Discrete Comput Geom 6:135–153 (1991)

Discrete & Computational Geometry

Minimum Ideal Triangulations of Hyperbolic 3-Manifolds*

Colin Adams[1] and William Sherman[2]

[1]Department of Mathematics, Williams College, Williamstown, MA 01267, USA

[2]Department of Mathematics, UCLA, Los Angeles, CA 90024, USA

Abstract. Let $\sigma(n)$ be the minimum number of ideal hyperbolic tetrahedra necessary to construct a finite volume n-cusped hyperbolic 3-manifold, orientable or not. Let $\sigma_{\text{or}}(n)$ be the corresponding number when we restrict ourselves to orientable manifolds. The correct values of $\sigma(n)$ and $\sigma_{\text{or}}(n)$ and the corresponding manifolds are given for $n = 1, 2, 3, 4$, and 5. We then show that $2n - 1 \leq \sigma(n) \leq \sigma_{\text{or}}(n) \leq 4n - 4$ for $n \geq 5$ and that $\sigma_{\text{or}}(n) \geq 2n$ for all n.

1. Introduction

An ideal tetrahedron in hyperbolic 3-space is a tetrahedron with its four vertices all lying on the boundary of H^3, its edges being geodesics and its faces lying in geodesic planes. Such a tetrahedron has the sum of its dihedral angles around an ideal vertex equal to 180° and its opposite dihedral angles equal. In [11], Thurston proves that a noncompact finite volume hyperbolic 3-manifold can always be decomposed into a finite set of ideal hyperbolic tetrahedra. The least number of ideal tetrahedra that such a manifold can be decomposed into is an invariant for the manifold.

Each finite volume hyperbolic 3-manifold M has a fixed finite number of cusps, which is nonzero if and only if M is noncompact (see Chapter 5 of [9]). We are interested in determining the minimum number of ideal tetrahedra necessary to construct a finite volume hyperbolic 3-manifold of n cusps. Our interest in this question is generated by the relationship between the hyperbolic volume of a manifold and the minimum number of tetrahedra in an ideal triangulation of M. Specifically, in [4], a lower bound on the number of ideal tetrahedra decomposing an n-cusped hyperbolic 3-manifold was used to show that the volume of an n-cusped hyperbolic 3-manifold is strictly greater than $n(1.01494\ldots)$, for $n \geq 3$.

* Both authors were supported by NSF Grants DMS-8711495, DMS-8802266 and Williams College Research Funds.

Moreover, although the minimum number of ideal tetrahedra in a decomposition of a hyperbolic 3-manifold of small volume need not be small, Thurston has proved that there exists a constant β such that if M is a hyperbolic manifold of volume v, M comes from Dehn filling a manifold which decomposes into less than βv ideal tetrahedra (cf. [12]). Hence, an understanding of hyperbolic manifolds obtained from few ideal tetrahedra may yield an understanding of the set of all hyperbolic manifolds of low volume.

Throughout what follows, we assume that all of the ideal hyperbolic tetrahedra involved have positive volume. Although a given ideal triangulation of a hyperbolic 3-manifold may be realized by tetrahedra which fold back on one another, giving us some tetrahedra with negative volumes, it is true that there always exists an ideal tetrahedralization with all tetrahedra having positive volume. Such a triangulation can be obtained by triangulating a fundamental domain which is dual to the classical Ford domain. See [6] for more details.

Let $\sigma(n)$ be the minimum number of ideal tetrahedra necessary to construct a connected n-cusped hyperbolic 3-manifold, orientable or not. Let $\sigma_{\mathrm{or}}(n)$ be the minimum number of ideal tetrahedra necessary to construct a connected orientable n-cusped hyperbolic 3-manifold.

The correct values of $\sigma(n)$ and $\sigma_{\mathrm{or}}(n)$ and examples of corresponding manifolds are given for $n = 1, 2, 3, 4$, and 5 in Tables 1 and 2 shown in Section 6. Additionally, the best-known upper bounds on $\sigma_{\mathrm{or}}(n)$ for $n = 6$, 7, 8, and 9 are given. In the following section we discuss the results indicated in these two tables.

In Section 3 we point out that the manifolds corresponding to $n = 2, 3, 4$, and 5 in the orientable case and $n = 3$ in the nonorientable case are all obtainable from the manifolds corresponding to one fewer cusp by the same type of geometric operation. Section 4 contains a proof of the fact that $2n - 1 \leq \sigma(n) \leq \sigma_{\mathrm{or}}(n) \leq 4n - 4$ for $n \geq 5$ and that $\sigma_{\mathrm{or}}(n) \geq 2n$ for all n, along with the proof of the particular results for $n = 1$, 2, 3, 4, and 5. Section 5 then proves uniqueness of the corresponding manifolds in the nonorientable cases for $n = 3$ and 4. Section 6 describes the tables.

Recently, a census of all cusped hyperbolic manifolds which can be obtained from five or fewer ideal tetrahedra has been completed by Weeks and Hildebrand (see [7]). Their results give an independent verification of our results for $n = 1$ and 2 in the orientable case and $n = 1$, 2, and 3 in the nonorientable case.

We expect that the lower bounds on $\sigma_{\mathrm{or}}(n)$ and $\sigma(n)$ can be improved upon, with a better understanding of the combinatorics involved. It remains to determine their actual values for $n \geq 6$. In particular, it might be asked if $\sigma_{\mathrm{or}}(n) = \sigma(n)$ for $n \geq 5$. Additionally, it would be of interest to know which of the above results hold if ideal tetrahedra with nonpositive volume are allowed.

In finding the results in this paper, we were looking for n-cusped manifolds which could be cut up into the fewest possible ideal tetrahedra. In another recent paper [13], Thurston *et al.* were looking for ideal hyperbolic polyhedra which must be cut up into the largest possible number of ideal tetrahedra. They show that, for sufficiently large m, it takes at least $2m - 10$ ideal tetrahedra in order to be able to cut up particular given ideal hyperbolic polyhedra with m ideal vertices. This has interesting applications to binary trees. In their result, it is not assumed that the tetrahedra have positive volume.

We wish to thank the Topology Undergraduate Research Group at Williams College, which included Manuel Alfaro, Mark Conger, Michael Gray, Karen von Haam, and Michael McDougall for their help in confirming and tabulating our results.

2. Small Numbers of Cusps

Restricting ourselves to one cusp, we find $\sigma(1) = 1$, with the unique corresponding manifold being the Gieseking manifold. It is not hard to check that this is the only hyperbolic manifold obtained by gluing faces on a single ideal hyperbolic tetrahedron. This manifold is also the unique noncompact hyperbolic manifold of minimum volume. For more details on this manifold, see [3].

For two cusps, we have $\sigma(2) = 2$, the corresponding manifold again being unique. A proof of uniqueness can be obtained by attempting all the possible gluings. The two possible gluings that are produced can be shown to yield the same manifold. This manifold is a nonorientable manifold with two nonorientable cusps and is the unique 2-cusped hyperbolic manifold of minimum volume (see [4]).

For three cusps, $\sigma(3) = 4$. The corresponding manifold is obtained from a single ideal octahedron, with all dihedral angles equal to $\pi/2$. It is nonorientable with two nonorientable cusps and one orientable cusp. A proof of its uniqueness appears in Section 5.

In the case of four cusps, we have $\sigma(4) = 6$. This is a nonorientable manifold obtained from six regular tetrahedra with all four cusps nonorientable. A proof of its uniqueness also appears in Section 5.

For five cusps, $\sigma(5) = 10$. This corresponds to an orientable manifold, a picture of which occurs in Table 1. There are at least two nonorientable manifolds with five cusps which can also be constructed with ten tetrahedra, one with three nonorientable cusps and one with four nonorientable cusps.

Each of the manifolds above, for $n = 2, 3, 4$, and 5, share their volume with a manifold of one fewer cusp. For $n > 5$, the correct value of $\sigma(n)$ is not known.

We now restrict ourselves to orientable manifolds. For one cusp, $\sigma_{or}(1) = 2$. There are two manifolds with this number of tetrahedra, the figure-eight knot complement, and its sibling manifold, which is obtained by a (5, 1) surgery on one component of the Whitehead link. Both of these manifolds are obtained by gluing together the faces of two ideal regular tetrahedra. Note that the smallest known closed orientable hyperbolic manifold comes from surgery on this sibling of the figure-eight knot complement (see [14]). By checking all possible gluings on two tetrahedra, we can prove these are the only two possibilities.

For two cusps, $\sigma_{or}(2) = 4$. Both the Whitehead link complement and the 6_2^2 link complement can be decomposed into four ideal tetrahedra. The census of [7] shows that in fact there are exactly two other orientable manifolds which have two cusps and which decompose into four ideal tetrahedra.

In the case of three cusps, $\sigma_{or}(3) = 6$. An example of a link complement corresponding to this appears in Table 2.

For four cusps, $\sigma_{or}(4) = 8$. A corresponding link complement which appears in Table 2 double covers the corresponding nonorientable 2-cusped manifold.

In the case of five cusps, $\sigma_{or}(5) = 10$ since the previous 5-cusped manifold which demonstrated that $\sigma(5) = 10$ was an orientable manifold.

Note that each of the orientable examples contains incompressible twice-punctured disks. Hence, by cutting these link complements open along a twice-punctured disk, twisting a full twist, and then reidentifying, we obtain distinct links but with homeomorphic complements. In fact, if we reidentify the two copies of the twice-punctured disk by any homeomorphism of the twice-punctured disk, we obtain a hyperbolic manifold with the same volume as the original (see [1]). In these examples, such a reidentification does not destroy the triangulation and hence the manifolds giving the correct answers for $\sigma_{or}(n)$ when $n = 1, 2, 3, 4$, and 5 are not unique.

When $n \geq 6$, the correct values for $\sigma_{or}(n)$ are still unknown. Table 2 gives the conjectured answer of 16 for $n = 6$. When $n = 7$ and 9, the conjectured answers are 20 and 30, respectively, the corresponding manifolds being obtained by taking a twofold cover and a threefold cover of the 5-cusped example.

The best known example for $n = 8$ is obtained by gluing the 5-cusped example to the 6-cusped example along a twice-punctured disk in each.

When $n \geq 10$, the best known upper bound, $\sigma_{or}(n) \leq 4(n - 1)$, is provided by the cyclic cover of one component of the Whitehead link complement (see Theorem 4.1).

3. Drilling

Let M be an n-cusped hyperbolic 3-manifold that is obtained from an ideal polyhedron P when pairs of faces on P are identified. Let A and A' be a pair of triangular faces or a pair of quadrilateral faces on P which are to be identified and which share exactly one vertex v. Assume further that there are exactly two other faces B and C which also share the vertex v.

Theorem 3.1. *If there exists an ideal triangulation of P with r tetrahedra such that the edges which are added to B and C do not go to v and in the case A and A' are quadrilaterals, the edges added to A and A' do go to v, then M comes from Dehn filling one cusp of an $(n + 1)$-cusped manifold which has a triangulation with $r + 2$ ideal tetrahedra.*

Proof. Add an ideal vertex to the center of A. Connect this central vertex to each of the vertices on the boundary of A by an edge. Do the same for A'. Let e and e' be the new edges on A and A' which run from the center of each face to v. Connect the two central vertices of A and A' by an unknotted edge f running through the interior of P. Let D be the disk in P which is bounded by e, e', and f. We then collapse D down to a single edge as we shrink f to a point which becomes a new ideal vertex. In the process, we identify e to e'. Call the resultant edge e''. If we add the edges to B and C corresponding to the tetrahedralization of P, we have one triangle from each sharing the vertex v. Hence there are two tetrahedra sharing the edge e''. Cut the two tetrahedra off P and call the resulting polyhedron P'. Then P' is combinatorially equivalent to P and hence can be cut up into r ideal tetrahedra. Thus, M' can be obtained from $r + 2$ ideal tetrahedra. □

Note that if the gluing of A to A' is orientation-preserving, we are removing a solid torus from M to obtain M'. If the gluing is orientation-reversing, we are removing a solid Klein bottle.

The manifold M' is not necessarily hyperbolic. In particular, if the map identifying A to A' fixes the vertex v, M will not be hyperbolic, since the new cusp together with the cusp corresponding to v will contain the boundary of an incompressible annulus.

However, in many cases, the manifold M' will be hyperbolic. In particular, this procedure yields a minimally triangulated orientable hyperbolic $(n + 1)$-cusped manifold from a minimally triangulated orientable hyperbolic n-cusped manifold for $n = 1, 2, 3$, and 4. For these values of n, the resulting manifold M' can be seen to be hyperbolic since, in each case, it is homeomorphic to an augmented alternating link complement, all of which are hyperbolic (see [2]).

The same type of drilling operation on a pair of triangular faces sharing an edge produces the minimally triangulated nonorientable hyperbolic manifold of three cusps from the corresponding nonorientable manifold of two cusps, however, no such drilling operation works to go from the minimally triangulated nonorientable hyperbolic manifold of $n + 1$ cusps for $n = 1, 3$, or 4.

4. Bounds

Theorem 4.1. *For all $n \geq 2$, $\sigma_{\mathrm{or}}(n) \leq 4n - 4$.*

Proof. We have already seen examples of manifolds which make this theorem true for $n = 2$. Hence, assume $n \geq 3$. Let L be a link obtained by taking an $(n - 1)$-fold cyclic cover over one component of the Whitehead link. Then L has n components and its complement is obtained from $n - 1$ ideal octahedra. Thus, $S^3 - L$ decomposes into $4n - 4$ ideal tetrahedra. □

We precede Theorem 4.6, which gives lower bounds for $\sigma(n)$ and $\sigma_{\mathrm{or}}(n)$, by some notation for ideal triangulations and several lemmas which may be useful in further investigations. Assume M is an n-cusped hyperbolic 3-manifold which decomposes into m ideal tetrahedra. Then, by Euler characteristic considerations, the number of edge types in the decomposition is also m.

We separate the ideal tetrahedra making up our manifold M into five types. Type I tetrahedra have all four vertices corresponding to distinct cusps. Type II tetrahedra have their four vertices corresponding to only three distinct cusps. Type III tetrahedra have three vertices corresponding to one cusp and the fourth vertex corresponding to a distinct cusp. Type IV tetrahedra have two vertices corresponding to one cusp and the remaining two vertices corresponding to a distinct cusp. Type V tetrahedra have all vertices corresponding to the same cusp. We denote cusps by capital letters and tetrahedra by the set of four capital letters coming from the cusps that correspond to their four vertices.

Lemma 4.2. *If M is orientable, a type II tetrahedron in the decomposition of M will have at least five distinct edge types on it.*

Proof. If not, there must be exactly four distinct edge types on such a tetrahedron T. By the labeling of the vertices on T, the two faces which each have a pair of edges identified would form a thrice-punctured sphere in M. If any of the three loops circling the punctures of the thrice-punctured sphere were trivial in the fundamental group of M, this tetrahedron would collapse to a triangle, contradicting the fact we are assuming all tetrahedra have positive volume. Hence, the thrice-punctured sphere is incompressible and must lift to a subset of H^3 with limit points on a circle in the sphere at ∞ by the results of [1]. However, in order that the four ideal vertices of the tetrahedron lie on a circle, the tetrahedron must be flat, that is, have a dihedral angle of $180°$. This again contradicts our assumption that all the tetrahedra have positive volume. □

Note that if M is not orientable, there is no such restriction on type II tetrahedra. See the 3-cusped manifold constructed from four ideal tetrahedra in Table 1 of Section 6, for instance.

Lemma 4.3. *If M is orientable, there cannot exist two type III tetrahedra, denoted $GBBB$ and $HBBB$, which share their BBB faces and which have only one GB edge type and one HB edge type.*

Proof. If such a situation did occur, any one of the GBB faces together with the HBB face which shares the BB edge with it would form a thrice-punctured sphere in M. Again, the thrice-punctured sphere would have to be incompressible and, as above, the dihedral angle between the two faces would have to be 180°. Since this would occur for all three GBB faces, we would have 540° of BB edge type on these two tetrahedra. But since the BB edges on these two tetrahedra should add up to 360°, 180° at the base of each of the two tetrahedra, this is a contradiction. □

Lemma 4.4. *In a minimum ideal triangulation of a 3-manifold M, there cannot exist an edge type which appears once on each of exactly three tetrahedra.*

Proof. If such an edge type existed, we could glue the three tetrahedra together around it and then drop it as an edge type, replacing the three tetrahedra by two tetrahedra which share a single face dual to the edge. This would contradict the minimality of the triangulation. □

Lemma 4.5. *If M is a 5-cusped hyperbolic 3-manifold and M is constructed out of type I tetrahedra, then there are at least ten tetrahedra.*

Proof. If the number of edge types were less than ten, then there exists two cusps D and E which are not connected by an edge. Hence, D and E can never occur on the same tetrahedron. Each cusp occurs on a type I tetrahedron and therefore occurs on at least three distinct edge types. Since the Euler characteristic of the boundary of a neighborhood of each vertex is 0, each of D and E must occur on at least six tetrahedra. Hence, there are at least 12 tetrahedra, contradicting our assumption that there were less than ten. □

Theorem 4.6. *For $n = 1$, we have $\sigma(1) = 1$. If $n = 2, 3$, or 4, $\sigma(n) = 2n - 2$. If $n \geq 5$, then $\sigma(n) \geq 2n - 1$. For all positive values of n, $\sigma_{\mathrm{or}}(n) \geq 2n$.*

Proof. Since there is only one manifold obtainable from a single ideal hyperbolic tetrahedron by attempting all the possible ways to glue faces and since there is a 2-cusped hyperbolic manifold obtained from two tetrahedra, we have $\sigma(1) = 1$ and $\sigma(2) = 2$. Hence, we restrict ourselves to $n \geq 3$ for the following.

Assume M is an n-cusped hyperbolic 3-manifold which has been decomposed into m ideal tetrahedra. We build a polyhedron by attaching the tetrahedra, one at a time, and we count edge types whenever a new cusp appears.

Assume first that there exists at least one type I tetrahedron. Then there are six distinct edge types on this tetrahedron interconnecting four cusps. Glue onto this tetrahedron all tetrahedra that do not involve new cusps. Then glue on any type I tetrahedron that does involve a new cusp. This will increase the number of edge types which have appeared on or in the resulting polyhedron by three. Continuing in this manner, we obtain $3(k - 2)$ edge types for each of the k new cusps appearing on a type I tetrahedron that we glue on.

Assume now that there exists a type II tetrahedron with one vertex corresponding to a cusp which has not yet appeared and we can glue this tetrahedron onto the current polyhedron P. There are at least four distinct edge types on a type II tetrahedron because of the three distinct cusps. Of these four edge types, we have already counted at most one. Hence, at this point we have $3(k - 2) + 3$ edge types coming from $k + 1$ cusps. Note that Lemma 4.2 implies that if M is orientable, we in fact have one more edge type than this at this point.

After gluing on all tetrahedra which do not involve new cusps and repeating the previous steps for any type I tetrahedra that we can now glue on, we glue on an additional type II tetrahedron involving a new cusp. This tetrahedron will have two new edge types involving the new cusp. Hence, after repeating this procedure for all type II tetrahedra that we can glue on, we have $3(k - 2) + 2r + 1$ edge types, where k is the number of new cusps first appearing on type I tetrahedra and r is the number of new cusps first appearing on type II tetrahedra. If M is orientable, we have at least one more edge type than this.

Assume now we have a new cusp coming from a type III tetrahedron. Call the new cusp G and the single cusp that the other three vertices are associated to, B. There is at least one GB edge type.

Rather than counting the edge types of BB edges, we count up the total angle around BB edges, where each 360° of BB angle is considered equivalent to the existence of a BB edge type. The type III tetrahedron must be attached to P along the BBB face. Since the edges on the face of P that we glue it to were not associated to any other cusp previously, we have a full 360° of edge type BB. Hence, we can associate a BB edge type and a GB edge type to G. Note that we have already counted at most one BB edge type when we were dealing with the type II tetrahedra. This single previously counted BB edge type, if it exists, will add some extra angle into our BB angle sum and thus can remain a separate edge type. In this manner, we obtain at least two additional edge types from each of the s new cusps coming from type III tetrahedra.

Note that if a type I, II, or III tetrahedron is present in the triangulation, then a new cusp can never occur on a type IV or type V tetrahedron. If only type IV or type V tetrahedra are present, we can have at most two cusps, hence we ignore this situation.

Thus, we have at least $3(k - 2) + 2r + 2s + 1$ edge types, if k and r are nonzero, where $k + r + s = n$. But if $k > 0$, then $k \geq 4$, and we have at least $2n - 1$ tetrahedra in this case, $2n$ if we assume M is orientable.

If both r and s are 0, then we have at least $3(k - 2)$ tetrahedra, which, for $k \geq 5$, yields at least $2n - 1$ tetrahedra. Note that when $n = k = 4$, there is a 4-cusped manifold obtained from $2n - 2$ type I tetrahedra.

In the orientable case, when $r = s = 0$, we know that, for $k \geq 6$, $3(k - 2) \geq 2k$ as desired, and, for $k = 5$, we can appeal to Lemma 4.5. Thus, since $k \geq 4$, we need only check that when $k = 4$ there must be eight tetrahedra. If there were six tetrahedra, there would be six edge types. All of the tetrahedra would be of the form $ABCD$ and so all the edge types on a given tetrahedron would have to be distinct. Take two tetrahedra sharing an ABC face and glue them together along this face. The two ABD faces, one from each tetrahedron, cannot be glued together as that would form an edge type of order two. Therefore, they will together form an incompressible thrice-punctured sphere in M. As mentioned previously, this forces the two faces to have dihedral angle 180°. Similarly for the pair of ACD faces and the pair of BCD faces. But this is not possible for two tetrahedra.

Again in the case $k = 4$, if there are seven tetrahedra, we have seven edge types, meaning there are, without loss of generality, two AB edge types. Each tetrahedron has exactly one of these two AB edge types on it. Hence, there exists a tetrahedron that is glued to a second tetrahedron that has the same AB edge type. Exactly as for the six tetrahedral case, this implies the existence of several incompressible thrice-punctured spheres from pairs of faces on the two tetrahedra and hence a contradiction.

If $k \neq 0$ and $s \neq 0$, but $r = 0$, then there must have been some type III tetrahedra which did not involve new cusps in order to get from the type I tetrahedra to the type III tetrahedra. As above, this forces the existence of at least one extra edge type if M is nonorientable and two extra edge types if M is orientable. Hence, we have $3(k - 2) + 2s + 1$ edge types which again yields at least $2n - 1$ tetrahedra in the nonorientable case and $2n$ tetrahedra in the orientable case.

If $k = 0$, but $r \neq 0$, then we start with a type II tetrahedron, say $ABBC$. This has three cusps and four (or five in the orientable case) distinct edge types. Since there are no type I tetrahedra, the ABC faces on this tetrahedron must glue to type II tetrahedra, where the fourth vertex in each case is A, B, or C. If in either case the fourth vertex is A or C, we pick up an extra AA or CC edge type without increasing the number of cusps counted so far. Since all the remaining cusps will contribute at least two edge types, we would have a total of $2n - 1$ or $2n$ edge types in the general and orientable cases, respectively.

Assume the fourth vertex in each case is B. There is a remaining ABC face on each of the new tetrahedra. Again, only in the case that the fourth vertex of each of the tetrahedra gluing to these ABC faces are B vertices do we not have an extra edge class. Continuing in this manner, we either obtain an extra edge class or we have a set of $ABBC$ tetrahedra which glue around the AC edge. In this last case, there must be at least four tetrahedra around this AC edge by Lemma 4.4. The BB edges opposite this central AC edge will have angle sum equal to the sum of their opposite angles, that is, 360°. The AB and BC edges on this wheel will each have angles summing to at least 360°. Assuming we have more than three cusps, the next type II tetrahedron which is glued onto the wheel will force the existence of either a second AB or BC edge class, and the existence of a second BB edge class. If there is no other type II tetrahedron, the next type III tetrahedron that we glue on will force the existence of an extra AB or BC edge type. Each new cusp after this point will add at least two edge types, and thus we end up, whenever $n \geq 4$, with at least $2n - 1$ tetrahedra. In the case that we have exactly three cusps, the example of a 3-cusped manifold obtained from four tetrahedra is realized by four type II tetrahedra.

In the orientable case, each of the original tetrahedra around the AC edge has five distinct edge types. If each of these tetrahedra does not have the same set of five edge types, we have at least six edge types from these three cusps and at least two edge types from each of the remaining cusps, yielding the requisite $2n$ edge types. Hence, we can assume all these tetrahedra have the same five edge types (again, counting angles in the case of the BB edge type, not actual edge types). Without loss of generality, we can assume there is only one BC edge type among these five and two AB edge types. If there were another AB or BC edge type in the manifold, we could count that and be done, since it would never be associated to any subsequent cusps. In order that there be only one BC edge type in the manifold, there must be exactly four tetrahedra in this wheel, by angle counting. The BBC faces on the wheel glue together in pairs. As above, we can assume there are no other type II tetrahedra. Similarly, there are no type IV or type V tetrahedra. In order for the angles to work out so that there are exactly two AB edge types, there must be exactly two $ABBB$ tetrahedra. This gives us an extra BB edge type. But then we have $2n$ edge types unless there is an additional new cusp D occurring for the first time on a type III tetrahedron which is associated to this new BB edge type. However, D must occur on at least two $DBBB$ tetrahedra and, hence, there are too many BB edge types.

Finally, if $k = 0$ and $r = 0$, we start with a type III tetrahedron $GBBB$. There must be at least one other $GBBB$ tetrahedron since the number of GBB faces on a $GBBB$ tetrahedron is odd while the number of GBB faces on a $GGBB$ tetrahedron is even. Hence, we have at least two edge types to associated to G, 360° of GB edge type and 360° of BB edge type. If there is a third $GBBB$ tetrahedron, there must be at least a fourth to match up the $GBBB$ faces, giving us two extra edge types. If there are any $GGBB$ tetrahedra in addition, we also have two more edge types, a GG edge type and a second GB edge type. Since there are no type II tetrahedra, any cusp other than B which appears three times on a tetrahedron must also appear once on some tetrahedron. As we argued for G, such a cusp has at least two associated edges. Hence, every cusp with the possible exception of B has at least two associated edges and we have at least $2n-2$ tetrahedra. If there are any other type III or type IV tetrahedra, we pick up at least two additional edge types and have $2n$ tetrahedra. Thus, the only way to have less than $2n$ tetrahedra is if every cusp J other than B appears on exactly two $JBBB$ tetrahedra with the possibility of the existence of one $BBBB$ tetrahedron.

Assume that in fact we do have exactly $2n-2$ tetrahedra forming a manifold as above. For each cusp J other than B, connect the two $JBBB$ tetrahedra along a JBB face, and call the resulting object a "pyramid," with the two BBB faces forming its base. Only the BBB faces glue to other pyramids. In fact, the manifold must consist of $n-1$ pyramids arranged in a cycle, meeting along BBB faces. Now, we require the existence of exactly $n-1$ BB edge types, since we have GB, HB, and so on, making $n-1$ edge types, and we must have a total of $2n-2$ edge types. The total number of BB edges on the tetrahedra is $6n-6$ and therefore there is an average of six BB edges in each BB edge class.

Consider our G pyramid. The BB edge types in the neighboring pyramids propagate themselves through our pyramid by the way the BBB faces on these pyramids are glued to the BBB faces on the G pyramid and by how the $GBBB$ faces on the G pyramid are glued. The G pyramid has one BB edge crossing the base diagonally, and four BB edges on the exposed GBB faces, which are identified in pairs. Thus there are at most three BB edge types with components in our pyramid (this applies, of course, to all pyramids). We call a BB edge type which appears on all the tetrahedra at least once a traverser. Note that there are therefore at most three traversers. Each traverser edge type must have at least $2n-2$ components.

We call a BB edge type which is not a traverser a reverser, since an edge type can avoid traversing the cycle of pyramids only by reversing direction at least twice as it travels in the cycle of pyramids. That is, if the two exposed GBB faces on a single tetrahedron in the G pyramid glue to each other, then one BB edge type from the neighboring pyramid on that side will not propagate onto the neighboring pyramid on the far side of the G pyramid, but will instead return to the neighboring pyramid whence it came. Hence, in order that an edge type be a reverser, and not exist on every tetrahedron in the cycle, it must turn around twice via two self-gluing tetrahedra (one at each U-turn). The two self-gluing tetrahedra cannot share a BBB face: if they did, there would be less than 180° of this reverse edge type on each tetrahedron and hence a total of less than the requisite 360° of this edge type. Hence, there must be at least one pyramid between these self-gluing tetrahedra in the cycle of pyramids. Thus, each reverser edge type contains at least eight components.

For $n \geq 5$, $2n-2 \geq 8$. Hence the number of components in either a traverser or reverser edge class is at least eight, making it impossible to reach the average of six components per BB edge class. When $n = 4$, $2n-2 = 6$ and hence we can construct a 4-cusped manifold out of six tetrahedra with three traverser edge types and no reverser edge types. Thus, the only cases in which we can have $2n - 2$ tetrahedra is with the 4-cusped 6-tetrahedral manifold, or with fewer cusps. Excepting those cases, we have our $2n - 1$ bound.

In the case that the manifold is orientable, assume we have $2n - 2$ (still possible for $n = 2$, 3, or 4) or $2n - 1$ tetrahedra. A BBB face on the bottom of a pyramid cannot be glued to the BBB face of any other pyramid by Lemma 4.3. Assuming more than two cusps, the pair of BBB faces on the bottom of a single pyramid cannot be identified to each other as that would disconnect the manifold. Hence, for more than two cusps, the manifold must consist of one type V $BBBB$ tetrahedron, and $n - 1$ pyramids.

All of the BBB faces on the bottoms of pyramids must glue to the $BBBB$ tetrahedron. This means there can be at most two pyramids and a total of three cusps. Therefore, we have at least $2n$ edge types for $n \geq 4$. In the case $n = 2$ or 3, we have to examine the possible gluings individually.

For $n = 2$, we could have either a single pyramid or we could have one pyramid and one $BBBB$ tetrahedron. The possible gluings are dictated by the orientability, the symmetry, and the arrangement of the cusps at the vertices. We can easily check that no gluing yields the requisite number of edge types.

For $n = 3$, we have two pyramids and one $BBBB$ tetrahedron. There must be a total of three BB edge types. Again, there are no gluings which yield this number of edge types. □

Theorem 4.7. *For $n = 5$, we have $\sigma(5) = 10$.*

Proof. The preceding theorem implies that we need only eliminate the possibility of a 5-cusped manifold constructed from nine tetrahedra. A careful and extended analysis of the dual graphs corresponding to the possible gluings is required to prove that no such gluing can succeed. Details appear in [8]. □

5. Uniqueness

Theorem 5.1. *There exists a unique 3-cusped hyperbolic 3-manifold composed of four ideal tetrahedra.*

Proof. Denote the three cusps by A, B, and C. Assume that there is a type II tetrahedron $ABCC$. Then the four edges AB, AC, BC, and CC are all the possible edges, since the number of edges equals the number of tetrahedra by an Euler characteristic argument. Furthermore, any tetrahedron with an AB edge must be an $ABCC$ tetrahedron in order to prevent the introduction of additional edge types. By Lemma 4.4, there must be at least four tetrahdra of this type.

We can assemble these four tetrahedra around the CC edge, obtaining an octahedron P such that each pair of antipodal vertices correspond to a cusp which is different from the cusps corresponding to any other antipodal pair. If a pair of faces on P sharing an edge were both glued to their opposite faces, we would obtain too many edge types in M. Hence, at most one pair of opposite faces are glued together. For the same reason, no face may glue to any of its neighbors. With these restrictions, in fact, no face may glue to its opposite face.

Pick two faces which share a vertex v but not an edge. Without loss of generality, we can assume that they will be glued to one another. The cusp labeling on their vertices completely determine their gluing. Their identification and our restrictions force the two faces which share the antipodal vertex but which do not share an edge with either of the first two faces to be glued to one another. Additionally, the remaining two faces sharing v cannot be identified or else we would end up with two BC edge types. The two possible ways to glue these two faces to the remaining unglued faces sharing the antipodal vertex are completely symmetric, and hence it does not matter which we choose. This gluing yields a valid manifold.

The other possibility is that there are no type II tetrahedra. If so, then there must be a pair of cusps not connected by any edges (or else we have AB, AC, BC, and some loop, such as AA, and then any tetrahedron with the BC edge must be an $AABC$ tetrahedron, which we have forbidden). So our edges are AC, BC, and two loops. There must be CC edges types or the manifold will be disconnected. Our tetrahedra are then two $ACCC$'s and two $BCCC$'s. We can note that an $ACCC$'s neighbor across the CCC face is a $BCCC$, so after gluing along the CCC faces, we have two pyramids with identical vertex labelings.

Now we have a total of six CC edges on our two pyramids, out of which we must form two edge classes. If we perform identifications on the ACC faces to make A a torus or Klein bottle cusps, the six CC edges fall into three CC edge classes, each containing two CC edges. When we perform the identifications on the BCC faces, two of these CC edge classes will be identified. Hence, there is a CC edge type which is made up of only one pair of CC edges from the two pyramids, or four tetrahedral edges, one from each tetrahedron. We can assemble our four tetrahedra around this edge class to obtain an octahedron. Discarding the interior CC edge of this octahedron and replacing it with an interior edge running from A to B allows us to cut the octahedron into four $ABCC$ tetrahedra. This puts us in the previously considered case. □

Theorem 5.2. *There exists a unique* 4-*cusped hyperbolic* 3-*manifold composed of exactly six ideal hyperbolic tetrahedra.*

Proof. We can assume that at least six tetrahedra are required, by Theorem 4.6. The example in Table 1 shows that there is at least one such manifold obtained from six ideal tetrahedra. We need only show that the example is unique.

Assume we have such a manifold with an ideal triangulation that includes at least one type I tetrahedron. Then the six distinct edge types in the triangulation all occur on this tetrahedron. Hence all six of the tetrahedra must be type I.

Label the cusps A, B, C, and D. Note that for any distinct pair of cusps, there must be exactly one edge type connecting the pair. We can assemble the six tetrahedra around the AB edge, obtaining an object as in Fig. 1. We henceforth call such an object a wheel.

We have to pair up the six ACD faces and the six BCD faces for gluing. No two adjacent faces can be identified, since no edge type can occur on just two tetrahedra. The two possible patterns of identification on one "hemisphere" are (1)

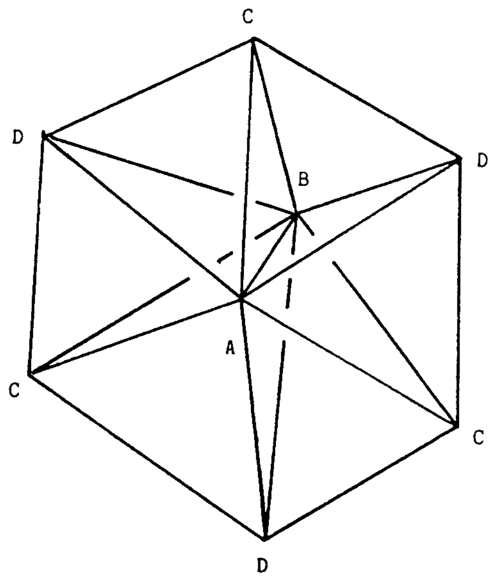


Fig. 1.

each face glues to its opposite and (2) one pair of opposite faces glue to each other and the other four glue to their opposite's free neighbor. The only combination of these patterns on the two hemispheres which yield only one CD edge is using pattern (2) on each hemisphere, such that the opposite-glued pairs are different. By symmetry of the object and gluings, there is only one such manifold.

Assume now that the triangulation of the 4-cusped manifold contains no type I tetrahedra but it does contain at least one type II tetrahedron. In the case $n = 4$, the proof of Theorem 4.6 shows that the number of tetrahedra would have to be at least seven and hence this case does not occur.

Finally, assume no tetrahedra of type I or II appear in the triangulation. In this case, the proof of Theorem 4.6 shows that the triangulation must consist only of type III tetrahedra. In particular, we can choose the cusp C so that the triangulation consists of two each of the tetrahedra labeled $ACCC$, $BCCC$, and $DCCC$.

The two $ACCC$ tetrahedra can be glued together along some ACC face, obtaining a "pyramid." There are only three ways to glue the remaining ACC faces so that the boundary of a neighborhood of A is a torus or Klein bottle. The same can be said for B and D. Furthermore, a wheel is formed by attaching the pyramids via the CCC faces. The only combination of gluings (up to symmetry) which does not yield a contradiction is obtainable by the following reassembly of the gluing obtained from the six type I tetrahedra in Fig. 1.

Beginning with the gluing from the type I tetrahedra, erase the "axis" AB edge. Add the three possible distinct CC edges and separate the wheel into two hemispheres, each of which decomposes into four tetrahedra. Separate all of these tetrahedra. Assemble the three tetrahedra with a BD edge around that edge, and likewise for the three with an AD edge. Then assemble our four polyhedra, two of which we just constructed and the two remaining tetrahedra, around any of one of the CC edge types, and we have the only valid gluing for the six type III tetrahedra. Hence, there is only one 4-cusped 6-tetrahedral manifold. □

6. Tables

In Tables 1 and 2 we give the correct values for $\sigma_{\mathrm{or}}(n)$ and $\sigma(n)$, some examples of corresponding manifolds, and a description of a gluing of ideal tetrahedra which

Table 1

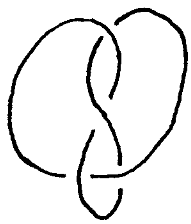
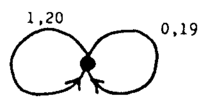
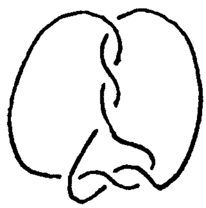
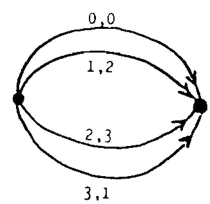
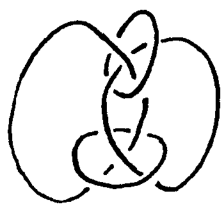
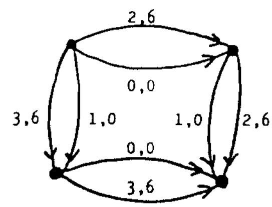
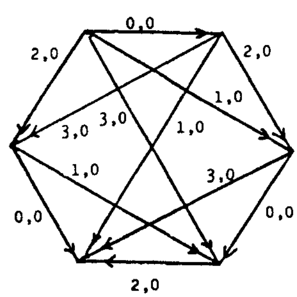
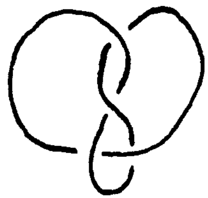
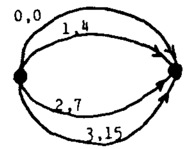
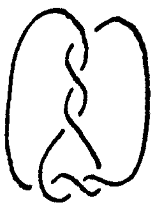
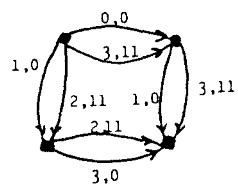
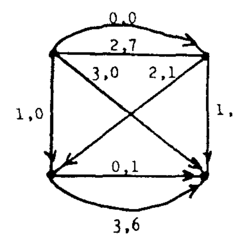
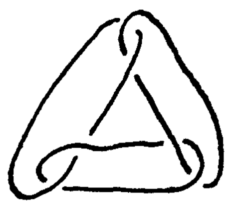
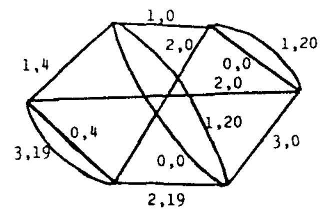
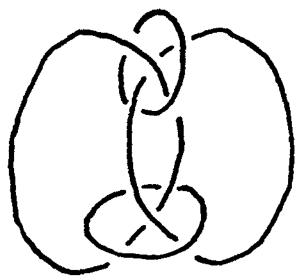
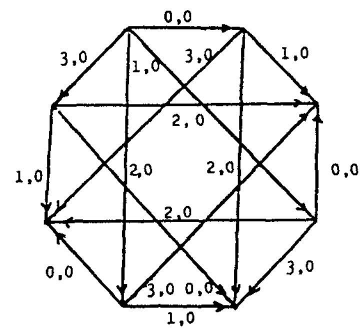
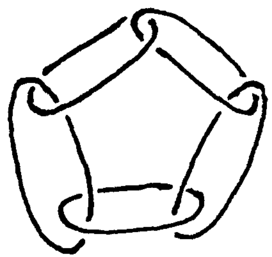
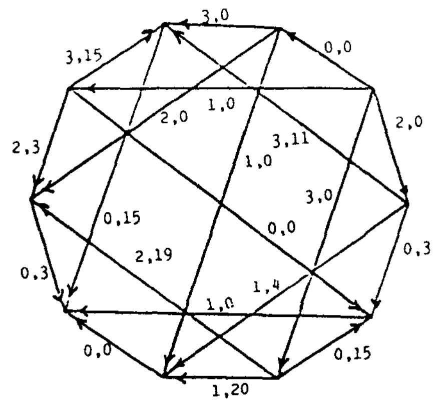
n	$\sigma(n)$	Manifolds		Gluing
1	1	Double covered by		
2	2	Double covered by		
3	4	Double covered by		
4	6	Double covered by		
5	10	See Table 2		

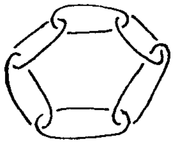
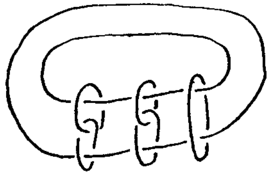
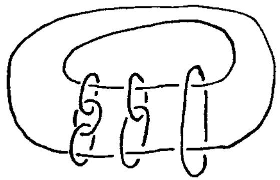
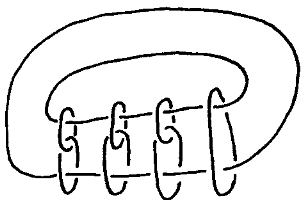
Table 2

n	$\sigma_{or}(n)$	Manifolds	Gluing
1	2		
2	4	 	 
3	6		

(*continued*)

Table 2 (*continued*)

n	$\sigma_{or}(n)$	Manifolds	Gluing
4	8		
5	10		

6	16?	
7	20?	
8	26?	
9	30?	

will yield the corresponding manifold. The method we employ to describe a gluing is based on a description of gluings due to Thurston. Each vertex in the oriented graph describing the gluing represents a tetrahedron and each edge represents a pair of faces which are glued together. The specific gluings are represented by the numbers indicated for each edge, in the following manner.

Given a tetrahedral representation of a three-manifold, we wish to represent it unambiguously and concisely via a graph. We number the vertices of each tetrahedron 0, 1, 2, and 3, and the faces of the tetrahedron are numbered as the vertices opposite them. The first step is to find a "base" tetrahedron, and to number its vertices. This is done by finding that edge type in the tetrahedralization which has the fewest tetrahedral edges as its members. The base tetrahedron is to have this edge type as the edge connecting vertices 0 and 1. Given this restriction, the next criterion is that the edge type of the edge connecting vertices 0 and 2 is to have as few members as possible, and so on, using the edges from 0 to 3, 1 to 2, 1 to 3, and 2 to 3. If these criteria yield a unique base tetrahedron with a unique numbering of its vertices, all well and good. Otherwise, using the criteria as much as possible, we simply choose a base tetrahedron from among the candidates. In the case when the base tetrahedron is not uniquely determined, the resulting graph representation of the gluing may not be unique.

We begin with the base tetrahedron, and list the four faces in order, as the first four elements in a queue of faces whose gluings are to be represented. We go through the queue, representing the gluings. If a gluing to be represented involves a tetrahedron not yet numbered and plotted, we number its vertices such that the numbers on the face involved match exactly with those of the face to which it is being glued. Furthermore, the free faces of the new tetrahedron are added to the end of our queue of faces to be glued.

In any case, the gluing is described by drawing a directed edge from the graph vertex representing the source tetrahedron to that of the other, and two numbers are specified. The first is the number of the face on the source tetrahedron which is involved in the gluing. The second represents the actual arrangement of how the vertices are identified. If we imagine matching the vertices on the faces, and also pairing the one omitted vertex, which is the name of the involved face, from the first tetrahedron with that of the second tetrahedron, we would have a permutation on the numbers of 0–3. These permutations are numbered in this way: 3210–3210 is gluing "0," 3210–3201 is gluing "1," and so on, in reverse lexicographical order, down to gluing "23" which is 3210–0123. Given one such permutation and the number of the face on the first tetrahedron, we can reproduce which vertices are identified to which. Note that the gluing by which any tetrahedron is first included, i.e., by which it is first given the numbering on its vertices, will be gluing "0."

For example, if an edge is labeled "3, 12," then we look up permutation 12 and find that it is 3210–1320. Thus we have face 3 of the first tetrahedron gluing to face 1 of the second, and the vertices are identified so: 0–0, 1–2, and 2–3. This information, along with the knowledge of the queue nature of gluings, allows a reconstruction of the triangulation from such a graph.

References

1. C. Adams, Thrice-punctured spheres in hyperbolic 3-manifolds, *Trans. Amer. Math. Soc.*, vol. 287, no. 2, Feb. 1985, pp. 645–656.
2. C. Adams, Augmented alternating link complements are hyperbolic, in *Low-Dimensional Topology and Kleinian Groups*, ed. D. Epstein, London Mathematical Society Lecture Notes, No. 112, Cambridge University Press, Cambridge, 1986, pp. 115–130.
3. C. Adams, The noncompact hyperbolic 3-manifold of minimal volume, *Proc. Amer. Math. Soc.*, vol. 100, no. 4, Aug. 1987, pp. 601–606.
4. C. Adams, Volumes of N-cusped hyperbolic 3-manifolds, *J. London Math. Soc*, vol. 38, 1988, pp. 555–565.
5. C. Adams, M. Hildebrand, and J. Weeks, Hyperbolic invariants of knots and links, *Trans. Amer. Math. Soc.*, to appear.
6. D. B. A. Epstein and R. C. Penner, Euclidean decompositions of noncompact hyperbolic manifolds, *J. Differential Geom.*, vol. 27, no. 1, Jan. 1988, pp. 67–80.
7. M. Hildebrand and J. Weeks, A Census of Cusped Hyperbolic Manifolds Obtained from 5 or Fewer Ideal Tetrahedra, Preprint, Geometry Supercomputer Project, Minnesota, 1989.
8. W. Sherman, Hyperbolic Manifolds from Ideal Tetrahedra, Undergraduate thesis, Williams College, 1988.
9. W. Thurston, The Geometry and Topology of 3-Manifolds, Lecture notes, Princeton University, 1978.
10. W. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.*, vol. 6, no. 3, May 1982, pp. 357–381.
11. W. Thurston, Hyperbolic structures on 3-manifolds, I: deformations of acylindrical manifolds, *Ann. of Math.*, vol. 124 1986, pp. 203–246.
12. W. Thurston, private communication.
13. W. Thurston, R. Tarjan, and D. Sleator, Rotation distance, triangulations and hyperbolic geometry, *J. Amer. Math. Soc.*, vol. 1, no. 3, July 1988, pp. 647–681.
14. J. Weeks, Hyperbolic Structures on Three-Manifolds, Ph.D. thesis, Princeton University, 1985.

Received July 15, 1988, and in revised form March 10, 1989.