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MINIMUM G_2-ABERRATION FOR NONREGULAR FRACTIONAL FACTORIAL DESIGNS

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Deng and Tang proposed generalized resolution and minimum aberration criteria for comparing and assessing nonregular fractional factorials, of which Plackett–Burman designs are special cases. A relaxed variant of generalized aberration is proposed and studied in this paper. We show that a best design according to this criterion minimizes the contamination of nonnegligible interactions on the estimation of main effects in the order of importance given by the hierarchical assumption. The new criterion is defined through a set of B values, a generalization of word length pattern. We derive some theoretical results that relate the B values of a nonregular fractional factorial and those of its complementary design. Application of this theory to the construction of the best designs according to the new aberration criterion is discussed. The results in this paper generalize those in Tang and Wu, which characterize a minimum aberration (regular) 2^{m-k} design through its complementary design.

1. Introduction. This paper is concerned with orthogonal factorial designs of two levels, denoted by $+$ and $-$, respectively. As usual, a two-level factorial design is represented by an $n \times m$ matrix with entries $+$ and $-$, in which each row corresponds to a run and each column to a factor. Orthogonality here implies that for every two columns the four level combinations $(++)$, $(+-)$, $(-+)$ and $(--)$ occur with the same frequency, and thus an orthogonal design is in fact an orthogonal array of strength two. When such an orthogonal design is used to study m factors, it permits uncorrelated estimation of all the main effects. Very often, we are also interested in some interaction effects. In this case, it is useful to distinguish *regular* factorials from *nonregular* factorials.

By regular fractional factorials, we mean those designs which are determined by defining relations. Such designs are often referred to as 2^{m-k} designs [Box, Hunter and Hunter (1978)]. A 2^{m-k} design has m factors and 2^{m-k} runs. The defining relation is the complete set of defining words. A word consists of letters that are labels of factors. Including I, the all $+$ column, a 2^{m-k} design has 2^k defining words. The number of letters in a word is the length of the word. The defining relation divides the 2^m effects into 2^{m-k} mutually exclusive and exhaustive groups of size 2^k, such that effects within a group are confounded with each other and effects from different groups are mutually orthogonal.

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The resolution of a regular design is given by the length of the shortest word in the defining relation [Box and Hunter (1961)]. In a resolution R design, no effect containing $r < R$ factors is confounded with any effect containing fewer than $R - r$ factors. Fries and Hunter (1980) refined the resolution criterion by proposing minimum aberration. For a 2^{m-k} design D, let $A_i(D)$ be the number of words of length i in the defining relation. Then $W(D) = [A_1(D), \dots, A_m(D)]$ is called the word length pattern. Now for two designs D_1 and D_2, let r be the smallest integer such that $A_r(D_1) \neq A_r(D_2)$. Then D_1 has less aberration than D_2 if $A_r(D_1) < A_r(D_2)$. If no design has less aberration than D_1, then D_1 has minimum aberration. Results on minimum aberration can be found in Franklin (1984), Chen and Wu (1991), Chen (1992), Tang and Wu (1996), Chen and Hedayat (1996), Suen, Chen and Wu (1997), Cheng, Steinberg and Sun (1999) and Mukerjee and Wu (1997).

Recall that in a regular factorial, any two effects are either orthogonal or *fully* aliased. However, in a nonregular factorial, there exist two effects that are *partially* aliased, meaning that they are neither orthogonal nor fully aliased; see Definition 2 below. Because of this complex aliasing structure, nonregular factorials were traditionally not advocated when some interactions are potentially important. However, they have been receiving increasing attention in the past several years. Hamada and Wu (1992) showed that some interactions could be detected using nonregular factorials. Lin and Draper (1992) and Wang and Wu (1995) studied the projection properties of some small Plackett–Burman (1946) and related designs. Cheng (1995) investigated the projection properties of two level orthogonal arrays. The methods for constructing supersaturated designs using Hadamard matrices given in Lin (1993) and Wu (1993) also rely heavily on partial aliasing. This was analyzed in some detail by Tang and Wu (1997). A Hadamard matrix H of order n is an $n \times n$ orthogonal matrix of ± 1, that is $H^T H = nE$, where E is the identity matrix.

Deng and Tang (1999) recently proposed generalized resolution and minimum aberration, and their use as systematic criteria to compare and assess the "goodness" of nonregular factorials was justified from projection and from minimizing bias. The criteria are aimed to capture the projection properties, in contrast with Webb's (1964) resolution, which concerns the estimability of lower order effects. As argued in Deng and Tang (1999), generalized resolution and minimum aberration provide fruitful criteria for ranking different designs while Webb's resolution is mainly useful as a classification rule. Previously, Box and Tyssedal (1996) introduced the notion of *projectivity* to quantify the projection properties. The generalized resolution as shown in Deng and Tang (1999) offers a more precise description of the projection properties than does projectivity.

In this paper, a new variant of generalized minimum aberration is proposed and studied. Its definition and statistical justification are the subject of Section 2. The new variant is defined through a set of B values, a generalization of word length pattern. Theoretical results that relate the B values of a fractional factorial and those of its complementary design are derived in Sec-

tion 3. In Section 4 we discuss the application of the theoretical results to the construction of the best nonregular factorials according to the new criterion.

2. Criteria for selecting fractional factorials. We first review in Section 2.1 the generalized resolution and minimum aberration criteria as proposed in Deng and Tang (1999). Then in Section 2.2, a new variant of the generalized aberration criterion is proposed and its statistical justification given.

2.1. *Generalized resolution and minimum aberration.* A fractional factorial D is regarded as a set of m columns $D = \{d_1, \dots, d_m\}$ or as an $n \times m$ matrix $D = (d_{ij})$, depending on our convenience. For any k-subset $s = \{c_1, \dots, c_k\}$ of D with $0 \le k \le m$, define

$$J_k(s) = J_k(c_1, \dots, c_k) = \left| \sum_{i=1}^{n} c_{i1} \cdots c_{ik} \right|, \tag{1}$$

where, for example, c_{i1} is the ith component of column c_1. Note that in (1) $k = 0$ corresponds to $s = \phi$, the empty set. We define $J_0(\phi) = n$. It is in fact meaningful to associate ϕ with the all $+1$ column. These $J_k(s)$ values are instrumental in our development of generalized resolution and minimum aberration criteria. For this reason, we give the following definition.

DEFINITION 1. The $J_k(s)$ values in (1) are called the J-characteristics of design D.

The concept of J-characteristics is a natural generalization of defining relation. For a regular fractional factorial, the defining relation consists of those subsets s of columns such that $J_k(s) = n$ [for all the other subsets s, $J_k(s) = 0$].

DEFINITION 2. A fractional factorial D is said to be regular if $J_k(s) = 0$ or n for all $s \subseteq D$. It is said to be nonregular if there exists an $s \subseteq D$ such that $0 < J_k(s) < n$.

Such a formal definition is necessary, for classification according to whether n is a power of 2 is not very useful. First, we would rather regard a design of 24 runs given by three replicates of a 2^{7-4} design as regular since it has the same confounding property as the 2^{7-4} design. Second, four of the five nonequivalent Hadamard matrices of order 16 [Hall (1961)] should be treated as nonregular because of their complex aliasing structure. Two Hadamard matrices are said to be equivalent if one can be obtained from the other by permuting the rows or columns, or switching the signs for each row or column, or a combination of the above.

For a design D, let r be the smallest integer such that $\max_{|s|=r} J_r(s) > 0$, where the maximization is taken over all the subsets s of size r. Then its

generalized resolution is defined as

$$R(D) = r + \delta \quad \text{where } \delta = 1 - \max_{|s|=r} J_r(s)/n. \tag{2}$$

Deng and Tang (1999) showed that for a regular design, its generalized resolution is the same as its traditional resolution. They provided a projection justification for the generalized resolution criterion.

Note that $J_k(s)$ is a multiple of 4 for orthogonal designs [Deng and Tang (1999)]. Let D be a design of size $n = 4t$. Let f_{kj} be the frequency of k column combinations that give $J_k(s) = 4(t + 1 - j)$ for $j = 1, \dots, t$. The *confounding frequency vector* of this design is then defined to be the vector of length $(m-2)t$,

$$F(D) = [(f_{31}, \dots, f_{3t}); (f_{41}, \dots, f_{4t}); \dots; (f_{m1}, \dots, f_{mt})].$$

This vector retains certain essential information contained in J-characteristics in the same way as the word length pattern does to the defining relation for a regular design. When D is regular, $f_{kj} = 0$ for $j \geq 2$ and the reduced vector $(f_{31}, f_{41}, \dots, f_{m1})$ is exactly the word length pattern.

Let $f_l(D_1)$ and $f_l(D_2)$ be the lth entries in the confounding frequency vectors of two designs D_1 and D_2, $l = 1, 2, \dots, (m-2)t$. Let l be the smallest integer such that $f_l(D_1) \neq f_l(D_2)$. If $f_l(D_1) < f_l(D_2)$, then following Deng and Tang (1999) we say D_1 has less G-aberration than D_2. If no design has less G-aberration than D_1, then D_1 is said to have minimum G-aberration. This criterion reduces to the usual minimum aberration for regular factorial designs.

2.2. *A new aberration criterion and its statistical justification.* Minimum G-aberration is very stringent and it attempts to control J-characteristics in a very strict manner. We now propose a relaxed version of minimum G-aberration. Let

$$B_k(D) = \sum_{|s|=k} \beta_k^2(s) \tag{3}$$

where $\beta_k(s) = J_k(s)/n$ are *normalized* J-characteristics. In terms of the confounding frequency vector, we have $B_k(D) = \sum_{j=1}^{t} f_{kj}[1 - (j-1)/t]^2$. For two designs D_1 and D_2, let r be the smallest integer such that $B_r(D_1) \neq B_r(D_2)$. Then we say that D_1 has less G_2-aberration than D_2 if $B_r(D_1) < B_r(D_2)$. If no design has less G_2-aberration than D_1, then D_1 is said to have minimum G_2-aberration. If D is regular, then $B_k(D) = A_k(D)$, where $A_k(D)$ is the number of defining words of length k, which implies that minimum G_2-aberration is equivalent to minimum aberration for regular designs. This relaxed variant has an important practical advantage over the original minimum G-aberration in that it is computationally much easier, a useful property when we deal with large designs. One can introduce minimum G_e-aberration for any $e > 0$ by replacing $\beta_k^2(s)$ in (3) by $\beta_k^e(s)$. The case $e = 2$ permits some nice mathematical treatment, and we therefore concentrate on this case in the paper.

Some comments on the consistency of the two criteria of minimum G-aberration and minimum G_2-aberration are in order. For this purpose, we revisit Deng and Tang (1998) who studied all the designs taken from Hadamard matrices of order 16 under the former criterion. Applying minimum G_2-aberration to the designs identified in Deng and Tang (1998), we have found that minimum G-aberration designs also have minimum G_2-aberration. (The reverse is not true and actually we often have more than one minimum G_2-aberration design. See Section 4 for examples.) In general, we expect that rankings of a list of designs, based on the two aberration criteria, should tend to be consistent with each other. Nevertheless, there are examples for which the two criteria produce conflicting results. Consider two designs D_1 and D_2 in Deng and Tang (1998), denoted by 9.57 and 9.58 in that paper. The two designs have confounding frequency vectors $F(D_1) = [(3, 0, 18, 0)_3; (5, 0, 18, 0)_4; \ldots]$ and $F(D_2) = [(4, 0, 0, 0)_3; (14, 0, 0, 0)_4; \ldots]$, respectively, where the subscript k for $k = 3, 4$ denotes the group of frequencies given by k columns. Clearly, D_1 has less G-aberration than D_2. However, one can easily check that D_2 has less G_2-aberration than D_1. Common sense would suggest that D_2 is better than D_1 because D_2 only has one more combination of three columns having $J_3 = 16$ than D_1 but D_1 has 18 more combinations of three columns having $J_3 = 8$ than D_2. We add that neither design has minimum G- or G_2-aberration among the 74 designs of 9 factors identified in Deng and Tang (1998). Finally, we note that the ranking of all these 74 designs given by one criterion is indeed quite consistent with that given by the other.

Deng and Tang (1999) provided a statistical justification for the generalized resolution by showing that a design of the highest generalized resolution minimizes the contamination of nonnegligible *two factor* interactions on the estimation of main effects. In the following, we extend that argument to give a statistical justification for minimum G_2-aberration. We refer to Deng and Tang [(1999), Section 2.4] for a discussion on a number of ways in which a design can be judged.

Consider the scenario in which main effects are of primary interest and that although some interactions may not be negligible, we are not interested in estimating them. The question is, how does the presence of these nonnegligible interactions affect the estimation of main effects? To estimate the main effects, we fit the main effect model $Y = \beta_0 I + X_1\beta_1 + \varepsilon$, where Y denotes the vector of n observations, β_0 is the grand mean and I the all $+1$ column, X_1 is the original design matrix D and β_1 is the vector of all main effects and ε is the vector of random errors, assumed to have zero mean and constant variance. The true model is

$$Y = \beta_0 I + X_1\beta_1 + X_2\beta_2 + \cdots + X_m\beta_m + \varepsilon, \tag{4}$$

where for example β_2 is the vector of $\binom{m}{2}$ two-factor interactions and X_2 is the corresponding matrix obtained by taking products for all pairs of columns of $X_1 = D$. In general, β_j in (4) denotes the $\binom{m}{j}$ interactions of order j, and X_j is given by the collection of products of j columns from $X_1 = D$. The least

square solution $\hat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T Y = n^{-1} X_1^T Y$ from the fitted model has expectation (under the true model),

$$E(\hat{\beta}_1) = \beta_1 + C_2\beta_2 + \cdots + C_m\beta_m,$$

where $C_j = n^{-1} X_1^T X_j$ for $j \geq 2$. Note that $C_j\beta_j$ is the contribution of β_j to the bias $C_2\beta_2 + \cdots + C_m\beta_m$. Let $C\beta$ be a generic term in the sum $C_2\beta_2 + \cdots + C_m\beta_m$. Since β is unknown, one can minimize $C\beta$ through minimizing $\|C\|^2 = \operatorname{trace}(C^T C) = \sum_{i,j} c_{ij}^2$ as defined, where $C = (c_{ij})$. Simple algebra gives

$$\|C_j\|^2 = (j+1)B_{j+1} + (m-j+1)B_{j-1} \tag{5}$$

for $2 \leq j \leq m-1$ and $\|C_m\|^2 = B_{m-1}$, where B_j's are defined in (3). Note that $\|C_2\|^2 = 3B_3$ and $\|C_3\|^2 = 4B_4$, because $B_1 = B_2 = 0$. Also note that $\|C_m\|^2$ is completely determined by $\|C_2\|^2, \ldots, \|C_{m-1}\|^2$, because from (5) we see that each of the two vectors $(\|C_2\|^2, \ldots, \|C_{m-1}\|^2)$ and $(B_3, \ldots, B_m)$ uniquely determines the other. Under the hierarchical assumption that lower order effects are more important than higher order effects, we should then sequentially minimize $\|C_2\|^2, \ldots, \|C_{m-1}\|^2$. We have thus established the following result.

THEOREM 1. *Sequentially minimizing $\|C_2\|^2, \ldots, \|C_{m-1}\|^2$ is equivalent to sequentially minimizing $B_3, \ldots, B_m$. This amounts to saying that minimum G_2-aberration is equivalent to a criterion that sequentially minimizes the contamination of nonnegligible interactions on the estimation of main effects, in the order of importance given by the hierarchical assumption.*

If the original design is regular, then $\|C_j\|^2$ becomes the number of interactions of order j confounded with the main effects. We obtain the following corollary of Theorem 1.

COROLLARY 1. *A minimum aberration (regular) design sequentially minimizes the number of interactions of order j confounded with the main effects in the order given by $j = 2, \ldots, m$.*

Cheng, Steinberg and Sun (1999) showed that a minimum aberration design minimizes the number of two-factor interactions confounded with the main effects. Clearly, Corollary 1 generalizes their result.

3. Theory of minimum G_2-aberration. In this section, we derive a relationship between the B values of a design D and those of its complementary design $\bar{D}$. Here complementation is with respect to a saturated design. We first introduce some more notation and provide a preliminary result so that we will be better prepared for the general theory.

Let $H^* = (h_0, h_1, \ldots, h_{n-1})$ be a Hadamard matrix of order n, where $h_0, h_1, \ldots, h_{n-1}$ denote its columns. We assume that H^* is normalized so that $h_0 = I$, the all + column. Our saturated design is given by $H = (h_1, \ldots, h_{n-1})$. Without loss of generality, let $D = (h_1, \ldots, h_m)$ and $\bar{D} = H \setminus D = (h_{m+1}, \ldots, h_{n-1})$.

Our objective is to establish a relationship between the two sets of B_k values, $\{B_k(D);\ k = 1, 2, 3, \ldots\}$ and $\{B_k(\bar{D});\ k = 1, 2, 3, \ldots\}$, where $B_k(D)$ is given by (3) and $B_k(\bar{D})$ is defined similarly. For the theoretical development, it is convenient to define, by generalizing $\beta_k(s)$ in (3),

$$\beta_{j_1 \cdots j_k} = \frac{1}{n}\left|\sum_{i=1}^{n} h_{ij_1} \cdots h_{ij_k}\right|, \tag{6}$$

for any k columns $h_{j_1}, \ldots, h_{j_k}$, not necessarily distinct, in $H^* = (h_0, h_1, \ldots, h_{n-1})$. The following properties of $\beta_{j_1 \cdots j_k}$ are immediate.

LEMMA 1. *Let $\beta_{j_1 \cdots j_k}$ be defined in* (6). *Then we have*:

(i) $\beta_{j_1 \cdots j_k}$ *is invariant under a permutation of* $j_1, \ldots, j_k$.
(ii) $\beta_{j_1 \cdots j_k} = \beta_{j_1 \cdots j_{k-1}}$ *if* $j_k = 0$.
(iii) $\beta_{j_1 \cdots j_k} = \beta_{j_1 \cdots j_{k-2}}$ *if* $j_{k-1} = j_k$.
(iv) $\sum_{j_k=0}^{n-1} \beta^2_{j_1 \cdots j_k} = 1$.

Parts (i), (ii) and (iii) are obvious. Part (iv) follows by noting that $(h_0/\sqrt{n}, \ldots, h_{n-1}/\sqrt{n})$ is an orthonormal basis and considering the unit vector $w = (w_1, \ldots, w_n)$ with $w_i = h_{ij_1} \cdots h_{ij_{k-1}}/\sqrt{n}$.

Applying Lemma 1(iv), we first have

$$\sum_{I}\sum_{II}\sum_{j_{k+1}=0}^{n-1} \beta^2_{j_1 \cdots j_k\, j_{k+1}} = \binom{m}{p}\binom{n-1-m}{k-p}, \tag{7}$$

where $\sum_I$ and $\sum_{II}$ stand for

$$\sum_{I} = \sum_{1 \le j_1 < \cdots < j_p \le m} \quad \text{and} \quad \sum_{II} = \sum_{m+1 \le j_{p+1} < \cdots < j_k \le n-1},$$

respectively. Now write

$$\sum_{j_{k+1}=0}^{n-1} = \sum_{j_{k+1} \in S_0} + \sum_{j_{k+1} \in S_1} + \sum_{j_{k+1} \in S_2} + \sum_{j_{k+1} \in S_3} + \sum_{j_{k+1} \in S_4}, \tag{8}$$

where $S_0 = \{0\}$, $S_1 = \{j_1, \ldots, j_p\}$, $S_2 = \{1, \ldots, m\} \setminus \{j_1, \ldots, j_p\}$, $S_3 = \{j_{p+1}, \ldots, j_k\}$, and $S_4 = \{m+1, \ldots, n-1\} \setminus \{j_{p+1}, \ldots, j_k\}$. For $p = 0, 1, \ldots, k$, let

$$B_{kp} = \sum_{|s \cap D| = p} \beta_k^2(s), \tag{9}$$

where the summation is taken over all the subsets of k columns with p columns from D and $k - p$ columns from $\bar{D}$. We therefore have

$$B_{kk} = \sum_{|s \cap D| = k} \beta_k^2(s) = B_k(D) \quad \text{and} \quad B_{k0} = \sum_{|s \cap D| = 0} \beta_k^2(s) = B_k(\bar{D}).$$

We are now ready to establish the following identity on the B_{kp} values.

THEOREM 2. *Let B_{kp} be as defined in* (9). *Then we have*

$$\binom{m}{p}\binom{n-1-m}{k-p} = B_{kp} + (m-p+1)B_{(k-1)(p-1)} + (p+1)B_{(k+1)(p+1)} + [(n-1-m)-(k-p)+1]B_{(k-1)p} + [(k-p)+1]B_{(k+1)p},$$

in which B_{kp} is defined to be zero if $k < p$.

PROOF. By Lemma 1(ii), we have

$$\sum_I \sum_{II} \sum_{j_{k+1}\in S_0} \beta^2_{j_1\cdots j_k j_{k+1}} = \sum_I \sum_{II} \beta^2_{j_1\cdots j_k} = B_{kp}. \tag{10}$$

Parts (i) and (iii) of Lemma 1 show that $\sum_I \sum_{II} \sum_{j_{k+1}\in S_1} \beta^2_{j_1\cdots j_k j_{k+1}}$ is a sum of terms $\beta^2_{k-1}(s)$ with $(p-1)$ columns of s from D and $k-p$ columns of s from $\bar{D}$. However, in the above sum, each column combination with $(p-1)$ columns from D and $k-p$ columns from $\bar{D}$ is counted $(m-p+1)$ times. This can be seen as follows. For each combination $1 \le j'_1 < \cdots < j'_{p-1} \le m$ and any $1 \le j' \le m$ with $j' \ne j'_1, \dots, j'_{p-1}$, there corresponds a unique assignment of $\{j_1, \dots, j_p\}$ to $\{j', j'_1, \dots, j'_{p-1}\}$ satisfying $1 \le j_1 < \cdots < j_p \le m$, where j_{k+1} is always assigned to j'. Since there are $(m-p+1)$ choices of $1 \le j' \le m$ with $j' \ne j'_1, \dots, j'_{p-1}$, we obtain

$$\sum_I \sum_{II} \sum_{j_{k+1}\in S_1} \beta^2_{j_1\cdots j_k j_{k+1}} = (m-p+1)B_{(k-1)(p-1)}. \tag{11}$$

Now consider $\sum_I \sum_{II} \sum_{j_{k+1}\in S_2} \beta^2_{j_1\cdots j_k j_{k+1}}$. It is a sum of terms $\beta^2_{k+1}(s)$ with $(p+1)$ columns of s from D and $k-p$ columns of s from $\bar{D}$. However, each column combination with $(p+1)$ columns from D and $k-p$ columns from $\bar{D}$ is counted $(p+1)$ times in the sum. This is because for each combination $1 \le j'_1 < \cdots < j'_p < j'_{p+1} \le m$, there are $(p+1)$ assignments of $\{j_1, \dots, j_p, j_{k+1}\}$ to $\{j'_1, \dots, j'_p, j'_{p+1}\}$ satisfying $1 \le j_1 < \cdots < j_p \le m$. We thus obtain

$$\sum_I \sum_{II} \sum_{j_{k+1}\in S_2} \beta^2_{j_1\cdots j_k j_{k+1}} = (p+1)B_{(k+1)(p+1)}. \tag{12}$$

Similar arguments show that

$$\sum_I \sum_{II} \sum_{j_{k+1}\in S_3} \beta^2_{j_1\cdots j_k j_{k+1}} = [(n-1-m)-(k-p)+1]B_{(k-1)p}, \tag{13}$$

$$\sum_I \sum_{II} \sum_{j_{k+1}\in S_4} \beta^2_{j_1\cdots j_k j_{k+1}} = [(k-p)+1]B_{(k+1)p}. \tag{14}$$

The theorem then follows by combining (7), (8), (10), (11), (12), (13) and (14). □

Now let

$$T_{kp} = \binom{m}{p}\binom{n-1-m}{k-p} - B_{kp} - (m-p+1)B_{(k-1)(p-1)} \tag{15}$$
$$-[(n-1-m)-(k-p)+1)]B_{(k-1)p}.$$

Then the identity in Theorem 2 can be rewritten as

$$(p+1)B_{(k+1)(p+1)} + (k-p+1)B_{(k+1)p} = T_{kp}$$

with T_{kp} given in (15). Note that T_{kp} is determined by $B_{k'p}$, with $0 \le p \le k'$ and $k' \le k$.

THEOREM 3. *Given the B values of design $\bar{D}$, $[B_3(\bar{D}), B_4(\bar{D}), \ldots] = (B_{30}, B_{40}, \ldots)$, the $B_{(k+1)(p+1)}$ values with $0 \le p \le k$ for $k = 2, 3, \ldots$, can be determined via*

$$B_{(k+1)(p+1)} = (-1)^{p+1}\binom{k+1}{p+1}B_{(k+1)0} + T^*_{kp} \quad \text{for } p = 0, 1, \ldots, k, \tag{16}$$

where $T^*_{kp} = (p+1)^{-1}\binom{k}{p}\sum_{j=0}^{p}(-1)^{p+j}T_{kj}/\binom{k}{j}$ depends on $B_{k'p}$ values with $k' \le k$ and $0 \le p \le k'$. For $p = k$ in (16) we have $B_{(k+1)(k+1)} = (-1)^{k+1}B_{(k+1)0} + T^*_{kk}$.

Applying Theorem 3 successively, we see that $B_{(k+1)(k+1)}$ can be expressed as a constant plus a linear combination of $(B_{30}, B_{40}, \ldots, B_{(k+1)0})$. Furthermore, we can prove that the two leading coefficients have the same value $(-1)^{k+1}$. These are summarized in Theorem 4.

THEOREM 4. *Let B_{kp} be as defined in (9). Then we have $B_{(k+1)(k+1)} = a_0 + \sum_{j=3}^{k+1} a_j B_{j0}$, or equivalently*

$$B_{k+1}(D) = a_0 + \sum_{j=3}^{k+1} a_j B_j(\bar{D}),$$

where $a_0, a_3, \ldots, a_{k+1}$ depend on n, m and k. Moreover, we have $a_{k+1} = a_k = (-1)^{k+1}$.

Theorems 3 and 4 combine to provide a relationship between the B values, $[B_3(D), B_4(D), \ldots]$, of design D and the B values, $[B_3(\bar{D}), B_4(\bar{D}), \ldots]$, of the complementary design $\bar{D}$. The corresponding results for regular factorials are given in Tang and Wu (1996). Theorems 3 and 4 are applicable to both regular and nonregular factorials. Based on Theorem 2, Theorems 3 and 4 can be proved by following the same arguments as in Tang and Wu (1996). Suen, Chen and Wu [(1997), Corollary 2] obtained some more comprehensive results for two-level regular designs than Tang and Wu (1996) using the theory of linear codes. Since the results of this paper are *formally* identical to those in Tang and Wu (1996), we conclude that Corollary 2 of Suen, Chen and Wu (1997) is also applicable to nonregular designs. It is possible to prove this result

directly from the identity in Theorem 2 of this paper. The theory of nonlinear codes does not seem to help here. As far as we know, there seem to be no counterparts for B values introduced here in the theory of nonlinear codes.

4. Application. In this section, we discuss the application of the theoretical results in searching for minimum G_2-aberration designs. We first note that a minimum G_2-aberration design is defined, as given in Section 2, within the whole class of two-level orthogonal designs. Construction of minimum G_2-aberration designs in this strong sense appears to be a very difficult problem. So far, we are able to do so for m factors with $m \leq 5$. We are in the process of writing up the results and the paper will be submitted elsewhere. We consider in this paper constructing minimum G_2-aberration designs within the class of orthogonal designs from Hadamard matrices. This is not as restrictive as it seems, at least for practical applications. Orthogonal designs from Hadamard matrices are rich and readily available and almost all orthogonal designs used in practical experiments or for other purposes are indeed from Hadamard matrices.

A design having minimum G_2-aberration within the class of orthogonal designs from Hadamard matrices may not have minimum G_2-aberration within the whole class of orthogonal designs. This is because it is not true that every orthogonal design can be embedded into a Hadamard matrix. Vijayan (1976) showed that any $n \times m'$ Hadamard submatrix can be embedded into a Hadamard matrix of order n if $m' \geq n - 4$. By a Hadamard submatrix, we mean a matrix of $+1$ and -1 such that its m' columns are orthogonal. An orthogonal design of m columns together with the all $+1$ column gives a Hadamard submatrix of $m + 1$ columns, and thus can be embedded into a Hadamard matrix of order n if $m + 1 \geq n - 4$. Therefore, we conclude that for $m \geq n - 5$, a design having minimum G_2-aberration within the class of orthogonal designs from the *complete set* of nonequivalent Hadamard matrices (to be discussed below) also has minimum G_2-aberration within the whole class of orthogonal designs.

Another complication is that for given order n, several nonequivalent Hadamard matrices may exist. It is known that a Hadamard matrix of order n is unique up to equivalence for $n \leq 12$. However, there are precisely five nonequivalent Hadamard matrices of order 16 [Hall (1961)] and the corresponding numbers for orders 20 and 24 are 3 [Hall (1965)] and 60 [Kimura (1989)], respectively. For order 28, there are precisely 487 nonequivalent Hadamard matrices [Spence (1995)]. This implies that it is impossible to consider all nonequivalent Hadamard matrices for $n \geq 32$, although one can do so for $n \leq 28$. One way out of this dilemma is to consider only several nonequivalent Hadamard matrices given by familiar construction methods such as those of Paley (1933) and Williamson (1944).

In the following, we discuss how to find minimum G_2-aberration designs from a *specific* Hadamard matrix of order n. For $m \leq n/2$, a direct search can be carried out. The number of designs under consideration is $\binom{n-1}{m}$. For $(n + 2)/2 \leq m \leq n - 1$, we can first search for the "worst" designs by se-

quentially maximizing $B_3, -B_4, B_5, -B_6$ and so on. By Theorem 4 minimum G_2-aberration designs can then be obtained by taking the complementary designs of the "worst" designs. The number of designs under consideration is also $\binom{n-1}{m} = \binom{n-1}{\bar{m}}$ where $\bar{m} = n - 1 - m$. In particular, we have the following simple results.

LEMMA 2. (i) *if* $\bar{m} = n-m-1 = |\bar{D}| = 1, 2$, *then any design* D *has minimum* G_2*-aberration and* (ii) *if* $\bar{m} = |\bar{D}| = 3$ *and* $J_3(\bar{D})$ *is maximized, then* D *has minimum* G_2*-aberration.*

We now look at some computational issues. For a fixed design, calculation of the whole vector $(B_3, B_4, \dots, B_m)$ may be cumbersome for large m since it involves calculation of $2^m - 1 - m - \binom{m}{2}$ J-characteristics. The amount of computation can be greatly reduced if we use the shortened vector (B_3, B_4, B_5) to compare different designs. In this case, the number of J-characteristics to be evaluated is $\binom{m}{3} + \binom{m}{4} + \binom{m}{5}$. The idea of using (B_3, B_4, B_5) as a surrogate of $(B_3, B_4, \dots, B_m)$ to compare designs is in line with that of MA-5 classifer in Deng and Tang (1998). For details on MA-5 classifer, see Deng and Tang (1998).

For $(n+2)/2 \le m \le n-1$ and thus $m > \bar{m} = n - 1 - m$, the method of complementing brings the number of J-characteristics to be evaluated from $2^m - 1 - m - \binom{m}{2}$ down to $2^{\bar{m}} - 1 - \bar{m} - \binom{\bar{m}}{2}$ if $(B_3, \dots, B_m)$ is used and from $\binom{m}{3} + \binom{m}{4} + \binom{m}{5}$ down to $\binom{\bar{m}}{3} + \binom{\bar{m}}{4} + \binom{\bar{m}}{5}$ if (B_3, B_4, B_5) is used.

Even though computation can be reduced by using (B_3, B_4, B_5), we will still need to compare $\binom{n-1}{m}$ designs if a complete search is desired. The combinatorial number $\binom{n-1}{m}$ can become exceedingly large and for example for $n = 24$ and $m = 12$, it equals 1,352,078. In future research we plan to develop an efficient computational algorithm without a complete search of all the $\binom{n-1}{m}$

TABLE 1

The third Hadamard matrix of order 16 given in Hall (1961)*, where the all* +1 *column is omitted*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
−	+	−	−	+	−	+	+	−	−	+	+	−	−	+
−	+	−	−	+	−	+	−	+	+	−	−	+	+	−
−	−	+	+	−	−	+	+	−	−	+	−	+	+	−
−	−	+	+	−	−	+	−	+	+	−	+	−	−	+
−	−	+	−	+	+	−	+	−	+	−	−	+	−	+
−	−	+	−	+	+	−	−	+	−	+	+	−	+	−

TABLE 2

Minimum G_2-aberration designs of 16 *runs for* $m = 3, 4, \ldots, 14$ *factors, constructed from the third Hadamard matrix of order* 16 *given in Table* 1. *If there are more than one minimum G_2-aberration design for a given m, the design given here also has minimum G-aberration*

m	B_3	B_4	B_5	Design columns
3	0	—	—	{8 9 10}
4	0	0	—	{5 12 14 15}
5	0	0	1	{3 4 6 14 15}
6	0	3	0	{4 5 6 7 12 13}
7	0	7	0	{8 9 10 11 12 13 14}
8	0	14	0	{8 9 10 11 12 13 14 15}
9	4	14	8	{2 8 9 10 11 12 13 14 15}
10	8	18	16	{2 3 6 7 8 9 10 11 12 13}
11	12	26	28	{2 3 4 5 8 9 10 12 13 14 15}
12	16	39	48	{2 3 4 5 8 9 10 11 12 13 14 15}
13	22	55	72	{2 3 4 5 6 8 9 10 11 12 13 14 15}
14	28	77	112	{2 3 4 5 6 7 8 9 10 11 12 13 14 15}

designs. The idea of forward selection and backward elimination from regression analysis might be useful in this regard.

We conclude the paper with a complete solution to the problem of finding minimum G_2-aberration from the third Hadamard matrix of order 16 given in Hall (1961). See Table 1 for a display of the matrix.

Based on (B_3, B_4, B_5), we have obtained minimum G_2-aberration designs for $3 \le m \le 15$ and the results are given in Table 2.

We want to mention that, in general, minimum G_2-aberration designs are not unique for given n and m. For example, all the designs of 16 runs and 14 factors have minimum G_2-aberration, and the design given in Table 2 also has minimum G-aberration. Complete ranking and classification of 16 run designs by considering all the five nonequivalent Hadamard matrices of order 16 is contained in Deng and Tang (1998).

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