetic energy and greatest potential energy. During each cycle of swing, it passes through a vertical position where it achieves its greatest kinetic energy and least potential energy. As the pendulum moves, energy is continuously passing back and forth between the two forms.

[7] When available energy $E$ declines to a certain amount, a pendulum will become motionless at a point at which it has the least height above the Earth. The amount of energy the pendulum has at this point is at a minimum for motion and can be defined as $E_{min}$. Taking the point as reference datum, $E_{min} = 0$ for the motion of a pendulum. When $E > E_{min}$, a pendulum cannot be stationary and in contrast experiences at that point a maximum velocity maximally converted from available potential energy. As seen in Figure 1, this stationary point is the axle center of the swing, thus a typical "attractor" of motion as defined in general physics [e.g., *Gleick*, 1998]. This means that when a pendulum cannot remain stationary because there is excess energy available, or $E > E_{min}$, motion will be in a dynamic form. Nevertheless, the point that characterizes the stationary state still plays a controlling role in the whole process of the dynamic movement. Eventually, the pendulum will stand still at the point when the excess energy is fully dissipated due to friction.

[8] Though relatively simple, the motion of a pendulum illustrates the importance of the role that a stationary state plays in a dynamic system. This is because of the law of energy conservation that makes the conversion of energy among its several types a limiting process. In other words, as long as the law of energy conservation is preserved, a stationary state will occur and will play a controlling role in a dynamic movement. This is why stationary state has long been recognized as one of essential features of a dynamic system such that the principle of minimum energy (PME) has gained application not only in classic mechanics but also in both quantum mechanics and thermodynamics [e.g., *Lanczos*, 1949; *Dugas*, 1957; *Stauffer and Stanley*, 1989; *Prigogine and Stengers*, 1984; *Kroemer*, 1994; *Thorn and Welford*, 1994]. However, for different materials, or for same material with different restrictions, stationary states vary considerably. In solid mechanics, a pendulum illustrates a simple case of motion in which the stationary state can be directly determined by the state of minimum potential energy. In contrast, particles like electrons in a quantum system present a complicated case of motion in which the stationary state occurs when the systems' total energy reaches a minimum, which is above the minimum of potential energy [e.g., *Kroemer*, 1994]. In thermodynamics, stationary state is characterized by minimum entropy pro-

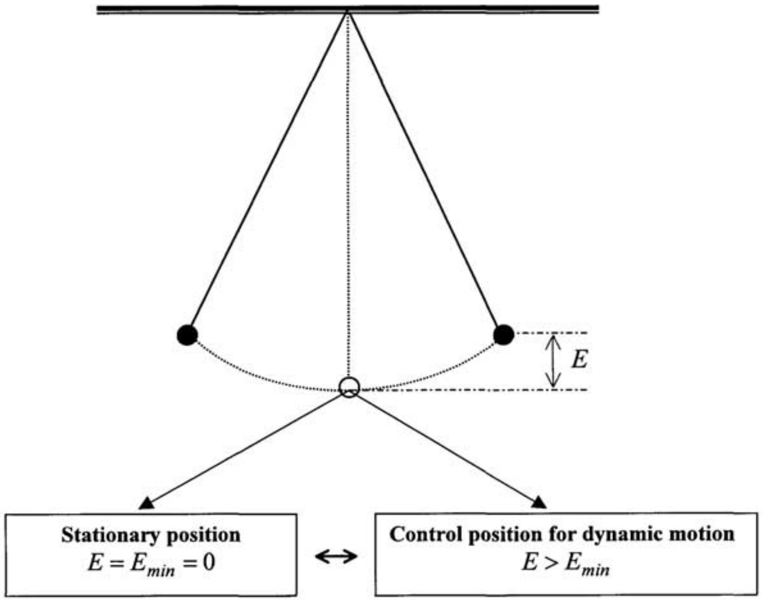


**Figure 1.** Position of controls in the swings of a pendulum in relation to the state of available energy $E$.

duction and is one toward which an open system evolves. Within the region of linear thermodynamics (near to equilibrium), an open system exhibits a single stationary state while, within the far-from-equilibrium region, it may produce energy fluxes that result in a nonlinear response in which many stationary states may appear [e.g., *Prigogine and Stengers*, 1984; *Thorn and Welford*, 1994]. PME has also been applied to study the structure of drainage networks [e.g., *Ijjasz-Vasquez et al.*, 1993; *Rodriguez-Iturbe and Rinaldo*, 1997]. In open channel flow systems, however, stationary states and the roles of PME in the dynamic movement of fluids have not been explicitly addressed.

## 3. Stationary State in Open Channel Flow

[9] In ideal frictionless open channel flow, a one-dimensional energy formulation applies in the form of:

$$E = h + \frac{V^2}{2g} \tag{3}$$

in which, $E$ is the specific energy of flow, $h$ is the vertical distance from channel bed to water surface, or flow depth, $V$ is the average velocity of flow, and $g$ is the acceleration due to gravity.

[10] Because of the necessity for fluids to maintain continuity, flow in a channel with a fixed width is most commonly subject to the restriction of:

$$q = hV \tag{4}$$

where $q$ is the fluid discharge per unit channel width.

[11] For given flow discharge $q$ and energy $E$, there are three mathematical solutions of flow depth $h$ that satisfy both equations (3) and (4). One of them, however, has a negative value and is thus impossible to achieve in a practical sense. There is even a unique solution of flow depth, $h_c$, at which the Froude number $F_r$ attains unity, or:

$$F_r = \frac{V}{\sqrt{gh}} = 1 \tag{5}$$

and energy $E$ reaches a minimum:

$$E_{min} = \frac{3}{2}\left(\frac{q^2}{g}\right)^{1/3} \tag{6}$$

[12] Consequently, a maximum flow discharge $q$ can also be identified from equations (3) and (4) for a given $E$ at exactly the same flow depth $h_c$. Furthermore, a minimum value of specific force $F$ has also been shown by Boussinesq in 1877 to occur at the flow depth $h_c$ [*Jaeger*, 1955; *Chow*, 1959]. In other words, the conditions of minimum specific energy, maximum efficiency of energy use and minimum specific force are equivalent, or:

$$\underset{h=h_c}{Min}\, E \propto \underset{h=h_c}{Max}\, q \propto \underset{h=h_c}{Min}\, F \tag{7}$$

[13] The optimal states illustrated in equation (7) and the use of Froude number $F_r$ to distinguish flow types from supercritical ($F_r > 1$) and subcritical ($F_r < 1$) to critical ($F_r = 1$) are the core of the Bélanger-Böss theorem of critical flow. Over the last century, this theorem has been widely applied in open channel hydraulics, however, further analysis is required in order to gain an understanding of the physics behind the adjustment of alluvial channel planform.

[14] As shown earlier in the interpretation of the pendulum movement, $E_{min}$ characterizes the stationary state in the motion of a solid material for it distinguishes the critical state for incipient motion, or $E_{min} = 0$. In open channel flow, however, equation (6) shows that $E_{min}$ is in direct proportion

to the fluid load $q$ and only when $q = 0$ does $E_{min} = 0$. This is because a certain amount of energy is always needed to transport a given amount of liquid as an entity, or even to keep the liquid standing still and contained in one place. This means that the stationary state defined in terms of the critical state for the incipient motion of solid materials is inappropriate for the motion of liquids. The meaningful definition of stationary state for the motion of liquids may better be defined in terms of $E_{min}$ in equation (6), which is normally greater than zero except in the situation of $q = 0$.

[15] This recognition of stationary state in open channel flow implies that the Bélanger-Böss theorem of critical flow illustrates only a special case of the widely applied principle of minimum energy (PME). Nevertheless, the Bélanger-Böss theorem of critical flow provides a basis for understanding how PME generally governs open channel flow. In terms of the theorem, the roles that PME plays in open channel flow differ at three energy states: $E < E_{min}$, $E = E_{min}$ and $E > E_{min}$. In the situation of $E < E_{min}$, no mathematical solutions of flow depth can be found to satisfy the laws of both energy conservation and fluid continuity. As noted in equation (6), the fluid load $q$ will be reduced in the motion in this situation. In other words, where $E < E_{min}$, available energy can only move part of the fluid load $q$ in the occupied channel. This illustrates a nonequilibrium flow regime.

[16] In the situation of $E = E_{min}$, available energy is just able to transport the imposed fluid load $q$ as a whole throughout the channel. Because $E_{min}$ characterizes the stationary state of motion in liquids, the case of $E = E_{min}$ defines a "stationary" equilibrium flow regime. In this situation, energy is used most efficiently for, in the situation of $E > E_{min}$, flow has excess energy to expend.

[17] As demonstrated earlier, in the situation of $E > E_{min}$ flow can expend excess energy ($E - E_{min}$) in two ways: (1) using it as increased energy of motion (kinetic energy) (supercritical flow state, $F_r > 1$) or (2) storing it as increased potential energy (subcritical flow state, $F_r < 1$). Although the law governing open channel flow is essentially the same as that for the movement of a pendulum (energy conservation), the conversion between potential energy and kinetic energy in frictionless open channel flow cannot be performed in the same way as that in the movement of a pendulum. Because of the restrictions of fluid continuity and the fixed channel boundary, the two resultant types of flow cannot exchange mutually and instead move uniformly downstream in either form. Only when all of the excess energy is dissipated do the two types of flow change into critical flow.

[18] This dissipation of energy can be achieved by some kind of channel contraction, such as a reduction in channel width or the placement of a step in channel bed. It is identifiable from both empirical observation and mathematical analysis that when the channel width reaches a minimum, or the step placed in the channel bed is uplifted to a maximum level, the two types of flow can convert from one to the other, given the flow is not so fully restricted as to be uniform in the transition region (i.e., the channel contraction is of localized nature) [*Henderson*, 1966, Figures 2–3, p. 32]. During each of these localized changes, critical flow occurs. This suggests that the achievement of PME in the situation of $E > E_{min}$ requires a maximal dissipation of the excess energy $E - E_{min}$, or:

$$\underset{h>h_c \text{ or } h<h_c}{Max\ E_{dis}} \rightarrow \underset{h=h_c}{E_{min}} \tag{8}$$

where $E_{dis}$ represents the energy dissipated by the flow. In quantity, $E_{dis}$ varies within the range of between 0 and $E - E_{min}$ for $E > E_{min}$.

[19] It is clear that although a "stationary" equilibrium cannot be achieved in the situation of $E > E_{min}$, the excess energy $E - E_{min}$ can be expended by adopting the alternative of two types of flow (supercritical and subcritical). Hence the situation of $E > E_{min}$ illustrates a dynamic equilibrium flow regime. In the dynamic movement, $E_{min}$ can be achieved only when the excess energy $E - E_{min}$ is fully dissipated.

[20] The occurrence of the three flow regimes in open channel flow is the response of energy conservation law to the restriction of fluid continuity. When a different restriction or more restrictions are imposed on the system, the responses of energy conservation law will be different. The following part of this study shows that when the restrictions of friction and sediment transport are imposed to constrain flow, the three flow regimes are still maintained but the values of $E_{min}$ and the mechanisms to expend excess energy are significantly different. Importantly, this recognition of minimum energy as a generally applicable principle is capable of clarifying the confusion over the applicable conditions for extremal hypotheses and to some extent for the development of other than simple straight river channel patterns, typically meandering and braiding.

## 4. Effect of Friction

[21] In the case where the effect of friction is significant and cannot be ignored, energy formulation in steady, uniform open channel flow applies the following D'Arcy-Weisbach relationship:

$$h_f = f \frac{L}{4R} \frac{V^2}{2g} \tag{9}$$

where $h_f$ is the frictional head of flow or the head loss in the flow, $f$ is a friction factor, $L$ is the length of channel, and $R$ is the hydraulic radius of channel cross section.

[22] For flow over fully rough boundaries, friction factor $f$ is a function of channel boundary roughness only and can be determined from the following relationship:

$$f = c_f \left( \frac{k_s}{R} \right)^{1/3} \tag{10}$$

where $k_s$ represents the height of roughness elements or the size of uniform sediment that forms the channel boundary and $c_f$ is a coefficient.

[23] For a channel with adjustable channel width and depth, the generally applied form of the continuity of fluid is:

$$Q = PRV \tag{11}$$

where $Q$ is the total discharge for the full channel width and $P$ is the wetted perimeter of cross section.

[24] As a result, incorporating equations (10) and (11) into equation (9) yields:

$$S_f = \frac{c_f k_s^{1/3}}{8g} \frac{V^2}{R^{4/3}} = \frac{c_f k_s^{1/3}}{8g} \frac{Q^2}{R^{4/3} A^2} \tag{12}$$

where $S_f$ is the gradient of frictional energy, or the frictional energy head per unit channel length, that is $S_f = h_f/L$, and $A$ is the cross-sectional area, thus $A = PR$.

[25] Equation (12) illustrates the general form of the response of the law of energy conservation to the restriction of fluid continuity in frictional open channel flow. It includes an optimal responsive condition, which has been illustrated using the concept of the best hydraulic section. By making hydraulic radius $R$ a maximum, or wetted perimeter $P$ a minimum, the best hydraulic section provides a minimum $S_f$ for given $Q$ and $A$ and a maximum flow discharge $Q$ for given $S_f$ and $A$. Among all kinds of sections with the same size, a semicircle has the least wetted perimeter and hence is the most hydraulically efficient of all sections [e.g., *Chow*, 1959].

[26] In the exactly same way, it is identifiable in equation (10) that friction factor $f$ takes a minimum value when $R$ is maximized, implying a condition of minimum boundary friction. As a whole, the following optimal conditions occur at the exactly same channel cross section and thus are equivalent:

$$Min\ S_f \propto Max\ Q \propto Max\ R \propto Min\ P \propto Min\ f \tag{13}$$

[27] Taking a rectangular cross section (commonly related to that for efficient bedload transport; *Pickup*, 1976] to be the form of channel section, letting $B$ and $\zeta$ respectively be channel width and width/depth ratio, or $\zeta = B/h$, and using the analytical methodology introduced by *Huang and Nanson* [2000, 2002], the following expression of hydraulic radius can be derived by incorporating the geometric relationships of $A = \zeta h^2$, $R = h\zeta/(\zeta + 2)$ and thus $R^2/A = \zeta/(\zeta + 2)^2$ into equation (12):

$$S_f = \frac{c_f k_s^{1/3} Q^2}{8gA^{8/3}} \frac{(\zeta + 2)^{4/3}}{\zeta^{2/3}} \tag{14}$$

[28] By letting $dS_f/d\zeta = 0$ for given $Q$ and $A$, it can be found from equation (14) that energy slope $S_f$ attains a minimum, or $S_{fmin}$, when channel shape factor $\zeta$ takes the value of 2. No matter whether channels are wide and shallow ($\zeta > 2$) or deep and narrow ($\zeta < 2$), the solutions of $S_f$ in equation (14) are all greater than $S_{fmin}$ for given $Q$ and $A$.

[29] It can be noted from equation (14) that uniform flow in a frictional open channel that has an adjustable width expends energy not in the direct forms of potential energy and kinetic energy but instead in the forms of overcoming friction from both channel bed and banks. A section with $\zeta > 2$ reflects a higher proportion of friction on the bed while a section with $\zeta < 2$ represents a higher proportion of friction on the banks. An increase either in the proportion of channel depth or in the proportion of channel width in the perimeter as a whole increases the total friction to flow. Only when $\zeta = 2$ does the total friction of the channel reach a minimum, which is equal to $S_{fmin}$. Hence $S_{fmin}$ defines the state of incipient motion for fluid to move as a whole (law of continuity) in a frictional open channel. In other words, $S_{fmin}$ characterizes the stationary state of the motion and so the situation of $S_f = S_{fmin}$ illustrates a "stationary" equilibrium flow regime.

[30] Consequently, the situation of $S_f < S_{fmin}$ can be regarded as reflecting a nonequilibrium flow regime, for flow has to drop part of the liquid it carries due to the insufficient supply of energy. The situation of $S_f > S_{fmin}$, on the other hand, can be regarded as defining a dynamic equilibrium flow regime because there are two types of cross section that are able to expend the excess energy $S_f - S_{fmin}$: wide and shallow ($\zeta > 2$) or deep and narrow ($\zeta < 2$). Although PME can be satisfied when the excess energy $S_f - S_{fmin}$ is fully expended, a maximum flow velocity comes with it. Hence the channel geometry that satisfies the condition of $S_{fmin}$ (the best hydraulic section) is the most unstable in the situation of $S_f > S_{fmin}$. To avoid this unstable situation in the design of a stable channel, two types of uneconomical channels needs to be adopted wide and shallow ($\zeta > 2$) or deep and narrow ($\zeta < 2$) [*Chow*, 1959]. In natural alluvial channels, however, the two types of channels cannot be developed without causing erosion or deposition on channel banks and/or bed. This, as detailed in the following, can lead to the development of different channel patterns in accordance with the balance between excess energy and the resistance of channel banks and bed to scour.

## 5. Effect of Sediment Transport

[31] Besides the restrictions of fluid continuity and resistance, flow in alluvial channels is subject to the additional constraint of the energy requirement for bedload transport. There are numerous bedload formulae, but tests of them against field observations show that different equations performed better than others under different criteria [e.g., *Gomez and Church*, 1989]. Nevertheless, most of the commonly applied bedload formulae can be written into the following generalized form:

$$\frac{Q_s}{B} = c_s \tau_o^i (\tau_o - \tau_c)^j \tag{15}$$

where $Q_s$ is the sediment discharge passing through the whole channel, $B$ is the channel width, $\tau_o$ is the flow shear stress, thus $\tau_o = \gamma R S_f$ for steady, uniform flow ($\gamma$ is the specific weight of water), $c_s$ is a coefficient, and $i$ and $j$ are exponents.

[32] It is clear that equation (15) covers a wide range of bedload transport conditions. For example, the *U.S. Waterways Experiment Station* [1935] suggests $i = 0$ and $1.5 < j < 1.8$, while the *Meyer-Peter and Muller* [1948] formula exhibits $i = 0$ and $j = 1.5$; the *DuBoys* [1879] formula proposes $i = j = 1.0$, while the *Parker* [1979] adopts $i = -3.0$ and $j = 4.5$.

[33] Because of the transport of sediment, flow resistance varies with different flow regimes or channel bedforms. However, this behavior can be generalized into the following form:

$$V = c_v R^x S_f^y \tag{16}$$

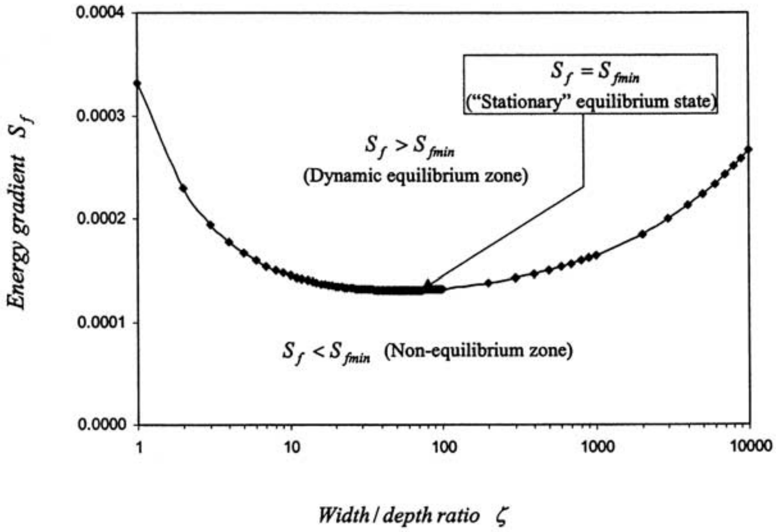


**Figure 2.** The relationship between energy gradient $S_f$ and channel cross-sectional shape $\zeta$ determined using the Meyer-Peter and Müller bedload transport formula and the Manning-Strickler flow resistance equation for given flow discharge (100 $m^3$ $s^{-1}$), bedload discharge (4 kg $s^{-1}$), and sediment size (0.3 mm).

where $c_v$ is a coefficient determined by sediment size, and exponents $x$ and $y$ are functions of flow regimes. For fixed-bed or flatbed flow regime, $x$ and $y$ take values of 2/3 and 1/2 respectively in terms of the Manning-Strickler formula. For the lower flow regime, $x$ and $y$ have values of 0.5293 and 0.3888 respectively, while for the upper flow regime, their respective values are 0.6005 and 0.4605 in terms of the study of *Brownlie* [1983].

[34] In accordance with the theory of energy expenditure on sediment transport proposed by *Bagnold* [1966], *Huang and Nanson* [2000, 2001, 2002] and *Huang et al.* [2002] have also found that the incorporation of the generalized bedload transport formula in equation (15) into flow resistance in equation (16) and fluid continuity relation $Q = VA$ yields the following generalized form of bedload transport relationship:

$$\frac{Q_s}{\Omega^{\alpha}} = \varepsilon\left(\zeta,\ \frac{\tau_o(\zeta)}{\tau_c}\right) \tag{17}$$

where $\Omega$ is the total steam power per unit channel length, or $\Omega = \gamma Q S_f$, $\alpha$ is an exponent that normally has a value of $0.65 \sim 0.85$, and $\varepsilon(\zeta)$ represents the efficiency of flow in transporting bedload by available energy and is a function of channel width-depth ratio $\zeta$.

[35] The specific expressions of the three constraints of fluid continuity $Q = VA$, bedload transport in equation (15) or in equation (17) and flow resistance in equation (16) have all been developed in a way that is able to apply to any form of channel section. As a result, they include at least an adjustable cross-section shape variable and can yield numerous channel cross sections. As demonstrated by *Huang and Nanson* [2002] and stated earlier, because the condition of $1.5 < i + j \leq 1.8$ is satisfied in a wide range of bedload transport conditions, a unique section always appears when energy slope $S_f$ reaches a minimum for given flow discharge $Q$ and sediment load $Q_s$ (Figure 2). On the exact same section, sediment load $Q_s$ gains a maximum for given $Q$ and $S_f$. This means that the following optimal conditions are equivalent [*Huang and Nanson*, 2000, 2001, 2002; *Huang et al.*, 2002]:

$$\underset{\zeta=\zeta_m}{Min}\ S_f \propto \underset{\zeta=\zeta_m}{Min}\ \Omega \propto \underset{\zeta=\zeta_m}{Max}\ Q_s \propto \underset{\zeta=\zeta_m}{Max}\ \varepsilon \tag{18}$$

where $\zeta_m$ is the optimal value of width-depth ratio $\zeta$ and $\zeta_m \geq 2$ for a rectangular form of cross section.

[36] As noted in Figure 2, there are two types of section that satisfy the laws of continuity, resistance and bedload transport: wider and shallower sections ($\zeta > \zeta_m$) and narrower and deeper sections ($\zeta < \zeta_m$). This mechanism is the same as that found earlier in frictional flow without transporting sediment. Because of the transport of sediment load, nevertheless, $\zeta_m > 2$ in most cases. Only when $Q_s = 0$ is the most efficient alluvial channel geometry exactly the same as the best hydraulic section defined in frictional flow without transporting sediment, for both occur at $\zeta_m = 2$. This means that $S_{fmin}$ defined in sediment-laded flow is just an extension of its counterpart defined earlier in frictional flow without transporting sediment.

[37] In alluvial channel systems, however, the "stationary" equilibrium flow regime that satisfies $S_f = S_{fmin}$ can be achieved only in special circumstances. Stable canals are such a special case because a high level of consistency is achieved between the theoretically derived most efficient alluvial channel geometry and the 'regime theory' developed empirically from stable canals [*Huang and Nanson*, 2000, 2001, 2002]. As well documented [e.g., *Lacey*, 1946; *Simons and Albertson*, 1960], the inputs of sediment load to those stable canals are artificially controlled such that the initially designed channels are allowed to adjust only slightly. Straight streams also appear to satisfy the condi-

tion of $S_f = S_{fmin}$, for the theoretically derived most efficient hydraulic geometry is also highly consistent with empirical hydraulic geometry relations developed from a wide range of geographical regions by *Julien and Wargadalam* [1995] and by *Huang and Warner* [1995] and *Huang and Nanson* [1995, 1998]. The observations from which those empirical hydraulic geometry relations have been developed are known to be carefully made only on reaches of stream that possess a single straight channel, usually of limited length.

[38] It is also only feasible in theory or through artificial design that a dynamic equilibrium flow regime that satisfies $S_f > S_{fmin}$ can be achieved through adopting two types of section in a straight single-thread channel system: wider and shallower sections ($\zeta > \zeta_m$) and narrower and deeper sections ($\zeta < \zeta_m$). Because of the possession of excess energy $S_f - S_{fmin}$ and the deformable nature of alluvium, flow is capable of using some of the excess energy to adjust not only channel cross-sectional shape but also planform. As a result, other than straight channel planforms develop such as meandering and braiding patterns.

## 6. Cause for Variations in River Channel Planform

[39] The energy that drives river channel flow comes from the valley gradient ($S_V$) and this is the product of historical geological processes (commonly fluvial, glacial and tectonic) [e.g., *Schumm*, 1977; *Schumm et al.*, 1987; *Knighton*, 1998]. As a consequence, a valley gradient is usually not directly the product of its present-day river but is in fact a condition imposed on the modern river and one that it must adjust to depending on imposed water and sediment loads from upstream. If the flow is aligned directly down the valley, the energy that a valley can provide ($\Omega_V$), can be described by:

$$\Omega_V = \gamma Q S_V \tag{19}$$

[40] According to studies by *Huang and Nanson* [2000, 2001, 2002], the minimum energy slope $S_{fmin}$ required for transporting imposed water load $Q$ and sediment load $Q_s$ in a straight single-thread alluvial channel can be determined from the following relationship:

$$S_{fmin} \propto Q_s^{0.708\sim0.522} Q^{-(0.788\sim0.702)} \propto \left(\frac{Q_s}{Q}\right)^{0.708\sim0.522} Q^{-(0.080\sim0.180)} \tag{20}$$

which leads to the determination of the required minimum energy as:

$$\Omega_{min} = \gamma Q S_{fmin} \propto Q_s^{0.708\sim0.522} Q^{(0.222\sim0.298)} \tag{21}$$

[41] Because $\Omega_V$ and $\Omega_{min}$ are determined by different mechanisms, a difference between $\Omega_V$ and $\Omega_{min}$ is almost always inevitable. Since both $\Omega_V$ and $\Omega_{min}$ include flow discharge $Q$, $S_V$ and $S_{fmin}$ can be used as their alternative expressions. As a result, there are three possibilities: (1) $S_V = S_{fmin}$; (2) $S_V > S_{fmin}$; and (3) $S_V < S_{fmin}$. Because $S_{fmin}$ represents the minimum amount of energy demanded by flow for transporting imposed water and sediment loads without causing erosion and deposition in a straight single-thread alluvial channel, the situation of $S_V < S_{fmin}$ means that equilibrium cannot be achieved in such a system. This is due to the insufficient supply of energy or, as noted in equation (20), due to the overload of sediments for available energy. As a result, sediment deposition and aggradation will be inevitable and channels may display an unstable braided and/or an unstable meandering pattern characterized by some wandering-planform rivers [e.g., *Chang*, 1979b; *Church*, 1983; *Bettess and White*, 1983; *Desloges and Church*, 1989; *Knighton*, 1998] or some anastomosing-planform rivers [e.g., *Makaske*, 2001]. These types of rivers can reach an equilibrium state only in circumstances in which their stream power (flow discharge and/or channel gradient) can be increased and/or their sediment load can be reduced. In other words, the equilibrium of these types of rivers relies on both flow and environmental conditions. Rivers traditionally recognized as ‘‘braided’’ rivers based on planar features can be formed due to either an overload of sediments or a very steep valley slope [e.g., *Leopold and Wolman*, 1957; *Parker*, 1976; *Chang*, 1979b; *Wang and Zhang*, 1989; *Knighton*, 1998; *Millar*, 2000]. Within the context of PME, however, only braided rivers due to excess energy (excess valley slope) are capable of achieving dynamic equilibrium through expending the *excess* part of the available energy. The behavior of these braided rivers depends fully on how flow expends its excess energy and for this reason we refer to ‘‘braided’’ rivers only as those formed due to excess energy in this study.

[42] In the situation of $S_V = S_{fmin}$, the valley gradient is exactly that required to transport the imposed water and sediment loads. Hence a straight single-thread channel is what the flow demands and the system not only uses energy in the most efficient way, but also will be both in equilibrium and laterally stable. The self-constructed bankfull geometry of alluvial channels in such a system will be highly compatible with the most efficient alluvial channel geometry. This is identifiable theoretically using the equivalent conditions of PME, such as minimum stream power, maximum sediment transporting capacity and maximum flow efficiency [e.g., *Chang*, 1979a, 1980a, 1980b; *White et al.*, 1982; *Huang and Nanson*, 2000, 2001, 2002].

[43] In the situation of $S_V > S_{fmin}$, however, energy provided by a valley is more than the minimum level that the flow demands. This is the case for valleys that are formed by processes more extreme than those required by the flow conditions of the present rivers (e.g., powerful tectonic forces, glaciation, extreme palaeofloods, etc.). In these valleys, normal bankfull flows aligned directly down over-steeped slopes will have excess energy to expend. For straight channels, Figure 2 shows that in theory either a wider and shallower or a deeper and narrower single-thread system can dissipate all of the excess energy over the increased boundary resistance. In reality, however, exceptionally deep channels are very rarely formed and maintained in a stable state because of severe bank collapse. Very wide channels are more common but develop localized zones of scour or deposition, forming bars or islands. In other words, to expend the excess energy, a straight single-thread alluvial channel will inevitably generate local scour laterally and/or vertically. Part of this adjustment process

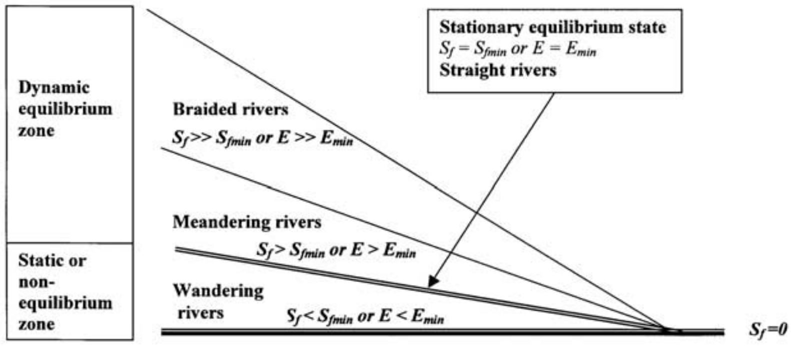


**Figure 3.** Schematic diagram of motion states and river channel patterns corresponding to the difference between energy supplied $S_f$ ($S_V$) or $E$ and energy minimally demanded by flow $S_{fmin}$ or $E_{min}$.

may limit channel widening to a degree but will also introduce a more sinuous path that reduces the energy gradient. As a result, channel cross-sectional shape and planform will mutually adjust in a complex way until a balance occurs between the excess energy and the resistance of channel bed and/or banks to scour.

[44] These changes in channel shape and planform that result from the expenditure of the excess energy can lead the flow away from the original uniform condition. Within the context of PME, the state of minimum energy occurs at a place at which these changes can make the flow's excess energy fully expended. A detailed investigation of the extent of the change in flow structure that arises from the adjustment in river channel shape and planform can lead to the determination of the conditions that satisfy PME. This, however, remains beyond the scope of this study.

[45] Nevertheless, relatively general illustrations of how an alluvial channel changes its planform when excess energy is supplied to a fluvial system have been given in several flume studies. *Ackers and Charlton* [1970, 1971] demonstrated that a flume channel initially stable and straight becomes sinuous as the flume gradient is increased. With increasing gradient the channel becomes progressively more sinuous, consuming progressively more excess energy at the bends in the form of distortion resistance. *Schumm and Khan* [1972] supported this observation and showed that, with further increases in flume gradient, the bank material proves unable to maintain a single meandering form, and a multiple braided-channel system results. In other words, braiding appears to consume more of the excess energy than does meandering.

[46] Field observations reveal that when the banks of alluvial channels contain silt and/or vegetation, local scouring by flow with excess energy will be confined within a single major channel that consequently meanders [*Schumm*, 1977, 1981, 1985; *Bettess and White*, 1983]. Importantly, two lines of evidence suggest that meandering channels normally show a tendency of maximally expending their excess energy. First, flume experiments have shown that water flowing through a bend has a higher energy loss than that flowing through a straight channel with the same water discharge [e.g., *Schukry*, 1950]. Second, field studies by *Nanson and Hickin* [1983, 1986] and *Hickin and Nanson* [1984] show that meandering rivers tend to maximize their rate of alluvial reworking (lateral shifting) and hence sediment transport by optimizing their bend curvatures. In addition to the evidence from meandering rivers, observations in flumes and field show that step-pool streams tend to adjust to the condition of maximum flow resistance or maximum energy expenditure [e.g., *Abrahams et al.*, 1995; *MacFarlane and Wohl*, 2003].

[47] For alluvial channels, particularly those with banks that are sandy and lack cohesive mud and/or binding vegetation, observations show that local scouring induced by flow with excess energy cannot be fully contained within a single major channel. The channel widens and as a result, sub-channels and associated in-channel bars are formed. By doing so, the braided channel system increases flow resistance and consumes excess energy [e.g., *Leopold and Wolman*, 1957; *Schumm and Khan*, 1972; *Schumm*, 1977; *Ashmore*, 1991; *Bristow and Best*, 1993; *Ferguson*, 1993].

[48] On the whole, it appears that the use of the following qualitative conditions to predict the potential formation of different river channel patterns for given bank material is a rational approach, for it provides the physical cause–energy and equilibrium:

| | |
|---|---|
| Braided channels | $S_V \gg S_{fmin};$ |
| Meandering channels | $S_V > S_{fmin};$ |
| Straight single-thread channels | $S_V = S_{fmin};$ |
| Nonequilibrium channels | $S_V < S_{fmin}.$ (22) |

[49] Letting $S_{Vcr}$ be the valley slope that distinguishes meandering channels from braided ones, the conditions in equation (22) can be more precisely written as:

| | |
|---|---|
| Braided channels | $S_V > S_{Vcr};$ |
| Meandering channels | $S_{fmin} < S_V < S_{Vcr};$ |
| Straight single-thread channels | $S_V = S_{fmin};$ |
| Nonequilibrium channels | $S_V < S_{fmin}.$ (23) |

[50] A schematic form of equation (23) is shown in Figure 3. It is interesting to note that our analytical results presented in equation (23) and Figure 3 are developed independently but are essentially consistent with the empir-

ical model proposed by *Bettess and White* [1983] for predicting river channel patterns. This consistency between two approaches suggests that PME forms a theoretical foundation for both. Importantly, the two approaches can be verified against field data from both natural and experimental streams using the methodology proposed by *Wang and Zhang* [1989].

[51] As analyzed earlier in this study, and observed previously by *Schumm* [1960, 1963, 1968, 1977, 1981, 1985], the capability of alluvial channels to resist lateral erosion determines the degree of sinuosity and the transition from a meandering channel to a braided one, when other flow conditions, such as flow discharge, sediment load and size, remain unchanged. In other words, the following relationship pertains:

$$S_{Vcr} \propto \tau_{bk} \tag{24}$$

where $\tau_{bk}$ represents the strength of bank material.

[52] Where different rivers are concerned, flow discharge $Q$, sediment load $Q_s$, sediment size $d$ and the minimally required channel slope $S_{fmin}$ all vary and so $S_{Vcr}$ needs to be replaced with the more general condition of stream power $\Omega_{Vcr}$ (=$\gamma Q S_{Vcr}$). Furthermore, previous studies have demonstrated that the general form of $S_{fmin}$ is a function of four variables: flow discharge $Q$, sediment load $Q_s$, channel roughness (or sediment size $d$ in relatively simple cases) and bank strength $\tau_{bk}$ [*Julien and Wargadalam*, 1995; *Huang and Nanson*, 1995, 1997, 1998, 2000, 2001, 2002; *Huang and Warner*, 1995; *Huang*, 1996]. As a consequence, equation (24) can be generally written as:

$$S_{Vcr} = f(Q,\ Q_s,\ d,\ \tau_{bk}) \tag{25}$$

[53] Letting $S$ be the finally formed channel slope, and because meandering rivers occur when $S_V < S_{Vcr}$ as shown in equations (22) and (23), $S$ is then less than $S_{Vcr}$, or $S < S_{Vcr}$, for meandering significantly increases channel length and so significantly reduces channel slope. In turn, braided rivers occur when $S_V > S_{Vcr}$ and thus $S > S_{Vcr}$, for this type of river takes an approximately straight path. As a consequence, a $S - Q$ relationship is inherent in equation (25) and can also be use to distinguish different channel patterns. The general form of such a relationship should be analogous to equation (25) and is so controlled by multiple variables, or:

$$S = f(Q,\ Q_s,\ d,\ \tau_{bk}) \tag{26}$$

[54] This explains why *Ferguson* [1987], *Dade* [2000], and *Bledsoe and Watson* [2001] found that the incorporation of sediment size into the bivariant based $S - Q$ relationship previously used by *Leopold and Wolman* [1957] usually improves the relationship. Further support for expressing the general form of the $S - Q$ relationship into the form of equation (26) comes from the detailed study of the influence of bank vegetation on alluvial channel patterns by *Millar* [2000]. By comparing the analytical results with a wide range of field observations, Millar demonstrated that the meandering-braiding transition slope is sensitive not only to flow discharge $Q$ but also to sediment size $d$ and bank vegetation. *Millar* [2000] also tried to distinguish wandering channel pattern from meandering and braided ones but found that the former shows no correlation with either flow discharge $Q$ or bank vegetation or sediment size $d$. As pointed out earlier, the "wandering" channel pattern is actually in the category of nonequilibrium rivers in equations (22) and (23). Hence *Millar*'s [2000] study suggests that the methods for predicting variations in channel patterns that are able to achieve dynamic equilibrium, such as meandering and braiding referred in this study, may not be appropriate for predicting variations in nonequilibrium channel forms.

[55] Because the general form of $S - Q$ relationship presented in equation (26) is the product of equation (23), both equations (23) and (26) can be used to predict the potential variations in river channel patterns. At this stage, however, there are difficulties in quantifying the effects of some environment-related variables on river flow in nonstraight channels at a general level, typically those of bank strength and channel roughness. These approaches therefore may best be applied to environments in which the effects of the environment-related variables can be determined or regarded as constants. On this aspect, the studies of *Wang and Zhang* [1989] and *Millar* [2000] shed light on the development of a suitable methodology for such an application. Furthermore, these approaches apply only to environments in which energy is oversupplied and so dynamically stable patterns can be achieved. For predicting variations in nonequilibrium channel form due to an overload of sediment, there has been no appropriate approach developed and further detailed study is required. Moreover, the approaches proposed here are derived only from a broad physical view of how fluvial systems tend to adjust their planform within the context of energy and equilibrium. A more detailed smaller-scale study of the way the available energy is expended in the situation of either $E > E_{min}$ or $E < E_{min}$ could lead to much more fruitful results. Further studies are underway.

## 7. General Form of Minimum Energy Principle (PME) and Clarifications for Extremal Hypotheses

[56] As demonstrated earlier, flow either in a nonadjustable channel that is either ideal frictionless or frictional or in an adjustable alluvial channel that transports sediment illustrates a "stationary" state when the supplied energy reaches a minimum, $E_{min}$. In all three cases, $E_{min}$ is determinable in the same way as solving the optimization problem:

$$E_{min}(\zeta) = Min\ \left[E_p(\zeta) + E_k(\zeta)\right] \tag{27}$$

subject to

$$\varphi_i(\zeta) = 0 \tag{28}$$

where $E_p$ is potential energy, $E_k$ is kinetic energy, and $\varphi_i(\zeta)$ represents $i$ number of constraints, including continuity of fluids, flow resistance and sediment transport, all of which are functions of a geometric variable of cross sections, $\zeta$ (either channel depth, or width or width and depth ratio).

[57] The determination of PME in equation (27) is exactly the same as finding the optimal condition of the

Hamilton function or Hamiltonia (sum of kinetic energy and potential energy), which is the most important operator of variational theorem in both classical and quantum mechanics [e.g., *Lanczos*, 1949; *Dugas*, 1957; *Stauffer and Stanley*, 1989; *Kroemer*, 1994]. This explains why PME possesses exactly the same physical meaning as the most generalized variational principle of least action (LAP), that of using the least possible amount of energy to complete a given task, such as transporting given water and sediment loads in fluvial systems. Hence PME can be regarded as a specific expression of LAP and this explains how LAP effectively governs open channel flow. Indeed, our recent studies have demonstrated that if there were no such principles as PME or LAP inherent in the Newtonian laws of motion, rivers would behave without such controls and there would be no regular channel geometry or channel patterns occurring across a wide range of geographical regions [e.g., *Huang and Nanson*, 2000, 2001, 2002; *Huang et al.*, 2002].

[58] As analyzed earlier, when a system possesses energy that is just equal to $E_{min}$, the motion will be in a perfect agreement with imposed restrictions in fluids or remain stationary in solid materials, so that no additional work or no work at all can be performed. This link between work and equilibrium is the base for *Leopold and Langbein*'s [1962] proposal of the extremal hypotheses of minimum variance and least work. In this sense, the two hypotheses can be regarded as essentially consistent with PME.

[59] When flow possesses excess energy, or $E > E_{min}$ ($E = E_p + E_k$), the law of energy conservation makes equation (27) or PME achievable only when the following condition is satisfied under the restriction of equation (28):

$$\underset{\zeta>\zeta_m \ or \ \zeta<\zeta_m}{Max} \quad E_{exp} \underset{\zeta=\zeta_m}{\rightarrow E_{min}} \tag{29}$$

in which $E_{exp}$ represents the energy expended by the flow on additional work and/or dissipation. In quantity, $E_{exp}$ varies within the range of between 0 and $E - E_{min}$ for $E > E_{min}$.

[60] This means that PME can be achieved only when the excess energy $E - E_{min}$ is maximally, that is fully, expended. However, excess energy is necessary for a system to be dynamic. Therefore most dynamic systems behave by maximizing energy expenditure. This explains why the extremal hypotheses of maximum energy expenditure or maximum friction factor or maximum flow resistance, or maximum rate of energy dissipation have been argued to be the fundamental principles governing alluvial channel flow [e.g., *Jefferson*, 1902; *Schoklitsch*, 1937; *Inglis*, 1947; *Davis and Sutherland*, 1980, 1983; *Huang*, 1983, 1988; *Abrahams et al.*, 1995; *Lamberti*, 1988; *Phillips*, 1991]. However, these extremal hypotheses are valid only if they deal specifically with the *excess* portion of available energy, or $E - E_{min}$, when $E > E_{min}$.

[61] During the process of maximizing the expenditure of excess energy, dynamic systems will inevitably encounter more frictional conditions than if they are static. Hence only when these dynamic systems are in a minimum energy state are they able to be frictionally minimal. Since friction represents energy dissipation into heat, the equivalence between PME and minimum friction shows that the principle of minimum energy dissipation (MED) derived in terms of entropy theory is indeed applicable to open channel flow, as argued by many scientists [e.g., *Yang*, 1971, 1987; *Yang and Song*, 1979; *Song and Yang*, 1980, 1982; *Yang et al.*, 1981]. However, among the many forms of MED that have been previously hypothesized, only those that are equivalent to PME, such as MFE (maximum flow efficiency), MSP (minimum stream power) and MSTC (maximum sediment transporting capacity), can be adopted. MED identified here from laws governing open channel flow might also be an extension of the Helmholtz and Korteweg's minimum energy dissipation theorems that were derived based on the assumption that the inertia terms (i.e., kinetic energy) in the Navier-Stokes equations are negligible. This extension, however, lacks proof and further detailed research is required.

[62] Furthermore, it is shown earlier that in flow with friction and/or sediment transport, available energy is expended on overcoming friction from both channel bed and banks for a given sediment load, while in frictionless conditions it is in the form of potential energy and kinetic energy. This means that Froude number is not a dominant controlling factor in alluvial channel flow and so the extremal hypothese of two forms of minimum Froude number proposed by *Jia* [1990] and by *Yalin and Silva* [1999, 2000] appear not to be appropriate. However, when channel width is constrained, Froude number can become a dominant controlling factor again even if there is friction and sediment transport. Steep mountain streams are typical examples of this kind of flow. Because these streams possess beds of coarse gravel or boulders or step pools, supercritical flow can be forced away from the uniform state. Where channel beds are so rough that the flow's excess energy can be fully expended, critical flow occurs. This may explain why critical flow and standing waves commonly occur in steep mountain streams [e.g., *Simon*, 1992; *Grant*, 1997; *Tinkler*, 1997a, 1999b].

[63] Finally, it needs to be pointed out here that PME only illustrates the simplest case of LAP for, as is known in both classic and quantum mechanics, LAP has been more often applied to solve dynamic problems. This means that a more detailed understanding of the behavior of other than straight rivers can be gained from a detailed investigation of the applicability of LAP in more complex environments where flow maybe nonuniform and even where there are secondary flows for the expenditure of excess energy. Such an investigation might be able to explain in what conditions meandering rivers exhibit a regular pattern [e.g., *Leopold and Langbein*, 1966; *Ferguson*, 1987] and why braided rivers tend to organize themselves into a critical state so as to exhibit dynamic scaling [e.g., *Sapozhnikov and Foufoula-Georgiou*, 1999].

## 8. Conclusions

[64] The most important finding of this study is that the widely applied *Bélanger-Böss* theorem of critical flow in open channel hydraulics can be generalized into a more versatile principle that has application, not only in ideal frictionless flow, but also in flow with friction (the best hydraulic section) and sediment transport (the most efficient alluvial channel geometry). This principle can be interpreted as the principle of minimum energy (PME) which elucidates open channel flow in terms of three scenarios based on the difference between available energy ($E$) and the minimum

level of energy ($E_{min}$) demanded by flow for motion: (1) $E < E_{min}$; (2) $E = E_{min}$; and (3) $E > E_{min}$.

[65] In the situation of $E < E_{min}$, there is insufficient energy available to satisfy the minimum demand of flow for transporting water with a sediment load. This produces a nonequilibrium situation whereby the flow has to drop part of its load causing the system to aggrade.

[66] Where $E = E_{min}$, no energy consuming adjustment is required to obtain an equilibrium flow regime in which the available energy is exactly that able to satisfy the minimum demand of the flow and is neither too little nor too much to transport the water with or without sediment supplied. Most importantly, this situation can be analogous to the state of incipient motion of solid materials, which has the least potential energy and characterizes the stationary state. The critical flow depth, the best hydraulic section and the most efficient alluvial channel geometry all occur in this state, where channel depth $h$ or shape factor $\zeta$ gains optimal values of $h_c$ or $\zeta_m$, respectively.

[67] In a situation where $E > E_{min}$, flow always performs additional work to expend the excess energy $E - E_{min}$. As a result, equilibrium is achieved in a dynamic (additional energy consuming) pattern. In ideal frictionless flow, the equilibrium is in the form of either supercritical ($F_r > 1$) or subcritical ($F_r < 1$) flow. In frictional and sediment-laded flow, the excess energy is fully consumed with friction from two theoretically possible types of channel geometry: wide and shallow ($\zeta > \zeta_m$) or deep and narrow ($\zeta < \zeta_m$). For natural streams, however, the theoretical channels, typically those in the range of $\zeta < \zeta_m$, are unlikely to be developed due to limited bank strength. In addition to changing their cross section, alluvial channels may also change energy consumption by changing their planform (pattern).

[68] Within the context of energy and equilibrium, the difference between the energy gradient of the valley ($S_V$) and the minimal energy gradient ($S_{fmin}$) demanded by flow in a straight single-thread channel provides a convincing causal explanation for the formation of commonly identified relatively stable river channel patterns. In the situation of $S_V = S_{fmin}$, a straight single-channel system will be both in equilibrium and stable, for the energy supplied by the valley gradient is in a complete agreement with that demanded by flow. In the situation where $S_V < S_{fmin}$, a straight single-channel system cannot achieve equilibrium in transporting imposed water and sediment loads due to the insufficient supply of energy. As a result, sediment deposition will occur in the form of splays and channel avulsion, as demonstrated in many vertically accreting wandering and anostomosing rivers. In the situation where $S_V > S_{fmin}$, excess energy is supplied and flow will generate scouring laterally and/or vertically while adjusting its channel' shape and size. When scouring can be confined within a major channel by relatively cohesive and/or vegetated banks, meandering channels occur. Weak banks, in contrast, cannot confine flow within a single well-defined channel and so excess energy is consumed by channel widening and the development of sub-channels or in-channel bars, leading to braiding. As a result, a causal approach that is based on the link between energy and equilibrium is proposed in equation (23) and in Figure 3 for use in predicting the formation of dynamically stable river channel patterns. Although derived purely from theoretical deduction, this approach is essentially consistent with the model proposed by *Bettess and White* [1983] and the commonly developed $S - Q$ relationships based on extensive field and laboratory evidence. Practically, this approach can be applied to predict river channel patterns in specific environments using the methodology proposed by *Wang and Zhang* [1989].

[69] Finally, it is argued that the existence of PME is widely accepted in both classic and quantum mechanics and its recognition in open channel flow enables the applicable conditions for extremal hypotheses to be clearly defined. While this study is largely limited to the behavior of uniform channel flow for gaining a general physical view of how alluvial channels tend to adjust themselves, it is also suggested that future studies following this approach may best be focused on the role of PME in nonuniform flow and/or the secondary flow for a better understanding of the behavior of rivers in other than straight single-channel systems.

[70] **Acknowledgments.** The authors are grateful to Greg Tucker, Trevor Hoey, Ted Hickin and anonymous reviewers for their helpful suggestions during the finalization of this manuscript.

## References

Abrahams, A., G. Li, and J. F. Atkinson (1995), Step-pool streams: Adjustment to maximum flow resistance, *Water Resour. Res.*, *31*, 2593–2602.
Ackers, P., and F. G. Charlton (1970), The geometry of small meandering channels, *Proc. Inst. Civil Eng., Suppl., London*, *12*, Pap. 7328S.
Ackers, P., and F. G. Charlton (1971), The slope and resistance of small meandering channels, *Proc. Inst. Civil Eng.*, *15*, suppl., Paper 7362S.
Ashmore, P. E. (1991), How do gravel bed rivers braid?, *Can. J. Earth Sci.*, *28*, 326–341.
Bagnold, R. A. (1966), An approach to the sediment transport problem from general physics, *U.S. Geol. Surv. Prof. Pap.*, *422-I*.
Bettess, R., and W. R. White (1983), Meandering and braiding of alluvial channels, *Proc. Inst. Civil Eng., Part 2*, *75*, 525–538.
Bettess, R., and W. R. White (1987), Extremal hypotheses applied to river regime, in *Sediment Transport in Gravel-Bed Rivers*, edited by C. R. Thorne, J. C. Bathurst, and R. D. Hey, pp. 767–789, John Wiley, Hoboken, N. J.
Bledsoe, B. P., and C. C. Watson (2001), Logistic analysis of channel pattern thresholds: Meandering, braiding, and incising, *Geomorphology*, *38*, 281–300.
Bristow, C. S., and J. L. Best (1993), Braided rivers: Perspective and problems, in *Braided Rivers*, edited by J. L. Best and C. S. Bristow, *Geol. Soc. Spec. Publ.*, *75*, 13–71.
Brownlie, W. R. (1983), Flow depth in sand-bed channels, *J. Hydraul. Eng.*, *109*, 959–990.
Chang, H. H. (1979a), Geometry of rivers in regime, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *105*, 691–706.
Chang, H. H. (1979b), Minimum stream power and river channel patterns, *J. Hydrol.*, *41*, 303–327.
Chang, H. H. (1980a), Stable alluvial canal design, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *106*, 873–891.
Chang, H. H. (1980b), Geometry of gravel streams, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *106*, 1443–1456.
Chang, H. H. (1985a), Design of stable alluvial canals in a system, *J. Irrig. Drain. Eng.*, *111*, 36–43.
Chang, H. H. (1985b), River morphology and thresholds, *J. Hydraul. Eng.*, *111*, 503–519.
Chang, H. H. (1988), *Fluvial Processes in River Engineering*, Krieger, Melbourne, Florida.
Chen, C. L. (1980), Discussion: Theory of minimum rate of energy dissipation, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *106*, 1541–1546.
Chow, V. T. (1959), *Open-Channel Hydraulics*, McGraw-Hill, New York.
Church, M. A. (1983), Pattern of instability in a wandering gravel bed channel, in *Modern and Ancient Fluvial System*, edited by J. D. Collinson and J. Lewin, *Spec. Publ. Int. Assoc. Sedimentol.*, *6*, 169–180.
Dade, W. B. (2000), Grain size, sediment transport and alluvial channel pattern, *Geomorphology*, *35*, 119–126.

Davis, T. R. H., and A. J. Sutherland (1980), Resistance to flow past deformable boundaries, *Earth Surf. Processes Landforms*, *5*, 175–179.

Davis, T. R. H., and A. J. Sutherland (1983), Extremal hypotheses for river behavior, *Water Resour. Res.*, *19*, 141–148.

Desloges, J. R., and M. A. Church (1989), Wandering gravel-bed rivers, *Can. Geogr.*, *33*, 360–364.

DuBoys, P. (1879), Le Rhone et les rivieres a lit affouillable, *Ann. Ponts Chaussees, Ser. 5*, *18*, 141–195.

Dugas, R. (1957), *A History of Mechanics*, translated from French by J. R. Maddox, Dover, Mineola, N. Y.

Ferguson, R. I. (1987), Hydraulic and sedimentary controls of channel pattern, in *River Channels: Environment and Process*, edited by K. S. Richards, pp. 129–158, Blackwell, Malden, Mass.

Ferguson, R. I. (1993), Understanding braiding processes in gravel-bed rivers, in *Braided Rivers*, edited by J. L. Best and C. S. Bristow, *Geol. Soc. Spec. Publ.*, *75*, 73–88.

Gleick, J. (1998), *Chaos: Making a New Science*, Vintage, London.

Gomez, B., and M. Church (1989), An assessment of bed load sediment transport formulae for gravel bed rivers, *Water Resour. Res.*, *25*, 1161–1186.

Grant, G. E. (1997), Critical flow constrains flow hydraulics in mobile-bed streams: A new hypothesis, *Water Resour. Res.*, *33*, 349–358.

Henderson, F. M. (1966), *Open Channel Flow*, Macmillan, Old Tappan, N. J.

Hickin, E. J., and G. C. Nanson (1984), Lateral migration rates of river bends, *J. Hydraul. Eng.*, *110*, 1557–1567.

Huang, H. Q. (1996), Discussion: Alluvial channel geometry: Theory and applications, *J. Hydraul. Eng.*, *122*, 750–751.

Huang, H. Q., and G. C. Nanson (1995), On a multivariate model of channel geometry, in *Proceedings of XXVIth IAHR Congress*, vol. 1, pp. 510–515, Thomas Telford, London.

Huang, H. Q., and G. C. Nanson (1997), Vegetation and channel variation; a case study of four small streams in southeastern Australia, *Geomorphology*, *18*, 237–249.

Huang, H. Q., and G. C. Nanson (1998), The influence of bank strength on channel geometry: An integrated analysis of some observations, *Earth Surf. Processes Landforms*, *23*, 865–876.

Huang, H. Q., and G. C. Nanson (2000), Hydraulic geometry and maximum flow efficiency as products of the principle of least action, *Earth Surf. Processes Landforms*, *25*, 1–13.

Huang, H. Q., and G. C. Nanson (2001), Alluvial channel-form adjustment and the variational principle of least action, in *Proceedings of XXIX IAHR Congress*, *Beijing*, pp. 410–415, Int. Assoc. for Hydraul. Res., Delft, Netherlands.

Huang, H. Q., and G. C. Nanson (2002), A stability criterion inherent in laws governing alluvial channel flow, *Earth Surf. Processes Landforms*, *27*, 929–944.

Huang, H. Q., and R. F. Warner (1995), The multivariate controls of hydraulic geometry; a causal investigation in terms of boundary shear distribution, *Earth Surf. Processes Landforms*, *20*, 115–130.

Huang, H. Q., G. C. Nanson, and S. D. Fagan (2002), Hydraulic geometry of straight alluvial channels and the principle of least action, *J. Hydraul. Res.*, *40*, 153–160.

Huang, W.-L. (1983), The extremity laws of hydro-thermodynamics, *Appl. Math. Mech.*, *4*(4), 499–510.

Huang, W.-L. (1988), The law of maximum rate of energy dissipation, paper presented at International Symposium on Hydraulics of High Dams, Chinese Hydraul. Eng. Soc., Beijing.

Ijjasz-Vasquez, E. J., R. L. Bras, and I. Rodriguez-Iturbe (1993), Hack's relation and optimal channel networks: The elongation of river basins as a consequence of energy minimization, *Geophys. Res. Lett.*, *20*, 1583–1586.

Inglis, C. C. (1947), *Meanders and Their Bearing on River Training*, Maritime and Waterways Eng. Div., Inst. of Civ. Eng., London.

Jaeger, C. (1955), *Engineering Fluid Mechanics*, Blackie Acad. and Prof., New York.

Jefferson, M. S. W. (1902), Limiting width of meander belts, *Natl. Geogr. Mag.*, *13*, 373–384.

Jia, Y. (1990), Minimum Froude number and the equilibrium of alluvial sand rivers, *Earth Surf. Processes Landforms*, *15*, 199–209.

Julien, P. Y., and J. Wargadalam (1995), Alluvial channel geometry: Theory and applications, *J. Hydraul. Eng.*, *121*, 312–325.

Kirkby, M. J. (1977), Maximum sediment transporting efficiency as a criterion for alluvial channels, in *River Channel Changes*, edited by K. J. Gregory, pp. 450–467, John Wiley, Hoboken, N. J.

Knighton, D. (1998), *Fluvial Forms and Processes: A New Perspective*, Edward Arnold, London.

Kroemer, H. (1994) *Quantum Mechanics: For Engineering, Materials Science, and Applied Physics*, Prentice-Hall, Old Tappan, N. J.

Lacey, G. (1946), A general theory of flow in alluvium, *Proc. Inst. Civ. Eng.*, *27*, 16–47.

Lamb, H. (1945), *Hydrodynamics*, Dover, Mineola, N. Y.

Lamberti, A. (1988), About extremal hypotheses and river regime, in *River Regime*, edited by W. R. White, pp. 121–134, John Wiley, Hoboken, N. J.

Lanczos, C. (1949), *The Variational Principles of Mechanics*, Univ. of Toronto Press, Toronto, Ontario, Canada.

Leopold, L. B., and W. B. Langbein (1962), The concept of entropy in landscape evolution, *U.S. Geol. Surv. Prof. Pap.*, *500A*.

Leopold, L. B., and W. B. Langbein (1966), River meanders, *Sci. Am.*, *214*, 60–70.

Leopold, L. B., and M. G. Wolman (1957), River channel patterns: Braided, meandering and straight, *U.S. Geol. Surv. Prof. Pap.*, *282-B*, 85 pp.

MacFarlane, W. A., and E. Wohl (2003), Influence of step composition on step geometry and flow resistance in step-pool streams of the Washington Cascades, *Water Resour. Res.*, *39*(2), 1037, doi:10.1029/2001WR001238.

Makaske, B. (2001), Anastomosing rivers: A review of their classification, origin and sedimentary products, *Earth Sci. Rev.*, *53*, 149–196.

Meyer-Peter, E., and R. Muller (1948), Formulas for bed load transport, paper presented at the 3rd Meeting of IAHR, Int. Assoc. for Hydraul. Res., Stockholm.

Millar, R. G. (2000), Influence of bank vegetation on alluvial channel patterns, *Water Resour. Res.*, *36*, 1109–1118.

Millar, R. G., and M. C. Quick (1993), Effect of bank stability on geometry of gravel rivers, *J. Hydraul. Eng.*, *119*, 1343–1363.

Millar, R. G., and M. C. Quick (1998), Stable width and depth of gravel-bed rivers with cohesive banks, *J. Hydraul. Eng.*, *124*, 1005–1013.

Nanson, G. C., and E. J. Hickin (1983), Channel migration and incision on Beatton River, *J. Hydraul. Eng.*, *109*, 327–337.

Nanson, G. C., and E. J. Hickin (1986), A statistical analysis of bank erosion and channel migration in western Canada, *Bull. Geol. Soc. Am.*, *97*, 497–504.

Parker, G. (1976), On the cause and characteristic scales of meandering and braiding in rivers, *J. Fluid Mech.*, *89*, 127–146.

Parker, G. (1979), Hydraulic geometry of active gravel rivers, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *105*, 1185–1201.

Phillips, J. D. (1991), Multiple modes of adjustment in unstable river channel cross-sections, *J. Hydrol.*, *123*, 39–49.

Pickup, G. (1976), Adjustment of stream-channel shape to hydrologic regime, *J. Hydrol.*, *30*, 365–373.

Prigogine, I., and I. Stengers (1984), *Order Out of Chaos*, Bantam Books, New York.

Richards, K. S. (1982), *Rivers: Form and Process in Alluvial Channels*, Methuen, New York.

Rodriguez-Iturbe, I., and A. Rinaldo (1997), *Fractal River Basins: Chance and Self-Organization*, Cambridge Univ. Press, New York.

Sapozhnikov, V. B., and E. Foufoula-Georgiou (1999), Horizontal and vertical self-organization of braided rivers toward a critical state, *Water Resour. Res.*, *35*, 843–851.

Schoklitsch, A. (1937), *Hydraulic Structure*, translated by Samuel Schulits, Am. Soc. of Mech. Eng., New York.

Schukry, A. (1950), Flow around bends in an open flume, *Trans. Am. Soc. Civ. Eng.*, *115*, 751–779.

Schumm, S. A. (1960), The shape of alluvial channels in relation to sediment type, *U.S. Geol. Surv. Prof. Pap.*, *352B*, 17–30.

Schumm, S. A. (1963), Sinuosity of alluvial rivers on the Great Plains, *Bull. Geol. Soc. Am.*, *74*, 1089–1100.

Schumm, S. A. (1968), River adjustment to altered hydrologic regime—Murrumbidgee River and paleochannels, Australia, *U.S. Geol. Surv. Prof. Pap.*, *598*.

Schumm, S. A. (1977), *The Fluvial System*, John Wiley, Hoboken, N. J.

Schumm, S. A. (1981), Evolution and response of the fluvial system, sedimentologic implications, *Spec. Publ. Soc. Econ. Paleontol. Mineral.*, *31*, 19–29.

Schumm, S. A. (1985), Patterns of alluvial rivers, *Annu. Rev. Earth Planet. Sci.*, *13*, 5–27.

Schumm, S. A., and H. R. Khan (1972), Experimental study of channel patterns, *Bull. Geol. Soc. Am.*, *83*, 1755–1770.

Schumm, S. A., M. P. Mosley, and W. E. Weaver (1987), *Experimental Fluvial Geomorphology*, John Wiley, Hoboken, N. J.

Simon, A. (1992), Energy, time, and channel evolution in catastrophically disturbed fluvial systems, *Geomorphology*, *5*, 345–372.

Simons, D. B., and M. L. Albertson (1960), Uniform water conveyance channels in alluvial materials, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *86*, 33–71.
Song, C. C. S., and C. T. Yang (1980), Minimum stream power theory, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *106*, 1477–1487.
Song, C. C. S., and C. T. Yang (1982), Minimum energy and energy dissipation rate, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *108*, 690–706.
Stauffer, D., and H. E. Stanley (1989), *From Newton to Mandelbrot: A Primer in Theoretical Physics*, Springer-Verlag, New York.
Thorn, C. E., and M. R. Welford (1994), The equilibrium concept in geomorphology, *Ann. Assoc. Am. Geogr.*, *84*, 666–696.
Tinkler, K. J. (1997a), Critical flow in rockbed streams with estimated values for Manning's *n*, *Geomorphology*, *20*, 147–164.
Tinkler, K. J. (1997b), Indirect velocity measurement from standing waves in rocked rivers, *J. Hydraul. Eng.*, *123*, 918–921.
U.S. Waterways Experiment Station (1935), Studies of river bed materials and their movement, with special reference to the lower Mississippi River, *USWES Pap. 17*, Vicksburg, Miss.
Wang, S., and R. Zhang (1989), Cause of formation of channel patterns and pattern prediction, in *Proceedings of the XXIIIth IAHR Congress, Ottawa, Ontario*, pp. B-131–B-136, Int. Assoc. for Hydraul. Res., Delft, Netherlands.
White, W. R., R. Bettess, and E. Paris (1982), Analytical approach to river regime, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *108*, 1179–1193.
Yalin, M. S., and A. M. F. Silva (1999), Regime channels in cohesionless alluvium, *J. Hydraul. Res.*, *37*, 725–742.
Yalin, M. S., and A. M. F. Silva (2000), Computation of regime channel characteristics on thermodynamics basis, *J. Hydraul. Res.*, *38*, 57–64.
Yang, C. T. (1971), On river meanders, *J. Hydrol.*, *13*, 231–253.
Yang, C. T. (1987), Energy dissipation rate approach in river mechanics, in *Sediment Transport in Gravel-Bed Rivers*, edited by C. R. Thorne, J. C. Bathurst, and R. D. Hey, pp. 735–766, John Wiley, Hoboken, N. J.
Yang, C. T., and C. C. S. Song (1979), Theory of minimum rate of energy dissipation, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *105*, 769–784.
Yang, C. T., C. C. S. Song, and M. J. Woldenberg (1981), Hydraulic geometry and minimum rate of energy dissipation, *Water Resour. Res.*, *17*, 1014–1018.

H. H. Chang, Department of Civil and Environmental Engineering, San Diego State University, San Diego, CA 92182, USA.

H. Q. Huang, School of Geography and the Environment, University of Oxford, Oxford OX1 3TB, UK. (he-qing.huang@geog.ox.ac.uk)

G. C. Nanson, School of Geosciences, University of Wollongong, Wollongong, NSW, 2522, Australia.